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ON GEOMETRIC PROPERTIES OF HARMONIC Lip_1-CAPACITY

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We shall study geometric properties of the harmonic Lip_1-capacity $\kappa'_n(E)$, $E \subset \mathbf{R}^n$. It is related to functions which are harmonic outside E and locally Lipschitzian everywhere. We shall show that $\kappa'_{n+1}(E \times I)$ is comparable to $\kappa'_n(E)$ for $E \subset \mathbf{R}^n$ and for intervals $I \subset \mathbf{R}$. We shall also show that if E lies on a Lipschitz graph, then $\kappa'_n(E)$ is comparable to the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}(E)$. Finally we give some general criteria to guarantee that $\kappa'_n(E) = 0$ although $\mathcal{H}^{n-1}(E) > 0$.

1. Introduction

We shall investigate some geometric properties of the harmonic Lipschitz and C^1 capacities κ'_n and κ_n in $\mathbf{R}^n$ which were introduced in [**P**]. For the definitions see Section 2. The compact null-sets of these capacities are exactly the removable sets for the corresponding classes of harmonic functions, see Section 2, and they appear very naturally in connection of harmonic approximation problems, cf. [**P**]. The analogs for them in theory of bounded analytic functions of the complex plane are the analytic capacity γ and the continuous analytic capacity α, see e.g. [**G2**].

In Section 3 we shall study sets $E \times I$ in $\mathbf{R}^{n+1}$ where E is a bounded set in $\mathbf{R}^n$ and I an interval in $\mathbf{R}$. We shall show that $\kappa'_{n+1}(E \times I)$ is comparable to $\kappa'_n(E)$ and $\kappa_{n+1}(E \times I)$ to $\kappa_n(E)$. This gives some information about the geometric measure-theoretic properties of the null-sets of κ'_n. First we note that, as for the analytic capacity, it is easy to see that if the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}(E)$ of E is zero, then $\kappa'_n(E) = 0$ and that if the Hausdorff dimension of E is greater than $n-1$, then $\kappa'_n(E) \geq \kappa(E) > 0$. Thus problems occur only when E has dimension $n-1$ and $\mathcal{H}^{n-1}(E) > 0$. Since the null-sets for γ are also null-sets for κ'_2, we can start from the many known examples where $\gamma(E) = 0$ and $\mathcal{H}^1(E) > 0$, see e.g. [**V**], [**G1**], [**G2**], [**M2**] and [**FX**], and take products with intervals to obtain various compact sets E in $\mathbf{R}^n$ with $\kappa'_n(E) = 0$ and $\mathcal{H}^{n-1}(E) > 0$. Earlier Uy in [**U2**] generalized the example and technique of Garnett from [**G1**] to find such a set. We shall also see that the null-sets for κ'_n and the $(n-1)$-dimensional integral-geometric (Favard) measure are different. For $n = 2$ this follows from [**M3**] and for general n by taking products with intervals.

In Section 4 we shall study κ'_n on sufficiently regular hypersurfaces, for example on Lipschitz graphs. Using the methods of singular integrals, as in [**C**, §VII] and [**U1**], we show that on such surfaces κ'_n is comparable to $\mathcal{H}^{n-1}$.

In the last section we shall give some general geometric measure-theoretic conditions on compact subsets E of $\mathbf{R}^n$ with $0 < \mathcal{H}^{n-1}(E) < \infty$ which imply $\kappa'_n(E) = 0$. Corresponding results for γ were found in [**M2**]. These conditions apply for example to $(n-1)$-dimensional self-similar sets satisfying Hutchinson's open set condition, see [**H**], yielding that such a set has zero κ'_n capacity unless it lies on a hyperplane. For sets lying on a hyperplane, κ'_n is comparable to $\mathcal{H}^{n-1}$ as follows from Section 4, or already from [**U1**, p. 298] and [**P**, Lemma 2.2 (8)].

2. Preliminaries

The norms $\|f\|$ and $\|f\|_\mu$ of a function f will stand for the L^∞ norms of f with respect to the Lebesgue measure and a Borel measure μ in $\mathbf{R}^n$, respectively. For a measure μ, $\|\mu\|$ is its variation norm. We denote by $B(x,r)$ or $B^n(x,r)$ the open ball with center $x \in \mathbf{R}^n$ and radius r.

Let $\mathrm{Lip}^1_{\mathrm{loc}}(\mathbf{R}^n)$ be the set of all real-valued locally Lipschitz functions (with exponent 1) on $\mathbf{R}^n$ and $C^1_{\mathrm{loc}}(\mathbf{R}^n)$ the set of all real-valued continuously differentiable functions on $\mathbf{R}^n$ (both without any assumption on the behavior at ∞). The fundamental solution Φ_n for the Laplace equation $\Delta_n f = 0$ in $\mathbf{R}^n$ is defined by

$$\Phi_n(x) = \begin{cases} -\dfrac{1}{2\pi}\log\dfrac{1}{|x|} & \text{for } n = 2, \\ -\dfrac{a_n}{|x|^{n-2}} & \text{for } n \geq 3, \text{ where } a_n > 0 \text{ is a constant.} \end{cases}$$

We now introduce the classes of admissible functions for the definitions of harmonic capacities. For a bounded set E in $\mathbf{R}^n$, set

$$\mathcal{U}_n(E) = \{f \in C^1_{\mathrm{loc}}(\mathbf{R}^n) : \mathrm{Supp}(\Delta_n f) \subset E,\ \|\nabla_n f\| \leq 1,\ \nabla_n f(\infty) = 0\},$$
$$\mathcal{U}'_n(E) = \{f \in \mathrm{Lip}^1_{\mathrm{loc}}(\mathbf{R}^n) : \mathrm{Supp}(\Delta_n f) \subset E,\ \|\nabla_n f\| \leq 1,\ \nabla_n f(\infty) = 0\},$$

where $\mathrm{Supp}(\Delta_n f)$ is the support of the distribution $\Delta_n f$. We shall consider functions modulo constants in $\mathcal{U}_n(E)$ and $\mathcal{U}'_n(E)$, that is, we shall write $f = g$ for functions f and g in $\mathcal{U}_n(E)$ and $\mathcal{U}'_n(E)$ if $f - g$ is constant. Note that the functions in $\mathcal{U}_n(E)$ and $\mathcal{U}'_n(E)$ are harmonic in $\mathbf{R}^n \setminus E$ and the defining conditions mean that $f = \Phi_n * (\Delta_n f) +$ constant. The C^1 and Lipschitz harmonic capacities of E are defined by

$$\kappa_n(E) = \sup\{ \langle \Delta_n f, 1 \rangle : f \in \mathcal{U}_n(E)\},$$
$$\kappa_n'(E) = \sup\{ \langle \Delta_n f, 1 \rangle : f \in \mathcal{U}_n'(E)\},$$

where, as usual, $\langle g, \varphi \rangle$ means the action of the distribution g on a smooth function φ.

Letting $\alpha(n-1)$ be the volume of the unit ball in $\mathbf{R}^{n-1}$, we define the $(n-1)$-dimensional (spherical) Hausdorff measure for a subset E of $\mathbf{R}^n$ by

$$\mathcal{H}^{n-1}(E) = \lim_{\delta \downarrow 0} \inf\Big\{ \sum_{i=1}^{\infty} \alpha(n-1)\, r_i^{n-1} : E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i),\ r_i \le \delta \Big\}.$$

Then $\mathcal{H}^{n-1}$ in $\mathbf{R}^{n-1}$ is the Lebesgue measure and, more generally, its restriction to sufficiently regular hypersurfaces gives the surface measure. We also let $\sigma(n) = \mathcal{H}^{n-1}(S^{n-1})$ be the area of the unit sphere in $\mathbf{R}^n$.

We shall now show that the null-sets for the above harmonic capacities are the same as the removable sets for the corresponding classes of harmonic functions. This fact was already noted in [**P**, Remark 2.4].

Definition 2.1. A subset E of $\mathbf{R}^n$ is called Lip_1-removable for harmonic functions, abbreviated L_1RH, if for each domain D in $\mathbf{R}^n$ every locally Lipschitz function $f : D \to \mathbf{R}$ which is harmonic in $D \setminus E$ is harmonic in D.

The C^1 removable sets for harmonic functions, C^1RH, are defined in a similar way.

Proposition 2.2. *A bounded subset E of $\mathbf{R}^n$ is*
(1) *L_1RH if and only if $\kappa_n'(E) = 0$,*
(2) *C^1RH if and only if $\kappa_n(E) = 0$.*

Proof. We shall prove (1); the proof of (2) is similar.

