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GENERALIZED FIXED-POINT ALGEBRAS OF CERTAIN ACTIONS ON CROSSED PRODUCTS

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Let G and H be two locally compact groups acting on a C*-algebra A by commuting actions λ and σ. We construct an action on $A \times_\lambda G$ out of σ and a unitary cocycle u. For A commutative, and free and proper actions λ and σ, we show that if the roles of λ and σ are reversed, and u is replaced by u^*, then the corresponding generalized fixed-point algebras, in the sense of Rieffel, are strong-Morita equivalent. This fact turns out to be a generalization of Green's result on the strong-Morita equivalence of the algebras $C_0(M/H) \times_\lambda G$ and $C_0(M/G) \times_\sigma H$. Finally, we use the Morita equivalence mentioned above to compute the K-theory of quantum Heisenberg manifolds.

Introduction.

Given two commuting actions λ and σ of locally compact groups G and H, respectively, on a C*-algebra A, we study the action $\gamma^{\sigma,u}$ of H on $A \times_\lambda G$ defined by

$$\left(\gamma_h^{\sigma,u}\Phi\right)(x) = u(x,h)\sigma_h(\Phi(x)),$$

where $\Phi \in C_c(G,A)$, $h \in H$, $x \in G$, $u(x,h)$ is a unitary element of the center of the multiplier algebra of A, and u satisfies the cocycle conditions

$$u(x_1x_2,h) = u(x_1,h)\lambda_{x_1}(u(x_2,h)) \quad \text{and} \quad u(x,h_1h_2) = u(x,h_1)\sigma_{h_1}(u(x,h_2)).$$

The study of this situation was originally motivated by the example of quantum Heisenberg manifolds ([**Rf5**]), which can be described as the generalized fixed-point algebras ([**Rf4**]) of actions of this form, when $A = C_0(R \times T)$, and $G = H = Z$.

This work is organized as follows. In Section 1 we define the action $\gamma^{\sigma,u}$ and show that for G and H second countable, and A separable, the crossed product $A \times_\lambda G \times_{\gamma^{\sigma,u}} H$ is isomorphic to a certain twisted crossed product of the algebra A by the group $G \times H$.

In Section 2 we assume that the algebra A is commutative and show that for free and proper actions λ and σ, the generalized fixed-point algebra of $A \times_\lambda G$ under $\gamma^{\sigma,u}$ and that of $A \times_\sigma H$ under γ^{λ,u^*} are strong-Morita equivalent.

In Section 3 we apply these results to show that the K-groups of the quantum Heisenberg manifolds do not depend on the deformation constant. This enables us to compute them, by calculating them in the commutative case.

In what follows, for a C*-algebra A, $\mathcal{M}(A)$ denotes its multiplier algebra, $\mathcal{Z}(A)$ its center, and $\mathcal{U}(A)$ the group of unitary elements in A. All actions of locally compact groups on C*-algebras are assumed to be strongly continuous. All integrations on a group G are with respect to a fixed left Haar measure μ_G with modular function Δ_G.

1. Actions on crossed products.

For locally compact groups G and H acting on a C*-algebra A by commuting actions λ and σ, respectively, and a cocycle on $G \times H$, we define an action $\gamma^{\sigma,u}$ of H on $A \times_\lambda G$. We show in Proposition 1.3 that, when A is separable, and G and H are second-countable, the crossed product $A \times_\lambda G \times_{\gamma^{\sigma,u}} H$ is a twisted crossed product of A by $G \times H$.

Proposition 1.1. *Let G be a group acting on a C*-algebra A by an action λ, and let $v : G \to \mathcal{UZM}(A)$ verify the cocycle condition*

$$v(xy) = v(x)\lambda_x(v(y)).$$

Let $\sigma \in \mathrm{Aut}(A)$ commute with λ, and, for $\Phi \in C_c(G, A)$, define

$$(\gamma^{\sigma,v}\Phi)(x) = v(x)\sigma(\Phi(x)).$$

Then $\gamma^{\sigma,v}$ extends to an automorphism on $A \times_\lambda G$.

Proof. Let (Π, V) be a covariant representation of the C*-dynamical system $C^*(G, A, \lambda)$ on a Hilbert space $\mathcal{H}$, and let $\Pi \times U$ denote its integrated form. Let Π^σ denote the representation of A on $\mathcal{H}$ defined by $\Pi^\sigma(a) = \Pi(\sigma(a))$, and let $\tilde{V}$ be the unitary representation of G on $\mathcal{H}$ given by $\tilde{V}_x = \Pi(v(x))V_x$, where Π also denotes its extension to $\mathcal{M}$. Then $(\Pi^\sigma, \tilde{V})$ is a covariant representation of $C^*(G, A, \lambda)$: for $x \in G$, and $a \in A$ we have

$$\begin{aligned}\tilde{V}_x\Pi^\sigma(a)\tilde{V}_{x^{-1}} &= \Pi(v(x))V_x\Pi(\sigma(a))\Pi(v(x^{-1}))V_{x^{-1}} \\ &= \Pi(v(x))\Pi(\lambda_x\sigma(a))V_x\Pi(v(x^{-1}))V_{x^{-1}} \\ &= \Pi(v(x))\Pi(\sigma\lambda_x(a))\Pi(\lambda_x v(x^{-1})) = \Pi^\sigma(\lambda_x(a)).\end{aligned}$$

We now show that for Φ in $C_c(G,A)$ we have that $(\Pi \times V)(\gamma^{\sigma,v}\Phi) = (\Pi^\sigma \times \tilde{V})(\Phi)$, which ends the proof: for any ξ in $\mathcal{H}$, we have

$$\begin{aligned}[(\Pi \times V)(\gamma^{\sigma,v}\Phi)](\xi) &= \int_G \Pi[(\gamma^{\sigma,v}\Phi)(x)]V_x\xi dx \\ &= \int_G \Pi(v(x))\Pi[(\sigma(\Phi(x))]V_x\xi dx \\ &= \int_G \Pi^\sigma[\Phi(x)]\tilde{V}_x\xi dx = [(\Pi^\sigma \times \tilde{V})(\Phi)](\xi).\end{aligned}$$

□

Proposition 1.2. *Assume that G, λ, and A are as in Proposition 1.1 and that H is a locally compact group acting on A by an action σ commuting with λ. Let*

$$u : G \times H \to \mathcal{U}ZM(A)$$

be continuous for the strict topology in $\mathcal{M}(A)$, and satisfy

$$u(xy,h) = u(x,h)\lambda_x u(y,h) \quad \text{and} \quad u(x,hg) = u(x,h)\sigma_h u(x,g),$$

for $x,y \in G$ and $h,g \in H$. For $h \in H$ and $\Phi \in C_c(G,A)$, let

$$\left(\gamma_h^{\sigma,u}\Phi\right)(x) = u(x,h)\sigma_h(\Phi(x)).$$

Then $h \mapsto \gamma_h$ is a (strongly continuous) action of H on $A \times_\lambda G$.

Proof. By Proposition 1.1 we have that $\gamma_h^{\sigma,u} \in \mathrm{Aut}(A \times_\lambda G)$, for all $h \in H$. Besides, the cocycle condition implies that $\gamma_{h_1h_2}^{\sigma,u}\Phi(x) = \gamma_{h_1}^{\sigma,u}\gamma_{h_2}^{\sigma,u}\Phi(x)$. Finally, $h \mapsto \gamma_h^{\sigma,u}\Phi$ is continuous for any $\Phi \in C_c(G,A)$:

$$\left\|\gamma_h^{\sigma,u}\Phi - \gamma_{h_0}^{\sigma,u}\Phi\right\|_{A\times_\lambda G} \leq \left\|\gamma_h^{\sigma,u}\Phi - \gamma_{h_0}^{\sigma,u}\Phi\right\|_{L^1(G,A)}$$

$$= \int_G \left\|u(x,h)\sigma_h(\Phi(x)) - u(x,h_0)\sigma_{h_0}(\Phi(x))\right\|_A dx \leq$$

$$\begin{aligned}\leq \int_{\mathrm{supp}(\Phi)} &\left\|\sigma_h(\Phi(x)) - \sigma_{h_0}(\Phi(x))\right\|_A \\ &+ \left\|(u(x,h) - u(x,h_0))\sigma_{h_0}(\Phi(x))\right\|_A dx,\end{aligned}$$

which converges to 0 when h goes to h_0, because u is continuous, and σ is strongly continuous. □

Next Proposition shows that the double crossed product $A \times_\lambda G \times_{\gamma^{\sigma,u}} H$ is isomorphic to a twisted crossed product. Since twisted crossed products are defined for separable algebras and second-countable groups, we add these conditions.

