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ITERATED LOOP MODULES AND A FILTERATION FOR VERTEX REPRESENTATION OF TOROIDAL LIE ALGEBRAS

S. ESWARA RAO

The purpose of this paper is two fold. The first one is to construct a continuous new family of irreducible (some of them are unitarizable) modules for Toroidal algebras. The second one is to describe the sub-quotients of the (integrable) modules constructed through the use of Vertex operators.

Introduction.

Toroidal algebras $\tau_{[d]}$ are defined for every $d \geq 1$ and when $d = 1$ they are precisely the untwisted affine Lie-algebras. Such an affine algebra $\mathcal{G}$ can be realized as the universal central extension of the loop algebra $\dot{\mathcal{G}} \otimes \mathbb{C}[t, t^{-1}]$ where $\dot{\mathcal{G}}$ is simple finite dimensional Lie-algebra over $\mathbb{C}$. It is well known that $\mathcal{G}$ is a one dimensional central extension of $\dot{\mathcal{G}} \otimes \mathbb{C}[t, t^{-1}]$. The Toroidal algebras $\tau_{[d]}$ are the universal central extensions of the iterated loop algebra $\dot{\mathcal{G}} \otimes \mathbb{C}[t_1^{\pm 1}, \cdots t_d^{\pm 1}]$ which, for $d \geq 2$, turnout to be infinite central extension. These algebras are interesting because they are related to the Lie-algebra of Map (X, G), the infinite dimensional group of polynomial maps of X to the complex algebraic group G where X is a d-dimensional torus.

For additional material on recent developments in the theory of Toroidal algebras one may consult [**BC**], [**FM**] and [**MS**].

In [**MEY**] and [**EM**] a countable family of modules (also integrable see [**EMY**]) are constructed for Toroidal algebras on Fock space through the use of Vertex Operators (Theorem 3.4, [**EM**]). However they are reducible and *not* completely reducible. In §5 we observe that the Fock space is a direct sum of certain $\underline{b}(\lambda)'s$. In (5.10) to (5.12) we prove that each $\underline{b}(\lambda)$ admits a filteration by an increasing sequence of modules such that the successive quotients are all isomorphic to V defined in §5. In (5.9) we prove that each V admits a filteration of decreasing sequence of modules such that the successive quotients are all irreducible. In our main Theorem 5.6 we will describe the irreducible modules as certain iterated loop modules twisted by an automorphism of $\tau_{[d]}$.

We now describe the contents of the paper.

In §1 we construct an iterated loop algebra of Kac-Moody Lie-algebra $\mathcal{G}$ and construct a family of completely reducible modules (Theorem (1.8)) using methods similar to [**E**]. In §2 we prove that some of the above modules are unitarizable (Proposition 2.3). In §3 we specialize these results for $\mathcal{G}$ finite dimensional (Theorem 3.3) and for $\mathcal{G}$ an affine Lie-algebra (Theorem 3.6) to get irreducible modules (some of them are unitarizable modules) for $\tau_{[d]}$. The modules considered in Theorem 3.6 are the first examples of irreducible modules for $\tau_{[d]}$ where part of the centre acts non-trivially. It should be mentioned that it is the unitarizable modules (and also integrable modules) which lift to the group.

In §4 we recall the construction of Vertex Operators and the Fock space. We also construct certain automorphisms of $\tau_{[d]}$ which are necessary in §5.

1.

Let d and k be positive integers. Let $\mathcal{G}$ be a Lie-algebra and let $\mathcal{G}_k = \oplus\mathcal{G}$ be k copies of $\mathcal{G}$. Let $A = A_d = \mathbb{C}[t_1^{\pm 1}, \cdots t_d^{\pm 1}]$ be Laurent polynomial ring in d variables. Then $\mathcal{G}_A = \mathcal{G} \otimes A$ is a Lie-algebra with Lie structure $[X \otimes a, Y \otimes b] = [X, Y] \otimes ab, X, Y \in \mathcal{G}\ a, b \in A$.

Let $\underline{n} = (n_1, n_2, \cdots n_d)$ be a $d-$tupple of integers and let

$$t^{\underline{n}} = t_1^{n_1} t_2^{n_2} \cdots t_d^{n_d}.$$

For $1 \leq i \leq k$ let $\underline{a}_i = (a_i(1), \cdots a_i(d))$ be a $d-$tupple of non-zero complex numbers. Let $a_i^{\underline{n}} = a_i^{n_1}(1) \cdots a_i^{n_d}(d)$ be the product. Consider the Lie-algebra homomorphism

$$\begin{aligned} \phi : \mathcal{G}_A &\to \mathcal{G}_k \\ X \otimes t^{\underline{n}} &\mapsto (X a_1^{\underline{n}}, \cdots, X a_k^{\underline{n}}). \end{aligned}$$

It is elementary to check that φ is *not* surjective if and only if $a_i(\ell) = a_j(\ell)$ for some $i \neq j$ and for all ℓ. First prove it for $d = 1$ and then using Vandermonde determinant for general d.

Define derivations $d_1, d_2 \cdots d_d$ on $\mathcal{G}_A$ by $[d_i, X \otimes t^{\underline{n}}] = n_i X \otimes t^{\underline{n}}$ and note $[d_i, d_j] = 0$. Let D be the linear span of $d_1, d_2, \cdots d_d$ and let $\tilde{\mathcal{G}}_A = \mathcal{G} \otimes A \oplus D$.

For any Lie-algebra $\mathcal{G}$, let $U(\mathcal{G})$ denote the universal enveloping algebra. Note that $A, \mathcal{G}_A$ and $U(\mathcal{G}_A)$ are obviously $\mathbb{Z}^d$ graded algebras.

1.1. From here onwards we will assume that $\mathcal{G}$ is a Kac-Moody Lie-algebra. Fix a Cartan subalgebra $\underline{h}$ of $\mathcal{G}$.

Let $\psi : U(\underline{h}_A) \to A$ be a $\mathbb{Z}^d$ graded homomorphism. Let A_ψ be the image of ψ which is a $\mathbb{Z}^d$ graded subalgebra of A.

A_ψ can be treated as an $\tilde{\underline{h}}_A-$ module by the ψ action,

$$h \otimes t^{\underline{n}}.t^{\underline{m}} = \psi(h \otimes t^{\underline{n}}).t^{\underline{m}}$$

$$d_i.t^{\underline{m}} = m_i t^{\underline{m}}.$$

1.2. Lemma. *A_ψ is an irreducible $\tilde{\underline{h}}_A-$ module if and only if homogeneous elements of A_ψ are invertible in A_ψ.*

Proof. Assume the homogeneous elements of A_ψ are invertible in A_ψ. To prove irreducibility it is sufficient to prove that given elements $t^{\underline{n}}$ and $t^{\underline{m}}$ there exists X in $U(\underline{h}_A)$ such that $X.t^{\underline{n}} = t^{\underline{m}}$. By assumption $t^{\underline{m}-\underline{n}}$ belongs to A_ψ. Since A_ψ is the image of ψ there exists X in $U(\underline{h}_A)$ such that $\psi(X) = t^{\underline{m}-\underline{n}}$ and clearly $X.t^{\underline{n}} = \psi(X)t^{\underline{n}} = t^{\underline{m}}$.

Now for the converse let $t^{\underline{n}}$ belong to A_ψ. First note that 1 belongs to A_ψ. There exists X in $U(\underline{h}_A)$ such that $\psi(X).t^{\underline{n}} = 1$ by irreducibility of A_ψ. Then clearly $\psi(X) = t^{-\underline{n}}$ and we are done. □

1.3. The purpose of this section is to construct irreducible modules for $\tilde{\mathcal{G}}_A$.

Let $V(\lambda_1), \cdots V(\lambda_k)$ be irreducible highest weight modules for $\mathcal{G}$ with highest weights $\lambda_1, \lambda_2, \cdots \lambda_k$ respectively. Then clearly $V = \otimes V(\lambda_i)$ is an irreducible module for $\mathcal{G}_k$.