Let E be L_1RH. If $f \in \mathcal{U}_n'(E)$, then f is harmonic on $\mathbf{R}^n$. Since $|f(x)| = O(\Phi_n(x))$, as $|x| \to \infty$, Liouville's theorem yields that f is constant. Hence $\langle \Delta f, 1 \rangle = 0$, which gives $\kappa_n'(E) = 0$.

Suppose E is not L_1RH. Then there exist a compact subset F of E, a domain D and a locally Lipschitz function f on D which is harmonic in $D \setminus F$ but not in D. Then f is not harmonic in the distributional sense which means that there exists $\varphi \in C_0^\infty(D)$ with $\langle \Delta f, \varphi \rangle = \langle f, \Delta\varphi \rangle > 0$. Set $f_\varphi = \Phi_n * (\varphi \Delta f)$. In the same way as in [**P**, Lemma 4.2] one can prove that $f_\varphi \in \mathrm{Lip}^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $\|\nabla f_\varphi\| = A < \infty$. Since $\mathrm{Supp}(\Delta f_\varphi) \subset F$, we have $f_\varphi / A \in \mathcal{U}_n'(F)$ and $\langle \Delta(f_\varphi/A), 1 \rangle = \langle \varphi \Delta f, 1 \rangle / A = \langle \Delta f, \varphi \rangle / A > 0$. Thus $\kappa_n'(F) > 0$ and so $\kappa_n'(E) > 0$. □

Remark. As noted in [**P**, Lemma 2.2 (9)], $\kappa_2'(E) \leq 2\pi\gamma(E)$ for $E \subset \mathbf{R}^2$. However, it is not known whether $\kappa_2'(E) \geq A\gamma(E)$ for some constant A, nor even whether $\kappa_2'(E) = 0$ implies $\gamma(E) = 0$. Another fundamental open problem is to decide if $\kappa_2'(K) \geq A\,\mathrm{diam}(K)$ for continua $K \subset \mathbf{R}^2$. For γ this holds, see [**G2**].

3. Harmonic capacities of product sets

We shall prove the following estimates:

Theorem 3.1. *Let r and δ be positive numbers, $E \subset B^n(0,r)$, $n \geq 2$, and $\widehat{E} = E \times [0,\delta] \subset \mathbf{R}^{n+1}$. Then*

$$\frac{A^{-1}\delta\kappa_n(E)}{\max\{1,(r/\delta)^2\}} \leq \kappa_{n+1}(\widehat{E}) \leq A\max\{\delta,r\}\kappa_n(E) \tag{1}$$

and

$$\frac{A^{-1}\delta\kappa_n'(E)}{\max\{1,(r/\delta)^2\}} \leq \kappa_{n+1}'(\widehat{E}) \leq A\max\{\delta,r\}\kappa_n'(E), \tag{2}$$

where A is a positive and finite constant depending only on n.

Proof. First we prove the left hand side inequality in (1). We can find $f \in \mathcal{U}_n(E)$ with $\langle \Delta_n f, 1\rangle = \kappa_n(E)/2$. Define $F \in C^1_{\mathrm{loc}}(\mathbf{R}^{n+1})$ by $F(x,x_{n+1}) = f(x)$. Obviously, $\|\nabla_{n+1}F\| \leq 1$. Choose a C^∞ function φ_1 such that $\mathrm{Supp}\,\varphi_1 \subset B^n(0,\max\{2\delta,2r\})$, $0 \leq \varphi_1 \leq 1$, $\varphi_1 = 1$ in some neigborhood of E and $\|\Delta_n\varphi_1\| \leq A_1/\delta^2$. Here and below in this proof $A_1, A_2, \ldots$ will be positive constants depending only on n. Choose also a C^∞ function φ_2 such that $\mathrm{Supp}\,\varphi_2 \subset (0,\delta)$, $0 \leq \varphi_2 \leq 1$, $\varphi_2 = 1$ on $(\delta/3, 2\delta/3)$ and $\|\varphi_2''\| \leq A_2/\delta^2$. Define φ by $\varphi(x,x_{n+1}) = \varphi_1(x)\,\varphi_2(x_{n+1})$ for $x \in \mathbf{R}^n$, $x_{n+1} \in \mathbf{R}$. Then $\|\Delta_{n+1}\varphi\| \leq A_3/\delta^2$.

Consider the localizing operator of Vitushkin:

$$F_\varphi = \Phi_{n+1} * (\varphi\Delta_{n+1}F).$$

According to [**P**, Lemma 4.2] one has

$$\|\nabla_{n+1}F_\varphi\| \leq A_4\max\{\delta^2,r^2\}/\delta^2 \equiv M.$$

Since

$$\begin{aligned}\Delta_{n+1}F_\varphi(x,x_{n+1}) &= \varphi_1(x)\,\varphi_2(x_{n+1})\Delta_{n+1}F(x,x_{n+1})\\ &= \varphi_1(x)\,\varphi_2(x_{n+1})\Delta_n f(x),\end{aligned}$$

we have $\mathrm{Supp}(\Delta_{n+1}F_\varphi) \subset \widehat{E}$ and $F_\varphi/M \in \mathcal{U}_{n+1}(\widehat{E})$. Finally

$$\begin{aligned}\langle \Delta_{n+1}F_\varphi, 1\rangle &= \langle \varphi_1(x)\,\varphi_2(x_{n+1})\Delta_n f(x), 1\rangle \\ &= \langle \Delta_n f, 1\rangle \int \varphi_2(t)\,dt \geq \delta\kappa_n(E)/6,\end{aligned}$$

which gives

$$\kappa_{n+1}(\widehat{E}) \geq \tfrac{1}{6}\delta\kappa_n(E)/M$$

as required.

Next we prove the right hand side inequality of (1). Choose $F \in \mathcal{U}_{n+1}(\widehat{E})$ with $\langle \Delta_{n+1}F, 1\rangle = \kappa_{n+1}(\widehat{E})/2$. Let $R = 2\max\{\delta, r\}$ so that $\widehat{E} \subset B^{n+1}(0,R)$. There exists $k > 1$, depending only on n, such that for $(x, x_{n+1}) \in \mathbf{R}^{n+1} \setminus B^{n+1}(0,kR)$ one has

$$F(x, x_{n+1}) = \sum_{|\alpha|\geq 0} c_\alpha \partial^\alpha \Phi_{n+1}(x, x_{n+1}). \tag{3}$$

Here $\alpha \in \mathbf{Z}_+^{n+1}$, $|\alpha|$, ∂^α and $\alpha!$ are as usual, cf. [**P**, § 2]. Note that $c_{(0,\ldots,0)} = \kappa_{n+1}(\widehat{E})/2$ and, by [**P**, (3.4)],

$$|c_\alpha| \leq A_5 \frac{(3R)^{|\alpha|}\kappa_{n+1}(\widehat{E})}{\alpha!}. \tag{4}$$

Define F_R by

$$F_R(x, x_{n+1}) = F(x, x_{n+1}) - F((x, x_{n+1}) + (5R, 0, \ldots, 0)),$$

and f by

$$f(x) = \int_{-\infty}^{\infty} F_R(x,t)\,dt.$$

(When $n \geq 3$ we can take $F_R = F$ and the computations below will be easier.) Evidently $f \in C^1_{\mathrm{loc}}(\mathbf{R}^n)$ and f is harmonic outside $E_R = E \cup E'_R$ where $E'_R = \{x \in \mathbf{R}^n : x + (5R, 0, \ldots, 0) \in E\}$.

We need to estimate $\|\nabla_n f\|$ and the behavior of $f(x^1)$, where $x^1 = (x_1, 0, \ldots, 0)$, as $x_1 \to \infty$. We obtain from the estimate [**P**, (3.5)] for $|(x, x_{n+1})| > kR$,

$$|\nabla_{n+1}F(x, x_{n+1})| \leq A_6\kappa_{n+1}(\widehat{E})|(x, x_{n+1})|^{-n},$$

and from the fact that $\|\nabla_{n+1}F_R\| \leq 2$ we derive

$$\begin{aligned}|\nabla_n f(x)| &\leq \int_{-kR}^{kR} 2\,dt + \int_{kR}^{\infty} \frac{A_6\kappa_{n+1}(\widehat{E})}{(|x|^2+t^2)^{n/2}}\,dt \\ &= 4kR + A_6\kappa_{n+1}(\widehat{E})|x|^{1-n}\int_{kR/|x|}^{\infty}(1+\tau^2)^{-n/2}d\tau \\ &\leq 4kR + A_7\kappa_{n+1}(\widehat{E})R^{1-n} \leq A_8 R,\end{aligned} \tag{5}$$

since $\kappa_{n+1}(\widehat{E}) \le A_9 R^n$ by [**P**, Lemma 2.2 (2) and (4)].

