PACIFIC JOURNAL OF MATHEMATICS
Vol. 172, No. 1, 1996

SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH FULLY NONLINEAR TWO POINT BOUNDARY CONDITIONS II

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We establish existence results for two point boundary value problems for second order ordinary differential equations of the form $y'' = f(x, y, y')$, $x \in [0,1]$, where f satisfies the Carathéodory measurability conditions and there exist lower and upper solutions. We consider boundary conditions of the form $G((y(0), y(1)); (y'(0), y'(1))) = 0$ for fully nonlinear, continuous G and of the form $(y(i), y'(i)) \in \mathcal{J}(i)$, $i = 0, 1$ for closed connected subsets $\mathcal{J}(i)$ of the plane. We obtain analogues of our results for continuous f. In particular we introduce compatibility conditions between the lower and upper solutions and : (i) G; (ii) the $\mathcal{J}(i)$, $i = 0, 1$. Assuming these compatibility conditions hold and, in addition, f satisfies assumptions guarenteeing a'priori bounds on the derivatives of solutions we show that solutions exist. As an application we generalise some results of Palamides.

1. Introduction.

In this paper we consider two point boundary value problem for second order ordinary differential equations of the form

$$y'' = f(x, y, y'), \text{ for almost all } x \in [0,1] \tag{1.1}$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Carathéodory conditions. By a solution of (1.1) we mean a function y with y' absolutely continuous and y satisfying (1.1) almost everywhere. The first class of boundary conditions we will consider are of the form

$$0 = G((y(0), y(1)); (y'(0), y'(1))), \tag{1.2}$$

where $G : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is continuous. We call boundary conditions of this form fully nonlinear boundary conditions. The second class of boundary conditions we will consider are of the form

$$(y(i), y'(i)) \in \mathcal{J}(i) \text{ for } i = 0, 1, \tag{1.3}$$

where $\mathcal{J}(i)$ are continuua. We will call boundary conditions of this form boundary set conditions.

We always assume that lower and upper solutions $\alpha \leq \beta$, respectively, exist for (1.1) (see Definition 1 below). We prove analogues of our existence results for the case f is continuous.

In paragraph 2 we introduce some notation, definitions and preliminary results. We define lower and upper solutions which are the natural analogues of those for continuous f. These cannot be used directly in maximum principle arguments. We define strong lower and strong upper solutions which can be used directly in maximum principle arguments and show how lower and upper solutions can be approximated by strong lower and strong upper solutions, respectively, for an approximating differential equation. We introduce the central notion of compatibility of the boundary conditions G with the lower and upper solutions. In the literature when lower and upper solutions are assumed to exist and the Picard, Neumann or Periodic boundary conditions are considered the assumptions usually made are equivalent to compatibility (see [**29**]).

In paragraph 3, we present our main existence results. If the boundary conditions G are compatible with α and β and f satisfies additional assumptions guarenteeing a'priori bounds for y' for solutions y of (1.1), then there exist solutions y of (1.1) and (1.2) satisfying $\alpha \leq y \leq \beta$ on $[0, 1]$. The existence proofs follow the same general lines as in the case that f is continuous (see [**29**]) but with an additional and more subtle modification argument (see [**28**]).

In paragraph 4 we give some applications generalising some results of Palamides [**24**].

In paragraph 5 we consider problem (1.1) and (1.3). We recall two types of compatibility of the boundary sets $\mathcal{J}(i),\ i = 0, 1$ with the lower and upper solutions (see the author [**29**]). These are satisfied by the usual boundary sets conditions considered in the literature. We give some existence results for problem (1.1) and (1.3) when the boundary sets are compatible.

The compatibility conditions are concrete conditions involving the given data which can be easily checked and are satisfied by just about every concrete existence result in the literature. Most existence results in the literature for (1.1) together with (1.2) or (1.3) which assume lower and upper solutions exist follow as a corollary to our results. In many cases our results can be used to significantly improve upon these results. This is especially true for results concerning fully nonlinear boundary conditions as can be seen for example in the application to Theorem 2.1 of Palamides [**24**] given in paragraph 4. Also the central notion of compatibility extends to systems with lower and upper solutions, to single equations and systems with lower and upper solutions replaced by other surfaces a'priori bounding solutions. We will discuss these extensions of our ideas and further applications of our results and their extensions in forthcoming papers.