Proposition 1.3. *Let G, H, A, u, λ, σ and $\gamma^{\sigma,u}$ be as in Proposition* 1.2. *If A is separable and H and G are second-countable, then $A \times_\lambda G \times_{\gamma^{\sigma,u}} H$ is isomorphic to the twisted crossed product $A \times_{(\lambda,\sigma),U} (G \times H)$, where*

$$(\lambda,\sigma)_{(x,h)}(a) = \lambda_x\sigma_h(a) \quad \textit{and} \quad U((x_0,h_0),(x_1,h_1)) = \lambda_{x_0}(u(x_1,h_0)).$$

Proof. First notice that $((\lambda,\sigma),U)$ is a twisted action of $G\times H$ on A: conditions a), b) and c) in [**PR**, Def. 2.1] are easily checked, and, for (x_0,h_0), (x_1,h_1), and (x_2,h_2) in $G\times H$, we have

$$\begin{aligned}(\lambda,\sigma)_{(x_0,h_0)}&[U((x_1,h_1),(x_2,h_2))]U((x_0,h_0),(x_1x_2,h_1h_2))\\ &= \lambda_{x_0}\sigma_{h_0}\lambda_{x_1}(u(x_2,h_1))\lambda_{x_0}(u(x_1x_2,h_0))\\ &= \lambda_{x_0x_1}(u(x_2,h_0h_1))\lambda_{x_0}(u(x_1,h_0)\\ &= U((x_0x_1,h_0h_1),(x_2,h_2))U((x_0,h_0),(x_1,h_1)).\end{aligned}$$

We now construct maps

$$i_A : A \to \mathcal{M}(A\times_\lambda G\times_{\gamma^{\sigma,u}} H)$$

and

$$i_{G\times H} : G\times H \to \mathcal{U}M(A\times_\lambda G\times_{\gamma^{\sigma,u}} H)$$

satisfying

$$i_A((\lambda,\sigma)_{(x,h)}(a)) = i_{G\times H}(x,h)i_A(a)i_{G\times H}(x,h)^* \quad \text{and}$$

$$i_{G\times H}(x_0,h_0)i_{G\times H}(x_1,h_1) = i_A(U((x_0,h_0),(x_1,h_1)))i_{G\times H}(x_0x_1,h_0h_1),$$

for all $x_i\in G$, $h_i\in H$, and $a\in A$.

If α is an action of a group K on a C*-algebra B, $b\in\mathcal{M}(B)$, and μ is a bounded complex Radon measure with compact support on G, , let $M(b,\mu)$ denote the multiplier of $B\times_\alpha K$ defined by

$$(M(b,\mu)f)(t) = b\int_K \alpha_s(f(s^{-1}t))d\mu(s),$$

for $f\in C_c(K,B)$.

Now define

$$i_A(a) = M(M(a,\delta_{1_G}),\delta_{1_H}) \quad \text{and} \quad i_{G\times H}(x,h) = M(M(1_A,\delta_x),\delta_h),$$

where δ_t denotes the point mass at t.

For $f \in C_c(G \times H, A)$, explicit formulas are given by:

$$(i_A(a)f)(x,h) = af(x,h), \text{and}$$

$$(i_{G\times H}(x_0,h_0)f)(x,h) = u^*(x_0,h_0)u(x,h_0)\lambda_{x_0}\sigma_{h_0}(f(x_0^{-1}x, h_0^{-1}h)).$$

It follows that

$$(i^*_{G\times H}(x_0,h_0)f)(x,h) = u(x,h_0^{-1})\sigma_{h_0^{-1}}\lambda_{x_0^{-1}}(f(x_0x, h_0h)).$$

The pair $(i_A, i_{G\times H})$ is covariant:

$$\begin{aligned}(i_{G\times H}&(x_0,h_0)i_A(a)i^*_{G\times H}(x_0,h_0)f)(x,h)\\ &= u^*(x_0,h_0)u(x,h_0)\lambda_{x_0}\sigma_{h_0}\left[au\left(x_0^{-1}x,h_0^{-1}\right)\sigma_{h_0^{-1}}\lambda_{x_0^{-1}}(f(x,h))\right]\\ &= (i_A(\lambda_{x_0}\sigma_{h_0}(a))f)(x,h),\end{aligned}$$

and

$$(i_{G\times H}(x_0,h_0)i_{G\times H}(x_1,h_1))(x,h)$$

$$\begin{aligned}&= u^*(x_0,h_0)u(x,h_0)\\ &\qquad \cdot \lambda_{x_0}\sigma_{h_0}\left[u^*(x_1,h_1)u\left(x_0^{-1}x,h_1\right)\lambda_{x_1}\sigma_{h_1}\left(f\left(x_1^{-1}x_0^{-1}x, h_1^{-1}h_0^{-1}h\right)\right)\right]\\ &= \lambda_{x_0}u(x_1,h_0)u^*(x_0x_1,h_0h_1)u(x,h_0h_1)\lambda_{x_0x_1}\sigma_{h_0h_1}\left(f\left(x_1^{-1}x_0^{-1}x, h_1^{-1}h_0^{-1}h\right)\right)\\ &= U((x_0,h_0),(x_1,h_1))i_{G\times H}((x_0x_1,h_0h_1)f)(x,y).\end{aligned}$$

We next show that for any covariant representation (Π, V) of

$$(A, G\times H, (\lambda,\sigma), U)$$

on a Hilbert space $\mathcal{H}$ there is an integrated form $\Pi \times V$ on $A \times_\lambda G \times_{\gamma^{\sigma,u}} H$. Let V_G and V_H be the restrictions of V to G and H, respectively. Then (Π, V_G) is a covariant representation of (A, G, λ) and, if $\Pi \times V_G$ denotes its integrated form, then $(\Pi \times V_G, V_H)$ is a covariant representation of $(A \times_\lambda G, H, \gamma^{\sigma,u})$. So $\Pi \times V_G \times V_H$ is a non-degenerate representation of $A \times_\lambda G \times_{\gamma^{\sigma,u}} H$ and

$$\Pi = \Pi \times V_G \times V_H \circ i_A \text{ and } V = \Pi \times V_G \times V_H \circ i_{G\times H}.$$

Finally, the set $\{i_A \times i_{G\times H}(f) : f \in L^1(G\times H, A)\}$, where

$$[i_A \times i_{G\times H}(f)](x,h) = \int_{G\times H} i_A[f(x,h)]i_{G\times H}(x,h)d(x,y)$$

is a dense subspace of $A \times_\lambda G \times_{\gamma^{\sigma,u}} H$, which ends the proof. □

Remark 1.4. Iain Raeburn pointed out to me how a simple proof of a weaker version of Theorem 2.12 can be obtained by using Proposition 1.3. If in Proposition 1.3 the roles of λ and σ are reversed and u is replaced by u^*, then we have that $A \times_\sigma H \times_{\gamma^{\lambda,u^*}} G$ is isomorphic to the twisted crossed product $A \times_{(\lambda,\sigma),W} (G \times H)$, where $W((x_0,h_0),(x_1,h_1)) = \sigma_{h_0}(u^*(x_0,h_1))$.