Let $x^1 = (x_1, 0, \dots, 0) \in \mathbf{R}^n$ with $x_1 > 5kR$. By the mean value theorem, for each $\alpha \in \mathbf{Z}_+^{n+1}$ there exists a number $x_1^\alpha(t) \in [x_1, x_1 + 5R]$ such that

$$\begin{aligned}&\partial^\alpha \Phi_{n+1}(x^1, t) - \partial^\alpha \Phi_{n+1}(x_1 + 5R, 0, \dots, 0, t)\\&= -5R\partial^{\alpha'} \Phi_{n+1}(x_1^\alpha(t), 0, \dots, 0, t)\end{aligned}$$

where $\alpha' = \alpha + (1, 0, \dots, 0) \in \mathbf{Z}_+^{n+1}$. We recall the elementary estimate

$$\left|\partial^\alpha \Phi_{n+1}(x, x_{n+1})\right| \le \alpha! k_1^{|\alpha|} \Big/ \left|(x, x_{n+1})\right|^{n-1+|\alpha|},$$

where k_1 depends only on n, cf. [**P**, (2.1)], and the following facts:

$$\begin{aligned}&\frac{\partial}{\partial y_1}\Phi_{n+1}(y) = \frac{a_{n+1}(n-1)\, y_1}{|y|^{n+1}},\\&1 \le |x_1(t)|/x_1 \le 2,\end{aligned}$$

where $x_1(t) = x_1^{(0,\dots,0)}(t)$ and $x_1 > 5kR$. From these observations using (3) and (4), one obtains

$$\begin{aligned}|f(x^1)| &\ge \tfrac{1}{2}\kappa_{n+1}(\widehat{E}) \int_{-\infty}^{\infty} 5Ra_n(n-1)\frac{x_1(t)}{(x_1(t)^2 + t^2)^{(n+1)/2}}\, dt\\&- \sum_{|\alpha|\ge 1} \int_{-\infty}^{\infty} A_5 \frac{(3R)^{|\alpha|}}{\alpha!}\kappa_{n+1}(\widehat{E})\, 5R(\alpha'!)\, k_1^{|\alpha|+1} \frac{1}{(x_1^\alpha(t)^2 + t^2)^{(n+|\alpha|)/2}}\, dt\\&\ge A_{10} R\kappa_{n+1}(\widehat{E})\, x_1^{1-n} \int_{-\infty}^{\infty} (1+\tau^2)^{-(n+1)/2} d\tau\\&- \sum_{|\alpha|\ge 1} A_{11}\kappa_{n+1}(\widehat{E})(|\alpha|+1)(5Rk_1)^{|\alpha|+1}\, 2^{n+|\alpha|}\, x_1^{-n-|\alpha|+1}\\&\cdot \int_{-\infty}^{\infty} (1+\tau^2)^{-(n+|\alpha|)/2} d\tau.\end{aligned}$$

The last integral may be estimated from above by $\int_{-\infty}^{\infty}(1+\tau^2)^{-1}d\tau = \pi$. From the elementary properties of geometric series one sees that for $x_1 > (5k + 10k_1)R$ the last series converges and the following estimate holds:

$$|f(x^1)| \ge A_{12} R\kappa_{n+1}(\widehat{E})\, x_1^{1-n} - A_{13} R^2 \kappa_{n+1}(\widehat{E})\, x_1^{-n}.$$

Hence for x^1 big enough

$$|f(x^1)| \ge A_{14} R\kappa_{n+1}(\widehat{E}) |x^1|^{1-n}. \tag{6}$$

In the same way we have for $|x| > (5k + 10k_1)R$ the estimate

$$|f(x)| \leq A_{15} R \kappa_{n+1}(\widehat{E}) |x|^{1-n}, \tag{7}$$

whence $f = \Phi_n * (\Delta_n f)$.

Write as in (3)

$$f(x) = \sum_{|\beta| \geq 0} b_\beta \partial^\beta \Phi_n(x). \tag{8}$$

Since for large $|x|$, $\Phi_n(x) \approx |x|^{2-n}$, when $n \geq 3$, and $\Phi_2(x) \approx \log |x|$, and since $|\partial_i \Phi_n(x)| \approx |x_i||x|^{-n}$, we see from (6), (7) and (8) that $b_{(0,\ldots,0)} = 0$ and

$$|b_1| \geq A_{16} R \kappa_{n+1}(\widehat{E}) \quad \text{where } b_1 = b_{(1,0,\ldots,0)}. \tag{9}$$

On the other hand, since by (5), $f/(A_8 R) \in \mathcal{U}_n(E_R)$, one finds from [**P**, Lemma 3.3] that

$$|b_1|/(A_8 R) \leq A_{17} R \kappa_n(E_R). \tag{10}$$

Using a partition of unity and [**P**, Lemma 4.2] one can easily prove that $\kappa_n(E_R) \leq A_{18} \kappa_n(E)$. From (9) and (10) we then have

$$A_{16} R \kappa_{n+1}(\widehat{E}) \leq A_{19} R^2 \kappa_n(E),$$

which completes the proof of (1).

By the definition of κ'_n it is enough to prove (2) for compact sets E. But for them one has by [**P**, Lemma 2.2 (1) and (7)],

$$\kappa'_n(E) = \inf\{\kappa'_n(G) : E \subset G,\ G \text{ is open}\}.$$

The rest is clear, since $\kappa'_n(G) = \kappa_n(G)$ for open sets G. □

As remarked before the following result was already obtained by Uy in [**U2**]:

Corollary 3.2. *For each $n \geq 2$ there exists a compact set E_n in $\mathbf{R}^n$ such that $\kappa'_n(E_n) = 0$ and $\mathcal{H}^{n-1}(E_n) > 0$.*

Proof. For $E \subset \mathbf{R}^2$, $\kappa'_2(E) \leq 2\pi\gamma(E)$, cf. [**P**, Lemma 2.2 (9)]. Examples of compact sets $E_2 \subset \mathbf{R}^2$ with $\gamma(E_2) = 0$ and $\mathcal{H}^1(E_2) > 0$ have been given in [**V**], [**G1**], [**G2**], [**M2**] and [**FX**]. Since $\mathcal{H}^n(E \times [0,1]) = \mathcal{H}^{n-1}(E)$ for $E \subset \mathbf{R}^{n-1}$, the result follows starting from such a set E_2 and taking products with intervals. □

Remark. The set E_n in 3.2 can also have non-σ-finite $\mathcal{H}^{n-1}$ measure since such an E_2 was shown to exist in [**G2**]. However, its Hausdorff dimension can be at most $n-1$.

The integral-geometric (Favard) measure $\mathcal{I}^{n-1}$ can be defined for Borel sets E in $\mathbf{R}^n$ by

$$\mathcal{I}^{n-1}(E) = \iint_V \operatorname{card}(E \cap P_V^{-1}\{y\})\, d\mathcal{H}^{n-1}y\, d\gamma_{n,n-1}V,$$

where card(F) gives the number of points in F, $\gamma_{n,n-1}$ is the natural invariant measure on the space of $(n-1)$-dimensional linear subspaces of $\mathbf{R}^n$, and $P_V : \mathbf{R}^n \to V$ denotes the orthogonal projection, see [**FH**, 2.10.5 and 15]. Thus $\mathcal{I}^{n-1}(E) = 0$ if and only if $\mathcal{H}^{n-1}(P_V E) = 0$ for $\gamma_{n,n-1}$ almost all V. By elementary linear algebra one sees that $\mathcal{I}^{n-1}(E) = 0$ if and only if $\mathcal{I}^n(E \times [0,1]) = 0$. It was shown in [**M3**] that the class of compact null-sets for $\mathcal{I}^1$ in $\mathbf{R}^2$ is not conformally invariant. Hence the compact null-sets for $\mathcal{I}^1$ and κ_2' are not the same. Using Theorem 3.1 we obtain this in any $\mathbf{R}^n$, $n \geq 2$:

Corollary 3.3. *The classes of compact null-sets for* $\mathcal{I}^{n-1}$ *and* κ_n' *are different.*

Remark. Jones and Murai showed in [**JM**] that there exists a compact set $E \subset \mathbf{R}^2$ with $\mathcal{I}^1(E) = 0$ and $\gamma(E) > 0$. It is not clear to us whether their proof also works for κ_2'.