The literature on problem (1.1) and (1.2) is vast and for further information we refer the interested reader to the excellent monographs by Bailey, Waltman and Shampine [**2**], Bernfeld and Lakshmikantham [**9**], Gaines and Mawhin [**11**], Guenther, Granas and Lee [**12**], Hartman [**13**], and Mawhin [**19**] and their references.

2. Background Notation, Definitions and Results.

In order to state our results we need some notation.

As usual we say a function $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Carathéodory conditions if

1. $f(\cdot, y, p)$ is measurable for each $(y,p) \in \mathbb{R}^2$
2. $f(x, \cdot, \cdot)$ is continuous for almost all $x \in [0,1]$
3. to each $l > 0$ there is an integrable function $r_l : [0,1] \to \mathbb{R}$ satisfying $|f(x,y,p)| \leq r_l(x)$ for all $(y,p) \in [-l,l]^2$ and almost all $x \in [0,1]$.

As usual, we denote the closure of a set T by $\bar{T}$ and its boundary by ∂T. We denote the space of m times continuously differentiable functions from A to B endowed with the maximum norm by $C^m(A;B)$. In the case of continuous functions we abreviate this to $C(A;B)$. We denote the space of measurable functions from A to B endowed with the usual norm by $L^m(A;B)$. In the case $B = \mathbb{R}$ we omitt the B. We denote by $W^{2,1}([a,b])$ the Sobolev space of functions $y : [a,b] \to \mathbb{R}$ with y' absolutely continuous and $y'' \in L^1([a,b])$ with the usual norm. If A is a bounded open subset of $\mathbb{R}^n$, $p \in \mathbb{R}^n$, $f \in C(\bar{A};\mathbb{R}^n)$ and $p \notin f(\partial A)$ we denote the Brouwer degree of f on A at p by $d(f,A,p)$. It is common in the proof of existence of solutions of two point boundary value problems for (1.1) to modify f. We do this making use of the following functions (see [**27**]).

If $c \leq d$ are given let $\pi : \mathbb{R} \to [c,d]$ be the retraction given by

$$\pi(y,c,d) = \max\{\min\{d,y\},c\}. \tag{2.1}$$

For each $\epsilon > 0$, let $K \in C(\mathbb{R} \times (0,\infty);[-1,1])$ satisfy

1. $K(\cdot,\epsilon)$ is an odd function,
2. $K(t,\epsilon) = 0$ iff $t = 0$ and
3. $K(t,\epsilon) = 1$ for all $t \geq \epsilon/4$.

If $c \leq d$ and $\epsilon > 0$ are given, let $T \in C(\mathbb{R})$ be given by

$$T(y,c,d,\epsilon) = K(y - \pi(y,c,d),\epsilon). \tag{2.2}$$

Let

$$\mathcal{Q}(x,t) = \begin{cases} (1-x)t, & \text{for } 0 \le t \le x \le 1 \\ (1-t)x, & \text{for } 0 \le x \le t \le 1, \end{cases}$$

and $w(y_0, y_1)(x) = y_0(1-x) + y_1 x$. Let $X = C^1([0,1]) \times \mathbb{R}^2$ with the usual product norm. Define $\mathcal{C} : C([0,1]) \to C^1([0,1])$ by

$$\mathcal{C}(\phi)(x) = -\int_0^1 \mathcal{Q}(x,t)\phi(t)dt, \quad \text{for all } (\phi, C, D) \in X.$$

Clearly $\mathcal{C}$ is completely continuous.