Now, a straightforward computation shows that the twisted actions $((\lambda,\sigma),U)$ and $((\lambda,\sigma),W)$ of $G\times H$ on A are exterior equivalent ([**PR**, 3.1]), the equivalence being implemented by the cocycle u.

Thus, under the assumptions of Proposition 1.3 the algebras

$$A \times_\lambda G \times_{\gamma^{\sigma,u}} H$$

and

$$A \times_\sigma H \times_{\gamma^{\lambda,u^*}} G$$

are isomorphic ([**PR**, 3.3]). This proves Theorem 2.12 when A is separable and G and H are amenable second countable groups.

2. The generalized fixed-point algebras.

With the example of quantum Heisenberg manifolds in mind, we now discuss the situation described in Section 1 in the case of some particular actions λ and σ on a commutative C*-algebra $C_0(M)$. We prove that if the action σ is proper, then so is $\gamma^{\sigma,u}$ (in the sense of [**Rf4**]), and that if σ is also free then $\gamma^{\sigma,u}$ is saturated ([**Rf4**]). Besides, for λ and σ free and proper, the generalized fixed-point algebras under $\gamma^{\sigma,u}$ and γ^{λ,u^*} respectively are strong-Morita equivalent.

More specifically, we show that the space $C_c(M)$ can be made into a dense submodule of an equivalence bimodule for the generalized fixed-point algebras. Part of this is done by adapting to our situation the techniques of [**Rf3**, Situation 10].

Assumptions and notation. Throughout this section M denotes a locally compact Hausdorff space, and βM its Stone-Cech compactification. The groups G and H act on M by commuting actions λ and σ, respectively. In this context, if T denotes the unit circle, the cocycle u of Section 1 consists of continuous functions $u(x,h): M \to T$, for $(x,h) \in G \times H$, such that, for any $f \in C_0(M)$ the map $(x,h) \to u(x,h)f$ is continuous for the supremum norm. As in Section 1 we require the cocycle conditions:

$$u(x_1x_2,h) = u(x_1,h)\lambda_{x_1}u(x_2,h) \quad\text{and}\quad u(x,h_1h_2) = u(x,h_1)\sigma_{h_1}u(x,h_2),$$

for $x, x_i \in G$ and $h, h_i \in H$. Notice that if these conditions are satisfied for u they also hold for u^*. We denote by $\gamma^{\sigma,u}$ and γ^{λ,u^*} the actions of H and G on $C_0(M) \times_\lambda G$ and $C_0(M) \times_\sigma H$ respectively, as defined in Proposition 1.2.

Proposition 2.1. *In the notation above, if σ is proper, so is the action $\gamma^{\sigma,u}$ of H on $C_0(M) \times_\lambda G$. The generalized fixed-point algebra $D^{\sigma,u}$ of $C_0(M) \times_\lambda G$ under $\gamma^{\sigma,u}$ consists of the closure in $\mathcal{M}(C_0(M) \times_\lambda G)$ of the linear span of the set $\{P_{\sigma,u}(E^* * F) : E, F \in C_c(M \times G)\}$, where $P_{\sigma,u}$ denotes the linear map $P_{\sigma,u} : C_c(M \times G) \to \mathcal{M}(C_0(M) \times_\lambda G)$ defined by*

$$(P_{\sigma,u}(F))(m,x) = \int_H (\gamma_h^{\sigma,u}(F))(m,x)dh,$$

for $F \in C_c(M \times G)$, and $(m,x) \in M \times G$.

Furthermore, $P_{\sigma,u}$ satisfies

i) $P_{\sigma,u}(F^*) = P_{\sigma,u}(F)^*$.

ii) $P_{\sigma,u}(F) \geq 0$, *for $F \geq 0$, where F and $P_{\sigma,u}(F)$ are viewed as elements of $\mathcal{M}(C_0(M) \times_\lambda G)$.*

iii) $P_{\sigma,u}(F * \Phi) = P_{\sigma,u}(F) * \Phi$ *and* $P_{\sigma,u}(\Phi * F) = \Phi * P_{\sigma,u}(F)$, *for any $\Phi \in \mathcal{M}(C_0(M) \times_\lambda G)$ carrying $C_c(M \times G)$ into itself and such that $\gamma_h^{\sigma,u}(\Phi) = \Phi$ for any $h \in H$.*

Proof. We check conditions 1) and 2) of [**Rf4**, Def. 1.2]. Let $B = C_c(M \times G)$. Then B is a dense *-subalgebra of $C_0(M) \times_\lambda G$, and it is invariant under $\gamma^{\sigma,u}$.

We now show that, for $E, F \in B$, the map $h \to E * \gamma_h^{\sigma,u}(F^*)$ is in $L^1(H, C_0(M) \times_\lambda G)$. For $(m,x) \in M \times G$ we have

$$\begin{aligned}&[E * \gamma_h^{\sigma,u}(F^*)]\,(m,x)\\ &\quad= \int_G E(m,y)\,[u(y^{-1}x,h)]\,(\lambda_{y^{-1}}m)\overline{F}\,(\lambda_{x^{-1}}\sigma_{h^{-1}}m, x^{-1}y)\,\Delta_G(x^{-1}y)dy.\end{aligned}$$

Since σ is proper and $\mathrm{supp}(E)$ and $\mathrm{supp}(F)$ are compact, then the set

$$\{h \in H : \sigma_{h^{-1}}\lambda_{x^{-1}}m \in \mathrm{supp}_M(F)$$

$$\text{for } (m,x) \in \mathrm{supp}_M(E) \times \mathrm{supp}_G(E)\mathrm{supp}_G(F)^{-1}\}$$

is compact. Therefore $h \to E * \gamma_h^{\sigma,u}(F^*)$ and $h \to \Delta_H^{-1/2}(h)E * \gamma_h^{\sigma,u}(F^*)$ are in $C_c(H,B) \subseteq L^1(H, \mathcal{M}(C_0(M) \times_\lambda G))$.

For $F \in B$ and $m_0 \in M$, let N be a neighborhood of m_0 with compact closure. Then there exists a compact set K in H such that

$$P_{\sigma,u}(F)(m,x) = \int_K (\gamma_h^{\sigma,u}F)\,(m,x)dh,$$

for all $(m,x) \in N \times G$, which shows that $P_{\sigma,u}(F)$ is continuous. Since $\mathrm{supp}_G(P_{\sigma,u}(F))$ is compact, then $P_{\sigma,u}(F)$ is bounded on $\mathrm{supp}_M(F) \times G$. Besides, for all $(m,x) \in M \times G$ and $h \in H$, we have $|P_{\sigma,u}F(m,x)| = |P_{\sigma,u}F(\sigma_h m, x)|$, and $\mathrm{supp}_M(P_{\sigma,u}(F)) \subset \sigma_H(\mathrm{supp}_M(F))$.

Therefore $P_{\sigma,u}(F) \in C_c(\beta M \times G) \subseteq \mathcal{M}(C_0(M) \times_\lambda G)$, and, as a multiplier, $P_{\sigma,u}(F)$ carries B into itself.

Notice now that the fact that $h \to E * \gamma_h^{\sigma,u}(F)$ is in $L^1(H, C_0(M) \times_\lambda G)$ implies that the integral $\int_H \gamma_h^{\sigma,u}(F)dh$ makes sense as an integral in the completion of $\mathcal{M}(C_0(M) \times_\lambda G)$, viewed as a locally convex linear space, for the topology induced by the set of seminorms $\{\| \ \|_F : F \in B\}$, where

$$\|\Phi\|_F = \|F * \Phi\|_{C_0(M)\times_\lambda G} + \|\Phi * F\|_{C_0(M)\times_\lambda G}$$

for $\Phi \in \mathcal{M}(C_0(M) \times_\lambda G)$.