4. Harmonic Lip_1-capacity on AD-regular sets

We shall say that a subset E of $\mathbf{R}^n$ is AD-regular (Ahlfors and David) if there exist positive and finite constants A_1 and A_2 such that

$$\begin{aligned} A_1 r^{n-1} &\leq \mathcal{H}^{n-1}(E \cap B(x,r)) \\ &\leq A_2 r^{n-1} \quad \text{for all } x \in E,\ 0 < r < \operatorname{diam}(E). \end{aligned} \tag{4.1}$$

We shall show that if the singular integral operators related to the Riesz kernels $|x|^{-n}x_i$, $i = 1, \dots, n$, are bounded on $L^2(E)$, then κ_n' and $\mathcal{H}^{n-1}$ are comparable on E. This assumption is valid on sufficiently regular hypersurfaces like Lipschitz graphs or bilipschitz images of $\mathbf{R}^{n-1}$. For the theory of singular integrals on AD-regular sets, see [**D2**] and [**DS**]. The results of this chapter are known in $\mathbf{R}^2$ for the analytic capacity, see e.g. [**C**].

We begin with a simple modification of the result [**C**, Theorem 23, p. 107] (a generalization of Uy's result [**U1**], cf. also [**VJ**, pp. 165–167]) on extremal problems for singular integrals. Let X be a locally compact Hausdorff space.

Denote by $C_0(X)$ the space of continuous functions on X vanishing at infinity, that is, the set of functions $f : X \to \mathbf{R}$ such that for every $\varepsilon > 0$ there is a compact set $K \subset X$ for which $|f(x)| < \varepsilon$ for $x \in X \setminus K$. We equip $C_0(X)$ with the supremum norm. Its dual is $\mathcal{M}(X)$, the space of all finite signed Radon measures on X equipped with the total variation norm. Let $T : \mathcal{M}(X) \to C_0(X)$ be a linear operator. We assume that its transpose T^* sends $\mathcal{M}(X)$ into $C_0(X)$, that is, $T^* : \mathcal{M}(X) \to C(X)$ is defined by

$$\int (T\nu_1)\,d\nu_2 = \int (T^*\nu_2)\,d\nu_1 \quad \text{for } \nu_1, \nu_2 \in \mathcal{M}(X).$$

Lemma 4.2. *Let μ be a positive Radon measure on a locally compact Hausdorff space X and let $T_i : \mathcal{M}(X) \to C(X)$, $i = 1, \dots, n$, be bounded linear operators. Suppose that each T_i^* sends $\mathcal{M}(X)$ into $C_0(X)$ and it is of weak type $(1,1)$ with respect to μ, that is there exists a constant A such that*

$$(1) \qquad \mu\{x : |T_i^*\nu(x)| > \alpha\} \le A\alpha^{-1}\|\nu\|$$

for $i = 1, \dots, n$, $\alpha > 0$, and $\nu \in \mathcal{M}(X)$. Then for $\tau > 0$ and any Borel set $E \subset X$ with $0 < \mu(E) < \infty$ there exists $h : X \to [0,1]$ in $L^\infty(\mu)$ satisfying $h(x) = 0$ for $x \in X \setminus E$,

$$(2) \qquad \int_E h\,d\mu \ge \mu(E)/2$$

and

$$(3) \qquad \|T_i(h\,d\mu)\| \le (n+\tau)A \quad \textit{for } i = 1, \dots, n.$$

Proof. Define an operator $T : \mathcal{M}(X) \to C(X)^n = Y$ by $T\nu = (T_1\nu, \dots, T_n\nu)$. For $\Psi = (\Psi_1, \dots, \Psi_n) \in Y$ put $\|\Psi\| = \max\{\|\Psi_i\| : i = 1, \dots, n\}$. Suppose we can find a Borel set $E \subset X$ and $\tau > 0$ contradicting the assertion of the lemma. Set

$$\begin{aligned} B_0 &= \Big\{ f \in L^\infty(\mu) : 0 \le f \le 1,\ f(x) = 0 \quad \text{for } x \in X \setminus E \\ &\qquad\qquad \text{and } \int_E f\,d\mu \ge \mu(E)/2 \Big\}, \\ B_1 &= \{T(f\,d\mu) : f \in B_0\}, \\ B_2 &= \{g \in Y : \|g\| \le (n+\tau)A\}. \end{aligned}$$

Then B_1 and B_2 are disjoint convex subsets of the Banach space Y and B_2 has non-empty interior. The dual Y^* of Y consists of λ of the form

$$\lambda(g_1, \dots, g_n) = \sum_{i=1}^n \int g_i\,d\lambda_i \quad \text{where } \lambda_i \in \mathcal{M}(X),\ g_i \in C(X),$$

with the norm $\|\lambda\| = \sum_{i=1}^{n} \|\lambda_i\|$. By a well-known separation lemma [**R**, p. 58] we can find such a $\lambda \in Y^*$ for which

$$\sum_{i=1}^{n} \int h_i \, d\lambda_i \geq \sum_{i=1}^{n} \int g_i \, d\lambda_i \quad \text{for } (h_1, \ldots, h_2) \in B_1,\ (g_1, \ldots, g_n) \in B_2.$$

This means that

$$(4) \qquad \sum_{i=1}^{n} \int T_i(f \, d\mu) \, d\lambda_i \geq \sum_{i=1}^{n} \int g_i \, d\lambda_i$$

for $f \in B_0$ and $g = (g_1, \ldots, g_n) \in B_2$.

Taking supremum over $g \in B_2$ in (4) one gets

$$(5) \qquad (n + \tau) A \|\lambda\| \leq \sum_{i=1}^{n} \int T_i(f \, d\mu) \, d\lambda_i \quad \text{for } f \in B_0.$$

Applying (1) with α replaced by $\alpha_i = 2nA\|\lambda_i\|/\mu(E)$ we can write for each $i = 1, \ldots, n$

$$\mu\{x : |T_i^* \lambda_i(x)| > \alpha_i\} \leq \mu(E)/(2n).$$

Hence for

$$E' = \{x \in E : |T_i^* \lambda_i(x)| \leq \alpha_i \quad \text{for } i = 1, \ldots, n\}$$

we have $\mu(E') \geq \mu(E)/2$. Define f by $f = \mu(E)/(2\mu(E'))$ on E' and $f = 0$ on $X \setminus E'$. Then $f \in B_0$, but

$$\Big|\sum_{i=1}^{n} \int T_i(f \, d\mu) \, d\lambda_i\Big| = \Big|\sum_{i=1}^{n} \int (T_i^* \lambda_i) f \, d\mu\Big| \leq \sum_{i=1}^{n} \alpha_i \mu(E)/2 = nA\|\lambda\|,$$

which contradicts (5). This completes the proof. □

Let now Γ be a closed AD-regular subset of $\mathbf{R}^n$ as defined in (4.1). Denote by μ_Γ the restriction of $\mathcal{H}^{n-1}$ to Γ;

$$\mu_\Gamma(E) = \mathcal{H}^{n-1}(\Gamma \cap E) \quad \text{for } E \subset \mathbf{R}^n.$$

For $i = 1, \ldots, n$ and $\varepsilon > 0$ we define the truncated singular integral operators $T_{i,\varepsilon}^\Gamma$ by

$$T_{i,\varepsilon}^\Gamma f(x) = b_n \int_{\mathbf{R}^n \setminus B(x,\varepsilon)} \frac{x_i - y_i}{|x - y|^n} f(y) \, d\mu_\Gamma y,$$

where $b_n = (n-2)a_n$ for $n \geq 3$, $b_2 = 1/(2\pi)$, with a_n as in the definition of Φ_n in Section 2. We shall consider AD-regular sets Γ for which these operators are bounded in $L^2(\mu_\Gamma)$ uniformly with respect to $\varepsilon > 0$. This means that there exists $A_1 < \infty$ such that

$$(4.3) \qquad \|T^\Gamma_{i,\varepsilon} f\|_2 \leq A_1 \|f\|_2 \quad \text{for } f \in L^2(\mu_\Gamma),\ \varepsilon > 0,\ i = 1, \dots, n.$$

Here and below $\| \ \|_p$ means the L^p-norm with respect to μ_Γ. For various consequences of (4.3), see [**CW**, Ch. 3] and [**C**, § 6]. For example, one can show that the maximal operators T^Γ_{i*}, $T^\Gamma_{i*}f(x) = \sup_{\varepsilon>0} |T^\Gamma_{i,\varepsilon} f(x)|$, are bounded in L^p for $1 < p < \infty$ and of weak type (1,1).