Definition 1. We call $\alpha(\beta)$ a lower (upper) solution for (1.1) if $\alpha \in W^{2,1}([0,1])$ $(\beta \in W^{2,1}([0,1]))$ and

$$\begin{aligned} &\alpha''(x) \ge f(x, \alpha(x), \alpha'(x)), \text{ for almost all } x \in [0,1], \\ &(\beta''(x) \le f(x, \beta(x), \beta'(x)), \text{ for almost all } x \in [0,1]). \end{aligned} \tag{2.3}$$

If α and β are lower and upper solutions for (1.1) on $[0,1]$ we will assume that $\alpha \le \beta$ and set $\omega = \{(x,y) \in [0,1] \times \mathbb{R}^2 : \alpha(x) < y < \beta(x)\}$ and $\bar{\omega} = \{(x,y) \in [0,1] \times \mathbb{R}^2 : \alpha(x) \le y \le \beta(x)\}$. We will call the pair nondegenerate if $\Delta = (\alpha(0), \beta(0)) \times (\alpha(1), \beta(1))$ is nonempty. Let $\pi_\Delta : \mathbb{R}^2 \to \bar{\Delta}$ be the retraction given by

$$\pi_\Delta(C, D) = (\pi(C, \alpha(0), \beta(0)), \pi(D, \alpha(1), \beta(1))).$$

Lower and upper solutions themselves cannot be used in maximum principle arguments consequently we introduce strong lower and upper solutions (cf Ako [**1**]).

Definition 2. We call $\alpha \in W^{2,1}([0,1])$ a strong lower solution for (1.1) if to each $c \in (0,1)$ there is an open interval $I_c \subseteq (0,1)$ satisfying $c \in I_c$ and a $\delta(c) > 0$ such that

$$\alpha''(x) > f(x, u(x), v(x)), \text{ for almost all } x \in I_c, \tag{2.4}$$

where $u, v : I_c \to \mathbb{R}$ are measurable and

$$(u(x), v(x)) \in (\alpha(x) - \delta, \alpha(x)] \times (\alpha'(c) - \delta, \alpha'(c) + \delta).$$

Similarly we define a strong upper solution β by substituting β in place of α and reversing the inequalities above; in this case

$$(u(x), v(x)) \in [\beta(x), \beta(x) + \delta) \times (\beta'(c) - \delta, \beta'(c) + \delta).$$

Let $\alpha \le \beta$ be lower and upper solutions, respectively. In the next lemma we show that there exist strong lower and strong upper solutions for an approximating equation which approximate the lower and upper solutions, respectively.

Define $g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x,y,p) = f(x, \pi(y,\alpha(x),\beta(x)), p) \text{ for all } x \in [0,1]. \tag{2.5}$$

Lemma 3. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfy the Carathéodory measurability conditions. Let $\alpha \le \beta$ be lower and upper solutions, respectively and g be given by* (2.5). *Given $\epsilon \in (0,1)$ there is a continuous function $\kappa_\epsilon : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ such that $|\kappa_\epsilon(x,y)| \le \epsilon$ for all $(x,y) \in [0,1] \times \mathbb{R}$. Moreover, setting*

$$j(x,y,p) = g(x,y,p+\kappa_\epsilon(x,y)), \text{ for all } (x,y,p) \in [0,1] \times \mathbb{R}^2,$$

there exist strong lower and strong upper solutions α^ϵ and β^ϵ, respectively, for

$$y'' = j(x,y,y'), \text{ for almost all } x \in [0,1] \tag{2.6}$$

satisfying

$$\alpha(x) - \epsilon/2 \le \alpha^\epsilon(x) \le \alpha(x) - \epsilon/6 \le \beta(x) + \epsilon/6 \le \beta^\epsilon(x) \le \beta(x) + \epsilon/2,$$

for all $x \in [0,1]$.

The proof of [**28**, Lemma 7] by the author applies. Moreover we see that $\alpha'(0) = \alpha^{\epsilon\prime}(0)$, $\beta'(0) = \beta^{\epsilon\prime}(0)$ and $|\alpha'(x) - \alpha^{\epsilon\prime}(x)|$, $|\beta'(x) - \beta^{\epsilon\prime}(x)| < \epsilon/2$ on $[0,1]$.