A straightforward application of Fubini's theorem shows that

$$\int_H (E * \gamma_h^{\sigma,u}(F))(m,x)dh = (E * P_{\sigma,u}(F))(m,x),$$

for any E, $F \in B$, $(m,x) \in M \times G$, and it follows that

$$\int_H \gamma_h^{\sigma,u}(F)dh = P_{\sigma,u}(F),$$

in the sense mentioned above.

Also, since the positive cone is closed, and involution and the extension of $\gamma^{\sigma,u}$ are continuous for the topology of $\mathcal{M}(C_0(M) \times_\lambda G)$ defined above, $P_{\sigma,u}$ satisfies i), ii), and iii) stated above.

Set now $\langle E, F\rangle_\sigma = P_{\sigma,u}(E^* * F)$, for E, $F \in B$. We have shown that $\gamma^{\sigma,u}$ is proper. The generalized fixed-point algebra $D^{\sigma,u}$ ([**Rf4**, Def.1.4]) of $C_0(M) \times_\lambda G$ under $\gamma^{\sigma,u}$ consists of the closure in $\mathcal{M}(C_0(M) \times_\lambda G)$ of the linear span of the set $\{\langle E, F\rangle_\sigma : E, F \in B\}$. □

Lemma 2.2. *Assume that σ is proper and let $\{\Phi_{N,\epsilon,K}\}$ be a net in $C_c(M \times G \times H)$, indexed by decreasing neighborhoods N of $1_{G\times H}$, decreasing $\epsilon > 0$, and increasing compact subsets K of M, satisfying*

i) $\mathrm{supp}_{G\times H}(\Phi_{N,\epsilon,K}) \subset N$

ii) $|\int_{G\times H} \Phi_{N,\epsilon,K}(m,x,h)dxdh - 1| < \epsilon$, *for all* $m \in K$

iii) *There exists a real number R such that*

$$\int_{G\times H} |\Phi_{N,\epsilon,K}(m,x,h)|dxdh \leq R,$$

for all $m \in K$, and for all K, ϵ and N.

Then $\{\Phi_{N,\epsilon,K}\}$ is an approximate identity for $C_c(M \times G \times H) \subset C_0(M) \times_\lambda G \times_{\gamma^{\sigma,u}} H$ in the inductive limit topology.

Proof. Let $\psi \in C_c(M \times G \times H)$ and $\delta > 0$ be given. Then

$$\begin{aligned}
&|(\Phi_{N,\epsilon,K} * \Psi - \Psi)(m,x,h)| \\
&\leq \Big| \int_{H\times G} [u^*(y,k)(m)u(x,k)(m) - 1] \\
&\qquad \Phi_{N,\epsilon,K}(m,y,k)\Psi\left(\sigma_{k^{-1}}\lambda_{y^{-1}}m, y^{-1}x, k^{-1}h\right) dkdy \Big| \\
&+ \Big| \int_{H\times G} \Phi_{N,\epsilon,K}(m,y,k)dydk - 1 \Big| \, |\Psi(m,x,h)| \\
&+ \Big| \int_{H\times G} \Phi_{N,\epsilon,K}(m,y,k)\left[\Psi\left(\sigma_{k^{-1}}\lambda_{y^{-1}}m, y^{-1}x, k^{-1}h\right) - \Psi(m,x,h)\right] dydk \Big| \\
&\leq \delta,
\end{aligned}$$

for appropriate choices of ϵ and N. □

Proposition 2.3. *If the action σ is free and proper, then $\gamma^{\sigma,u}$ is saturated.*

Proof. Let J denote the ideal of $C_r^*(H, C_0(M \times_\lambda G))$ consisting of maps $h \mapsto \Delta_H^{-1/2}(h)E * \gamma_h^{\sigma,u}(F^*)$, for $E, F \in C_c(M \times G)$. In order to show that J is dense in $C_r^*(H, C_0(M) \times_\lambda G)$ we prove that J contains an approximate identity for $C_c(M \times G \times H)$.

Let N, ϵ, and K as in Lemma 2.2 be given. We assume without loss of generality that the closure of N is compact. Fix an open set U with compact closure such that $K \subset U$. Choose neighborhoods N_G and N_H of 1_G and 1_H, respectively, such that $N_G \times N_H \subset N$, $|\Delta_G(x) - 1| < \epsilon_1$ for all $x \in N_G$ and $|u^*(y,h)(m)u(x,h)(m) - 1| < \epsilon_2$, for all $h \in N_H, m \in U$, $x, y \in V$, V being a fixed open set with compact closure containing N_G, and for some ϵ_1 and ϵ_2 to be chosen later.

The action of $G \times H$ on $M \times G$ defined by $(x,h)(m,y) = (\lambda_x\sigma_h m, xy)$ is free and proper, so for each $(m,y) \in K \times \overline{N_G}$ we can choose ([**Rf3**, Situation 10]) a neighborhood $U_{(m,y)} \subset U \times V$ of (m,y) such that

$$\{(x,h) : (x,h)(U_{(m,y)}) \cap U_{(m,y)} \neq \emptyset\} \subset N_G \times N_H.$$

Take a finite subcover $\{U_1, U_2, ..., U_n\}$ of $\{U_{(m,y)}\}_{(m,y)\in K\times\overline{N_G}}$ and, for each $i = 1, ..., n$, let $F_i \in C_c^+(M \times G)$ be such that $\mathrm{supp}(F_i) \subset U_i$, and

$$\int_G \sum_i F_i(m,x)dx = 1$$

for all $m \in K$.

Now we can find ([**Rf3**, Situation 10]) functions $G_i \in C_c^+(M \times G)$ such that $\operatorname{supp}(G_i) \subset \operatorname{supp}(F_i)$, and

$$\left| F_i(m,y) - G_i(m,y) \int_{G \times H} G_i(\lambda_{x^{-1}}\sigma_{h^{-1}}m, x^{-1}y)\,dx\,dh \right| < \epsilon_3,$$

for all $(m,y) \in M \times G$, and some ϵ_3 to be chosen later.

Now set

$$\Phi_{N,\epsilon,K}(m,x,h) = \sum_i \Delta_H^{-1/2}(h) G_i * \gamma_h^{\sigma,u}(G_i^*)(m,x).$$

Then,

$$\begin{aligned}
&\left| \int_{H \times G} \Phi_{N,\epsilon,K}(m,x,h)\,dx\,dh - 1 \right| \\
&\quad = \sum_i \int_{G \times G \times H} \Delta_G(x^{-1}y)[u^*(y,h)u(x,h)](m) G_i(m,y) \\
&\qquad\qquad \cdot G(\lambda_{x^{-1}}\sigma_{h^{-1}}m, x^{-1}y)\,dx\,dy\,dh \\
&\qquad\qquad\qquad - \sum_i \int_G F_i(m,y)\,dy \\
&\quad \leq \left| \sum_i \int_V \Big(\quad [u^*(y,h)(m)u(x,h)(m)\Delta_G\,(x^{-1}y) - 1] \right. \\
&\qquad\qquad \left. G_i(m,y) \int_{G \times H} G_i\,(\lambda_{x^{-1}}\sigma_{h^{-1}}m, x^{-1}y)\,dx\,dh \Big)\,dy \right| \\
&\quad + \left| \sum_i \int_V G_i(m,y) \int_{G \times H} G_i\,(\lambda_{x^{-1}}\sigma_{h^{-1}}m, x^{-1}y)\,dx\,dh - F_i(m,y)\,dy \right| < \epsilon,
\end{aligned}$$

for appropriate choices of ϵ_1, ϵ_2, and ϵ_3.

