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MATCHING THEOREMS FOR TWISTED ORBITAL INTEGRALS

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Let F be a ρ-adic field and E a cyclic extension of F of degree d corresponding to the character κ of $F^{\times}$. For any positive integer m, we consider $H = GL(m, E)$ as a subgroup of $G = GL(md, F)$. In this paper we discuss matching of orbital integrals between H and G. Specifically, ordinary orbital integrals corresponding to regular semisimple elements of H are matched with orbital integrals on G which are twisted by the character κ. For the general situation we only match functions which are smooth and compactly supported on the regular set. If the extension E/F is unramified, we are able to match arbitrary smooth, compactly supported functions.

§1. Introduction.

Let F be a locally compact, non-discrete, nonarchimedean local field of characteristic zero. Let κ be a unitary character of $F^{\times}$ of order d, and let E be the cyclic extension of F corresponding to κ. Let m and n be positive integers with $n = md$ and write $G = GL(n, F)$, $\quad H = GL(m, E)$. H can be identified with a subgroup of G. In this paper we discuss matching of orbital integrals between H and G. Specifically, ordinary orbital integrals corresponding to regular semisimple elements of H are matched with orbital integrals on G which are twisted by the character κ. For the general situation we only match functions which are smooth and compactly supported on the regular set. If the extension E/F is unramified, we are able to match arbitrary smooth, compactly supported functions.

Extend κ to a character of G by $\kappa(g) = \kappa(\det g)$ and let

$$G_0 = \{g \in G : \kappa(g) = 1\}.$$

G_0 is an open normal subgroup of G of finite index and $H \subset G_0$. Let $C_c^{\infty}(G)$ denote the set of locally constant, compactly supported, complex-valued functions on G. For any $\gamma \in G$ we let G_γ denote the centralizer of $\gamma \in G$. If $G_\gamma \subset G_0$, let

$$\Lambda_\kappa^G(f, \gamma) = \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)\kappa(x)dx, f \in C_c^{\infty}(G),$$

be the twisted orbital integral of f over the orbit of γ. If $G_\gamma \not\subset G_0$, set $\Lambda_\kappa^G(f,\gamma) = 0$. Clearly for all $x, \gamma \in G, f \in C_c^\infty(G)$,

$$\Lambda_\kappa^G(f, x\gamma x^{-1}) = \kappa(x)\Lambda_\kappa^G(f,\gamma).$$

Similarly we define

$$\Lambda^H(f,\gamma) = \int_{H_\gamma \backslash H} f(x^{-1}\gamma x)dx, f \in C_c^\infty(H), \gamma \in H,$$

the ordinary orbital integral of f over the H-orbit of γ.

The main results of this paper are the following theorems. Let G' denote the set of regular semisimple elements of G and $C_c^\infty(G')$ the subset of all $f \in C_c^\infty(G)$ with support in G'.

Theorem 1.1.

(i) *Let $f_G \in C_c^\infty(G')$. Then there is $f_H \in C_c^\infty(H \cap G')$ such that for all $\gamma \in H \cap G'$,*

$$\Lambda_\kappa^G(f_G, \gamma) = \Lambda^H(f_H, \gamma).$$

(ii) *Conversely, suppose $f_H \in C_c^\infty(H \cap G')$ such that*

$$\Lambda^H(f_H, x\gamma x^{-1}) = \kappa(x)\Lambda^H(f_H,\gamma)$$

for all $x \in G, \gamma \in H \cap G'$ such that $x\gamma x^{-1} \in H$. Then there is $f_G \in C_c^\infty(G')$ such that for all $\gamma \in H \cap G'$,

$$\Lambda_\kappa^G(f_G, \gamma) = \Lambda^H(f_H, \gamma).$$

In the case that κ is unramified, a stronger version of Theorem 1.1 can be proven using results of [**W2, Hn**]. Let Δ_G^H be the transfer factor defined as in [**W2**].

Theorem 1.2. *Assume that κ is unramified.*

(i) *Let $f_G \in C_c^\infty(G)$. Then there is $f_H \in C_c^\infty(H)$ such that for all $\gamma \in H \cap G'$,*

$$\Delta_G^H(\gamma)\Lambda_\kappa^G(f_G, \gamma) = \Lambda^H(f_H, \gamma).$$

(ii) *Conversely, suppose $f_H \in C_c^\infty(H)$ such that*

$$\Lambda^H(f_H, x\gamma x^{-1}) = \Delta_G^H(x\gamma x^{-1})\Delta_G^H(\gamma)^{-1}\kappa(x)\Lambda^H(f_H,\gamma)$$

for all $x \in G, \gamma \in H \cap G'$ such that $x\gamma x^{-1} \in H$. Then there is $f_G \in C_c^\infty(G)$ such that for all $\gamma \in H \cap G'$,

$$\Delta_G^H(\gamma)\Lambda_\kappa^G(f_G, \gamma) = \Lambda^H(f_H, \gamma).$$

The matching theorems for twisted orbital integrals will be used in another paper to prove character formulas relating twisted characters on G to ordinary characters on H. These will generalize the lifting theorem proven by Kazhdan [**K**] in the case that $m = 1$. It will be shown in that paper that

$$\Delta_G^H\left(x\gamma x^{-1}\right) = \Delta_G^H(\gamma)\kappa(x)^{-1}$$

for all $x \in N_G(H), \gamma \in H \cap G'$. Thus when $x \in N_G(H)$, the condition on f_H in Theorem 1.2, (ii), is just

$$\Lambda^H\left(f_H, x\gamma x^{-1}\right) = \Lambda^H\left(f_H, \gamma\right)$$

for all $\gamma \in H \cap G'$. Since Λ^H is an ordinary orbital integral, this is automatic when $x \in H$.

The proof of Theorem 1.1 is routine using an easy extension of results in [**V**] to the twisted case and techniques as in [**A-C**, 1.3]. The proof of Theorem 1.2 uses the fundamental lemma proven by [**W2**, **Hn**]. Assume that κ is unramified. Let $K = GL(n, R)$ where R is the ring of integers of F and let $\mathcal{H}(G)$ denote the Hecke algebra of functions in $C_c^\infty(G)$ which are K bi-invariant. Similarly, we define $\mathcal{H}(H)$, the Hecke algebra of H. Let $b : \mathcal{H}(G) \to \mathcal{H}(H)$ be the homomorphism defined in [**W2**]. The following theorem was proven by Waldspurger [**W2**] when the algebra $F(\gamma)$ is a product of tamely ramified extensions of F and was extended to the general case (as well as to the case of characteristic F not zero) by Henniart [**Hn**].

Theorem 1.3 (Waldspurger, Henniart). *Let $\phi \in \mathcal{H}(G), \gamma \in H \cap G'$. Then*

$$\Delta_G^H(\gamma)\Lambda_\kappa^G(\phi, \gamma) = \Lambda^H(b\phi, \gamma).$$

Theorem 1.2 follows from Theorem 1.3 as follows. First, using standard techniques, it is enough to prove a matching of orbital integrals in a neighborhood of each semisimple element s of H. Further, by passing to centralizers, it is easy to reduce to the case that $s = 1$. The matching in a neighborhood of $s = 1$ is a result of the following theorems which show that all germs in a neighborhood of the identity come from Hecke functions.