We associate with these strong lower and strong upper solutions α^ϵ and β^ϵ the function $\gamma : \mathbb{R} \to \mathbb{R}$ given by

$$\gamma(D,\alpha^\epsilon,\beta^\epsilon) = \begin{cases} (\alpha(1) - D)(\alpha'(1) - \alpha^{\epsilon\prime}(1))/(\alpha(1) - \alpha^\epsilon(1)), & \text{for } D \le \alpha(1) \\ 0, & \text{for } \alpha(1) \le D \le \beta(1) \\ (D - \beta(1))(\beta'(1) - \beta^{\epsilon\prime}(1))/(\beta^\epsilon(1) - \beta(1)), & \text{for } D \ge \beta(1). \end{cases}$$

Definition 4. Let $\alpha \le \beta$ be lower and upper solutions for (1.1) on $[0,1]$. We say f satisfies the Bernstein–Nagumo–Zwirner condition if there exist $L > 0$, $h \in C([0,\infty);(0,\infty))$, $\bar{h} \in L^1([\alpha_m,\beta_M];(0,\infty))$ and $r \in L^1([0,1];(0,\infty))$ such that

$$|f(x,y,p)| \le h(|p|)\bar{h}(y) + r(x), \text{ for all } (x,y) \in [0,1] \times (\alpha(x),\beta(x)) \text{ and} \tag{2.7}$$

$$\int_\sigma^L \frac{s\,ds}{h(s)} > \int_{\alpha_m}^{\beta_M} \bar{h}(s)ds + K\int_0^1 r(x)dx, \tag{2.8}$$

where $K = \sup\{s/h(s) : s \in [\sigma, L]\}$, $\beta_M = \max\{\beta(x) : x \in [0,1]\}$, $\alpha_m = \min\{\alpha(x) : x \in [0,1]\}$ and $\sigma = \max\{|\beta(1) - \alpha(0)|, |\beta(0) - \alpha(1)|\}$.

See Bernstein [**10**], Nagumo [**20**], Scorza Dragoni [**26**], Zwirner [**30**] and Thompson [**28**].

Remark 5. In the special case $\bar{h} = 1$ and $r = 0$ this has been called the Bernstein–Nagumo condition by some authors (see Granas et al [**12**]).

For the convenience of the reader and the sake of completeness we recall some notation and definitions from [**29**].

Definition 6. We call the vector field $\Psi = (\psi^0, \psi^1) \in C(\bar{\Delta}; \mathbb{R}^2)$ strongly inwardly pointing on $\bar{\Delta}$ if for all $(C, D) \in \partial\Delta$

$$\begin{aligned} &\psi^0(\alpha(0), D) > \alpha'(0), \psi^0(\beta(0), D) < \beta'(0) \text{ and} \\ &\psi^1(C, \alpha(1)) < \alpha'(1), \psi^1(C, \beta(1)) > \alpha'(1). \end{aligned} \tag{2.9}$$

We call Ψ inwardly pointing if the strict inequalities are replaced by weak inequalities.

Definition 7. Let $G \in C(\bar{\Delta} \times \mathbb{R}^2; \mathbb{R}^2)$. We say G is strongly compatible with α and β if for all strongly inwardly pointing Ψ on $\bar{\Delta}$

$$\mathcal{G}(C, D) \neq 0 \text{ for all } (C, D) \in \partial\Delta \text{ and} \tag{2.10}$$

$$d(\mathcal{G}, \Delta, 0) \neq 0, \tag{2.11}$$

where

$$\mathcal{G}(C, D) = G((C, D); \Psi(C, D)) \text{ for all } (C, D) \in \bar{\Delta}. \tag{2.12}$$

We say G is compatible with α and β if there is a sequence $\{G_i\}_{i=1}^{\infty}$ strongly compatible with α and β which converges to G uniformly on compact subsets of $\bar{\Delta} \times \mathbb{R}^2$.

3. Existence of Solutions.

Theorem 1. *Assume that f satisfies the Carathéodory measurability conditions, that there exist nondegenerate lower and upper solutions $\alpha \leq \beta$ for* (1.1) *and f satisfies the Bernstein–Nagumo–Zwirner condition. If $G \in C(\bar{\Delta} \times \mathbb{R}^2; \mathbb{R}^2)$ is compatible with α and β, then problem* (1.1) *and* (1.2) *has a solution y lying between α and β.*

Proof. Assume first that G is strongly compatible with α and β.