1.4. Also V can be treated as an irreducible $\mathcal{G}_A$-module via the Lie-algebra homomorphism ϕ. But it is not a $\mathbb{Z}^d-$ graded module. That is, it cannot be extended to $\tilde{\mathcal{G}}_A$.

Consider $V_A := V \otimes A$ which will be given $\tilde{\mathcal{G}}_A$ module structure.

$$\begin{aligned} X \otimes t^{\underline{m}}(v \otimes t^{\underline{n}}) &= \varphi(X \otimes t^{\underline{m}})v \otimes t^{\underline{m}+\underline{n}} \\ d_i(v \otimes t^{\underline{n}}) &= n_i v \otimes t^{\underline{n}} \ (X \in \mathcal{G}, v \in V). \end{aligned} \tag{1.5}$$

We denote the $\tilde{\mathcal{G}}_A$ module by (V_A, π).

Let $\psi : U(\underline{h}_A) \to A$ be $h \otimes t^{\underline{n}} \mapsto \sum_i \lambda_i(h) \underline{a}_i^{\underline{n}} t^{\underline{n}}$ be a $\mathbb{Z}^d-$ graded homomorphism of algebras.

1.6. We believe that the conditions for ϕ to be *surjective* are sufficient to prove that A_ψ is irreducible, but we could not prove that. Instead we will give a continuous family of examples where $A_\psi = A$. In particular for these examples A_ψ is irreducible.

1.7. Assume $1 \leq \ell \leq d, a_i(\ell)/a_j(\ell)$ is not a $k-$th root of unity. Then by Lemma 4.4 of [**CP**] the following $\mathbb{Z}-$graded algebra homomorphism is surjective for all ℓ,

$$U(\underline{h} \otimes \mathbb{C}[t_\ell, t_\ell^{-1}]) \to \mathbb{C}[t_\ell, t_\ell^{-1}]$$

$$h \otimes t_\ell^n \mapsto \sum_{j=1}^{k} \lambda_j(h) a_j^n(\ell) t_\ell^n.$$

Hence A_ψ contains $\mathbb{C}[t_\ell, t_\ell^{-1}]$ for all ℓ. So that $A_\psi = A$.

From here on we will assume that A_ψ is irreducible $\tilde{\underline{h}}_A-$ module.
We will also note ψ and the module (V_A, π) depends on the choice of λ's and $a_i(\ell)'s$.

For any $v \in V$ let $v(\underline{n}) = v \otimes t^{\underline{n}}$. We will now prove that (V_A, π) as defined in (1.5) is a completely reducible $\tilde{\underline{\mathcal{G}}}_A-$ module.

1.8. Theorem. *Let $G \subseteq \mathbb{Z}^d$ be such that $\{t^{\underline{m}}, \underline{m} \in G\}$ is a set of coset representatives of A/A_ψ. Let $v = v_1 \otimes \cdots \otimes v_k$ where each v_i is a highest weight vector of $V(\lambda_i)$. Then*

(1) *$V_A = \oplus_{\underline{m} \in G} U(v(\underline{m}))$ as $\tilde{\underline{\mathcal{G}}}_A$- module where $U(v(\underline{m}))$ is the $\tilde{\mathcal{G}}_A$ submodule generated by the vector $v(\underline{m})$.*

(2) *Each $U(v(\underline{m}))$ is an irreducible $\tilde{\mathcal{G}}_A-$ module.*
In particular (V_A, π) is an irreducible $\tilde{\mathcal{G}}_A$ -module whenever $A = A_\psi$.

Before we prove the theorem we prove some lemmas.

1.9. Lemma. *Any non-zero $\tilde{\mathcal{G}}_A-$ submodule of V_A contains $v(\underline{m})$ for some $\underline{m}$.*

Proof. Consider the map $S : V_A \to V$ defined by $S(w(\underline{m})) = w$ (extend linearly to V_A). Then clearly S is a surjective $\mathcal{G}_A$- module map (it is not a $\tilde{\mathcal{G}}_A$-module map).

Claim. *$S(W) = V$ for any non-zero $\tilde{\mathcal{G}}_A$ module W of (V_A, π).*

Since V is an irreducible $\mathcal{G}_A$ -module and $S(W)$ is a submodule of V, to see the claim it is sufficient to prove that $S(W) \neq 0$. But that is clear. Since W is a $\tilde{\mathcal{G}}_A$-module it therefore contains vector of the form $w(\underline{m}), w \in V$ and $S(w(\underline{m})) = w \neq 0$. This proves the claim.

Now let w in W be such that $S(w) = v$ where v is the vector defined in the statement of the theorem. Since S is a $\underline{h}-$ module map, w and v are the same weight. $\{v(\underline{m}), \underline{m} \in \mathbb{Z}^d\}$ are the only such weight vectors of $\check{V}_A$ and hence $w = \sum_i C_i v(\underline{m}^i)$ for some complex numbers C_i and some $\underline{m}^i \in \mathbb{Z}^d$. But W is a $\tilde{\mathcal{G}}_A$ module so is ($\mathbb{Z}^d$-graded) and it follows that $v(\underline{m}^i)$ belongs to W. □

1.10. Lemma. *The following are true.*
(1) $v(\underline{m}) \in U(\tilde{\underline{h}}_A)v(\underline{n})$ *if and only if* $t^{\underline{m}-\underline{n}} \in A_\psi$.
(2) $v(\underline{m}) \in U(\tilde{\mathcal{G}}_A)v(\underline{n})$ *if and only if* $t^{\underline{m}-\underline{n}} \in A_\psi$.
(3) $U(\tilde{\mathcal{G}}_A)v(\underline{m}) = U(\tilde{\mathcal{G}}_A)v(\underline{n})$ *if and only if* $t^{\underline{m}-\underline{n}} \in A_\psi$.
(4) $N = U(\tilde{\mathcal{G}}_A)v(\underline{m}) \cap U(\tilde{\mathcal{G}}_A)v(\underline{n}) \neq \{0\}$ *if and only if* $t^{\underline{m}-\underline{n}} \in A_\psi$.

Proof. (1) Clear. (2) The "if" part follows from (1). For the "only if" part write $\mathcal{G} = N^- \oplus \underline{h} \oplus N^+$ where N^+ is sum of positive root spaces and N^- is sum of negative root spaces. Now note that $U(\tilde{\mathcal{G}}_A)v(\underline{m}) = U(N_A^-)U(\underline{h}_A)v(\underline{m})$ which follows from the fact that $U(N_A^+)v(\underline{m}) = 0$ and the Poincare-Birkhoff-witt theorem. Now it is easy to see by a weight argument that $v(\underline{m}) \in U(\tilde{\mathcal{G}}_A)v(\underline{n})$ implies that $v(\underline{m}) \in U(\tilde{\underline{h}}_A)v(\underline{n})$. So by (1),(2) is complete. (3) follows from (2). (4) Assume $N \neq \{0\}$. By Lemma (1.9) there is a $\underline{k}$ such that $v(\underline{k}) \in N$. Hence by (2) $t^{\underline{k}-\underline{m}}, t^{\underline{k}-\underline{n}} \in A_\psi$. But $t^{\underline{m}-\underline{k}} \in A_\psi$ and therefore $t^{\underline{m}-\underline{n}} \in A_\psi$. Converse follows from (3).

Proof of the Theorem 1.8. (2) Let W be a non-zero $\tilde{\mathcal{G}}_A$ submodule of $U(\tilde{\mathcal{G}}_A)v(\underline{m})$. Then W contains $v(\underline{n})$ for some $\underline{n}$ (by Lemma 1.9). So that $U(\tilde{\mathcal{G}}_A)v(\underline{n}) \subseteq W$. But by Lemma (1.10), $t^{\pm(\underline{m}-\underline{n})} \in A_\psi$. Now by Lemma 1.10 (3) it will follow that $U(\tilde{\mathcal{G}}_A)v(\underline{n}) = U(\tilde{\mathcal{G}}_A)v(\underline{m})$. In particular $W = U(\tilde{\mathcal{G}}_A)v(\underline{m})$.