Theorem 1.4 [**W1**, **Hr**]. *Let $u_1, ..., u_p$ be a complete set of representatives for the unipotent conjugacy classes of H. Then there are $\phi_1, ..., \phi_p \in \mathcal{H}(H)$ such that*

$$\Lambda^H\left(\phi_i, u_j\right) = \begin{cases} 1, & \text{if } 1 \le i = j \le p; \\ 0, & \text{if } 1 \le i \ne j \le p. \end{cases}$$

Using the results of [**V**] we obtain the following corollary.

Corollary 1.5. *Let $u_1, ..., u_p, \phi_1, ..., \phi_p$ be as above. Let $f \in C_c^\infty(H)$. Then there is a neighborhood U of 1 in H so that for all $\gamma \in U$,*

$$\Lambda^H(f,\gamma) = \sum_{i=1}^{p} \Lambda^H(f,u_i)\,\Lambda^H(\phi_i,\gamma).$$

Let u be a unipotent element of G. If $G_u \not\subset G_0$, then $\Lambda_\kappa^G(f,u) = 0$ for all $f \in C_c^\infty(G)$. It is easy to show that the unipotent conjugacy classes $\mathcal{O}(u)$ of G for which $G_u \subset G_0$ are in bijective correspondance with the unipotent conjugacy classes of H.

Theorem 1.6 [Hr]. *Let $v_1, ..., v_p$ be a complete set of representatives for the unipotent conjugacy classes in G such that $G_{v_i} \subset G_0$. Then there are $\psi_1, ..., \psi_p \in \mathcal{H}(G)$ such that*

$$\Lambda_\kappa^G(\psi_i, v_j) = \begin{cases} 1, & \text{if } 1 \leq i = j \leq p; \\ 0, & \text{if } 1 \leq i \neq j \leq p. \end{cases}$$

An easy extension of germ expansions to the twisted case yields the following corollary.

Corollary 1.7. *Let $v_1, ..., v_p, \psi_1, ..., \psi_p$ be as above. Let $f \in C_c^\infty(G)$. Then there is a neighborhood U of 1 in G so that for all $\gamma \in U$,*

$$\Lambda_\kappa^G(f,\gamma) = \sum_{i=1}^{p} \Lambda_\kappa^G(f,v_i)\,\Lambda_\kappa^G(\psi_i,\gamma).$$

The organization of the paper is as follows.

In §2 we extend many of the results of Vignéras [**V**] to the case of twisted orbital integrals.

In §3 we use the results of §2 to prove Theorems 1.1 and 1.2.

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§2. Twisted Orbital Integrals.

Let $G = GL(n, F)$ and let κ be a unitary character of $F^\times$ of order d, d a divisor of n. In this section we do not assume that κ is unramified. We extend κ to a character of G by setting $\kappa(g) = \kappa(\det g), g \in G$. Let $G_0 = \{g \in G : \kappa(g) = 1\}$. Then G_0 is an open normal subgroup of finite index in

G. For any $x \in G$ we let G_x denote the centralizer of $x \in G$. If $G_x \subseteq G_0$, we let

$$\Lambda_\kappa(f,x) = \int_{G_x \backslash G} f\left(g^{-1}xg\right)\kappa(g)dg, f \in C_c^\infty(G), x \in G$$

be the twisted orbital integral of f over the orbit of x. If $G_x \not\subseteq G_0$, we let $\Lambda_\kappa(f,x) = 0$ for all $f \in C_c^\infty(G)$. (We assume measures are normalized as in [**V**, 1.h].)

In this section we will extend results of Vignéras on orbital integrals to the twisted case. For $x \in G$, define the normalizing factor $d(x)$ as in [**V**, 1.g]. We will also write

$$F_\kappa(f,x) = d(x)\Lambda_\kappa(f,x).$$

Let s be a semisimple element in G. Then as in [**V**, 1.j] we write A_s for the set of all elements x of G with semisimple part (of the Jordan decomposition of x) conjugate to s. Let $A_s = \cup\mathcal{O}\left(su_i\right), 1 \leq i \leq m$, be the standard decomposition as in [**V**, 1.j] where $\mathcal{O}(x)$ denotes the G orbit of $x \in G$. For $x \in G_0$ we will write $\mathcal{O}_0(x)$ for the G_0 orbit of x.

Lemma 2.1. *Fix $1 \leq i \leq m$ and suppose that $G_{su_i} \subseteq G_0$. Then there is $f_i \in C_c^\infty(G)$ such that*

$$F_\kappa\left(f_i, su_j\right) = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

Proof. As in [**V**, 1.k], for each $1 \leq i \leq m$ there is a compact open subset K_i in G so that $su_i \in K_i$, and $K_i \cap \mathcal{O}\left(su_j\right) = \emptyset, 1 \leq j \leq i-1$. Now suppose that $G_{su_i} \subseteq G_0$. Then

$$\mathcal{O}\left(su_i\right) \approx G_{su_i}\backslash G \approx G_{su_i}\backslash G_0 \times G_0\backslash G \approx \mathcal{O}_0\left(su_i\right) \times G_0\backslash G$$

so that $\mathcal{O}_0\left(su_i\right)$ is open and closed in $\mathcal{O}\left(su_i\right)$. Thus there is $K_i' \subseteq K_i$ compact open in G so that $su_i \in K_i', K_i' \cap \mathcal{O}\left(su_i\right) \subseteq \mathcal{O}_0\left(su_i\right)$. Now if f_i' is the characteristic function of K_i', then $F_\kappa\left(f_i', su_i\right) \neq 0$ because there can be no cancellation in the integral, and $F_\kappa\left(f_i', su_j\right) = 0, 1 \leq j \leq i-1$. Now using a standard Graham-Schmidt type procedure we can obtain f_i's as in the lemma. □

Lemma 2.2. *Let $s \in G$ be semisimple and suppose that $f \in C_c^\infty(G)$ satisfies $F_\kappa(f,x) = 0$ for all $x \in A_s$. Then there is a neighborhood V_f of s in G such that $F_\kappa(f,x) = 0$ for all $x \in V_f$.*

Proof. We follow the proof of [**K**, 3.8]. Let $S = C_c^\infty\left(A_s\right)$. Since A_s is G-invariant, G acts on S by $g\ \tilde{f}(x) = \tilde{f}\left(g^{-1}xg\right), g \in G, x \in A_s, \tilde{f} \in S$. Since

A_s is closed in G, restriction gives a mapping $\pi : C_c^\infty(G) \to S$. Let S' be the dual of S and let $\Lambda = \left\{\lambda \in S' : \lambda\left(g \cdot \tilde{f}\right) = \kappa(g)\lambda\left(\tilde{f}\right), \forall g \in G, \tilde{f} \in S\right\}$. Then since G has only a finite number of orbits in A_s we see that Λ is generated by the $\lambda_i, 1 \leq i \leq m$, where $\lambda_i(\pi(f)) = F_\kappa(f, su_i)$. Let $S_\kappa = \left\{\tilde{f} \in S : \lambda\left(\tilde{f}\right) = 0, \forall \lambda \in \Lambda\right\}$. Then S_κ is the set of all finite sums of functions of the form $g \cdot \tilde{f} - \kappa(g)\tilde{f}$.