We approximate the lower and upper solutions by strong lower and strong upper solutions α^ϵ and β^ϵ, respectively, for the approximating differential equation (2.6). We modify this equation for y not between α and β to obtain a second pair of constant strong lower and strong upper solutions α_ϵ and β_ϵ, respectively, satisfying $\alpha_\epsilon < \alpha^\epsilon < \alpha \leq \beta < \beta^\epsilon < \beta_\epsilon$. We also modify the boundary conditions so that they are compatible with α^ϵ and β^ϵ. We reformulate the approximating problem as a coupled system of integral and boundary condition equations and show that a solution of the modified problem lies in the region where j is unmodified and hence is solution of the approximating equation and modified boundary conditions. We obtain the required solution by using compactness to find a subsequence converging to the desired solution. We use Schauder degree theory to prove existence for the modified problem and compute the degree using a homotopy; it is easier to construct a suitable homotopy for the modified equation and boundary conditions.

Extend $\bar{h}$ to $\mathbb{R}$ by $\bar{h}(y) = \bar{h}(\pi(y, \alpha_m, \beta_M))$. By (2.8) and the Monontone convergence theorem there exist $\epsilon_0 > 0$ such that

$$(3.1) \qquad \int_{\sigma+2\epsilon_0}^{L} \frac{s\,ds}{h(s)+\epsilon_0} > \int_{\alpha_m-\epsilon_0}^{\beta_M+\epsilon_0} \bar{h}(s)ds + K\int_0^1 (r(x)+\epsilon_0)dx.$$

Choose $N > \max\{|\alpha'(x)|, |\beta'(x)|, L : x \in [0,1]\} + \epsilon_0$, and let

$$(3.2) \qquad k(x,y,p) = j(x, \pi(y, \alpha^\epsilon(x), \beta^\epsilon(x)), \pi(p, -N, N)), \text{ and}$$

$$\begin{aligned} l(x,y,p) =& (1 - |T(y, \alpha^\epsilon(x), \beta^\epsilon(x), \epsilon)|)k(x,y,p) + \\ (3.3) \qquad & T(y, \alpha^\epsilon(x), \beta^\epsilon(x), \epsilon)(|k(x,y,p)| + \epsilon), \end{aligned}$$

where π and T are given by (2.1) and (2.2), respectively. Let $\alpha_\epsilon = \alpha_m - \epsilon$ and $\beta_\epsilon = \beta_M + \epsilon$. Thus l satisfies the Carathéodory conditions on $[0,1] \times \mathbb{R}^2$ and by continuity, for almost all $x \in [0,1]$ and ϵ small enough

$$(3.4) \qquad |l(x,y,p)| \leq (h(|p|) + \epsilon_0)\bar{h}(y) + (r(x) + \epsilon_0),$$

for all $(y,p) \in [\alpha_\epsilon, \beta_\epsilon] \times [-N, N]$.

Consider

$$(3.5) \qquad y'' = l(x, y, y'), \text{ for almost all } x \in [0,1]$$

together with

$$(3.6) \qquad G(\pi_\Delta(y(0), y(1)); (y'(0), y'(1) + \gamma(y(1), \alpha^\epsilon, \beta^\epsilon))) = 0.$$

Suppose that (3.5) and (3.6) has a solution y^ϵ satisfying $\alpha^\epsilon \leq y^\epsilon \leq \beta^\epsilon$ and $|y^{\epsilon\prime}| \leq L$ on $[0,1]$. Then by compactness there is a subsequence y^{ϵ_i} converging in $W^{2,1}([0,1])$ to y, say, as ϵ_i converges to 0 and y is the required solution. To see this proceed as follows. First $\alpha \leq y \leq \beta$ and $|y'| \leq L$ on $[0,1]$. Since

κ_{ϵ_i} converges uniformly to 0, letting ϵ_i go to 0 in the the integral equation satisfied by y^{ϵ_i} and noting that f and l coincide for $(x,y) \in \bar{\omega}$ it follows that y is a solution of the (1.1). Now $\gamma(y^{\epsilon_i}(1), \alpha^{\epsilon_i}, \beta^{\epsilon_i})$ converges to 0 and thus y satisfies the boundary conditions.

We show that there is such a solution y^ϵ. First

$$\begin{aligned} \alpha_\epsilon{}'' &= 0 > -(|k(x,u(x),v(x))| + \epsilon) \\ &= l(x,u(x),v(x)), \text{ for almost all } x \in [0,1], u(x) \leq \alpha_m - 2\epsilon/3. \end{aligned}$$

Thus α_ϵ is a strong lower for

$$y'' = \theta l(x,y,y') \text{ for almost all } x \in [0,1] \tag{3.7}$$

for each $\theta \in (0,1]$. Similarly β_ϵ is a strong upper solution for (3.7).