(1) Let $w(\underline{m}) \in V_A, w \in V, \underline{m} \in \mathbb{Z}^d$. Since V is irreducible $\mathcal{G}_A-$ module (see 1.4) there exist $X \in U(\mathcal{G}_A)$ such that $\varphi(X)v = w$. Write $X = \sum_{\underline{n}\in\mathbb{Z}^d} X_{\underline{n}}$ where $[d_i, X_{\underline{n}}] = n_i X_{\underline{n}}$.

Consider $\sum \pi(X_{\underline{n}})v(-\underline{n}+\underline{m}) = \sum \varphi(X_{\underline{n}})v(\underline{m}) = \varphi(X)v(\underline{m}) = w(\underline{m})$. Hence we have proved that

$$V_A \subseteq \sum_{\underline{m}\in\mathbb{Z}^d} U(\tilde{\mathcal{G}}_A)v(\underline{m}). \tag{1.11}$$

We also have by Lemma 1.10 (3)

$$\sum_{\underline{m}\in\mathbb{Z}^d} U(\tilde{\mathcal{G}}_A)v(\underline{m}) = \sum_{\underline{m}\in G} U(\tilde{\mathcal{G}}_A)v(\underline{m}). \tag{1.12}$$

In view of (1.11) and (1.12), to see (1) of the Theorem we only have to prove that the sum of RHS in (1.12) is direct.

For that it is sufficient to prove that for all $\underline{m} \in G$.

$$U(\tilde{\mathcal{G}}_A)v(\underline{m}) \cap \sum_{\underline{n}\in G, \underline{m}\neq\underline{n}} U(\tilde{\mathcal{G}}_A)v(\underline{n}) = \{0\}. \tag{1.13}$$

Suppose (1.13) does not hold for some $\underline{m} \in G$. Then

$$U(\tilde{\mathcal{G}}_A)v(\underline{m}) \subseteq \sum_{\underline{n}\in G, \underline{m}\neq\underline{n}} U(\tilde{\mathcal{G}}_A)v(\underline{n})$$

(since $U(\tilde{\mathcal{G}}_A)v(\underline{m})$ is irreducible). Since all modules under consideration are $\underline{h} \oplus D-$ modules $v(\underline{m})$ has to be linear combinations of $v(\underline{k})$ where $v(\underline{k}) \in U(\tilde{\mathcal{G}}_A)v(\underline{n}), \underline{n} \neq \underline{m}$ and $\underline{n} \in G$. But this is *not* possible since none of the $\underline{k}$ can be equal to $\underline{m}$ by Lemma 1.10 (2) and the choice of G. This proves that the sum at RHS in (1.12) is direct and that completes the proof of the theorem.

2.

In this section we will prove that modules constructed earlier for $\tilde{\mathcal{G}}_A$ are unitarizable subject to some conditions on λ's and $a_i(\ell)$'s.

We will start with

2.1. Definition. A conjugate-linear anti-involution of a Lie-algebra $\mathcal{G}'$ is a map $D : \mathcal{G}' \to \mathcal{G}'$ such that

$$D(X+Y) = D(X) + D(Y), D(\lambda X) = \overline{\lambda} D(X)$$

$$D[X,Y] = [D(Y), D(X)], D^2 = \mathrm{Id}.$$

for all $X, Y \in \mathcal{G}'$ and $\lambda \in C$.

Such maps are also called forms.

2.2. Definition. A $\mathcal{G}'-$ module V is said to be unitarizable with respect to conjugate linear anti-involution D of $\mathcal{G}'$ if there exists a positive definite hermitian form $\langle , \rangle$ on V satisfying,

$$\langle X v_1, v_2 \rangle = \langle v_1, D(X) v_2 \rangle$$

for all $v_1, v_2 \in V$ and $X \in \mathcal{G}'$.

Let w be a conjugate-linear anti-involution of $\mathcal{G}$. Then $w_k = \oplus_k w$ is conjugate-linear anti-involution of $\mathcal{G}_k$. Define $\overline{w}$ on $\tilde{\mathcal{G}}_A$ by $\overline{w} X(\underline{n}) = w(X)(-\underline{n})$ and $\overline{w}(d_i) = d_i$ and verify that it extends uniquely to a conjugate-linear anti-involution.

Let $V(\lambda_i)$ be highest weight unitarizable $\mathcal{G}-$ module with respect to w. Then it is a standard fact that $V = \otimes V(\lambda_i)$ is a unitarizable $\mathcal{G}_k-$ module with respect to w_k.

2.3. Proposition. *(V_A, π) is a unitarizable $\tilde{\mathcal{G}}_A-$ module with respect to $\overline{w}$ if $\mid a_i(\ell) \mid = \mid a_j(\ell) \mid$ for all i, j and ℓ.*

Proof. Define an automorphism $\mathcal{O} : \mathcal{G}_A \to \mathcal{G}_A$ by $\mathcal{O}(X(\underline{n})) = \mid a_1^{-2n} \mid X(\underline{n})$. Then the following diagram commutes.

$$(2.4) \qquad \begin{array}{ccc} \mathcal{G}_A & \overset{\overline{\varphi}}{\to} & \mathcal{G}_k \\ \overline{w} \downarrow & & \downarrow w_k \\ \mathcal{G}_A & \underset{\phi o \mathcal{O}}{\to} & \mathcal{G}_k \end{array}.$$

2.5. Hence we have $\varphi \circ \mathcal{O} \circ \overline{w} = w_k \circ \overline{\varphi}$. Also we have $\varphi \circ \mathcal{O} \circ \overline{w} X(\underline{k})$ $= \mid a_1^{2\underline{k}} \mid \varphi \circ \overline{w} X(\underline{k})$.

Let $(,)$ be a positive definite hermitian form on V satisfying

$$(Xv_1, v_2) = (v_1, w_k X v_2), X \in \mathcal{G}_k, v_1, v_2 \in V. \tag{2.6}$$

Define a positive definite hermitian form on V_A by

$$\langle v_1(\underline{m}), v_2(\underline{n})\rangle = \begin{cases} \mid a_1^{-2\underline{m}} \mid (v_1, v_2) \text{ if } \underline{m} = \underline{n} \\ 0 \text{ otherwise.} \end{cases}$$

The following can be easily seen.

(1) $\langle \pi d_i v_1(\underline{n}), v_2(\underline{m})\rangle = \langle v_1(\underline{n}), \pi d_i v_2(\underline{m})\rangle$

$$= \begin{cases} 0 \text{ if } \underline{n} \neq \underline{m} \\ n_i \langle v_1(\underline{n}), v_2(\underline{n})\rangle \text{ if } \underline{n} = \underline{m} \end{cases}$$

(2) $\langle \pi X(\underline{k}) v_1(\underline{n}), v_2(\underline{m})\rangle = \langle v_1(\underline{n}), \pi o \overline{w} X(k) v_2(\underline{m})\rangle = 0$ if $\underline{k} + \underline{n} \neq \underline{m}$.

Let $v_1, v_2 \in V, X \in \mathcal{G}, \underline{k}, \underline{n} \in \mathbb{Z}^d$ and $\underline{m} = \underline{k} + \underline{n}$.
Consider

$$\begin{aligned} \langle \pi X(\underline{k}).v_1(\underline{n}), v_2(\underline{m})\rangle &= \langle (\varphi X(\underline{k}) v_1)(\underline{k} + \underline{n}), v_2(\underline{m})\rangle \\ &= \mid a_1^{-2(\underline{n}+\underline{k})} \mid (\varphi X(\underline{k}) v_1, v_2) \text{ (by def)} \\ &= \mid a_1^{-2(\underline{n}+\underline{k})} \mid (v_1, w_k \varphi X(\underline{k}) v_2) \text{ (by 2.6)} \\ &= \mid a_1^{-2(\underline{n}+\underline{k})} \mid (v_1, \varphi o \mathcal{O} o \overline{w} X(\underline{k}) v_2) \text{ (by (2.5))} \\ &= \mid a_1^{-2\underline{n}} \mid (v_1, \varphi o \overline{w} X(\underline{k}) v_2) \text{ (by (2.5))} \\ &= \langle v_1(\underline{n}), (\varphi o \overline{w} X(\underline{k})) v_2(\underline{n})\rangle \text{(by def)} \\ &= \langle v_1(\underline{n}), \pi \overline{w} X(\underline{k}) v_2(\underline{n} - \underline{k})\rangle \text{ (by def of } \pi). \end{aligned}$$

It now follows that V_A is unitarizable as $\tilde{\mathcal{G}}_A$- module.