Now let $f \in C_c^\infty(G)$ such that $F_\kappa(f, su_i) = 0, 1 \leq i \leq m$. Then $\tilde{f} = \pi(f) \in S_\kappa$ so there are $g_1, ..., g_k \in G, \tilde{f}_1, ..., \tilde{f}_k \in S$, such that $\tilde{f} = \sum_{i=1}^k g_i \cdot \tilde{f}_i - \kappa(g_i)\tilde{f}_i$. Let $f_i \in C_c^\infty(G)$ such that $\pi(f_i) = \tilde{f}_i$, and let $\phi = f - \sum_{i=1}^k g_i \cdot f_i + \kappa(g_i) f_i$. Then $\pi(\phi) = 0$ so by [**V**, 2.4] there is an open, G-invariant neighborhood V_f of s such that ϕ is zero on V_f. Thus $F_\kappa(\phi, x) = 0$ for all $x \in V_f$. But for all $x \in G$, $F_\kappa(f, x) = F_\kappa(\phi, x)$. □

Renumber $u_1, ..., u_m$ so that $su_i, 1 \leq i \leq k$, are the orbits of A_s such that $G_{su_i} \subseteq G_0, 1 \leq i \leq k$. Suppose $f_1, ..., f_k \in C_c^\infty(G)$ satisfy $F_\kappa(f_i, su_j) = \delta_{ij}, 1 \leq i, j \leq k$, and $f_1', ..., f_k' \in C_c^\infty(G)$ satisfy $\Lambda_\kappa(f_i', su_j) = \delta_{ij}, 1 \leq i, j \leq k$.

Lemma 2.3. *Let $f \in C_c^\infty(G)$. Then there is a neighborhood V_f of s in G so that*

$$F_\kappa(f, x) = \sum_{i=1}^k F_\kappa(f, su_i) F_\kappa(f_i, x)$$

and

$$\Lambda_\kappa(f, x) = \sum_{i=1}^k \Lambda_\kappa(f, su_i) \Lambda_\kappa(f_i', x)$$

for all $x \in V_f$.

Proof. Let $f' = f - \sum_{i=1}^k F_\kappa(f, su_i) f_i$. Then $F_\kappa(f', su_j) = 0, 1 \leq j \leq k$. Thus by Lemma 2.2 there is a neighborhood V_f of s such that $F_\kappa(f', x) = 0$ for all $x \in V_f$. □

As in [**V**, 1.m], for any $s \in G$ semisimple, we let T be the center of $M = G_s$. Let $u \in Z_G(T)$ be unipotent. Then (T, u) is called a standard couple. For any subset X of G, let X^{reg} denote the subset of elements $x \in X$ such that the dimension of the conjugacy class of x is greater than or equal to the dimension of the conjugacy class of any $y \in X$.

We can now extend Theorems A and B of [**V**, 1.n] to the twisted case.

Theorem 2.4. *(A) Let $f \in C_c^\infty(G)$ and let $F(x) = F_\kappa(f, x), x \in G$. Let (T, u) be any standard couple. Then F has the following properties.*

(i) *$F(gxg^{-1}) = \kappa(g)F(x), \forall x, g \in G$;*

(ii) *the restriction of F to Tu^{reg} is locally constant;*

(iii) *the restriction of F to Tu has compact support;*

(iv) *for every $s \in T$ there is a neighborhood V_F of s in T such that for $t \in V_F \cap T^{reg}$,*

$$F(tu) = \sum_{i=1}^{k} F(su_i) F_\kappa(f_i, tu)$$

where $su_i, f_i, 1 \leq i \leq k$ are defined as in Lemma 2.3.

(B) Conversely, if F is a function on G satisfying (i)-(iv) *above, then there is $f \in C_c^\infty(G)$ such that $F(x) = F_\kappa(f, x)$ for all $x \in G$.*

Proof. Part (A) follows from Lemma 2.3 and [**V**, 2.7]. It also follows easily from [**V**, 2.7] that if $f' \in C_c^\infty(T^{reg})$ transforms according to κ under the action of $W(Tu) = N_G(Tu)/Z_G(Tu)$, then there is $f \in C_c^\infty(\mathcal{O}(T^{reg}))$ such that $f'(t) = F_\kappa(f, t)$ for all $t \in T^{reg}$. Now the proof of (B) follows by an induction argument as in [**V**, 2.8]. □

We can use Theorem 2.4 to obtain the following localization result. Let $T_1, ..., T_r$ be a complete set of Cartan subgroups of G, up to G-conjugacy. Let $X = \cup_{i=1}^r T_i \subseteq G$.

Lemma 2.5. *Let V be a closed and open subset of X such that $\mathcal{O}(V) \cap X = V$. Then given $f \in C_c^\infty(G)$ there is $f_V \in C_c^\infty(G)$ such that*

$$F_\kappa(f, \gamma) = F_\kappa(f_V, \gamma), \gamma \in V$$

and

$$F_\kappa(f_V, \gamma) = 0, \gamma \in X \backslash V.$$

Proof. Let $F(x) = F_\kappa(f, x), x \in G$. For any $x \in G$, write $x = s(x)u(x)$ for the Jordan decomposition of x. Define

$$F_V(x) = \begin{cases} F(x), & \text{if } s(x) \in \mathcal{O}(V); \\ 0, & \text{otherwise.} \end{cases}$$

Then for any $x, g \in G, s(gxg^{-1}) = gs(x)g^{-1} \in \mathcal{O}(V)$ if and only if $s(x) \in \mathcal{O}(V)$. Thus if $s(x) \notin \mathcal{O}(V)$ we have $F_V(x) = F_V(gxg^{-1}) = 0$. If $s(x) \in \mathcal{O}(V)$ we have $F_V(gxg^{-1}) = F(gxg^{-1}) = \kappa(g)F(x) = \kappa(g)F_V(x)$. Thus F_V satisfies (i) of Theorem 2.4.

Let (T, u) be any standard couple. We can assume that $T \subseteq T_i \subseteq X$ for some T_i. Let $V_T = V \cap T$. It is open and closed in T. Let χ_V be the characteristic function of $V_T u$. It is a locally constant function. Further $F_V|_{Tu} = F|_{Tu} \cdot \chi_V$ since, using our assumption that $\mathcal{O}(V) \cap X = V$, for every $tu \in Tu, t \in \mathcal{O}(V)$ if and only if $t \in V_T$. Thus F_V satisfies (ii) and (iii) of Theorem 2.4.

Finally, fix $s \in T$. If $s \notin V_T$, there is a neighborhood U of s in T such that $U \cap V_T = \emptyset$. Now $F_V(su_i) = 0$ for all i and $F_V(tu) = 0$ for all $t \in U$. Thus F_V satisfies the germ expansion in U. If $s \in V_T$, then let V_F be a neighborhood of s in T such that for all $t \in V_F \cap T^{reg}$,

$$F(tu) = \sum_i F(su_i) F_\kappa(f_i, tu).$$

Let $V_{F_V} = V_F \cap V_T$. Then for all $t \in V_{F_V}, F_V(tu) = F(tu)$. Also $F_V(su_i) = F(su_i)$ for all i. Thus F_V also satifies (iv). □

Let $s \in G$ be an arbitrary semisimple element. Let $\{T_1, ..., T_r\}$ be representatives for the Cartan subgroups of G, up to G-conjugacy, such that $s \in T_i, 1 \le i \le r$. Let M be the centralizer of s in G. Then $T_i \subseteq M, 1 \le i \le r$, and for any $\psi \in C_c^\infty(M), \gamma \in T_i \cap G'$, we can define

$$\Lambda_\kappa^M(\psi, \gamma) = \int_{T_i \backslash M} \psi(m^{-1}\gamma m)\, \kappa(m) dm$$

if $T_i \subset G_0$ and $\Lambda_\kappa^M(\psi, \gamma) = 0$ if $T_i \not\subset G_0$.