As mentioned before the condition (4.3) is known to hold for many hypersurfaces parametrized by Lipschitz maps. For such results see [**D2**] and [**DS**]. However, it is not known, even when $n = 2$, whether (4.3) implies some kind of rectifiability of Γ.

We shall now prove that $\mathcal{H}^{n-1}$ and κ'_n are comparable on AD-regular sets satisfying (4.3).

Theorem 4.4. *Let Γ be a closed AD-regular subset of $\mathbf{R}^n$ satisfying* (4.3). *Then there exists a positive and finite constant A depending only on Γ such that for all closed sets $E \subset \Gamma$,*

$$(1) \qquad A^{-1}\mu_\Gamma(E) \leq \kappa'_n(E) \leq A\mu_\Gamma(E).$$

Proof. The scheme is similar to that in [**C**], pp. 105, 107–111, the proofs of Theorems 17 and 18.

Fix a radial function $\varphi \in C^\infty(\mathbf{R}^n)$ such that $\varphi = 0$ on $B(0, 1/2)$ and $\varphi = 1$ on $\mathbf{R}^n \setminus B(0,1)$. For $\varepsilon > 0$ define

$$\widetilde{T}^\Gamma_{i,\varepsilon} f(x) = b_n \int \varphi\Big(\frac{x-y}{\varepsilon}\Big) \frac{x_i - y_i}{|x-y|^n} f(y)\, d\mu_\Gamma y \quad \text{for } f \in L^1(\mu_\Gamma).$$

From the regularity of Γ one easily checks that for $x \in \Gamma$ and $\varepsilon > 0$

$$|\widetilde{T}^\Gamma_{i,\varepsilon} f(x) - T^\Gamma_{i,\varepsilon} f(x)| \leq A_1 M f(x),$$

where Mf is the Hardy–Littlewood maximal function corresponding to μ_Γ;

$$Mf(x) = \sup_{r>0} \frac{1}{\mu_\Gamma(B(x,r))} \int_{B(x,r)} f\, d\mu_\Gamma.$$

Here and below $A_1, A_2, \dots$ are finite constants depending only on Γ and φ. It is well-known that M is bounded in L^p for $1 < p \leq \infty$ and of weak type (1,1). For example the method of [**S**, § 1] generalizes readily from Lebesgue measure to our case, or see [**CW**, Theorem 2.1, Ch. 3] for the weak type (1,1). Thus the operators $\widetilde{T}^{\Gamma}_{i,\varepsilon}$ are uniformly bounded in $L^2(\mu_\Gamma)$. The kernels $k_{i,\varepsilon}$,

$$k_{i,\varepsilon}(x,y) = b_n \varphi\Big(\frac{x-y}{\varepsilon}\Big)\frac{x_i - y_i}{|x-y|^n}$$

satisfy the conditions of [**CW**, Theorem 2.4, Ch. 3], which implies that there exists a constant A_2 such that

$$\mu_\Gamma\{x \in \Gamma : |\widetilde{T}^{\Gamma}_{i,\varepsilon} f(x)| > \alpha\} \le A_2 \alpha^{-1} \|f\|_1 \quad \text{for } f \in L^1(\mu_\Gamma).$$

Since the kernel $k_{i,\varepsilon}$ is smooth, we can extend the operator $\widetilde{T}^{\Gamma}_{i,\varepsilon}$ from $L^1(\mu_\Gamma)$ to $\mathcal{M}(\Gamma)$ with

$$\mu_\Gamma\{x \in \Gamma : |\widetilde{T}^{\Gamma}_{i,\varepsilon} \nu(x)| > \alpha\} \le A_2 \alpha^{-1} \|\nu\| \quad \text{for } \nu \in \mathcal{M}(\Gamma). \tag{2}$$

Now we can apply Lemma 4.2 to $T_i = \widetilde{T}^{\Gamma}_{i,\varepsilon}$. Observe that then $T_i^* = -T_i$. Fix any compact set E in Γ with $0 < \mu(E) < \infty$. We can find for each $\varepsilon > 0$ a function $h_\varepsilon \in L^\infty(\mu_\Gamma)$ such that $0 \le h_\varepsilon \le 1$, $h_\varepsilon = 0$ outside E,

$$\int h_\varepsilon \, d\mu_\Gamma \ge \mu_\Gamma(E)/2$$

and

$$\big\|\widetilde{T}^{\Gamma}_{i,\varepsilon} h_\varepsilon\big\|_{\mu_\Gamma} \le 2nA_2. \tag{3}$$

One can easily prove, as in [**C**, p. 110], that (3) yields

$$\big\|\widetilde{T}^{\Gamma}_{i,\varepsilon} h_\varepsilon\big\|_{L^\infty(U_\varepsilon)} \le A_3,$$

where U_ε is the ε-neighborhood of Γ. Since the functions $\widetilde{T}^{\Gamma}_{i,\varepsilon} h_\varepsilon$ are continuous, harmonic outside U_ε (as $k_{i,\varepsilon}(x,y) = (\partial \Phi_n/\partial x_i)(x-y)$ for $|x-y| > \varepsilon$) and vanish at ∞, we have by the maximum principle

$$\big\|\widetilde{T}^{\Gamma}_{i,\varepsilon} h_\varepsilon\big\|_{L^\infty(\mathbf{R}^n)} \le A_3 \quad \text{for } i = 1, \dots, n.$$

Put $f_\varepsilon = \Phi_n * (h_\varepsilon \, d\mu_\Gamma)$. Since

$$\nabla f_\varepsilon(x) = (\widetilde{T}^{\Gamma}_{1,\varepsilon} h_\varepsilon(x), \dots, \widetilde{T}^{\Gamma}_{n,\varepsilon} h_\varepsilon(x))$$

for $x \in \mathbf{R}^n \setminus U_\varepsilon$, we have

$$|\nabla f_\varepsilon(x)| \le \sqrt{n} A_3 \quad \text{for } x \in \mathbf{R}^n \setminus U_\varepsilon. \tag{4}$$

There exists a sequence $\varepsilon_k \downarrow 0$ such that $h_{\varepsilon_k} d\mu_\Gamma$ converges weakly to some measure ν_0. Trivially $\nu_0 = h_0 d\mu_\Gamma$ where $0 \le h_0 \le 1$, $h_0 = 0$ outside E, and $\int h_0 d\mu_\Gamma \ge \mu_\Gamma(E)/2$.

Put $f_0 = \Phi_n * (h_0\, d\mu_\Gamma)$. Since $f_0(x) = \lim_{k\to\infty} f_{\varepsilon_k}(x)$ for $x \in \mathbf{R}^n \setminus E$, f_0 is harmonic in $\mathbf{R}^n \setminus E$ and we obtain from (4) that

$$|\nabla f_0(x)| \le \sqrt{n} A_3 \quad \text{for } x \in \mathbf{R}^n \setminus E. \tag{5}$$

Furthermore, $\nu_0(B(a,\delta)) \le A_4 \delta^{n-1}$ implies that f_0 is continuous. Since $\mathcal{H}^{n-1}(E) < \infty$, almost all lines in any fixed direction meet E in a finite set, see [**FH**, 2.10.25]. These facts together with (5) easily yield that $f \in \mathrm{Lip}^1_{\mathrm{loc}}(\mathbf{R}^n)$. Hence $f_0/(\sqrt{n}A_3) \in \mathcal{U}'_n(E)$, but

$$\langle \Delta f_0, 1\rangle = \int h_0\, d\mu_\Gamma \ge \mu_\Gamma(E)/2,$$

which gives the left hand side of (1). The right hand side is elementary, see [**P**, Lemma 2.2(1)]. □

From the proof of Theorem 4.4 we get more.

Definition 4.5. For a bounded set E in $\mathbf{R}^n$ define

$$\begin{aligned}\kappa'_+(E) = \sup\{\nu(E) : \nu \text{ is a positive Radon}\\ \text{measure with } \operatorname{Supp}\nu \subset E \text{ and } \|\nabla\Phi_n * \nu\| \le 1\}.\end{aligned}$$

Corollary 4.6. *In the inequalities* (1) *of Theorem* 4.4 κ'_n *can be replaced by* κ'_+.