3.

We will now use the results of Section 1 and 2 to produce irreducible unitarizable modules for Toroidal algebras.

First recall the construction of a Toroidal algebra $\tau_{[d]}$ from Section 2 of [**MEY**].

$\tau_{[d]} = \mathcal{G} \otimes A \oplus \Omega_A / d_A$ where $\mathcal{G}$ is finite dimensional simple Lie-algebra and A is a Laurent polynomial ring in d variables. $\Omega_d := \Omega_A / d_A$ is central in $\tau_{[d]}$. Ω_d

is a linear space spanned by $d(t^{\underline{m}})t^{\underline{n}}$ with the relation $d(t^{\underline{m}})t^{\underline{n}} + d(t^{\underline{n}})t^{\underline{m}} = 0$. The Lie-algebra structure on $\tau_{[d]}$ is defined as

$$[X(\underline{m}), Y(\underline{n})] = [X, Y](\underline{m} + \underline{n}) + (X, Y)d(t^{\underline{m}})t^{\underline{n}}$$

where (,) is a Killing form on $\mathcal{G}$.

Let $d_1, d_2, \cdots d_d$ be derivations on $\tau_{[d]}$ defined by

$$[d_i, X(\underline{m})] = m_i X(\underline{m}), [d_i, d(t^{\underline{m}})t^{\underline{n}}] = (m_i + n_i)d(t^{\underline{m}})t^{\underline{n}}, [d_i, d_j] = 0.$$

Let D be the linear span of $d_1, d_2, \cdots d_d$. Then $\tilde{\tau}_{[d]} := \tau_{[d]} \oplus D$ is a Lie-algebra and Ω_d is an *abelian ideal.*

3.1. Let w be any conjugate-linear anti-involution of $\mathcal{G}$ and let $\overline{w}$ be a conjugate-linear anti-involution of $\tilde{\mathcal{G}}_A$ as defined in Section 2. Extend $\overline{w}$ to $\tilde{\tau}_{[d]}$ by

$$\overline{w}.d(t^{\underline{m}})t^{\underline{n}} = d(t^{-\underline{m}})t^{-\underline{n}}.$$

We also note that $\tilde{\tau}_{[d]}/\Omega_d \tilde{=} \tilde{\mathcal{G}}_A$. Let S denote the quotient map. So that any $\tilde{\mathcal{G}}_A-$ module can be treated as $\tilde{\tau}_{[d]}$ module via S.

Now let $(V(\lambda_i), \tau_i)$ be the highest weight module for $\mathcal{G}$. Let $V = \otimes_i V(\lambda_i)$ and $V_A = V \otimes A$. Then (V_A, π) is a $\tilde{\tau}_{[d]}-$ module in the following way.

(3.2) $\pi X(\underline{m}).(v_1 \otimes \cdots \otimes v_k)(\underline{n}) = \sum a_i^{\underline{m}}(v_1 \otimes \cdots \otimes \tau_i(X)v_i \otimes \cdots \otimes v_k)(\underline{n} + \underline{m})$

$$\pi d_i(v_1 \otimes \cdots \otimes v_k)(\underline{n}) = n_i(v_1 \otimes \cdots \otimes v_k)(\underline{n})$$
$$\pi \Omega_d.V_A = 0.$$

This is precisely the definition given in Section 1.

As earlier let ψ be the $\mathbb{Z}^d-$ graded algebra homomorphism from $U(\underline{h}_A) \to A$ given by $h(\underline{m}) \mapsto \sum_i \lambda_i(h) a_i^{\underline{m}} t^{\underline{m}}$. Denote the image by A_ψ.

3.3. Theorem. (1)(V_A, π) *is a completely reducible* $\tau_{[d]}-$ *module if* A_ψ *is irreducible as* $\underline{h} \otimes A-$ *module.*

(2) (V_A, π) *is irreducible as* $\tilde{\tau}_{[d]}-$*module if* $a_i(\ell)/a_j(\ell), i \neq j$, *is not a* $k-$*th root of unity.*

(3) *If* $V(\lambda_i)$ *is unitarizable highest weight module for* $\mathcal{G}$ *with respect to the form* w *and* $| a_i(\ell) |=| a_j(\ell) |$ *for all* i, j *and* ℓ, *then* (V_A, π) *is unitarizable* $\tilde{\tau}_{[d]}-$ *module with respect to* $\overline{w}$.

Proof. (1) and (2) follows from Theorem (1.8) by taking Kac-Moody Lie-algebra $\mathcal{G}$ to be the finite dimensional simple Lie-algebra $\mathcal{G}$.

(3) Follows from Proposition (2.3). □

The special case where $d = 1$ and w is a compact form of $\mathcal{G}$ is due to [**CP**]. The case $d = 1$ and w *any* form, including twisted affine Lie-algebras is due to [**E**].

3.4. Let $\mathcal{G}$ be finite dimensional simple Lie-algebra and let

$$\overline{\mathcal{G}} = \mathcal{G} \otimes \mathbb{C}[t_{d+1}, t_{d+1}^{-1}] \oplus \mathbb{C}c$$

be the corresponding nontwisted affine Lie-algebra. Let $\tilde{\overline{\mathcal{G}}} = \overline{\mathcal{G}} \oplus \mathbb{C}d$ where d is a derivation. Consider the Lie-algebra $\overline{\mathcal{G}}_A$ and observe that the center Ω^1 of $\overline{\mathcal{G}}_A$ is spanned by $\{c(\underline{m}), \underline{m} \in \mathbb{Z}^d\}$. Let $A^1 = A_{d+1}$ be the Laurent polynomials ring in $d+1$ variables. Clearly $\overline{\mathcal{G}}_A/\Omega^1 \simeq \mathcal{G}_{A^1}$. Let ϕ denote the quotient map from $\overline{\mathcal{G}}_A \to \mathcal{G}_{A^1}$ which is clearly a central extension. Since $\tau_{[d+1]}$ is the universal central extension of $\mathcal{G}_{A^1}$ (see Proposition 2.2 of [**MEY**]), there exists a homomorphism $\varphi^1 : \tau_{[d+1]} \to \overline{\mathcal{G}}_A$ such that $\ker \varphi^1$ is central. It is not difficult to write down the homomorphism (see (5.4)) and then one can see that the $\ker \varphi^1$ consists of homogeneous elements with respect to the $\mathbb{Z}^{d+1}-$ grading. Hence we can extend the homomorphism $\varphi^1 : \tilde{\tau}_{[d+1]} \to \tilde{\overline{\mathcal{G}}}_A$ by sending $d_i \mapsto d_i (1 \le i \le d)$ and $d_{d+1} \mapsto d$. Also any $\tilde{\overline{\mathcal{G}}}_A$ - module can be treated as $\tilde{\tau}_{[d+1]}$ via φ^1.

3.5. Now let $(V(\lambda_i), \tau_i)$ be an highest weight module for $\tilde{\overline{\mathcal{G}}}$ and let $V = \otimes_i V(\lambda_i)$ and $V_A = V \otimes A$. Then (V_A, π^1) is a $\tilde{\tau}_{[d+1]}-$ module in the following way.

$$\pi^1 X(\underline{m})(v_1 \otimes \cdots \otimes v_k)(\underline{n}) = \sum a_i^{\underline{m}} (v_1 \otimes \cdots \otimes \tau_i(X) v_i \otimes \cdots \otimes v_k)(\underline{n} + \underline{m})$$

$$\pi^1 d_i (v_1 \otimes \cdots v_k)(\underline{n}) = n_i (v_1 \otimes \cdots \otimes v_k)(\underline{n}) \ (1 \le i \le d)$$

$$\pi^1 d (v_1 \otimes \cdots \otimes v_k)(\underline{n}) = \sum_i (v_1 \otimes \cdots \otimes \tau_i(d) v_i \otimes \cdots \otimes v_k)(\underline{n})$$

$$\pi^1 \ker \varphi^1 V_A = 0$$

for all $X \in \overline{\mathcal{G}}, v_i \in V$ and $\underline{m}, \underline{n} \in \mathbb{Z}^d$.