Besides, $\operatorname{supp}(\Phi_{N,\epsilon,K}) \subset N_G \times N_H \subset N$. Finally, a similar argument shows that from some N_0 and ϵ_0 on we have

$$\int_{H \times G} |\Phi_{N,\epsilon,K}(m,x,h)|\,dx\,dh \leq R,$$

for some real number R, and all $m \in K$. $\square$

Assumptions. We next compare the generalized fixed-point algebras obtained when the roles of σ and λ are reversed. That is why we require symmetric conditions on these two actions. So, we assume from now on that both λ and σ are free and proper actions.

Notation. Let $C^{\sigma,u}$ denote the subalgebra of $\mathcal{M}(C_0(M) \times_\lambda G)$ consisting of functions $\Phi \in C_c(\beta M \times G)$ such that the projection of $\mathrm{supp}_M(\Phi)$ on M/H is precompact and $\gamma_h^{\sigma,u}\Phi = \Phi$ for all $h \in H$.

Remark 2.4. When the cocycle u is the identity, then $C^{\sigma,u}$ can be identified with $C_c(M/H \times G)$, as a subalgebra of $C_0(M/H) \times_\lambda G$.

Remark 2.5. Notice that, for $F \in C_c(M \times G)$, we have that

$$\mathrm{supp}_M(P_{\sigma,u}F) \subset \sigma_H(\mathrm{supp}_M(F)),$$

and therefore $C^{\sigma,u}$ contains the image of $P_{\sigma,u}$.

Lemma 2.6. *Let $\{\Phi_{N,\epsilon}\}$ be a net in $C^{\sigma,u}$, indexed by decreasing neighborhoods N of 1_G, increasing compact subsets K of M, and decreasing $\epsilon > 0$, and such that*

1) $\mathrm{supp}_G(\Phi_{N,\epsilon,K}) \subseteq N$.

2) $\left|\int_G \Delta_G^{1/2}(x)\Phi_{N,\epsilon}(m,x)dx - 1\right| < \epsilon$ *for all* $m \in K$.

3) *There is a real number R such that $\int_G |\Phi_{N,\epsilon}(m,x)|dx \leq R$, for all $m \in K$, and for all N and ϵ from some N_0 and ϵ_0 on.*

Then $\{\Phi_{N,\epsilon,K}\}$ is an approximate identity for $C^{\sigma,u}$.

Proof. Let $\Psi \in C^{\sigma,u}$ and $\delta > 0$ be given. Fix a neighborhood N' of 1_G with compact closure, and let $K' \subset M$ be a compact set such that $\Pi_H(\mathrm{supp}_M \Psi) \subset \Pi_H(K')$, where Π_H denotes the canonical projection on M/H.

As in Lemma 2.2, we can find $N_0 \subset N'$, ϵ_0, and K_0 such that, from N_0, ϵ_0, and K_0 on, we have

$$|(\Phi_{N,\epsilon,K} * \Psi - \Psi)(m,x)| < \delta,$$

for all $m \in \lambda_{N'}(K')$.

Therefore, if $m \in \mathrm{supp}(\Phi_{N,\epsilon,K} * \Psi - \Psi)$, then we have that $\sigma_h m \in \lambda_{N'}(K')$, for some $h \in H$. On the other hand we have that

$$|(\Phi_{N,\epsilon,K} * \Psi - \Psi)(\sigma_h m, x)| = |(\Phi_{N,\epsilon,K} * \Psi - \Psi)(m,x)|,$$

for all $h \in H$, $m \in M$, and $x \in G$, because $\Phi_{N,\epsilon,K}$ and $\Psi \in C^{\sigma,u}$. This shows that $|(\Phi_{N,\epsilon,K} * \Psi - \Psi)(m,x)| < \delta$ for all $m \in M$. Therefore $\Phi_{N,\epsilon,K} * \Psi$ converge to Ψ in the multiplier algebra norm. □

Remark 2.7. Notice that Lemma 2.6 above also holds, with a similar proof, if condition 2) is replaced by

2') $|\int_G \Phi_{N,\epsilon,K}(m,x)dx - 1| < \epsilon$ for all $m \in K$.

Proposition 2.8. *The generalized fixed point algebra $D^{\sigma,u}$ is the closure in $\mathcal{M}(C_0(M) \times_\lambda G)$ of $C^{\sigma,u}$.*

Proof. In view of property iii) in Proposition 2.1, it suffices to show that the span of the set

$$\{P_{\sigma,u}(E^* * F) : E, F \in C_c(M \times G)\}$$

contains an approximate identity for $C^{\sigma,u}$.

For a given compact set $K \subset M$, let us fix an open set U of compact closure containing K. Then the set $L = \{h \in H : \sigma_h m \in \overline{U} \text{ for some } m \in K\}$ is compact.

Let N be a given neighborhood of 1_G and $\epsilon > 0$. As in [**Rf3**, Sit. 10, first lemma], we can take an open cover $\{U_1, U_2, ..., U_n\}$ of K, such that $U_i \subseteq U$ and $U_i \cap \lambda_x U_i \neq \emptyset$ only if $x \in N$. For each $i = 1, ..., n$, let $H_i \in C_c^+(M \times G)$ be such that $\text{supp}(H_i) \subset U_i \times N$, and $\sum_i H_i$ is strictly positive on $K \times 1_G$. Then $\sum_i \int_{H \times G} H_i(\sigma_{h^{-1}} m, y) dh dy > 0$ for all $m \in K$. Therefore, we can find functions $F_i \in C_c^+(M \times G)$ such that $\text{supp}(F_i) \subset \text{supp}(H_i)$ and $\int_{H \times G} F_i(\sigma_{h^{-1}} m, y) dh dy = 1$ for all $m \in K$. Now, the action of G on $M \times G$ given by $\alpha_x(m, y) = (\lambda_x m, xy)$ is free and proper, so the second lemma in [**Rf3**, Situation 10] applies and for each $i = 1, ..., n$ we can find $G_i \in C_c^+(M \times G)$ such that $\text{supp}(G_i) \subseteq \text{supp}(F_i)$ and

$$\left| F_i(m, y) - G_i(m, y) \int_G G_i\left(\lambda_{x^{-1}} m, x^{-1} y\right) dx \right| < \delta / n,$$

for all $m \in M, y \in G$, and some positive number δ to be chosen later. Set now $\Phi_{N,\epsilon,K} = \sum_{i=1}^{i=n} P_{\sigma,u}(G_i * J_i)$, where $J_i(m, x) = G_i(\lambda_{x^{-1}} m, x^{-1})$. We have

$$\Phi_{N,\epsilon,K}(m, x) = \sum_i \int_H u(x, h) \int_G G_i(\sigma_{h^{-1}} m, y) G_i(\sigma_{h^{-1}} \lambda_{x^{-1}} m, x^{-1} y) dy,$$

so, since $\text{supp}(G_i) \subseteq \text{supp}(F_i)$, it follows that $\text{supp}_G(\Phi_{N,\epsilon,K}) \subseteq N$.

Besides, if $m \in K$,

$$\left| \int_G \Phi_{N,\epsilon,K}(m, x) dx - 1 \right|$$

$$= \left| \sum_i \int_H \int_G \left[u(x, h) G_i(\sigma_{h^{-1}} m, y) \int_G G_i\left(\sigma_{h^{-1}} \lambda_{x^{-1}} m, x^{-1} y\right) dx \right.\right.$$

$$\left.\left. - F_i(\sigma_{h^{-1}} m, y) \right] dy dh \right| < \epsilon,$$

for a suitable choice of δ, if N is chosen to have $|u(x, h) - 1|$ small enough for all $x \in N$ and $h \in L$.

Finally, from some ϵ_0 and N_0 on, $\int_G |\Phi_{N,\epsilon,K}(m,x)|dx \leq R$, for some real number R and all $m \in K$.

Then, by Remark 2.7, $\{\Phi_{N,\epsilon,K}\}$ is an approximate identity for $C^{\sigma,u}$. □

We will later make use of the following variation of the construction in the proof of Theorem 2.8.