Remarks. We say that a subset E of $\mathbf{R}^n$ is $(n-1)$-rectifiable if $\mathcal{H}^{n-1}(E) < \infty$ and there are $(n-1)$-dimensional C^1 submanifolds of $\mathbf{R}^n$ $M_1, M_2, \dots$ such that $\mathcal{H}^{n-1}(E \setminus \cup_{i=1}^\infty M_i) = 0$. A set $E \subset \mathbf{R}^n$ is called purely $(n-1)$-unrectifiable if $\mathcal{H}^{n-1}(E \cap M) = 0$ for all $(n-1)$-dimensional C^1 submanifolds M. In both of these definitions C^1 submanifolds can be replaced by Lipschitz images of $\mathbf{R}^{n-1}$, see [**FH**, § 3.2]. If E is an $\mathcal{H}^{n-1}$ measurable $(n-1)$-rectifiable subset of $\mathbf{R}^n$ with $\mathcal{H}^{n-1}(E) > 0$, E contains a closed subset F with $\mathcal{H}^{n-1}(F) > 0$ which lies on a Lipschitz graph. Thus by Theorem 4.4 and the aforementioned validity of (4.3) on Lipschitz graphs, $0 < \kappa'_n(F) \le \kappa'_n(E)$. It seems plausible that the converse might also hold, which would mean that the answer to the following question is affirmative:

Is it true that if E is a closed subset of $\mathbf{R}^n$ with $\mathcal{H}^{n-1}(E) < \infty$, then $\kappa'_n(E) = 0$ if and only if E is purely $(n-1)$-unrectifiable?

The answer is not known even for $n = 2$. The examples of sets E with $\kappa'_n(E) = 0$ in Chapter 3 as well as those presented in the next chapter are purely $(n-1)$-unrectifiable.

5. A class of sets with zero κ'_n capacity

In this chapter we shall develop the method of [**M2**] to find a rather large class of compact subsets of $\mathbf{R}^n$ having positive $\mathcal{H}^{n-1}$ measure and zero κ'_n capacity. We first present some preliminary results.

Lemma 5.1. *Let $E \subset \mathbf{R}^n$ be $\mathcal{H}^{n-1}$ measurable with $\mathcal{H}^{n-1}(E) < \infty$. Then*

$$(1) \qquad 2^{1-n} \leq \limsup_{r \downarrow 0} \alpha(n-1)^{-1} r^{1-n} \mathcal{H}^{n-1}(E \cap B(x,r)) \leq 1$$

for $\mathcal{H}^{n-1}$ almost all $x \in E$, and

$$(2) \qquad \lim_{r \downarrow 0} r^{1-n} \mathcal{H}^{n-1}(E \cap B(x,r)) = 0$$

for $\mathcal{H}^{n-1}$ almost all $x \in \mathbf{R}^n \setminus E$.

For a proof see [**FH**, 2.10.19].

Lemma 5.2. *Let $E \subset \mathbf{R}^n$ be $\mathcal{H}^{n-1}$ measurable with $\mathcal{H}^{n-1}(E) < \infty$. Then for $\mathcal{H}^{n-1}$ almost all $x \in A$ there exists $v \in S^{n-1}$ such that*

$$(1) \qquad \liminf_{r \downarrow 0} r^{1-n} \mathcal{H}^{n-1}\{y \in E \cap B(x,r) : (y-x) \cdot v < -\eta |y-x|\} = 0$$

for all $\eta > 0$.

By a simple limiting argument one sees that it is sufficient to prove that whenever a fixed $\eta > 0$ is given, then for $\mathcal{H}^{n-1}$ almost all $x \in E$ there is $v \in S^{n-1}$ such that (1) holds. This can be proven with a modification of the argument given by Marstrand in [**MJ**, pp. 295–297].

Lemma 5.3. *Let X be a compact subset of $\mathbf{R}^n$ with $\mathcal{H}^{n-1}(X) < \infty$ and let μ be the restriction of $\mathcal{H}^{n-1}$ to X normalized so that $\mu(A) = \frac{\sigma(n)}{\alpha(n-1)} \mathcal{H}^{n-1}(X \cap A)$ for $A \subset \mathbf{R}^n$. Suppose that $f \in \mathcal{U}'_n(X)$. Then there exists $h \in L^\infty(\mu)$ with $\|h\|_\mu \leq 1$, such that $\Delta f = h\, d\mu$ in the distributional sense, that is, $f = \Phi_n * (h\, d\mu)$.*

(Recall our convention to identify two functions which differ by a constant.)

Proof. For each $m = 1, 2, \ldots$ we can find a finite number j_m of balls $B_{m,j}$, $j = 1, \ldots, j_m$, with radii $r_{m,j}$ such that

$$(1) \qquad X \subset \bigcup_{j=1}^{j_m} B_{m,j}, \quad r_{m,j} \leq 1/m, \quad \sum_{j=1}^{j_m} \sigma(n) r_{m,j}^{n-1} \leq \mu(X) + 1/m.$$

Let $G_m = \cup_{j=1}^{j_m} B_{m,j}$ and $\delta_m = \operatorname{dist}(X, \partial G_m) > 0$. Choose a radial function $\varphi \in C_0^\infty(B(0,1))$ with $0 \le \varphi \le 1$, $|\Delta\varphi| \le A_1$ and $\int \varphi(x)\,dx = 1$. Here and later in this proof $A_1, A_2, \dots$ will be constants depending only on n. For $\delta > 0$ write $\varphi_\delta(x) = \delta^{-n}\varphi(x/\delta)$. Then $\int \varphi_\delta(x)\,dx = 1$. Let $f_m = \varphi_{\delta_m/2} * f$. One can easily check that $f_m = f$ in a neighborhood of $\mathbf{R}^n \setminus G_m$ and that $f_m \in C^\infty(\mathbf{R}^n)$ with $|\nabla f_m| \le 1$. Then we have $f_m = \Phi_n * \Delta f_m$. Moreover, since $\Delta f_m = \Delta f = 0$ in $\mathbf{R}^n \setminus G_m$, Gauss formula gives for all $x \in \mathbf{R}^n \setminus \overline{G}_m$,

$$
\begin{aligned}
(2)\qquad f(x) = f_m(x) &= \int_{G_m} \Phi_n(x-y)\Delta f_m(y)\,dy \\
&= \int_{G_m} \big(\operatorname{div}_y(\Phi_n(x-y)\nabla_y f_m(y)) - \nabla_y\Phi_n(x-y)\cdot\nabla_y f_m(y)\big)\,dy \\
&= \int_{\partial G_m} \Phi_n(x-y)\frac{\partial f_m}{\partial \nu_y}\,d\sigma_m y + \int_{G_m} \nabla\Phi_n(x-y)\cdot\nabla f(y)\,dy,
\end{aligned}
$$

where ν_y is the outer unit normal to ∂G_m and σ_m is the surface measure on ∂G_m, that is, $\sigma_m = \mathcal{H}^{n-1}|\partial G_m$.

Write $\mu_m = \frac{\partial f_m}{\partial \nu_y}\sigma_m = \frac{\partial f}{\partial \nu_y}\sigma_m$. Then by (1),

$$
(3)\qquad \|\mu_m\| \le \|\nabla f\|\|\sigma_m\| \le \mu(X) + 1/m.
$$

Let μ_0 be the weak limit of some subsequence (μ_{m_k}) of (μ_m). Then $\operatorname{Supp}\mu_0 \subset X$ and $\|\mu_0\| \le \mu(X)$ by (3). Fix any $x \in \mathbf{R}^n \setminus X$. By (2) we can write for k big enough,

$$
(4)\qquad f(x) = \Phi_n * \mu_{m_k}(x) + \int_{G_{m_k}} \nabla\Phi_n(x-y)\cdot\nabla f(y)\,dy.
$$

By (1), $\mathcal{H}^n(G_{m_k}) \to 0$, whence

$$
\Big|\int_{G_{m_k}} \nabla\Phi_n(x-y)\cdot\nabla f(y)\,dy\Big| \le A_2 \int_{G_{m_k}} |x-y|^{1-n}dy \to 0.
$$

Thus (4) yields

$$
f(x) = \Phi_n * \mu_0(x)
$$

for $x \in \mathbf{R}^n \setminus X$. In particular $f = \Phi_n * \mu_0$ $\mathcal{H}^n$ almost everywhere. Since $f \in \mathrm{Lip}^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $\Phi_n * \mu_0 \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, this implies that f and $\Phi_n * \mu_0$ agree as distributions, and so

$$
\Delta f = \Delta\Phi_n * \mu_0 = \mu_0.
$$

It remains to prove that $h = d\mu_0/d\mu \in L^\infty(\mu)$ and $\|h\|_\mu \le 1$. To this end it is enough to prove that for each open ball B and its closure $\overline{B}$,

$$
(5)\qquad |\mu_0|(B) \le \mu(\overline{B}).
$$

In fact, given a closed ball $\overline{B}$ we can apply (5) to open balls $B_i \downarrow B$ to see that

$$|\mu_0|(\overline{B}) \leq \lim_{i\to\infty} |\mu_0|(B_i) \leq \lim_{i\to\infty} \mu(\overline{B}_i) = \mu(\overline{B}),$$

which gives $\|h\|_\mu \leq 1$.