This definition is precisely the one given at (1.5) by taking the Kac-Moody Lie-algebra $\mathcal{G}$ to be the affine Lie-algebra $\overline{\mathcal{G}}$.

Let $\underline{h}$ be a Cartan subalgebra of $\overline{\mathcal{G}}$ and let ψ be $\mathbb{Z}^d-$graded algebra homomorphism from $U(\underline{h}_A) \to A$ given by $h(\underline{m}) \mapsto \sum \lambda_i(h) a_i^{\underline{m}} t^{\underline{m}}$ and denote the image by A_ψ.

3.6. Theorem. (1)(V_A, π) *is a completely reducible* $\tilde{\tau}_{[d+1]}$ *module if* A_ψ *is an irreducible* $\underline{h}_A-$*module.*

(2) (V_A, π) *is irreducible as a* $\tilde{\tau}_{[d+1]}-$ *module if* $a_i(\ell)/a_j(\ell), i \ne j$ *is not a* k*th root of unity.*

(3) *If* $V(\lambda_i)$ *is a unitarizable highest weight module for* $\overline{\mathcal{G}}$ *with respect to the form* w *and* $| a_i(\ell) | = | a_j(\ell) |$ *for all* i, j *and* ℓ, *then* (V_A, π^1) *is unitarizable* $\tilde{\tau}_{[d+1]}-$ *module with respect to* $\overline{w}$.

Proof. (1) and (2) follows from Theorem (1.8) by taking the Kac-Moody Lie-algebra $\mathcal{G}$ to be the affine Lie-algebra $\overline{\mathcal{G}}$.

(3) Follows from Proposition (2.3).

3.7. Remark. (1) The modules considered in Theorem (3.3) and (3.6) for Toroidal algebras are different. In the first case the full centre acts trivially where as in the second case a part of the centre acts non trivially.

(2) These are the first known irreducible module for Toroidal Lie-algebras where part of the center acts non trivially. The faithful modules constructed in [**MEY**] and [**EM**] through the use of vertex operators are all reducible. We will prove towards the end of the paper that the modules constructed in [**MEY**] and [**EM**] admits a filteration such that the successive irreducible quotient modules are isomorphic to the one considered here upto an automorphism of the Toroidal Lie-algebra.

4.

In this section we briefly review the construction of Fock space and the vertex operators that act on it. The theory is due to Frenkel-Kac [**FK**]. For further details one may also consult [**FLM**], [**GO**] and [**MEY**]. We will closely follow the notation from [**EM**]. We will also construct some automorphisms of Toroidal Lie-algebras.

4.1. Let Q be root lattice of the type ADE and let $\alpha_1, \alpha_2, \cdots \alpha_\ell$ be a $\mathbb{Z}$-basis. Let $A = (a_{ij})$ be the cartan matrix and $a_{ij} = (\alpha_i \mid \alpha_j)$. Let Γ be a free $\mathbb{Z}$- module on generators $\alpha_1, \alpha_2, \cdots \alpha_\ell, \delta_1, \delta_2, \cdots \delta_{n-1}, \; d_1, d_2, \cdots d_{n-1}$. Let $(.\mid.)$ be a $\mathbb{Z}-$valued symmetric bilinear form such that $(\delta_i \mid \delta_j) = (d_i \mid d_j) = (\alpha_i \mid \delta_j) = (\alpha_i \mid d_j) = 0, (\delta_i \mid d_j) = \delta_{i,j}$ and $(\alpha_i \mid \alpha_j) = a_{ij}$. Let Q be the sublattice generated by $\alpha_1, \alpha_2, \cdots \alpha_\ell, \delta_1, \cdots \delta_{n-1}$. Here n is any positive integer.

Let $\underline{t} = \mathbb{C} \otimes_{\mathbb{Z}} \Gamma, \underline{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q$ and $\underline{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q$. We will define a Heisenberg algebra structure $\hat{\underline{b}} = \oplus \underline{t}(k) \oplus \mathbb{C}c$ where each $\underline{t}(k)$ is an isomorphic copy of $\underline{t}$ and the isomorphism is by $\alpha \mapsto \alpha(k)$. The Lie-algebra structure is defined by

$$[\alpha(k), \beta(m)] = k(\alpha \mid \beta)\delta_{k+m,o}c.$$

Let $\hat{\underline{a}} = \oplus_{k \in \mathbb{Z}} \underline{h}(k) \oplus \mathbb{C}c$. Define $\underline{b} = \oplus_{k \neq o} \underline{t}(k) \oplus \mathbb{C}c \subseteq \hat{\underline{b}}$ and $\underline{b}_{\pm} = \oplus_{k \gtrless o} t(k)$.

Similarly define $\underline{a}, \underline{a}_{\pm}$ by replacing $\underline{t}$ by $\underline{h}$ and $\underline{a}$ by replacing $\underline{t}$ by $\underline{h}$.

The Fock space representation of $\underline{b}$ is the symmetric algebra $S(\underline{b}-)$ of $\underline{b}-$ together with an action of $\underline{b}$ on $S(\underline{b}-)$ defined by the following:

c acts as 1, $a(-m)$ acts as multiplication by $a(-m), m > o$
$a(m)$ acts as the unique derivation on $S(\underline{b}-)$ for which $b(-n) \to \delta_{m,-n} m(a \mid b).(a, b \in \underline{t}, m, n > 0)$.

$S(\underline{b}-)$ affords an irreducible representation of $\underline{b}$. However $S(\underline{a}-)$ does not afford an irreducible representation of $\underline{a}$ since the form is degenerate on $\underline{h}$.

A vector δ in Q is called a null root if $\delta = n_1\delta_1 + n_2\delta_2 + \cdots + n_{n-1}\delta_{n-1}$ for some integers n_i. Note that $(\delta \mid \delta) = 0$.

Following [**EMY**] we let $\epsilon : Q \times Q \to \{\pm 1\}$ be a bi- multiplicative map satisfying

(1) $\epsilon(\alpha, \alpha) = (-1)^{(\alpha|\alpha)}$

(2) $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{(\alpha|\beta)}$

(3) $\epsilon(\alpha, \delta) = 1$,

where $\alpha, \beta \in Q$ and δ a null root. Extend ϵ to a bimultiplicative map of $Q \times \Gamma \to \{\pm 1\}$ and form the vector space $\mathbb{C}[\Gamma]$, with $\mathbb{C}$ basis $\{e^\nu, \nu \in \Gamma\}$. Then $C[\Gamma]$ contains the space $\mathbb{C}[Q]$ similarly defined. Following Borcherds [**B**] make $\mathbb{C}[\Gamma]$ a $\mathbb{C}[Q]$ module by defining

$$e^\alpha . e^\nu = \epsilon(\alpha, \nu) e^{\alpha+\nu} (\alpha \in Q, \gamma \in \Gamma).$$

Let $M \subseteq S(\underline{b}-)$ be any $\underline{a}-$ submodule (with respect to the Fock space action). We define $V(\Gamma, M) = \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} M$. Of particular interest in the sequal will be $V(\Gamma, S(\underline{a}-))$ and $V(\Gamma, S(\underline{b}-))$ which we simply denote by $V(\Gamma)$ and $V(\Gamma, \underline{b})$ respectively. We extend the action of $\underline{a}$ on M to $\hat{\underline{a}}$ on $V(\Gamma, M)$ by

$$\begin{aligned} a(m)(e^\gamma \otimes u) &= e^\gamma \otimes a(m)u, m \neq 0 \\ a(0)e^\gamma \otimes u &= (a \mid \gamma) e^\gamma \otimes u. \end{aligned}$$

Vertex operators: Let z be a complex valued variable and let $\alpha \in Q$. Define

$$T_\pm(\alpha, z) = -\sum_{n \gtrless o} \frac{1}{n} \alpha(n) z^{-n}.$$

Then the vertex operator for α in Q is defined as $X(\alpha, z) = z^{\frac{(\alpha|\alpha)}{2}} \exp T(\alpha, z)$ where $\exp T(\alpha, z) = \exp T_-(\alpha, z) e^\alpha z^{\alpha(o)} \exp T_+(\alpha, z)$ and the operators $z^{\alpha(0)}$ is defined by

$$z^{\alpha(0)} e^\gamma \otimes u = z^{(\alpha|\gamma)} e^\gamma \otimes u.$$

$X(\alpha, z)$ can be formally expanded in powers of z to give

$$X(\alpha, z) = \sum_{n \in \mathbb{Z}} X_n(\alpha) z^{-n}$$

and the moments $X_n(\alpha) = \frac{1}{2\pi i} \int X(\alpha, z) z^n \frac{dz}{z}$ are operators on $V(\Gamma, M)$.