Let $\bar{\omega}^\epsilon = \{(x,y^\epsilon) \in [0,1] \times \mathbb{R} : \alpha^\epsilon(x) \leq y^\epsilon \leq \beta^\epsilon(x)\}$ and $\Delta^\epsilon = (\alpha^\epsilon(0), \beta^\epsilon(0)) \times (\alpha^\epsilon(1), \beta^\epsilon(1))$.

Suppose that y^ϵ is a solution of (3.7) with $\alpha_\epsilon \leq y^\epsilon \leq \beta_\epsilon$ on $[0,1]$ and $\theta \in [0,1]$. Then $|y^{\epsilon\prime}| < L$ by the standard argument (see for example the author [**28**]). Suppose that y^ϵ is a solution of (3.5) and (3.6) satisfying $\alpha_\epsilon \leq y^\epsilon \leq \beta_\epsilon$ and $(y^\epsilon(0), y^\epsilon(1)) \in \bar{\Delta}^\epsilon$. We show that $\alpha^\epsilon \leq y^\epsilon \leq \beta^\epsilon$ on $[0,1]$ and hence y^ϵ is the required solution. Suppose for example that $y^\epsilon(t) < \alpha^\epsilon(t)$ for some $t \in [0,1]$. From the boundary conditions and continuity we may assume that $\alpha^\epsilon - y^\epsilon$ attains its positive maximum at $t \in (0,1)$. Thus $\alpha^{\epsilon\prime}(t) = y^{\epsilon\prime}(t)$. From (3.3) and the continuity of $\alpha^{\epsilon\prime}$ and $y^{\epsilon\prime}$ there is an interval $J_t \subseteq I_t$ such that we have

$$\begin{aligned} y^{\epsilon\prime\prime}(x) &= l(x, y^\epsilon(x), y^{\epsilon\prime}(x)) \\ &< l(x, \alpha^\epsilon(x), \alpha^{\epsilon\prime}(x)) < \alpha^{\epsilon\prime\prime}(x) \text{ for almost all } x \in J_t \end{aligned}$$

a contradiction. Similarly $y^\epsilon \leq \beta^\epsilon$ on $[0,1]$. Thus y^ϵ is the required solution.

We show that y^ϵ exists. As the proof is similar to that in [**29**, Theorem 1] we only sketch the proof highlighting the differences.

Let $\Omega_\epsilon = \{y \in C^1([0,1]) : \alpha_\epsilon < y < \beta_\epsilon, |y'| < N, \text{ on } [0,1]\}$ and $\Gamma_\epsilon = \Omega_\epsilon \times \Delta^\epsilon \subseteq X$.

Define $\mathcal{L} : C^1([0,1]) \to L^1([0,1])$ by

$$\mathcal{L}(\phi) = l(\cdot, \phi, \phi').$$

Let Ψ be a strongly inwardly pointing vector field on $\bar{\Delta}$ and let $\mathcal{A} : \Delta^\epsilon \to \mathbb{R}^2$ be given by

$$\mathcal{A}(C,D) = G(\pi_\Delta(C,D); \Psi(\pi_\Delta(C,D))). \tag{3.8}$$

Then $\mathcal{A} \neq 0$ on $\partial\Delta^\epsilon$, since G is strongly compatible with α and β. Define $\mathcal{H} : \bar{\Gamma}_\epsilon \times [0,1] \to X$ by