Lemma 2.6.

(i) *Let $f \in C_c^\infty(G)$. Then there are neighborhoods V_i of s in T_i and $\psi \in C_c^\infty(M)$ so that for all $1 \le i \le r, \gamma \in V_i \cap G'$,*

$$\Lambda_\kappa^G(f, \gamma) = \Lambda_\kappa^M(\psi, \gamma).$$

(ii) *Let $\psi \in C_c^\infty(M)$. Then there are neighborhoods V_i of s in T_i and $f \in C_c^\infty(G)$ so that for all $1 \le i \le r, \gamma \in V_i \cap G'$,*

$$\Lambda_\kappa^G(f, \gamma) = \Lambda_\kappa^M(\psi, \gamma).$$

Proof. The proof is an easy generalization of the argument used in [**V**, 2.5]. Define $su_j, f_j, 1 \le j \le k$ as in Theorem 2.4. Let T be the center of M.

Fix $f \in C_c^\infty(G)$ and let $\Omega = supp\ f$. Then using [**HC**], there are neighborhoods V_i of s in T_i and an open, compact subset $\omega \subseteq M\backslash G$ so that $g^{-1}V_i g \cap \Omega = \emptyset, 1 \le i \le r$, unless $Mg \in \omega$. Further, as in [**V**, 2.5], there is a neighborhood V of s in T and an open, compact subset $C \subseteq M\backslash G$ so that $g^{-1}Vu_j g \cap \Omega = \emptyset, 1 \le j \le k$, unless $Mg \in C$. Choose $\alpha \in C_c^\infty(G)$ so that

$$\tilde{\alpha}(g) = \int_M \alpha(mg) dm = \begin{cases} 1, & \text{if } Mg \in C \cup \omega; \\ 0, & \text{if } Mg \notin C \cup \omega. \end{cases}$$

Define

$$\psi(m) = \int_G \alpha(x)\kappa(x)f\left(x^{-1}mx\right)dx, m \in M.$$

Then $\psi \in C_c^\infty(M)$, and it is easy to check that for all $1 \leq i \leq r, \gamma \in V_i \cap G'$,

$$\Lambda_\kappa^G(f,\gamma) = \Lambda_\kappa^M(\psi,\gamma).$$

Further, for all $1 \leq j \leq k, \gamma \in V$,

$$\Lambda_\kappa^G\left(f,\gamma u_j\right) = \Lambda_\kappa^M\left(\psi,\gamma u_j\right).$$

This proves part (i) of the Lemma.

Define $su_j, f_j, 1 \leq j \leq k$ as above and let $f_j' = d\left(su_j\right)f_j$. Then the f_j' satisfy $\Lambda_\kappa^G\left(f_j', su_l\right) = \delta_{jl}, 1 \leq j,l \leq k$. To prove part (ii), we use (i) to choose neighborhoods V_i of s in $T_i, 1 \leq i \leq r$ and V of s in T, and functions $\psi_j \in C_c^\infty(M), 1 \leq j \leq k$, so that for all $1 \leq j \leq k, 1 \leq i \leq r, \gamma \in V_i \cap T_i'$,

$$\Lambda_\kappa^G\left(f_j',\gamma\right) = \Lambda_\kappa^M\left(\psi_j,\gamma\right).$$

Further, for all $1 \leq l \leq k, \gamma \in V$,

$$\Lambda_\kappa^G\left(f_j',\gamma u_l\right) = \Lambda_\kappa^M\left(\psi_j,\gamma u_l\right).$$

Thus the functions ψ_j satisfy

$$\Lambda_\kappa^M\left(\psi_j, su_l\right) = \begin{cases} 1, & \text{if } j = l; \\ 0, & \text{if } j \neq l. \end{cases}$$

Now fix $\psi \in C_c^\infty(M)$. As in [**V**, 2.5], the orbital decomposition of $A_{s,M}$ and A_s can be represented by the same elements $su_1, ..., su_m$. Also $M_{su_i} = G_{su_i}$ and $M_0 = M \cap G_0$, so that $M_{su_i} \subseteq M_0$ if and only if $G_{su_i} \subseteq G_0$. Thus we can also take $su_1, ..., su_k$ the same for M and G. Thus using Lemma 2.3 applied to M there is a neighborhood U of s in M so that for all $m \in U$,

$$\Lambda_\kappa^M(\psi,m) = \sum_{j=1}^k \Lambda_\kappa^M\left(\psi_j,m\right)\Lambda_\kappa^M(\psi,su_j).$$

Define $f \in C_c^\infty(G)$ by

$$f(g) = \sum_{j=1}^k \Lambda_\kappa^M\left(\psi,su_j\right)f_j'(g), g \in G.$$

Then for all $\gamma \in G$,

$$\Lambda_\kappa^G(f,\gamma) = \sum_{j=1}^{k} \Lambda_\kappa^M(\psi, su_j)\, \Lambda_\kappa^G\left(f_j', \gamma\right).$$

But now we have

$$\Lambda_\kappa^G\left(f_j',\gamma\right) = \Lambda_\kappa^M(\psi_j,\gamma), \gamma \in V_i \cap T_i', 1 \leq i \leq r, 1 \leq j \leq k,$$

so that

$$\Lambda_\kappa^G(f,\gamma) = \sum_{j=1}^{k} \Lambda_\kappa^M(\psi, su_j)\, \Lambda_\kappa^M(\psi_j, \gamma).$$

Thus for $\gamma \in V_i \cap U \cap T_i', 1 \leq i \leq r$, we have

$$\Lambda_\kappa^G(f,\gamma) = \Lambda_\kappa^M(\psi,\gamma).$$

□

§3. Matching Theorems.

Let $G = GL(n,F), K = GL(n,R)$, and let κ be a unitary character of $F^\times$ of order d, d a divisor of n. Unless otherwise noted we will assume that κ is unramified.

As in Theorem 1.6 we let $u_1, ..., u_k$ represent the unipotent conjugacy classes with $G_{u_i} \subset G_0$, and $\phi_1, ..., \phi_k \in \mathcal{H}(G)$ satisfy $\Lambda_\kappa(\phi_i, u_j) = \delta_{ij}$. The following lemma is a special case of Lemma 2.3.

Lemma 3.1. *Let $f \in C_c^\infty(G)$. Then there is a neighborhood U of 1 in G so that*

$$\Lambda_\kappa(f,\gamma) = \sum_{i=1}^{k} \Lambda_\kappa(f, u_i)\, \Lambda_\kappa(\phi_i, \gamma)$$

for all $\gamma \in U \cap G'$.

Now let E be the cyclic extension of order d of F corresponding to κ and let $H = GL(m,E), md = n$. Fix an embedding of H in G as in [**W2**]. Then for $\gamma \in H$ we can define both the ordinary orbital integral $\Lambda^H(f,\gamma), f \in C_c^\infty(H)$, and the twisted orbital integral $\Lambda_\kappa^G(f,\gamma), f \in C_c^\infty(G)$.

Write $\mathcal{H}(G), \mathcal{H}(H)$ for the Hecke algebras of G and H respectively. Let $b: \mathcal{H}(G) \to \mathcal{H}(H)$ be the homomorphism of $\mathcal{H}(G)$ onto $\mathcal{H}(H)$ defined as in [**W2**], and define the transfer factor Δ_G^H as in [**W2, Hl**]. The following theorem was proven by Waldspurger [**W2**] for F of characteristic zero and $F(\gamma)$ tamely ramified over F, and was extended by Henniart [**Hn**].