Let $\triangle$ be a root system and let $X_\alpha, \alpha \in \triangle, h \in \underline{h}$ be a Chevalley basis. Let $\underline{r} = (r_1, r_2, \cdots r_{n-1}) \in \mathbb{Z}^{n-1}$ and $\overline{r} = (\underline{r}, r_n) \in \mathbb{Z}^n$. Let $\delta_{\underline{r}} = r_1\delta_1 + \cdots + r_{n-1}\delta_{n-1}$.

We will now recall the main Theorem 3.14 of [**EM**].

4.2. Theorem. *The Lie-algebra generated by operators $X_m(\alpha + \delta)$, ($\alpha \in\Delta, m \in \mathbb{Z}$ and δ a null root) on $V(\Gamma, \underline{b})$ is isomorphic to the Toroidal Lie-algebra $\tau_{[n]}$. The isomorphism is given by*

$$\begin{aligned} X_{r_n}(\alpha + \delta_{\underline{r}}) &\mapsto X_\alpha \otimes t^{\overline{r}} \ , \alpha \in\Delta \\ T^h_{r_n}(\delta_{\underline{r}}) &\mapsto h \otimes t^{\overline{r}} \\ T^{\delta_i}_{r_n}(\delta_{\underline{r}}) &\mapsto t^{\overline{r}} t_i^{-1} dt_i \ (1 \le i \le n-1) \\ X_{r_n}(\delta_{\underline{r}}) &\mapsto t^{\overline{r}} t_n^{-1} dt_n \end{aligned}$$

where $t^{\overline{r}} = t_1^{r_1} t_2^{r_2} \cdots t_n^{r_n}$ *and*

$$T^g_m(\alpha) = \frac{1}{2\pi i} \int : g(z) X(\alpha, z) : \frac{dz}{z} z^m.$$

4.3. Here we construct certain automorphisms $\tau_{[d]}$ which are needed in our main result in section 5. Let $SL(d, \mathbb{Z})$ be the group of matrices of order $d \times d$ with integral entries and determinate 1.

Let $A = (a_{ji})_{\substack{1 \le i \le d \\ 1 \le j \le d}}$ be an element of $SL(d, \mathbb{Z})$. Let $\underline{r} = (r_1, r_2, \cdots r_d)$ and $\underline{s} = (s_1, s_2, \cdots s_d)$ be such that $r_i, s_i \in \mathbb{Z}$. Let $e_i = (0, \cdots, 1, \cdots 0)$ where 1 is at the ith place. Let $A\underline{r}^T = \underline{m}^T, A\underline{s}^T = \underline{n}^T$ and $Ae_i^T = a(i)^T$ where $\underline{m} = (m_1, m_2, \cdots m_d), \underline{n} = (n_1, n_2, \cdots n_d)$ and $a(i) = (a_{1i}, a_{2i}, \cdots a_{di})$ and T denotes the transpose. We now define an automorphism of $\tau_{[d]}$ again denoted by A.

$$\begin{aligned} A(X(\underline{r})) &= X(\underline{m}) \\ A(d(t^{\underline{r}})t^{\underline{s}}) &= d(t^{\underline{m}})t^{\underline{n}}. \end{aligned}$$

It is straightforward to check that A is an automorphism of $\tau_{[d]}$. We will now extend it to $\tilde{\tau}_{[d]}$. Let $(d_1^1, d_2^1, \cdots d_d^1)^T = A^{-1}(d_1, d_2, \cdots, d_d)^T$. Define $A(d_i) = d_i^1$ and check that it defines an automorphisms of $\tilde{\tau}_{[d]}$.

Remark. A does not quite preserve the *natural* $\mathbb{Z}^d$–gradation of $\tau_{[d]}$.

5.

The purpose of this section is to describe the subquotients of $V(\Gamma) = \mathbb{C}[\Gamma] \otimes S(\underline{a}-)$. We will follow the notation of §5 of [**MEY**].

Let $\lambda \in \Gamma$ and define $V(\lambda) := e^{\lambda+Q} \otimes S(\underline{a}-)$. Note that $V(\Gamma) = \oplus_{\lambda \in \Gamma/Q} V(\lambda)$. Each $V(\lambda)$ is a cyclic $\tau_{[n]}-$ module with generator $e^\lambda \otimes 1$. Towards the end of the section we will exhibit a decreasing filteration of modules and describe its successive irreducible quotients.

Fix $\lambda \in \Gamma$ and let $W(\lambda) := W = e^{\lambda+Q} \otimes S(\underline{a}-)$. This is a module for the non-twisted affine Lie-algebra $\mathcal{G}_{aff} = \mathcal{G} \otimes \mathbb{C}[t_n, t_n^{-1}] \oplus \mathbb{C}c$ generated by the operators $X_n(\alpha), \alpha \in \Delta$. But we know by Frenkel-Kac theory [**FK**] that this is an irreducible level 1 representation of $\mathcal{G}_{aff}$. Let $V = V(\lambda)$.

The following Lemma is trivial to see.

5.1. Lemma.

$$V = \oplus_{n_1,n_2,\cdots n_k>0,k\geq 0} \ \delta_{i_1}(-n_1)\cdots\delta_{i_k}(-n_k)e^{\lambda+Q}\otimes S(\underline{a}\ -).$$

Let $M = \oplus_{n_1,n_2,\cdots n_k>0,k>0} \ \delta_{i_1}(-n_1)\cdots\delta_{i_k}(-n_k)e^{\lambda+\dot{Q}}\otimes S(\underline{\dot{a}}\ -)$.

By Lemma (5.1) it is easy to verify that M is a $\tau_{[n]}-$ module.

We will recall the following Lemma (5.2) of [**MEY**], which though stated for the $n = 2$ case there, is true for any n.

5.2. Lemma . *Let* $\lambda \in \Gamma$, *let* δ *be a null root, let* $N = (\lambda \mid \delta)$ *and let* $m \in \mathbb{Z}$. *Then*

$$X_m(\delta)e^{\lambda}\otimes 1 = \begin{cases} \epsilon(\delta,\lambda)e^{\lambda+\delta}\otimes S_{-m-N}(\delta), m+n<0 \\ \epsilon(\delta,\lambda)e^{\lambda+\delta}\otimes 1, m+n=0 \\ 0\ , m+n>0 \end{cases}$$

where the operators $S_p(\delta)$ *are defined by* $\exp T_-(\delta,z) = \sum_{p=0}^{\infty} S_p(\delta)z^p$.

5.3. Set $\tau_i = X_{-N_i}(\delta_i)$ where $N_i = (\lambda \mid \delta_i)$. We note that $\tau_i^k = X_{-kN_i}(k\delta_i)$. Let $\tau^{\underline{m}} = \tau_1^{m_1}\tau_2^{m_2}\cdots\tau_{n-1}^{m_{n-1}}$ and let

$$W\otimes t^{\underline{m}} := \tau^{\underline{m}}W = e^{\lambda+\sum m_i\delta_i+\dot{Q}}\otimes S(\underline{\dot{a}}\ -).$$

Lemma. $V/M \cong W\otimes A_{[n-1]}$ *as Vector spaces.*

Our aim is to describe V/M as a $\tilde{\tau}_{[n]}-$ module.

5.4. Note that $\mathcal{G}_{aff}\otimes A_{[n-1]} = \dot{\mathcal{G}}\otimes A_{[n]} \oplus \sum_{\underline{m}\in\mathbb{Z}^{n-1}} \mathbb{C}c\otimes t^{\underline{m}}$.