Suppose that we could find an open ball B and $\varepsilon > 0$ such that $|\mu_0|(B) > (\mu(\overline{B}) + \varepsilon)$. There exists a compact set $K \subset X \setminus \overline{B}$ such that

$$\mu(K) > \mu(X \setminus \overline{B}) - \varepsilon/4. \tag{6}$$

Let $\delta_\varepsilon = \operatorname{dist}(K, \overline{B}) > 0$. For all k big enough,

$$\max_j r_{m_k,j} \leq 1/m_k < \delta_\varepsilon/2, \quad \text{and} \tag{7}$$

$$\sum_{j=1}^{j_m} \sigma(n)(r_{m_k,j})^{n-1} \leq \mu(X) + \varepsilon/2. \tag{8}$$

Let

$$J_k' = \{j : B_{m_k,j} \cap \overline{B} \neq \emptyset\}, \quad J_k'' = \{j : B_{m_k,j} \cap K \neq \emptyset\}.$$

For all k big enough we have also

$$\sum_{j\in J_k''} \sigma(n)\, r_{m_k,j}^{n-1} \geq \mu(K) - \varepsilon/4. \tag{9}$$

From (7) we have $\overline{B}_{m_k,j_1} \cap \overline{B}_{m_k,j_2} = \emptyset$ for $j_1 \in J_k'$ and $j_2 \in J_k''$ so that by (8), (9) and (6),

$$\begin{aligned}\sum_{j\in J_k'} \sigma(n)\, r_{m_k,j}^{n-1} &\leq \mu(X) + \varepsilon/2 - \sum_{j\in J_k''} \sigma(n)\, r_{m_k,j}^{n-1} \\ &\leq \mu(X) + \varepsilon/2 - \mu(K) + \varepsilon/4 \\ &< \mu(X) - \mu(X \setminus \overline{B}) + \varepsilon = \mu(\overline{B}) + \varepsilon.\end{aligned}$$

By the definition of μ_{m_k}, since $\|\nabla f\| \leq 1$, we then have

$$\begin{aligned}|\mu_{m_k}|(X \cap \overline{B}) &\leq \sum_{j\in J_k'} \sigma_m(\partial B_{m_k,j}) \\ &= \sigma(n) \sum_{j\in J_k'} r_{m_k,j}^{n-1} \leq \mu(\overline{B}) + \varepsilon.\end{aligned}$$

Since $\mu_{m_k} \to \mu_0$, we obtain

$$|\mu_0|(B) \leq \liminf_{k\to\infty} |\mu_{m_k}|(B) \leq \mu(\overline{B}) + \varepsilon,$$

which contradicts our first assumption. The lemma is proven. □

Let $K(x) = |x|^{-n}x = \text{constant} \cdot \nabla\Phi_n(x)$. For a signed Borel measure μ in $\mathbf{R}^n$ and $\varepsilon > 0$ define

$$K^\mu(x) = \int K(x-y)\, d\mu y, \quad \text{when the integral exists,}$$

$$K^\mu_\varepsilon(x) = \int_{\mathbf{R}^n \setminus B(x,\varepsilon)} K(x-y)\, d\mu y,$$

and

$$K^\mu_*(x) = \sup_{\varepsilon>0} |K^\mu_\varepsilon(x)|.$$

The proof of the following lemma was suggested by S. Semmes.

Lemma 5.4. *Let μ be a signed Borel measure in $\mathbf{R}^n$ such that $|\mu|B(x,r) \leq r^{n-1}$ for $x \in \mathbf{R}^n$, $r > 0$. Then*

$$|K^\mu_*(x)| \leq \|K^\mu\| + A \quad \textit{for } x \in \mathbf{R}^n,$$

where A is a constant depending only on n.

Proof. Suppose $L = \|K^\mu\| < \infty$. For $\varepsilon > 0$ and $x \in \mathbf{R}^n$ we estimate the average

$$\begin{aligned}
&\frac{1}{\alpha(n)(\varepsilon/2)^n} \int_{B(x,\varepsilon/2)} \int_{B(x,\varepsilon)} |z-y|^{1-n} d|\mu|y\, dz \\
&\leq \int_{B(x,\varepsilon)} \frac{2^n}{\alpha(n)\varepsilon^n} \int_{B(y,2\varepsilon)} |z-y|^{1-n} dz\, d|\mu|y \\
&\leq A_1 \varepsilon^{1-n} |\mu| B(x,\varepsilon) \leq A_1.
\end{aligned}$$

Here and below the constants $A_1, A_2, \ldots$ depend only on n. Hence there is $z \in B(x,\varepsilon/2)$ with $|K^\mu(z)| \leq L$ and

$$\int_{B(x,\varepsilon)} |z-y|^{1-n} d|\mu|y \leq A_1.$$

Thus

$$\left|K_\varepsilon^\mu(x) - K^\mu(z)\right| = \left| \int\limits_{\mathbf{R}^n \setminus B(x,\varepsilon)} \frac{x-y}{|x-y|^n}\, d\mu y - \int \frac{z-y}{|z-y|^n}\, d\mu y \right|$$

$$\leq A_2 \int\limits_{\mathbf{R}^n \setminus B(x,\varepsilon)} \frac{|x-z|(|x-y|^n + |z-y|^n)}{|x-y|^n|z-y|^n}\, d|\mu|y + \int\limits_{B(x,\varepsilon)} |z-y|^{1-n} d|\mu|y$$

$$\leq A_2\varepsilon \left(\int\limits_{\mathbf{R}^n \setminus B(x,\varepsilon)} |z-y|^{-n} d|\mu|y + \int\limits_{\mathbf{R}^n \setminus B(x,\varepsilon)} |x-y|^n d|\mu|y \right) + A_1.$$

Both of the last two integrals can be estimated in the same way. For example, as $z \in B(x, \varepsilon/2)$,

$$\int\limits_{\mathbf{R}^n \setminus B(x,\varepsilon)} |z-y|^{-n} d|\mu|y = \sum_{j=0}^{\infty} \int\limits_{B(x,2^{j+1}\varepsilon)\setminus B(x,2^j\varepsilon)} |z-y|^{-n} d|\mu|y$$

$$\leq \sum_{j=0}^{\infty} (2^{j-1}\varepsilon)^{-n} |\mu| B(x, 2^{j+1}\varepsilon)$$

$$\leq \sum_{j=0}^{\infty} (2^{j-1}\varepsilon)^{-n} (2^{j+1}\varepsilon)^{n-1} = 2^{2n-1}\varepsilon^{-1} \sum_{j=0}^{\infty} 2^{-j} = 2^{2n}\varepsilon^{-1}.$$

Thus

$$\left|K_\varepsilon^\mu(x)\right| \leq \left|K_\varepsilon^\mu(x) - K^\mu(z)\right| + \left|K^\mu(z)\right| \leq A_3 + L,$$

which proves the lemma. □

Theorem 5.5. *Let X be a compact subset of $\mathbf{R}^n$ with $\mathcal{H}^{n-1}(X) < \infty$ such that for some constant A,*

(1) $$\mathcal{H}^{n-1}(X \cap B(a,r)) \leq A r^{n-1} \quad \text{for } a \in \mathbf{R}^n, r > 0.$$

Suppose that the following holds at $\mathcal{H}^{n-1}$ almost all points $a \in X$: For every $v \in S^{n-1}$ there is $\delta > 0$ such that

(2) $$\liminf_{r \downarrow 0} r^{1-n} \mathcal{H}^{n-1}\{y \in X \cap B(a,r) : |(y-a)\cdot v| > \delta|y-a|\} > 0.$$

Then $\kappa_n'(X) = 0$.