Theorem 3.2 (Waldspurger, Henniart). *Let $f \in \mathcal{H}(G), \gamma \in H \cap G'$. Then*

$$\Delta_G^H(\gamma)\Lambda_\kappa^G(f,\gamma) = \Lambda^H(bf,\gamma).$$

Write Z_G for the center of G.

Theorem 3.3. *Let $z \in Z_G$.*

(i) *Let $f_G \in C_c^\infty(G)$. Then there are a neighborhood U of z in H and $f_H \in C_c^\infty(H)$ so that*

$$\Delta_G^H(\gamma)\Lambda_\kappa^G(f_G,\gamma) = \Lambda^H(f_H,\gamma)$$

for all $\gamma \in U \cap G'$.

(ii) *Let $f_H \in C_c^\infty(H)$. Then there are a neighborhood U of z in H and $f_G \in C_c^\infty(G)$ so that*

$$\Delta_G^H(\gamma)\Lambda_\kappa^G(f_G,\gamma) = \Lambda^H(f_H,\gamma)$$

for all $\gamma \in U \cap G'$.

Proof. Suppose first that $z = 1$ is the identity. Define $u_1, ..., u_k \in G$, $\phi_1, ..., \phi_k \in \mathcal{H}(G)$, as in Lemma 3.1. Let $f_G \in C_c^\infty(G)$ and let V be a neighborhood of 1 in G so that

$$\Lambda_\kappa^G(f_G,\gamma) = \sum_{i=1}^k \Lambda_\kappa^G(f_G,u_i)\,\Lambda_\kappa^G(\phi_i,\gamma)$$

for all $\gamma \in V \cap G'$. Define $f_H \in C_c^\infty(H)$ by

$$f_H = \sum_{i=1}^k \Lambda_\kappa^G(f_G,u_i)\,(b\phi_i).$$

Let $U = V \cap H$. Then using Theorem 3.2, for all $\gamma \in U \cap G'$,

$$\begin{aligned}\Delta_G^H(\gamma)\Lambda_\kappa^G(f_G,\gamma) &= \sum_{i=1}^k \Lambda_\kappa^G(f_G,u_i)\,\Delta_G^H(\gamma)\Lambda_\kappa^G(\phi_i,\gamma)\\ &= \sum_{i=1}^k \Lambda_\kappa^G(f_G,u_i)\,\Lambda^H(b\phi_i,\gamma) = \Lambda^H(f_H,\gamma).\end{aligned}$$

Now let $u_1', ..., u_k' \in H, \phi_1', ..., \phi_k' \in \mathcal{H}(H)$ be defined as in Theorem 1.4 so that $u_1', ..., u_k'$ represent the unipotent conjugacy classes in H and

$\Lambda^H\left(\phi_i', u_j'\right) = \delta_{ij}$. Let $f_H \in C_c^\infty(H)$ and let U be a neighborhood of 1 in H so that

$$\Lambda^H(f_H,\gamma) = \sum_{i=1}^{k} \Lambda^H(f_H,u_i')\,\Lambda^H(\phi_i',\gamma)$$

for all $\gamma \in U \cap H'$. Choose $\phi_1, \dots, \phi_k \in \mathcal{H}(G)$ so that $b\phi_i = \phi_i'$, $1 \le i \le k$, and define

$$f_G = \sum_{i=1}^{k} \Lambda^H(f_H,u_i')\,\phi_i.$$

Then as above

$$\begin{aligned}\Delta_G^H(\gamma)\Lambda_\kappa^G(f_G,\gamma) &= \sum_{i=1}^{k} \Lambda^H(f_H,u_i')\,\Delta_G^H(\gamma)\Lambda_\kappa^G(\phi_i,\gamma)\\ &= \sum_{i=1}^{k} \Lambda^H(f_H,u_i')\,\Lambda^H(\phi_i',\gamma) = \Lambda^H(f_H,\gamma)\end{aligned}$$

for all $\gamma \in U \cap G'$.

To extend the result to arbitrary $z \in Z_G$, we use right translation by z as in [**V**, 2.5]. □

We want to extend the matching of Theorem 3.3 to a matching which is valid for every $\gamma \in H \cap G'$. In order to do this, we need to be able to match orbital integrals in the neighborhood of any semisimple element of H.

Let $s \in H$ be an arbitrary semisimple element. Let M_G be the centralizer of s in G and let M_H be the centralizer of s in H.

Lemma 3.4.

(i) *Let $\psi_G \in C_c^\infty(M_G)$. Then there are a neighborhood U of s in M_H and $\psi_H \in C_c^\infty(M_H)$ so that for all $\gamma \in U \cap G'$,*

$$\Delta_H^G(\gamma)\Lambda_\kappa^{M_G}(\psi_G,\gamma) = \Lambda^{M_H}(\psi_H,\gamma).$$

(ii) *Let $\psi_H \in C_c^\infty(M_H)$. Then there are a neighborhood U of s in M_H and $\psi_G \in C_c^\infty(M_G)$ so that for all $\gamma \in U \cap G'$,*

$$\Delta_H^G(\gamma)\Lambda_\kappa^{M_G}(\psi_G,\gamma) = \Lambda^{M_H}(\psi_H,\gamma).$$

Proof. Write $M_G = \prod_{i=1}^k GL(n_i, F_i)$ where the F_i are extensions of degree r_i of F and $\sum_{i=1}^k n_i r_i = n$. For each $1 \le i \le k$, let κ_i be the character of $F_i^\times$ given by $\kappa_i(\lambda) = \kappa\left(N_{F_i/F}(\lambda)\right)$. Now the center T_G of M_G is isomorphic to

$\prod_{i=1}^{k} F_i^{\times}$. For $\lambda_i \in F_i^{\times}, 1 \leq i \leq k$, write $a(\lambda_1, ..., \lambda_k)$ for the corresponding element of T_G. Then

$$\kappa\left(a\left(\lambda_1, ..., \lambda_k\right)\right) = \prod_{i=1}^{k} \kappa_i\left(\lambda_i^{n_i}\right).$$

Let d_i be the order of κ_i. Then if there is $1 \leq i \leq k$ such that d_i does not divide n_i, there is $a \in T_G$ so that $\kappa(a) \neq 1$. But since $s \in H$ is semisimple, it is contained in some Cartan subgroup T of H. But every Cartan subgroup of H is a Cartan subgroup of G so that $T_G \subseteq T$. Thus $T_G \subseteq H$ so that $\kappa(a) = 1$ for all $a \in T_G$. Thus d_i divides n_i for all i. Write $n_i = m_i d_i, 1 \leq i \leq k$ and let E_i be the extension of F_i corresponding to κ_i. It is the minimal extension of F_i containing E. Now $M_H = \prod_{i=1}^{k} GL(m_i, E_i)$.