See Section (3.4) where we considered such a Lie-algebra.

Define a Lie-algebra homomorphism $\varphi^1 : \tau_{[n]} \to \mathcal{G}_{aff}\otimes A_{[n-1]}$ by

1) φ^1 is Id. on $\dot{\mathcal{G}}\otimes A_{[n]}$

2) $\varphi^1(t^{\overline{m}}t_i^{-1}dt_i) = \begin{cases} 0 \text{ if } m_n \neq 0 \\ 0 \text{ if } m_n = 0, 1\leq i\leq n-1 \\ c\otimes t^{\overline{m}} \text{ if } m_n = 0 \text{ and } i = n. \end{cases}$

5.5. Let $(\overline{V}, \pi)$ be an irreducible highest weight module for $\mathcal{G}_{aff}$. Then $\overline{V} \otimes A_{[n-1]}$ admits an action by $\mathcal{G}_{aff} \otimes A_{[n-1]}$ in the following way,
$X(\underline{m})v(\underline{n}) = Xv(\underline{m} + \underline{n}), (X \in \mathcal{G}_{aff}, v \in \overline{V})$.

Now we extend the module action to one of $\tau_{[n]}$ via the surjective homomorphism φ^1. That is

$$X.v(\underline{m}) = \varphi^1(X).v(\underline{m}).$$

It is easy to see that $\overline{V} \otimes A_{[n-1]}$ admits a natural $\mathbb{Z}^{n-1}$- gradation and is graded irreducible as a $\tau_{[n]}-$ module. In other words $\tilde{\tau}_{[n]}$ irreducible (one can apply Theorem 3.6(2) taking $d+1 = n$ and $k = 1$).

We can twist the module action of $\tilde{\tau}_{[n]}$ by an automorphism A considered in (4.3), so that $(\overline{V} \otimes A_{[n-1]},\ Ao\varphi^1 o\pi)$ is a $\tilde{\tau}_{[n]}$ irreducible module.

With the notations in (5.3) we have the following result.

5.6. Theorem. *V/M is isomorphic to $(W \otimes A_{[n-1]}, Ao\varphi^1 o\pi)$ as $\tau_{[n]}-$ module where $A = (b_{ij})$ and $b_{ii} = 1, b_{in} = (\lambda \mid \delta_i), 1 \leq i \leq n-1, b_{ij} = 0$ otherwise.*

We need some lemmas before we give a proof of the Theorem.

5.7. Lemma. *The following equality holds as operators on V/M.*

(1) $\delta_{\underline{m}}(n) = \begin{cases} 0 \text{ if } n \neq o \\ (\lambda \mid \delta_{\underline{m}}) \text{ if } n = 0 \end{cases}$

(2) $X_n(\delta_{\underline{m}}) = \begin{cases} 0 \text{ if } n + (\lambda \mid \delta_{\underline{m}}) \neq 0 \\ \tau^{\underline{m}} \text{ if } n + (\lambda \mid \delta_{\underline{m}}) = 0 \end{cases}$

(3) $X_n(\alpha + \delta_{\underline{m}}) = X_{-(\lambda|\delta_{\underline{m}})}(\delta_{\underline{m}}) X_{n+(\lambda|\delta_{\underline{m}})}(\alpha) = X_{n+(\lambda|\delta_{\underline{m}})}(\alpha)\tau^{\underline{m}}, \alpha \in \dot{\Delta}$.

(4) $T_n^h(\delta_{\underline{m}}) = h(n + (\lambda \mid \delta_{\underline{m}})) X_{-(\lambda|\delta_{\underline{m}})}(\delta_{\underline{m}}) = h(n + (\lambda \mid \delta_{\underline{m}}))\tau^{\underline{m}}, h \in \dot{h}$.

Proof. First observe that $\delta_{\underline{m}}(n)$ and $X_n(\delta_{\underline{m}})$ are central operators on V and hence are determined on the generator $e^\lambda \otimes 1$.

1. By definition $\delta_{\underline{m}}(n)e^\lambda \otimes 1 = 0$ for $n > 0$ and $\delta_{\underline{m}}(0)e^\lambda \otimes 1 = (\lambda \mid \delta_{\underline{m}})e^\lambda \otimes 1$. Hence $\delta_{\underline{m}}(n)$ on V/M is zero for $n > 0$ and $(\lambda \mid \delta_{\underline{m}})$ for $n = 0$. For $n < 0, \delta_{\underline{m}}(n)e^\lambda \otimes 1 = e^\lambda \otimes \delta_{\underline{m}}(n) \in M$ and hence $\delta_{\underline{m}}(n)$ is zero on V/M.
2. The case $n + (\lambda \mid \delta_{\underline{m}}) > 0$ follows from the Lemma (5.2). Also for the case $n + (\lambda \mid \delta_{\underline{m}}) < 0$, note that

$$X_n(\delta_{\underline{m}})e^\lambda \otimes 1 \in M \text{ (by Lemma 5.2)}.$$

Hence $X_n(\delta_{\underline{m}})$ is zero on V/M. Now let $n + (\lambda \mid \delta_{\underline{m}}) = 0$. First we will note that it will follow from the definition of vertex operators that

$X(\alpha+\beta, z) = X(\alpha, z)X(\beta, z)$ whenever $(\alpha \mid \beta) = 0$. Hence we have the following:

$$X_n(\delta_{\underline{m}}) = \sum_{k_1+k_2+\cdots+k_{n-1}=n} X_{k_1}(m_1\delta_1)\cdots X_{k_{n-1}}(m_{n-1}\delta_{n-1})$$

$$= \sum_{\substack{k_1+k_2+\cdots+k_{n-1}= \\ n, k_i+m_i(\lambda \mid \delta_i)\le 0}} X_{k_1}(m_1\delta_1)\cdots X_{k_{n-1}}(m_{n-1}\delta_{n-1}) \text{ (by Lemma 5.2).}$$

But $0 = n + (\lambda \mid \delta_{\underline{m}}) = \sum_i (k_i + m_i(\lambda \mid \delta_i)) \le 0$. It follows that $k_i + m_i(\lambda \mid \delta_i) = 0$ for all i. Now by Lemma 5.2 (2) we have $X_{k_i}(m_i(\lambda \mid \delta_i)) = \tau_i^{m_i}$. Hence $X_n(\delta_{\underline{m}}) = \tau^{\underline{m}}$.

3. Since $(\alpha \mid \delta_{\underline{m}}) = 0, \alpha \in \triangle$ we have

$$\begin{aligned} X_n(\alpha+\delta_{\underline{m}}) &= \sum_k X_{m-k}(\alpha)X_k(\delta_{\underline{m}}) \\ &= X_{n+(\lambda|\delta_{\underline{m}})}(\alpha)\ X_{-(\lambda|\delta_{\underline{m}})}(\delta_{\underline{m}}) \text{ on } V/M \text{ (by(2))} \\ &= X_{n+(\lambda|\delta_{\underline{m}})}(\alpha)\ \tau^{\underline{m}} \text{ (by (2)).} \end{aligned}$$

4. The proof is similar to (3). We only note that

$$T_n^h(\delta_{\underline{m}}) = \sum_k h(n-k)X_n(\delta_{\underline{m}}) \text{ (by def).}$$

□

Proof of the Theorem 5.6. We will first describe the $\tau_{[n]}-$ module $(W \otimes A_{[n-1]}, Ao\varphi^1 o\pi)$.
$X(\overline{r})v(\underline{k}) = (\pi X \otimes t_n^{r_n+(\lambda|\delta_{\underline{r}})}v)(\underline{r}+\underline{k})$.

$$d(t^{\overline{r}})t^{\overline{s}}v(\underline{k}) = \begin{cases} 0 \text{ if } r_n+s_n+(\lambda \mid \delta_{\underline{r}+\underline{s}}) \ne 0 \\ (r_n + (\lambda \mid \delta_{\underline{r}}))v(\underline{k}+\underline{r}+\underline{s}) \text{ if } r_n+s_n+(\lambda \mid \delta_{\underline{r}+\underline{s}}) = 0 \end{cases}$$

This follows from definition of A, φ^1 and π.