Proof. Suppose that $\kappa_n'(X) > 0$. Using the definition of κ_n' and Lemma 5.3 we find $h \in L^\infty(\mathcal{H}^{n-1}|X)$ such that $0 < \|h\| \leq 1$ and, with $K = \text{constant} \cdot \nabla\Phi_n$ as before,

(3) $$\left\|K * (h\mathcal{H}^{n-1}|X)\right\|_{\mathcal{H}^{n-1}|X} \leq 1.$$

By Lusin's theorem there is a compact set $F \subset \{x \in X : h(x) \neq 0\}$ such that $\mathcal{H}^{n-1}(F) > 0$ and $h|F$ is continuous. Applying part (2) of Lemma 5.1 to $E = X \setminus F$ and Lemma 5.2 to $E = X$, we find a point $a \in F$ and $v \in S^{n-1}$ such that

$$\lim_{r\downarrow 0} r^{1-n}\mathcal{H}^{n-1}\Big((X \setminus F) \cap B(a,r)\Big) = 0, \tag{4}$$

$$\liminf_{r\downarrow 0} r^{1-n}\mathcal{H}^{n-1}\{y \in X \cap B(a,r) : (y-a)\cdot v < -\eta|y-a|\} = 0 \tag{5}$$

for any $\eta > 0$, and that the assumption of the theorem holds at a. We shall prove that for $\mu = h\mathcal{H}^{n-1}|X$,

$$\big|K_*^{\mu}(a)\big| = \infty, \tag{6}$$

which will contradict Lemma 5.4.

To establish (6) we may assume $a = 0$, $h(0) = \beta > 0$ and $v = (1, 0, \dots, 0)$. Let $\delta > 0$ be as in (2) corresponding to 0 and v and let α be the lower limit in (2) corresponding to 0, v and δ. We introduce some notation: Let m be a positive integer and set

$$\begin{aligned}
&t = \big(\alpha/(8A)\big)^{1/(n-1)},\\
&\varepsilon = \tfrac{1}{64}\alpha\beta\delta t^{m(n-1)},\\
&\eta = \tfrac{1}{32}A^{-1}\alpha\beta\delta t^{n-1},\\
&B(r) = B(0,r),\\
&C(r) = \{x \in B(r) : \delta|x| \le |x_1|\},\\
&C^+ = \{x : \delta|x| \le x_1\},\\
&C^- = \{x : x_1 \le -\eta|x|\},\\
&D^+ = \{x : 0 \le x_1 < \delta|x|\},\\
&D^- = \{x : -\eta|x| < x_1 \le 0\},\\
&E = \{x \in X : h(x) \ge \beta/2\}.
\end{aligned}$$

Using the continuity of $h|F$, (4), (5) and (2), we find $S > 0$ such that for all $0 < r \le S$,

$$\mathcal{H}^{n-1}((X \setminus E) \cap B(a,r)) \le \varepsilon r^{n-1}, \tag{7}$$

$$\mathcal{H}^{n-1}(X \cap C^- \cap B(S)) \le \varepsilon S^{n-1}, \tag{8}$$

$$\mathcal{H}^{n-1}(X \cap C(r)) \ge \tfrac{1}{2}\alpha r^{n-1}. \tag{9}$$

Put $s = t^m S$ and for $j = 1, \dots, m$,

$$R_j = X \cap B(t^{j-1}S) \setminus B(t^j S).$$

Then

$$\begin{aligned}
&\mathcal{H}^{n-1}(E \cap C^+ \cap R_j) \\
&\quad \geq \mathcal{H}^{n-1}(X \cap C(t^{j-1}S)) - \mathcal{H}^{n-1}(X \cap B(t^jS)) \\
&\quad - \mathcal{H}^{n-1}((X \setminus E) \cap B(S)) - \mathcal{H}^{n-1}(X \cap C^- \cap B(S)) \\
&\quad \geq \tfrac{1}{2}\alpha(t^{j-1}S)^{n-1} - A(t^jS)^{n-1} - 2\varepsilon S^{n-1} \\
&\quad = (\alpha/2 - At^{n-1} - 2\varepsilon t^{(n-1)(1-j)})(t^{j-1}S)^{n-1} \\
&\quad \geq \tfrac{1}{4}\alpha(t^{j-1}S)^{n-1}.
\end{aligned}$$

Let φ, $\varphi(x) = |x|^{-n}x_1$, be the first coordinate function of K. From the last estimate we get

$$\begin{aligned}
\int_{E\cap C^+\cap R_j} \varphi(x)h(x)\,d\mathcal{H}^{n-1}x &\geq \tfrac{1}{2}\beta\delta \int_{E\cap C^+\cap R_j} |x|^{1-n}d\mathcal{H}^{n-1}x \\
&\geq \tfrac{1}{2}\beta\delta(t^{j-1}S)^{1-n}\mathcal{H}^{n-1}(E \cap C^+ \cap R_j) \\
&\geq \tfrac{1}{8}\alpha\beta\delta.
\end{aligned}$$

By (7) and (8),

$$\begin{aligned}
&\mathcal{H}^{n-1}(R_j \setminus E) \leq \varepsilon(t^{j-1}S)^{n-1} \leq \varepsilon S^{n-1}, \\
&\mathcal{H}^{n-1}(E \cap R_j \cap C^-) \leq \varepsilon S^{n-1},
\end{aligned}$$

so that

$$\begin{aligned}
\left| \int_{(R_j\setminus E)\cup(E\cap R_j\cap C^-)} \varphi(x)h(x)\,d\mathcal{H}^{n-1}x \right| &\leq (t^jS)^{1-n}2\varepsilon S^{n-1} \\
&= 2\varepsilon t^{j(1-n)} \leq \tfrac{1}{32}\alpha\beta\delta.
\end{aligned}$$

For $x \in R_j \setminus C^-$, $\varphi(x) \geq -\eta|x|^{1-n}$, whence by (1)

$$\begin{aligned}
\int_{E\cap R_j\setminus(C^+\cup C^-)} \varphi(x)h(x)\,d\mathcal{H}^{n-1}x &\geq -\eta(t^jS)^{1-n}\mathcal{H}^{n-1}(R_j) \\
&\geq -\eta(t^jS)^{1-n}A(t^{j-1}S)^{n-1} \\
&= -\eta At^{1-n} \geq -\tfrac{1}{32}\alpha\beta\delta.
\end{aligned}$$

Putting these estimates together we have

$$\int_{R_j} \varphi(x)h(x)\,d\mathcal{H}^{n-1}x \geq \tfrac{1}{16}\alpha\beta\delta.$$

Summing over j,

$$\int_{X\cap B(S)\setminus B(s)} \varphi(x)h(x)\,d\mathcal{H}^{n-1}x \geq \tfrac{1}{16}\alpha\beta\delta m.$$

Since we can choose m as large as we please, independently of α, β and δ, (6) follows, and the theorem is proven. □

Remarks. The assumption (1) in Theorem 5.5 is actually superfluous. It was introduced in order that we could apply Lemma 5.4. Without that a more complicated argument using Lemma 5.1 would work as in [**M2**].

In [**U2**] Uy showed that $\kappa'_n(X) = 0$ where X is the n-fold product of the ordinary Cantor set in $\mathbf{R}$ with dissection ratio $2^{-n/(n-1)}$ (so that $0 < \mathcal{H}^{n-1}(X) < \infty$). The assumptions of Theorem 5.5 hold in that case. They hold also for many other self-similar constructions.

Following Hutchinson [**H**] we say that a compact subset X of $\mathbf{R}^n$ is self-similar satisfying the open set condition if there exist contracting similarity maps $S_i : \mathbf{R}^n \to \mathbf{R}^n$, $i = 1, \ldots, N$, $N \geq 2$, and a bounded non-empty open set O such that

$$X = \bigcup_{i=1}^{N} S_i(X),$$
$$\bigcup_{i=1}^{N} S_i(O) \subset O \quad \text{and}$$
$$S_i(O) \cap S_j(O) = \emptyset \quad \text{for } i \neq j.$$

Corollary 5.6. *Let X be as above. If $\mathcal{H}^{n-1}(X) < \infty$ and X does not lie in any $(n-1)$-plane, then $\kappa'_n(X) = 0$.*

Proof. The assumption (1) of Theorem 5.5 follows from the proof of [**H**, Theorem 5.1]. The assumption (2) follows from [**M1**, Theorem 4.2]. □

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Notes added in proof: (1) V. Eiderman has observed that there is an error in [**G2**] so that we cannot use this reference in the remark following Corollary 3.2. However Ivanov has given an example of such a set E_2 in "*On sets of analytic capacity zero*, in Linear and Complex Analysis Problem Book 3, Part II, Lecture Notes in Math., **1574**, Springer-Verlag, 1994."
(2) Recently it has been proved in "P. Mattila, M.S. Melnikov and J. Verdera, *The Cauchy integral, analytic capacity and uniform rectifiability*, to appear in Ann. of Math." that for 1-dimensional AD-regular set E the condition (4.3) holds if and only if E is contained in an AD-regular curve.