Thus $M_G = \prod_{i=1}^{k} GL(n_i, F_i)$ and $M_H = \prod_{i=1}^{k} GL(m_i, E_i)$ are products of groups $G_i = GL(n_i, F_i)$, $H_i = GL(m_i, E_i)$ of the same type as our original groups G and H. Further, if $g = (g_1, g_2, ..., g_k) \in M_G = \prod G_i$, then $\det_G g = \prod N_{F_i/F}(\det_{G_i} g_i)$ so that $\kappa(g) = \prod \kappa_i(g_i)$. Thus κ-twisted orbital integrals on M_G are the products of κ_i-twisted orbital integrals on the factors G_i. Now since $s \in M_H$ is central in M_G, we can apply Theorem 3.3 to match functions $\psi_G \in C_c^{\infty}(M_G)$ in a neighborhood of s with functions $\psi'_H \in C_c^{\infty}(M_H)$ using the transfer factor $\Delta_{M_G}^{M_H}$. Thus to complete the proof of the lemma it suffices to show that there is a neighborhood U of s in M_H so that $\Delta_G^H \left(\Delta_{M_G}^{M_H}\right)^{-1}$ is constant and non-zero on $U \cap G'$, so we can also match using the transfer factor Δ_G^H. This is proven in Lemmas 3.5 and 3.6 below. □

In order to complete the proof of Lemma 3.4, we must define the transfer factors. For $\gamma, \delta \in H$, let $c_1, ..., c_m$, respectively $d_1, ..., d_m$ denote the eigenvalues of γ, resp. δ, in some extension of E. As in [**W2, Hl**] we set

$$r(\gamma, \delta) = \prod_{i,j=1}^{m} (c_i - d_j).$$

Then for all $\gamma \in H \cap G'$, we define

$$\Delta_G^{H,1}(\gamma) = \left| \prod_{\sigma,\tau \in \mathcal{G}(E/F), \sigma \neq \tau} r(\sigma\gamma, \tau\gamma) \right|_F^{\frac{1}{2}} \left| \det_G(\gamma) \right|_F^{\frac{(m-n)}{2}}$$

where $\mathcal{G}(E/F)$ denotes the Galois group of E/F. Further, we set

$$\Delta_G^{H,2}(\gamma) = 1$$

for all $\gamma \in H$ if d is odd. If d is even, let σ_+ be the unique element of order 2 in $\mathcal{G}(E/F)$ and let ν_E denote the valuation in E. Then we define

$$\Delta_G^{H,2}(\gamma) = (-1)^{\nu_E(r(\gamma,\sigma_+\gamma))}$$

for all $\gamma \in H$. Finally, for all $\gamma \in H \cap G'$, we define

$$\Delta_G^H(\gamma) = \Delta_G^{H,1}(\gamma)\Delta_G^{H,2}(\gamma).$$

We now return to the notation of Lemma 3.4 so that $s \in H$ is an arbitrary semisimple element with centralizers M_G and M_H in G and H respectively.

Lemma 3.5. *There is a neighborhood U of s in M_H so that $\Delta_G^{H,1}\left(\Delta_{M_G}^{M_H,1}\right)^{-1}$ is constant and non-zero on $U \cap G'$.*

Proof. For $\gamma \in H \cap G'$, let $c_1, ..., c_m$ denote the eigenvalues of γ considered as an element of $H = GL(m, E)$ and let $d_1, ..., d_n$ denote its eigenvalues considered as an element of $G = GL(n, F)$. Define

$$\Delta_H(\gamma) = \prod_{1 \leq i \neq j \leq m} (c_i - c_j), \quad \Delta_G(\gamma) = \prod_{1 \leq i \neq j \leq n} (d_i - d_j).$$

Fix $\gamma \in H \cap G'$. For each $\sigma \in \mathcal{G}(E/F)$, let $c(i, \sigma), 1 \leq i \leq m$, denote the eigenvalues of $\sigma\gamma$ as an element of $H = GL(m, E)$. Then as an element of $G = GL(n, F), \gamma$ has eigenvalues $c(i, \sigma), 1 \leq i \leq m, \sigma \in \mathcal{G}(E/F)$. Thus we can rewrite

$$\prod_{\sigma,\tau \in \mathcal{G}(E/F), \sigma \neq \tau} r(\sigma\gamma, \tau\gamma) = \Delta_G(\gamma) N_{E/F} \Delta_H(\gamma)^{-1}$$

and

$$\Delta_G^{H,1}(\gamma) = |\Delta_G(\gamma)|_F^{\frac{1}{2}} |\Delta_H(\gamma)|_E^{\frac{-1}{2}} \left|\det_G(\gamma)\right|_F^{\frac{(m-n)}{2}}.$$

Now use the notation in the proof of Lemma 3.4 so that we have $M_H = \prod H_i, M_G = \prod G_i$, where for $1 \leq i \leq k$, $H_i = GL(m_i, E_i), G_i = GL(n_i, F_i)$. Then for any $\gamma = \prod \gamma_i \in M_H \cap M_G'$, we have

$$\begin{aligned}\Delta_{M_G}^{M_H,1}(\gamma) &= \prod \Delta_{G_i}^{H_i,1}(\gamma_i) \\ &= \prod_i |\Delta_{G_i}(\gamma_i)|_{F_i}^{\frac{1}{2}} |\Delta_{H_i}(\gamma_i)|_{E_i}^{\frac{-1}{2}} \left|\det_{G_i}(\gamma_i)\right|_{F_i}^{\frac{(m_i-n_i)}{2}}.\end{aligned}$$

Thus

$$\Delta_G^{H,1}(\gamma)\ \Delta_{M_G}^{M_H,1}(\gamma)^{-1} = |\Delta_G(\gamma)|_F^{\frac{1}{2}} \prod_i |N_{F_i/F}\Delta_{G_i}(\gamma_i)|_F^{\frac{-1}{2}} \times |\Delta_H(\gamma)|_E^{\frac{-1}{2}} \prod_i |N_{E_i/E}\Delta_{H_i}(\gamma_i)|_E^{\frac{1}{2}} \times \left|\det_G(\gamma)\right|_F^{\frac{(m-n)}{2}} \prod_i \left|\det_{G_i}(\gamma_i)\right|_{F_i}^{\frac{(n_i-m_i)}{2}}.$$

We can index the eigenvalues of γ in $GL(n,F)$ as $c(i,j,t), 1 \le i \le k, 1 \le j \le n_i, 1 \le t \le r_i$, so that

$$\prod_i N_{F_i/F}\Delta_{G_i}(\gamma_i) = \prod_i \prod_t \prod_{j\neq j'} [c(i,j,t) - c(i,j',t)].$$

Then we have

$$\Delta_G(\gamma) \prod_i N_{F_i/F}\Delta_{G_i}(\gamma_i)^{-1} = \prod_{(i,t)\neq(i',t')} \prod_{j,j'} [c(i,j,t) - c(i',j',t')].$$

Thus $\gamma \mapsto \Delta_G(\gamma) \prod_i N_{F_i/F}\Delta_{G_i}(\gamma_i)^{-1}$ extends to a continuous function on M_G. Further, when $\gamma = s$, γ_i is central in G_i for all i so that $c(i,j,t) = c(i,j',t)$ for all i,j,j',t. But since $M_G = \prod G_i$ is the full centralizer of s in G we have $c(i,j,t) \neq c(i',j',t')$ if $(i,t) \neq (i',t')$. Thus $\Delta_G(\gamma)\prod_i N_{F_i/F}\Delta_{G_i}(\gamma_i)^{-1}$ is non-zero at $\gamma = s$. Similarly, we see that $\gamma \mapsto \Delta_H(\gamma)\prod_i N_{E_i/E}\Delta_{H_i}(\gamma_i)^{-1}$ extends to a continuous function on M_H which is non-zero at $\gamma = s$. Finally, the determinant factors are certainly continuous and non-zero on all of M_H. Thus

$$\gamma \mapsto \Delta_G^{H,1}(\gamma)\Delta_{M_G}^{M_H,1}(\gamma)^{-1}$$

extends to a function which is constant in a neighborhood of s in M_H. □

Lemma 3.6. *There is a neighborhood U of s in M_H so that $\Delta_G^{H,2}\left(\Delta_{M_G}^{M_H,2}\right)^{-1}$ is constant and non-zero on $U \cap G'$.*