Remark 2.9. The span of the set

$$\left\{P_{\sigma,u}(F) : F(m,x) = \Delta_G^{-1/2}(x)e_i(m)\overline{e}_i(\lambda_{x^{-1}}m), e \in C_c(M)\right\}$$

contains an approximate identity for $C^{\sigma,u}$.

Proof. In the notation of Proposition 2.8, let $\{f_i\} \subset C_c^+(M)$ be such that $\text{supp}(f_i) \subset U_i$, and $\int_H \sum_i f_i(\sigma_{h^{-1}}m) > 0$, for all $m \in K$. Since the action λ is proper we can get $g_i \in C_c^+(M)$ such that $\text{supp}(g_i) \subseteq \text{supp}(f_i)$ and $|f_i(m) - g_i(m)\int_G g_i(\lambda_{x^{-1}}m)dx| < \delta$ for all $m \in M$ and a given positive number δ. Then, if we let $L_i(m,x) = \Delta_G^{-1/2}(x)g_i(m)g_i(\lambda_{x^{-1}}m)$ we have that, for an appropriate choice of δ in terms of ϵ, the function $\Phi_{N,\epsilon,K} = \sum_i P_{\sigma,u}(L_i)$ can be shown (by an argument quite similar to that in Proposition 2.8) to satisfy the hypotheses of Lemma 2.6. □

Notation. We denote by ${}_\lambda\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_\lambda$ the $C_c(M \times G)$-valued maps defined on $C_c(M) \times C_c(M)$ by

$$\begin{aligned} {}_\lambda\langle f,g\rangle(m,x) &= \Delta_G^{-1/2}(x)f(m)\overline{g}(\lambda_{x^{-1}}m) \\ \text{and } \langle f,g\rangle_\lambda(m,x) &= \Delta_G^{-1/2}(x)\overline{f}(m)g(\lambda_{x^{-1}}m), \end{aligned}$$

where $f, g \in C_c(M)$.

Remark 2.10. It is a well known result ([**Rf3**, Situation 2]) that $C_c(M)$ is a left (resp. right) $C_c(M \times G)$-rigged module for ${}_\lambda\langle\ ,\ \rangle$ (resp. $\langle\ ,\ \rangle_\lambda$) and the actions given by:

$$\begin{aligned} (\Phi \cdot f)(m) &= \int_G \Delta_G^{1/2}(y)\Phi(m,y)f(\lambda_{y^{-1}}m)dy \\ \text{and } (f \cdot \Phi)(m) &= \int_G \Delta_G^{-1/2}(y)\Phi(\lambda_{y^{-1}}m, y^{-1})f(\lambda_{y^{-1}}m)dy, \end{aligned}$$

for $\Phi \in C_c(M \times G)$ It is easily checked that, by taking $\Phi \in C_c(\beta M \times G)$ in the formulas above, one makes $C_c(M)$ into a $C_c(\beta M \times G)$-module with inner product. Of course it is no longer a rigged space because the condition of density fails.

Proposition 2.11. *Let $C^{\sigma,u} \subseteq C_c(\beta M \times G)$ act on $C_c(M)$ on the left and on the right as in Remark* 2.10. *For $f, g \in C_c(M)$ define*

$$\langle f, g\rangle_{D^{\sigma,u}} = P_{\sigma,u}(\langle f,g\rangle_\lambda) \quad \text{and} \quad {}_{D^{\sigma,u}}\langle f, g\rangle = P_{\sigma,u}({}_\lambda\langle f,g\rangle).$$

Then $C_c(M)$ is a left (resp. right) $C^{\sigma,u}$-rigged space with respect to ${}_{D^{\sigma,u}}\langle\ ,\ \rangle$ (resp. $\langle\ ,\ \rangle_{D^{\sigma,u}}$).

Proof. The density condition follows from Remark 2.9. All other properties follow immediately from the fact that ${}_\lambda\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_\lambda$ are inner products and from Remark 2.5 and properties i), ii), and iii) of $P_{\sigma,u}$ shown in Proposition 2.1. □

We are now ready to show the main result of this section.

Theorem 2.12. *Let λ and σ be free and proper commuting actions of locally compact groups G and H respectively on a locally compact space M. Let u be a cocycle as in Proposition* 1.2. *Then the generalized fixed-point algebras $D^{\sigma,u}$ and D^{λ,u^*} of the actions $\gamma^{\sigma,u}$ and γ^{λ,u^*} on $C_0(M) \times_\lambda G$ and $C_0(M) \times_\sigma H$, respectively, are strong-Morita equivalent.*

Proof. By Proposition 2.11, $C_c(M)$ is a left $C^{\sigma,u}$-rigged space and a right C^{λ,u^*}-rigged space under

$$(\Phi \cdot f)(m) = \int_G \Delta_G^{1/2}(y)\Phi(m,y)f(\lambda_{y^{-1}}m)dy \quad , \quad {}_{D^{\sigma,u}}\langle f,g\rangle = P_{\sigma,u}({}_\lambda\langle f,g\rangle),$$

$$(f \cdot \Psi)(m) = \int_H \Delta_H^{-1/2}(h)\Psi(\sigma_{h^{-1}}m, h^{-1})f(\sigma_{h^{-1}}m)dh,$$

$$\text{and } \langle f,g\rangle_{D^{\lambda,u^*}} = P_{\lambda,u^*}(\langle f,g\rangle_\sigma),$$

where $f,g \in C_c(M)$, $\Phi \in C^{\sigma,u}$ and $\Psi \in C^{\lambda,u^*}$.

Then $C_c(M)$ is an $C^{\sigma,u}$-C^{λ,u^*} bimodule: for Φ, Ψ and f as above we have

$$[(\Phi \cdot f)\cdot \Psi](m)$$

$$= \int_H \int_G \Delta_H^{-1/2}(h)\Delta_G^{1/2}(y)\Psi\left(\sigma_{h^{-1}}m, h^{-1}\right)\Phi(\sigma_{h^{-1}}m, y)f(\sigma_{h^{-1}}\lambda_{y^{-1}}m)dydh$$

$$= \int_H \int_G \Delta_H^{-1/2}(h)\Delta_G^{1/2}(y)\Psi\left(\sigma_{h^{-1}}\lambda_{y^{-1}}m, h^{-1}\right)\Phi(m,y)f(\sigma_{h^{-1}}\lambda_{y^{-1}}m)dydh$$

$$= [\Phi \cdot (f \cdot \Psi)](m).$$

Besides, for $e, f, g \in C_c(M)$, we have

$$({}_{D^{\sigma,u}}\langle e, f\rangle \cdot g)(m) = \int_G \int_H u(y,h) e(\sigma_{h^{-1}} m) \overline{f}(\lambda_{y^{-1}}\sigma_{h^{-1}} m) g(\lambda_{y^{-1}} m) dh dy =$$

$$= (e\langle f, g\rangle_{D^{\lambda,u^*}})(m).$$

We now prove the continuity of the module structures with respect to the inner products.

Fix a measure μ of full support on M. Then, by [**Ph**, 6.1] and [**Pd**, 7.7.5], we have faithful representations Π of $C^{\sigma,u}$ on $L^2(M \times G)$ and Θ of C^{λ,u^*} on $L^2(M \times H)$ given by

$$(\Pi_\Phi \xi)(m, x) = \int_G \Phi(\lambda_x m, y) \xi(m, y^{-1}x) dx,$$

$$\text{and } (\Theta_\Psi \eta)(m, h) = \int_H \Psi(\sigma_h m, k) \eta(m, k^{-1}h) dk,$$

where $\Phi \in C^{\sigma,u}$, $\Psi \in C^{\lambda,u^*}$, $\xi \in L^2(M \times G)$ and $\eta \in L^2(M \times H)$.