We will now compute the action of $\tau_{[n]}$ on V/M using Lemma (5.7) and then verify it to be exactly as above. That will complete the proof.

(a) $X_\alpha(\overline{r}).w(\underline{k}) = X_{r_n}(\alpha+\delta_{\underline{r}})w(\underline{k})$ (by Theorem 4.1)
$= X_{r_n+(\lambda|\delta_{\underline{r}})}(\alpha)w(\underline{r}+\underline{k})$ (by Lemma 5.7 and Notation at 5.3)
$= \pi(X_\alpha \otimes t_n^{r_n+(\lambda|\delta_{\underline{r}})})w(\underline{r}+\underline{k})$ (by [FK]),

(b) $h(\overline{r}).w(\underline{k}) = T^{h}_{r_n}(\alpha + \delta_{\underline{r}})w(\underline{k})$ (by Theorem 4.1)
$= h(r_n + (\lambda \mid \delta_{\underline{r}}))w(\underline{r} + \underline{k})$ (by Lemma 5.7 and Notation 5.3)
$= \pi(h \otimes t^{r_n + (\lambda|\delta_{\underline{r}})})w(\underline{r} + \underline{k})$(by [FK]),

(c) $d(t^{\overline{r}}).t^{\overline{s}}w(\underline{k}) = (T^{\delta_{\underline{r}}}_{r_n+s_n}(\delta_{\underline{r}+\underline{s}}) + r_n X_{r_n+s_n}(\delta_{\underline{r}+\underline{s}}))w(\underline{k})$ (by Theorem 4.1)
$= \delta_{\underline{r}}(r_n + s_n + (\lambda \mid \delta_{\underline{r}+\underline{s}}))w(\underline{k} + \underline{r} + \underline{s})$
$+ r_n X_{r_n+s_n}(\delta_{\underline{r}+\underline{s}})w(\underline{k} + \underline{r} + \underline{s})$ (by Lemma 5.7 and Notation at 5.3)
$= \begin{cases} ((\lambda \mid \delta_{\underline{r}}) + r_n)w(\underline{k} + \underline{r} + \underline{s}) \text{ if } r_n + s_n + (\lambda \mid \delta_{\underline{r}+\underline{s}}) = 0 \\ 0 \text{ otherwise} \end{cases}$

(follows from Lemma 5.7 (1) and 5.7 (2)). □

5.8. Remark. Note that the derivations $d_1, d_2, \cdots d_{n-1}$ act on V as $d_i(0).e^{\lambda} \otimes a = (\lambda \mid d_i)e^{\lambda} \otimes a$ and remain the same on the quotient V/M. Also note that $A^{-1}d_i = d_i^1 = d_i (1 \leq i \leq n-1)$ (see 4.3). Hence the isomorphism in the theorem preserves the natural $\mathbb{Z}^{n-1}$ gradation.

5.9. Remark. Let $V_N = \oplus_{k \geq N} \delta_{i_1}(-n_1) \cdots \delta_{i_k}(-n_k)e^{\lambda+Q} \otimes S(\underline{a}-)$ which is a $\tau_{[n]}$ submodule of V. Further $V = V_o \supseteq V_1 \supseteq V_2 \supseteq \cdots$ is decreasing sequence of $\tau_{[n]}-$ modules. Consider $V_N/V_{N+1} = \oplus \delta_{i_1}(-n_1) \cdots \delta_{i_N}(-n_N)e^{\lambda+Q} \otimes S(\underline{a}\ -)$ as $\tau_{[n]}-$ module. Then for a fixed $i_1, i_2 \cdots, i_N$ and $n_1, n_2, \cdots, n_N$, $F = \delta_{i_1}(-n_1) \cdots \delta_{i_N}(-n_N)e^{\lambda+Q} \otimes S(\underline{a}\ -)$ is a $\tau_{[n]}$ submodule of V_N/V_{N+1}. It is also easy to see that $F \cong V/M$ as $\tau_{[n]}-$ module. In other words there exists a filteration of $\tau_{[n]}$ submodules of V such that the successive quotients are $\mathbb{Z}^{n-1}-$ graded irreducible and isomorphic to V/M.

5.10. Recall $d_1, d_2, \cdots d_{n-1}$ from Section 4. Let $N \geq 0$ and let

$$W_N = \oplus_{0 \leq k \leq N} d_{i_1}(-n_1) \cdots d_{i_k}(-n_k)V.$$

Clearly $V = W_o \subseteq W_1 \subseteq W_2 \subseteq \cdots$. We will first prove that W_N is a $\tau_{[n]}-$ module. Since $X_n(\alpha)$ generate $\tau_{[n]}$ as a Lie-algebra (see Theorem 3.14 of [**EM**]) it is sufficient to prove the

5.11. Lemma. *For α, m, k and for all v in V*

$$X_m(\alpha)d_{i_1}(-n_1) \cdots d_{i_k}(-n_k)v \in W_k.$$

Proof. By induction on k. Clearly this is true for $k = 0$. For $k = 1$,

$$X_m(\alpha)d_{i_1}(-n_1)v = d_{i_1}(-n_1)X_m(\alpha)v + (\alpha \mid d_{i_1})X_{m-n_1}(\alpha)v$$

(by 3.8 (1) of [**EM**]). Clearly $d_{i_1}(-n_1)X_n(\alpha)v \in W_1$ and $X_{m-n_1}(\alpha) \in W_o \subseteq W_1$. Hence we are done.

We will now assume the Lemma for all $\ell, 1 \leq \ell \leq k$ and prove it for $\ell = k+1$.

Consider

$$X_m(\alpha)d_{i_1}(-n_1)\cdots d_{i_{k+1}}(-n_{k+1})v$$
$$= \sum_{1\leq j\leq k+1} T_j \; + \; d_{i_1}(-n_1)\cdots d_{i_{k+1}}(-n_{k+1})X_m(\alpha)v$$

where $T_j = d_{i_1}(-n_1)\cdots[X_m(\alpha), d_{i_j}(-n_j)]\cdots d_{i_{k+1}}(-n_{k+1})v$. For $1 \leq j \leq k+1$, we have $k+1-j \leq k$ and by induction hypothesis we have

$$(\alpha \mid d_j)X_{m-n_j}(\alpha)d_{i_{j+1}}(-n_{j+1})\cdots d_{i_{k+1}}(-n_{k+1})v \in W_{k+1-j}.$$

Hence $T_j \in W_k \subseteq W_{k+1}$. (See 3.8 (1) of [**EM**]). This completes the proof of the Lemma. □

From the above we also have

$$X_m(\alpha)d_{i_1}(-n_1)\cdots d_{i_{k+1}}(-n_{k+1})v = d_{i_1}(-n_1)\cdots d_{i_{k+1}}(-n_{k+1})X_m(\alpha)v \text{ in } W_{k+1}/W_k. \tag{5.12}$$

Further $W_{k+1}/W_k = \oplus d_{i_1}(-n_1)\cdots d_{i_{k+1}}(-n_{k+1})V$ and from (5.12) each $d_{i_1}(-n_1)\cdots d_{i_{k+1}}(-n_{k+1})V$ is a submodule of W_{k+1}/W_k isomorphic to V.

Put $\cup_{i\geq 0}W_i = \underline{b}(\lambda)$ and remember each W_i depends on V and V in turn depends on λ.

5.13. Remark. (1) The full Fock space $V(\Gamma, \underline{b}) = \oplus_{\lambda\in\Gamma/Q}\underline{b}(\lambda)$ as $\tau_{[n]}-$ module.

(2) Each $\underline{b}(\lambda)$ admits a filteration by an increasing sequence of modules whose successive quotients are isomorphic to V (see above).

(3) Each V above admits a filtration by a decreasing sequence of modules such that the successive quotients are irreducible (see Remark 5.9).

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TATA INSTITUTE OF FUNDAMENTAL RESEARCH
HOMI BHABHA ROAD
BOMBAY - 5
INDIA
E-mail address: Senapati@tifrvax.bitnet