Proof. We first need to derive an alternate formula for $\Delta_G^{H,2}$. Let σ_0 be a generator of $\mathcal{G}(E/F)$. For all $\gamma \in H$ we define

$$\tilde{\Delta}(\gamma) = \prod_{0\le i<j\le d-1} r\left(\sigma_0^i\gamma, \sigma_0^j\gamma\right).$$

Then for each $\gamma \in H \cap G', \tilde{\Delta}(\gamma)$ is an element of $E^\times$. Clearly $r(\delta,\gamma) = (-1)^m r(\gamma,\delta)$ for all $\gamma,\delta \in H$. Thus it is easy to see that

$$\sigma_0\tilde{\Delta}(\gamma) = (-1)^{m(d-1)}\tilde{\Delta}(\gamma),\ \forall\gamma \in H.$$

If $m(d-1)$ is even we let $e_0 = 1$. Suppose that $m(d-1)$ is odd. Then d is even. Define $E_2 = \{e \in E : \sigma_0^2 e = e\}$. Then E_2/F is a cyclic extension of degree 2 and we can choose a unit $e_0 \in E_2$ such that $E_2 = F[e_0], \sigma_0 e_0 = -e_0$. With these choices of e_0 we have $e_0\tilde{\Delta}(\gamma) \in F$ for all $\gamma \in H$. We now claim that

$$\Delta_G^{H,2}(\gamma) = \kappa\left(e_0\tilde{\Delta}(\gamma)\right), \quad \gamma \in H \cap G'.$$

Let η be an unramified character of $E^\times$ which extends κ. Thus for all $e \in E^\times, \eta(e) = \zeta^{\nu_E(e)}$ where ζ is a primitive d^{th} root of unity. Now since e_0 is a unit we have

$$\Delta_G^{H,2}(\gamma) = \eta\left(e_0\tilde{\Delta}(\gamma)\right) = \eta\left(\tilde{\Delta}(\gamma)\right).$$

Now

$$\prod_{\sigma \neq \tau} r(\sigma\gamma, \tau\gamma) = \pm\tilde{\Delta}(\gamma)^2$$

so that

$$\nu_E\left(\tilde{\Delta}(\gamma)\right) = \frac{1}{2}\nu_E\left(\prod_{\sigma\neq\tau} r(\sigma\gamma,\tau\gamma)\right).$$

But

$$\prod_{\sigma\neq\tau} r(\sigma\gamma,\tau\gamma) = N_{E/F}\left(\prod_{\tau\neq 1} r(\gamma,\tau\gamma)\right).$$

Thus

$$\nu_E\left(\tilde{\Delta}(\gamma)\right) = \frac{1}{2}\prod_{1\leq i\leq d-1} \nu_E\left(N_{E/F} r\left(\gamma, \sigma_0^i\gamma\right)\right).$$

But for any $1 \leq i \leq d-1$,

$$\begin{aligned}\nu_E\left(N_{E/F} r\left(\gamma, \sigma_0^{d-i}\gamma\right)\right) &= \nu_E\left(N_{E/F}\sigma_0^{d-i} r\left(\sigma_0^i\gamma, \gamma\right)\right)\\ &= \nu_E\left(N_{E/F} r\left(\gamma, \sigma_0^i\gamma\right)\right).\end{aligned}$$

Further, $\nu_E\left(N_{E/F}(e)\right) = d\nu_E(e)$ for all $e \in E^\times$. Thus, calculating modulo d, we have

$$\nu_E\left(\tilde{\Delta}(\gamma)\right) \equiv \begin{cases} \frac{d}{2}\nu_E\left(r\left(\gamma, \sigma_0^{\frac{d}{2}}\gamma\right)\right), & \text{if } d \text{ is even;}\\ 0, & \text{if } d \text{ is odd.}\end{cases}$$

Now when d is even $\sigma_0^{\frac{d}{2}} = \sigma_+$ so we can conclude that

$$\eta\left(\tilde{\Delta}(\gamma)\right) = \begin{cases} (-1)^{\nu_E(r(\gamma,\sigma_+\gamma))}, & \text{if } d \text{ is even;}\\ 1, & \text{if } d \text{ is odd.}\end{cases}$$

This completes the proof that $\Delta_G^{H,2}(\gamma) = \kappa\left(e_0\tilde{\Delta}(\gamma)\right), \quad \gamma \in H \cap G'$.

Similarly, for all $\gamma = \prod_i \gamma_i \in \prod H_i$, we have

$$\Delta_{M_G}^{M_H,2}(\gamma) = \prod \kappa_i\left(e_{0,i}\tilde{\Delta}_i(\gamma_i)\right)$$

where $e_{0,i}, \tilde{\Delta}_i$ are defined for the pair H_i, G_i. Since $\kappa_i = \kappa \circ N_{F_i/F}$, for $\gamma = \prod \gamma_i \in M_H \cap G'$ we have

$$\Delta_G^{H,2}(\gamma)\Delta_{M_G}^{M_H,2}(\gamma)^{-1} = \kappa\left(e_0\tilde{\Delta}(\gamma)\prod N_{F_i/F}\left(e_{0,i}\tilde{\Delta}_i(\gamma_i)\right)^{-1}\right).$$

Thus $\Delta_G^{H,2}\left(\Delta_{M_G}^{M_H,2}\right)^{-1}$ will extend to a function which is constant and nonzero in a neighborhood of s if we can show that

$$\gamma \mapsto \tilde{\Delta}(\gamma)\prod N_{F_i/F}\left(e_{0,i}\tilde{\Delta}_i(\gamma_i)\right)^{-1}$$

extends to a continuous function on M_H which is not zero at $\gamma = s$. Note that, using the notation in the proof of Lemma 3.6, we have

$$\tilde{\Delta}(\gamma)^2 = \pm\prod_{\sigma\neq\tau} r(\sigma\gamma,\tau\gamma) = \pm\Delta_G(\gamma)N_{E/F}\Delta_H(\gamma)^{-1}.$$

Thus the analysis proceeds exactly as in Lemma 3.6. That is, $\prod N_{F_i/F}\left(e_{0,i}\tilde{\Delta}_i(\gamma_i)\right)^{-1}$ cancels out exactly the terms in $\tilde{\Delta}(\gamma)$ which are zero when $\gamma = s$. □

Let $T_1, ..., T_k$ denote the Cartan subgroups of H containing s, up to G-conjugacy.

Lemma 3.7.

(i) *Let $f_G \in C_c^\infty(G)$. Then there are neighborhoods V_i of s in T_i and $f_H \in C_c^\infty(H)$ so that for all $1 \leq i \leq k, \gamma \in V_i \cap G'$,*

$$\Delta_H^G(\gamma)\Lambda_\kappa^G(f_G,\gamma) = \Lambda^H(f_H,\gamma).$$

(ii) *Let $f_H \in C_c^\infty(H)$. Then there are neighborhoods V_i of s in T_i and $f_G \in C_c^\infty(G)$ so that for all $1 \leq i \leq k, \gamma \in V_i \cap G'$,*

$$\Delta_H^G(\gamma)\Lambda_\kappa^G(f_G,\gamma) = \Lambda^H(f_H,\gamma).$$

Proof. This follows easily from combining Lemmas 3.4 and 2.6. □

Locally there is no obstruction to matching twisted orbital integrals on G with ordinary orbital integrals on H. However, if $f_H \in C_c^\infty(H)$ is to match orbital integrals with f_G for all $h \in H \cap G'$, we must have

$$(*) \qquad \Lambda^H\left(f_H, xhx^{-1}\right) = \kappa(x)\Delta_H^G\left(xhx^{-1}\right)\Delta_H^G(h)^{-1}\Lambda^H\left(f_H, h\right)$$

for all $h \in H \cap G'$ and $x \in G$ such that $xhx^{-1} \in H$.