Now, for $f \in C_c(M)$ and $\eta \in L^2(M \times H)$

$$\langle \Theta_{\langle f,f\rangle_{D^{\lambda,u^*}}} \eta, \eta\rangle_{L^2(M\times H)}$$

$$= \int_{M\times G\times H\times H} \sigma_{h^{-1}}(u^*(y,k)) \Delta_H^{-1/2}(k) \overline{f}(\lambda_{y^{-1}}\sigma_h m)$$

$$\cdot f(\lambda_{y^{-1}}\sigma_{k^{-1}h}) \eta(m, k^{-1}h) \overline{\eta}(m, h) dk dh dy dm$$

$$= \|\xi(f,\eta)\|^2_{L^2(M\times G)},$$

where $\xi(f, \eta) \in L^2(M \times G)$ is given by

$$(\xi(f,\eta))(m,x) = \int_H u^*(x, h^{-1}) \Delta_H^{-1/2}(h) f(\lambda_{x^{-1}}\sigma_h m) \eta(m, h) dh.$$

Then, if $\Phi \in C^{\sigma,u}$

$$[\xi(\Phi \cdot f, \eta)](m, x) =$$

$$= \int_G \int_H u^*(x, h^{-1}) \Delta_H^{-1/2}(h) \Delta_G^{1/2}(y) \Phi(\lambda_{x^{-1}}\sigma_h m, y)$$

$$\cdot f(\lambda_{y^{-1}x^{-1}}\sigma_h m) \eta(m, h) dh dy =$$

$$= (U \Pi_\Phi U \xi(f, \eta))(m, x),$$

where U denotes the unitary operator on $L^2(M \times G)$ defined by

$$(U\xi)(m,x) = \Delta_G^{-1/2}(x)\xi(m,x^{-1}).$$

Thus we have

$$\begin{aligned}\langle\Theta_{\langle\Phi\cdot f,\Phi\cdot f\rangle_{D^{\lambda,u^*}}}\eta,\eta\rangle_{L^2(M\times H)} &= \|\xi(\Phi\cdot f,\eta)\|^2 = \|U\Pi_\Phi U\xi(f,\eta)\|^2 \\ &\leq \|\Phi\|^2\|\xi(f,\eta)\|^2 = \|\Phi\|^2\langle\Theta_{\langle f,f\rangle_{D^{\lambda,u^*}}}\eta,\eta\rangle_{L^2(M\times H)},\end{aligned}$$

and it follows that

$$\langle\Phi\cdot f,\Phi\cdot f\rangle_{D^{\lambda,u^*}} \leq \|\Phi\|^2\langle f,f\rangle_{D^{\lambda,u^*}},$$

as elements of D^{λ,u^*}. Analogously, one shows that, for $f \in C_c(M)$ and $\xi \in L^2(M\times G)$

$$\langle\Pi_{D^{\sigma,u}\langle f,f\rangle}\xi,\xi\rangle_{L^2(M\times G)}\|\eta(f,\xi)\|^2,$$

for some $\eta(f,\xi) \in L^2(M\times H)$, and that, for $\Psi \in C^{\lambda,u^*}$ one has

$$\eta(f\cdot\Psi,\xi) = (V\Theta_{\Psi^*}V)(\eta(f,\xi)),$$

where V denotes the unitary operator in $L^2(M \times H)$ defined by $(V\eta)(m,h) = \Delta_H^{-1/2}(h)\eta(m,h^{-1})$. It follows that

$${}_{D^{\sigma,u}}\langle f\cdot\Psi,f\cdot\Psi\rangle \leq \|\Psi\|^2_{D^{\sigma,u}}\langle f,f\rangle,$$

as elements of $D^{\sigma,u}$.

Thus, we have proven that $C_c(M)$ is a $C^{\sigma,u} - C^{\lambda,u^*}$ equivalence bimodule. Now, if we define on $C_c(M)$ the norms

$$\|f\|^2_{D^{\sigma,u}} = \|{}_{D^{\sigma,u}}\langle f,f\rangle\| \quad\text{and}\quad \|f\|^2_{D^{\lambda,u^*}} = \|\langle f,f\rangle_{D^{\lambda,u^*}}\|,$$

it follows from [**Rf1**, 3.1] that $\|\ \|_{D^{\sigma,u}} = \|\ \|_{D^{\lambda,u^*}}$ and that the completion of $C_c(M)$ with respect to this norm gives, by continuity, an equivalence bimodule between $D^{\sigma,u}$ and D^{λ,u^*}. □

Remark 2.13. In view of Remark 2.4, when the cocycle u is the identity, Theorem 2.12 becomes Green's result: the algebras $C_0(M/H)\times_\lambda G$ and $C_0(M/G)\times_\sigma H$ are strong-Morita equivalent.

Corollary 2.14. *Under the assumptions of Theorem 2.12, the algebras $C_r^*(H, C_0(M)\times_\lambda G)$ and $C_r^*(G, C_0(M)\times_\sigma H)$ are strong-Morita equivalent.*

Proof. The proof follows from Proposition 2.3, Theorem 2.12, and [**Rf4**, 1.7]. □

3. Applications to quantum Heisenberg manifolds.

In this section we apply the previous results to the computation of the K-groups of the quantum Heisenberg manifolds. We recall the basic results and definitions concerning those algebras. We refer the reader to [**Rf5**] for further details.

For each positive integer c, the Heisenberg manifold M_c consists of the quotient G/D_c, where G is the Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} ; \text{ for real numbers } x, y, z \right\}$$

and D_c is the discrete subgroup obtained when x, y and cz above are integers.

The set of non-zero Poisson brackets on M_c that are invariant under the action of G by left translation can be parametrized by two real numbers μ and ν, with $\mu^2 + \nu^2 \neq 0$. A deformation quantization $\{D^{c,\hbar}_{\mu\nu}\}_{\hbar \in R}$ of M_c in the direction of a given invariant Poisson bracket $\Lambda_{\mu\nu}$ was constructed in [**Rf5**].

The algebra $D^{c,\hbar}_{\mu\nu}$ can be described as a generalized fixed-point algebra as follows. Let $M = R \times T$ and $\lambda^\hbar$ and σ be the commuting actions of Z on M induced by the homeomorphisms

$$\lambda^\hbar(x,y) = (x + 2\hbar\mu, y + 2\hbar\nu) \text{ and } \sigma(x,y) = (x-1, y).$$

Consider the action ρ of Z on $C_0(R \times T) \times_{\lambda^\hbar} Z$ given by

$$(\rho_k \Phi)(x,y,p) = e(ckp(y - \hbar p\nu))\Phi(x+k, y, p),$$

where $e(x) = exp(2\pi i x)$ for any real number x. The action ρ defined above corresponds to the action ρ defined in [**Rf5**, p. 539], after taking Fourier transform in the third variable to get the algebra denoted in that paper by $A_\hbar$, and viewing $A_\hbar$ as a dense *-subalgebra of $C_0(R \times T) \times_{\lambda^\hbar} Z$ via the embedding J defined in [**Rf5**, p. 547].

Notice that, for $M = R \times T$, $G = H = Z$, and $\hbar \neq 0$, the actions $\lambda^\hbar$ and σ satisfy the hypotheses of Section 2 and that the action ρ defined above corresponds, in that context, to the action we denoted by $\gamma^{\sigma,u}$, where $u : Z \times Z \to \mathcal{Z}UM(C_0(R \times T))$ is the cocycle defined by

$$u(p,k) = e(ckp(y - \hbar p\nu)),$$

for $p, k \in Z$. Besides, [**Rf5**, Theorem 5.4] shows that the algebra $D^{c,\hbar}_{\mu\nu}$ is the generalized fixed-point algebra of $C_0(R \times T) \times_{\lambda^\hbar} Z$ under the action ρ, and it follows from the proof of that theorem that $D^{c,\hbar}_{\mu\nu}$ is the algebra that we denote, in the context of Section 2, by $D^{\sigma,u}$.