$$\begin{aligned}
\mathcal{H}(\phi, C, D, \theta) &= (\phi + C\mathcal{L}(\phi) - w(C,D), \mathcal{V}(\phi, C, D, \theta)) \text{ for } \theta \in [2/3, 1],\\
\mathcal{H}(\phi, C, D, \theta) &= (\phi + 3(\theta - 1/3)C\mathcal{L}(\phi) - w(C,D), \mathcal{A}(C,D))\\
&\qquad \text{for } \theta \in [1/3, 2/3] \text{ and}\\
\mathcal{H}(\phi, C, D, \theta) &= (\phi - 3\theta w(C,D) - (1 - 3\theta)(\alpha_\epsilon + \beta_\epsilon)/2, \mathcal{A}(C,D))\\
&\qquad \text{for } \theta \in [0, 1/3], \text{ where}\\
\mathcal{V}(\phi, C, D, \theta) &= G(\pi_\Delta(C,D); \mathcal{S}(\phi, C, D, \theta)) \text{ and}\\
\mathcal{S}(\phi, C, D, \theta) &= 3(\theta - 2/3)(\phi'(0), \phi'(1) + \gamma(D, \alpha^\epsilon, \beta^\epsilon))\\
&\qquad + 3(1-\theta)\Psi(\pi_\Delta(C,D)).
\end{aligned}$$

Clearly $\mathcal{H}$ is completely continuous. Now $\mathcal{H}(y^\epsilon, y^\epsilon(0), y^\epsilon(1), 1) = 0$ iff y^ϵ is a solution of (3.5) and (3.6) with $(y^\epsilon, y^\epsilon(0), y^\epsilon(1)) \in \bar{\Gamma}_\epsilon$. If there is a solution $(y^\epsilon, y^\epsilon(0), y^\epsilon(1))$ in $\partial\Gamma_\epsilon$ we are through.

Assume that there is no such solution. We show that $\mathcal{H}$ is a homotopy for Schauder degree on Γ_ϵ at 0. Assume that there is a solution (y^ϵ, C, D) of $\mathcal{H}(y^\epsilon, C, D, \theta) = 0$ in $\partial\Gamma_\epsilon$ with $\theta \in [0,1)$. From the formula for $\mathcal{H}$ we see that $\theta \notin [0, 1/3)$.

Assume that $\theta \in [2/3, 1)$. As in the case $\theta = 1$, $\alpha^\epsilon(x) \le y^\epsilon(x) \le \beta^\epsilon(x)$ on $[0,1]$, $y^\epsilon(0) = C$ and $y^\epsilon(1) = D$.

Assume $(y^\epsilon(0), y^\epsilon(1)) \in \partial\Delta^\epsilon$. If $y^\epsilon(1) = \alpha^\epsilon(1)$, then $y^{\epsilon\prime}(1) \le \alpha^{\epsilon\prime}(1)$. This leads to the contradiction that $\mathcal{S}(y^\epsilon, y^\epsilon(0), y^\epsilon(1), \theta) < \alpha'(1)$ and $\mathcal{V}(y^\epsilon, y^\epsilon(0), y^\epsilon(1), \theta) \neq 0$. The other cases $(y^\epsilon(0), y^\epsilon(1)) \in \partial\Delta^\epsilon$ follow similarly.

Now $\alpha_\epsilon < y^\epsilon < \beta_\epsilon$ on $(0,1)$ since α_ϵ and β_ϵ are strong lower and strong upper solutions, respectively, for (3.5). Thus $y^\epsilon \notin \partial\Omega_\epsilon$ and there are no solutions of $\mathcal{H}(y^\epsilon, C, D, \theta) = 0$ with $\theta \in [2/3, 1]$ and $(y^\epsilon, C, D) \in \partial\Gamma_\epsilon$.

Assume that $\theta \in [1/3, 2/3)$. Since $\mathcal{A}(C,D) \neq 0$ on $\partial\Delta^\epsilon$ there are no solutions (y^ϵ, C, D) with $(C,D) \in \partial\Delta^\epsilon$. The proof that $y^\epsilon \notin \partial\Omega_\epsilon$ is similar to that for $\theta \in [2/3, 1)$.

Thus $\mathcal{H}$ is a homotopy for the Schauder degree. Now $\mathcal{H}(\phi, C, D, 0) = (\phi - c, \mathcal{A}(C,D))$ where $c \in \Omega_\epsilon$ is a constant and $\mathcal{A} \neq 0$ in $\bar{\Delta}^\epsilon \setminus \Delta$. Thus by the Homotopy invariance, Reduction and Excision properties of Schauder degree

$$\begin{aligned}
d(\mathcal{H}(\cdot, 1), \Gamma_\epsilon, 0) &= d(\mathcal{H}(\cdot, 0), \Gamma_\epsilon, 0)\\
&= d(\mathcal{A}, \Delta^\epsilon, 0)\\
&= d(\mathcal{A}, \Delta, 0) \neq 0.
\end{aligned}$$

Thus there is a solution y^ϵ of (3.5) and (3.6) and hence a solution y of (1.1) and (1.2).