Theorem 3.8.

(i) *Let $f_G \in C_c^\infty(G)$. Then there is $f_H \in C_c^\infty(H)$ so that for all $\gamma \in H \cap G'$,*

$$\Delta_H^G(\gamma)\Lambda_\kappa^G\left(f_G, \gamma\right) = \Lambda^H\left(f_H, \gamma\right).$$

(ii) *Let $f_H \in C_c^\infty(H)$ satisfying (*). Then there is $f_G \in C_c^\infty(G)$ so that for all $\gamma \in H \cap G'$,*

$$\Delta_H^G(\gamma)\Lambda_\kappa^G\left(f_G, \gamma\right) = \Lambda^H\left(f_H, \gamma\right).$$

Proof. (i) Let $T_1, ..., T_k$ be a complete set of Cartan subgroups of H up to H-conjugacy. For each i, let Ω_i be the support of $\Lambda_\kappa^G\left(f_G, \cdot\right)$ restricted to T_i. Let $X = \cup T_i$ and $\Omega = \cup \Omega_i$. Then Ω is a compact subset of X. For each $s \in X$, use Lemma 3.7 to find $U(s)$, a compact open neighborhood of s in X, and $f_s \in C_c^\infty(H)$ such that

$$\Delta_H^G(\gamma)\Lambda_\kappa^G\left(f_G, \gamma\right) = \Lambda^H\left(f_s, \gamma\right), \gamma \in U(s) \cap G'.$$

Note that since both sides are invariant under H-conjugacy, the equality is in fact valid for all $\gamma \in \mathcal{O}_H(U(s)) \cap G'$. Write $U'(s) = \mathcal{O}_H(U(s)) \cap X$.

Since Ω is compact, there are $s_1, ..., s_p$ so that $\Omega \subseteq \cup_{i=1}^p U'\left(s_i\right)$. By shrinking if necessary we can assume that the $U'\left(s_i\right)$ are disjoint. Now by Lemma 2.5 applied to ordinary orbital integrals on H, there are $f_i \in C_c^\infty(H), 1 \leq i \leq p$, so that

$$\Lambda^H\left(f_i, \gamma\right) = \begin{cases} \Lambda^H\left(f_{s_i}, \gamma\right), & \text{if } \gamma \in U'\left(s_i\right); \\ 0, & \text{if } \gamma \in X \backslash U'\left(s_i\right). \end{cases}$$

Let $f_H = \sum_{i=1}^p f_i$. Then for $\gamma \in X \cap G'$, if $\gamma \in U'\left(s_i\right)$, then

$$\Lambda^H\left(f_H, \gamma\right) = \Lambda_H\left(f_{s_i}, \gamma\right) = \Delta_H^G(\gamma)\Lambda_\kappa^G\left(f_G, \gamma\right).$$

If $\gamma \notin \cup_{i=1}^p U'\left(s_i\right)$, then $\gamma \notin \Omega$ so that

$$\Lambda_\kappa^G\left(f_G, \gamma\right) = 0 = \Lambda^H\left(f_H, \gamma\right).$$

(ii) Let $T_1, ..., T_k$ be a complete set of Cartan subgroups of H up to G-conjugacy. For each i, let Ω_i be the support of $\Lambda^H(f_H, \cdot)$ restricted to T_i. Let $X = \cup T_i$ and $\Omega = \cup \Omega_i$. Then Ω is a compact subset of X. For each $s \in X$, use Lemma 3.7 to find $U(s)$, a compact open neighborhood of s in X, and $f_s \in C_c^\infty(G)$ such that

$$\Delta_H^G(\gamma)\Lambda_\kappa^G(f_s, \gamma) = \Lambda^H(f_H, \gamma), \gamma \in U(s) \cap G'.$$

Note that since both sides transform in the same way with respect to G-conjugacy, the equality is in fact valid for all $\gamma \in \mathcal{O}_G(U(s)) \cap H \cap G'$. Write $U'(s) = \mathcal{O}_G(U(s)) \cap X$. Now the proof is finished in the same way as that of (i) using Lemma 2.5. □

If we drop the assumption that E/F is unramified, we can obtain a weaker version of Theorem 3.8 as follows. Let s be a semisimple element of H and as before let $T_1, ..., T_r$ be the Cartan subgroups of G which contain s, up to G-conjugacy. Suppose that $M_G = M_H$. Then $T_i \subseteq M_G = M_H \subseteq H$ for all $1 \leq i \leq r$. We can use the results of §2 to prove the following lemma.

Lemma 3.9. *Suppose $s \in H$ is a semisimple element such that $M_G = M_H$.*

(i) *Let $f_G \in C_c^\infty(G)$. Then there are neighborhoods V_i of s in T_i and $f_H \in C_c^\infty(H)$ so that for all $1 \leq i \leq r, \gamma \in V_i \cap G'$,*

$$\Lambda_\kappa^G(f_G, \gamma) = \Lambda^H(f_H, \gamma).$$

(ii) *Let $f_H \in C_c^\infty(H)$. Then there are neighborhoods V_i of γ_0 in T_i and $f_G \in C_c^\infty(G)$ so that for all $1 \leq i \leq r, \gamma \in V_i \cap G'$,*

$$\Lambda_\kappa^G(f_G, \gamma) = \Lambda^H(f_H, \gamma).$$

Proof. For part (i), use Lemma 2.6 to match $f_G \in C_c^\infty(G)$ with $\psi_G \in C_c^\infty(M_G)$. Now use Vignéras's version of Lemma 2.6 [**V**] applied to H and ordinary orbital integrals to match $\psi_H = \psi_G \in C_c^\infty(M_H)$ with $f_H \in C_c^\infty(H)$. For part (ii) go backwards. □

Suppose that $s \in H \cap G'$. Then $M_G = M_H$ is a Cartan subgroup of H and G, so that we can apply Lemma 3.9 in a neighborhood of s. Thus if we restrict our attention to functions supported on such points, we can use Lemmas 3.9 and 2.5 to prove the following theorem.

Theorem 3.10.

(i) *Let $f_G \in C_c^\infty(G')$. Then there is $f_H \in C_c^\infty(H \cap G')$ so that for all $\gamma \in H \cap G'$,*

$$\Lambda_\kappa^G(f_G, \gamma) = \Lambda^H(f_H, \gamma).$$

(ii) *Let $f_H \in C_c^\infty(H \cap G')$ such that*

$$\Lambda^H\left(f_H, x\gamma x^{-1}\right) = \kappa(x)\Lambda^H\left(f_H, \gamma\right)$$

for all $\gamma \in H \cap G', x \in G$ such that $x\gamma x^{-1} \in H$. Then there is $f_G \in C_c^\infty(G')$ so that for all $\gamma \in H \cap G'$,

$$\Lambda_\kappa^G\left(f_G, \gamma\right) = \Lambda^H\left(f_H, \gamma\right).$$

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