Remark 3.1. We will also use the fact that the algebra $\tilde{D}^{c,\hbar}_{\mu\nu}$ consisting of functions $\Phi \in C_c(\beta(R \times T) \times Z)$ satisfying $\rho_k(\Phi) = \Phi$ for all $k \in Z$ is a dense *-subalgebra of $D^{c,\hbar}_{\mu\nu}$. This follows from Remark 2.5, Proposition 2.8, and from the fact that $(R \times T)/\sigma$ is compact.

Theorem 3.2. *For $\hbar \neq 0$ the K-groups of $D^{c,\hbar}_{\mu\nu}$ do not depend on $\hbar$.*

Proof. It follows from Theorem 2.12 that, for $\hbar \neq 0$, $D^{c,\hbar}_{\mu\nu}$ is strong-Morita equivalent to the generalized fixed-point algebra $E^{c,\hbar}_{\mu\nu}$ of $C_0(R \times T) \times_\sigma Z$ under the action $\gamma^{\lambda^\hbar}$ of Z defined by

$$(\gamma^{\lambda^\hbar}_p \Phi)(x, y, k) = e(-ckp(y - \hbar p\nu))\Phi(x - 2p\hbar\mu, y - 2p\hbar\nu, k).$$

Now, by Proposition 2.3, $\gamma^{\lambda^\hbar}$ is saturated, so we have ([**Rf4**, Corollary 1.7]) that $D^{c,\hbar}_{\mu\nu}$ is strong-Morita equivalent to $C_0(R\times T) \times_\sigma Z \times_{\gamma^{\lambda^\hbar}} Z$.

Besides, $\hbar \mapsto \lambda^\hbar$ is a homotopy between the $\lambda^\hbar$'s, which shows ([**Bl**, 10.5.2]) that the K-groups of $C_0(R\times T)\times_\sigma Z\times_{\gamma^{\lambda^\hbar}} Z$ do not depend on $\hbar$. On the other hand, since strong-Morita equivalent separable C*-algebras are stably isomorphic ([**BGR**]) and therefore have the same K-groups, we have proven that the K-groups of $D^{c,\hbar}_{\mu\nu}$, for $\hbar \neq 0$, do not depend on $\hbar$. □

Notation. Since the algebras $D^{c,\hbar}_{\mu\nu}$ and $D^{c,1}_{\hbar\mu,\hbar\nu}$ are isomorphic, we drop from now on the constant $\hbar$ from our notation and absorb it into the parameters μ and ν.

Remark 3.3. Notice that, since for any pair of integers k and l the algebras $D^c_{\mu\nu}$ and $D^c_{\mu+k,\nu+l}$ are isomorphic ([**Ab**]), the assumption $\hbar \neq 0$ in Theorem 3.2 can be dropped.

Theorem 3.4. *$K_0(D^c_{\mu\nu}) \cong Z^3 + Z_c$ and $K_1(D^c_{\mu\nu}) \cong Z^3$.*

Proof. In view of Theorem 3.2 and Remark 3.3, it suffices to prove the theorem for the commutative case where $D^c_{\mu\nu} = C(M_c)$.

After reparametrizing the Heisenberg group we get that $M_c = G/H_c$ where

$$G = \left\{ \begin{pmatrix} 1 & y & z/c \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in R \right\}$$

and

$$H_c = \left\{ \begin{pmatrix} 1 & m & p/c \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} : m, p, q \in Z \right\}.$$

We first use [**Ro**, Corollary 3] to reduce the proof to the computation of the K-theory of $C^*(H_c)$.

The group C*-algebra $C^*(H_c)$ is strong-Morita equivalent to $C(G/H_c) \times G$, where G acts by left translation [**Rf2**, Example 1]. Now, G is nilpotent and simply connected so we have

$$G = R \rtimes R \times R$$

as a semi-direct product.

Therefore

$$C(G/H_c) \times G \simeq C(G/H_c) \times R \rtimes R \times R,$$

and Connes'-Thom isomorphism ([**Bl**, 10.2.2]) gives

$$K_i(C^*(H_c)) = K_i(C(G/H_c) \times G) = K_{1-i}(C(G/H_c)) = K_{1-i}(C(M_c)).$$

So it suffices to compute $K_i(C^*(H_c))$. The computation was made in [**AP**, Prop. 1.4] for the case c=1, and the general case can be obtained with slight modifications to their proof. We first write H_c as a semi-direct product, so its group C^*-algebra can be expressed as a crossed product algebra. Then, by using the Pimsner-Voiculescu exact sequence ([**Bl**, 10.2.1]), we get its K-groups.

Let

$$N = \left\{ \begin{pmatrix} 1 & m & p/c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : m, p, \in Z \right\} \text{ and } K = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} : q \in Z \right\}.$$

Then $H_c = N \times_{\alpha_c} K$, where α_c is conjugation. If we identify in the obvious way N and K with Z^2 and Z respectively, we have that $H_c \simeq Z^2 \times_{\alpha_c} Z$, where $\alpha_c(q)(m, p) = (m, p - cmq)$. Then the Pimsner-Voiculescu exact sequence yields:

$$\begin{array}{ccccc} K_0(C(T^2)) & \xrightarrow{id - \alpha_{c*}} & K_0(C(T^2)) & \xrightarrow{i_*} & K_0(H_c) \\ \delta \uparrow & & & & \downarrow \delta \\ K_1(H_c) & \xleftarrow{i_*} & K_1(C(T^2)) & \xleftarrow{id - \alpha_{c*}} & K_1(C(T^2)) \end{array} .$$

It was shown on [**AP**, Prop.1.4] that $id = \alpha_{1_*}$ on $K_0(C(T^2))$ and, since $\alpha_{c_*} = \alpha_{1_*}^c$ it follows that $id = \alpha_{c_*}$ on $K_0(C(T^2))$ for any c. Thus we get the following short exact sequences:

$$0 \longrightarrow Z^2 \longrightarrow K_0(H_c) \stackrel{\delta}{\longrightarrow} \mathrm{Ker}(id - \alpha_{c_*}) \longrightarrow 0$$

$$0 \longrightarrow K_1(C(T^2))/\mathrm{Ker}(id - \alpha_{c_*}) \longrightarrow K_1(H_c) \stackrel{\delta}{\longrightarrow} Z^2 \longrightarrow 0,$$

where $id - \alpha_{c_*}$ is the map on $K_1(C(T^2))$.

Let us now compute $id - \alpha_{c_*}$ on $K_1(C(T^2))$. We have identified $C(T^2)$ with $C^*(Z^2)$ via Fourier transform, so the automorphism α_c on $C(T^2)$ becomes $(\alpha_c f)(x, y) = f(x - cy, y)$. Now, $K_1(C(T^2)) = Z^2$ if we identify $[u_1]_{K_1}$ and $[u_2]_{K_2}$ with (1,0) and (0,1) in Z^2, respectively, where $u_1(x, y) = e(x)$, $u_2(x, y) = e(y)$ for all $(x, y) \in T^2$ Then, for $(a, b) \in Z^2$ we have

$$(id - \alpha_{c_*})(a, b) = (a, b) - (a, b - ac) = (0, ac).$$

This shows that

$$\mathrm{Ker}(id - \alpha_{c_*}) = Z \oplus \{0\} \subset Z^2, \mathrm{Im}(id - \alpha_{c_*}) = \{0\} \oplus cZ \subset Z^2.$$

So the exact sequences above become:

$$0 \longrightarrow Z^2 \longrightarrow K_0(H_c) \longrightarrow Z \longrightarrow 0$$

$$0 \longrightarrow Z + Z_c \longrightarrow K_1(H_c) \longrightarrow Z^2 \longrightarrow 0.$$

Therefore

$$K_1(D^c_{\mu\nu}) = K_0(H_c) = Z^3 \text{ and } K_0(D^c_{\mu\nu}) = K_1(H_c) = Z^3 + Z_c.$$

□

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