Suppose now that G is compatible with α and β. Then there is a sequence $\{G_i\}_{i=1}^{\infty}$ strongly compatible with α and β and converging uniformly to G on compact subsets of $\bar{\Delta} \times \mathbb{R}^2$. Let y_i be the corresponding solutions. By compactness there is a subsequence of the y_i converging in $W^{2,1}([0,1])$ to the desired solution. $\square$

Remark 8. In the case Δ is degenerate we have to modify the result. Let $\alpha \le \beta$ be lower and upper solutions for (1.1), respectively and suppose, for example, that $\alpha(0) = \beta(0)$. Then we set $\Delta = (\alpha(1), \beta(1))$ and change the other conditions as follows.

We call the vector field $\Psi \in C(\bar{\Delta})$ strongly inwardly pointing on $\bar{\Delta}$ if for all $D \in \partial\Delta$

$$\Psi(\alpha(1)) < \alpha'(1), \Psi(\beta(1)) > \beta'(1).$$

We call Ψ inwardly pointing if the strict inequalities are replaced by weak ones.

Let $G \in C(\bar{\Delta} \times \mathbb{R})$ and $\mathcal{G}(D) = G(D, \Psi(D))$ for all $D \in \bar{\Delta}$. We say G is strongly compatible with α and β if for all strongly inwardly pointing Ψ on $\bar{\Delta}$

$$\mathcal{G}(D) \neq 0 \text{ for all } D \in \partial\Delta \text{ and}$$
$$d(\mathcal{G}, \Delta, 0) \neq 0.$$

We define compatible as before. Theorem 1 and its proof are modified in the obvious way. In the degenerate case $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$ strong compatibility implies that there are no solutions to the problem.

As mentioned earlier our central idea leads to existence results provided f is such that there are a'priori bounds on y' for solutions y satisfying $\alpha \le y \le \beta$. We now discuss the case where f satisfies the Nagumo–Knobloch–Schmitt condition.

Definition 9. Let $\alpha \le \beta$ be lower and upper solutions for (1.1) on $[0,1]$. We say f satisfies the Nagumo–Knobloch–Schmitt conditions relative to α and β if there exists $\Phi < \Upsilon \in C^1([0,1] \times \mathbb{R})$ such that

$$f(x, y, \Phi(x,y)) \ge \Phi_x(x,y) + \Phi_y(x,y)\Phi(x,y) \text{ and} \tag{3.9}$$

$$f(x, y, \Upsilon(x,y)) \le \Upsilon_x(x,y) + \Upsilon_y(x,y)\Upsilon(x,y) \tag{3.10}$$

for almost all $x \in [0,1]$ and all $y \in [\alpha(x), \beta(x)]$.

See Nagumo [**21**, **22**], Knobloch [**16**, **17**] and Schmitt [**25**].

Theorem 2. *Assume that there exist nondegenerate lower and upper solutions $\alpha \leq \beta$ for (1.1), that f satisfies the Nagumo–Knobloch–Schmitt condition, that $G \in C(\bar{\Delta} \times \mathbb{R}^2; \mathbb{R}^2)$ is compatible with α and β, that $\alpha'(x) \geq \Phi(x, \alpha(x))$ and $\Upsilon(x, \beta(x)) \geq \beta'(x)$ almost everywhere and moreover $G((C, D); (E, F)) = 0$ only if $E \in [\Phi(0, C), \Upsilon(0, C)]$. Then problem (1.1) and (1.2) has a solution y lying between α and β.*

Proof. Again we modify f. Choose

$$N > \max\{|\Phi(x,y)|, |\Upsilon(x,y)|, |\alpha'(x)|, |\beta'(x)| : (x,y) \in \bar{\omega}\}$$

and let

$$l(x,y,p) = \begin{cases} \max\{f(x,y,\Phi(x,y)) + (\Phi(x,y) - p), f(x,y,p)\}, & \text{for } p \leq \Phi(x,y) \\ \min\{f(x,y,\Upsilon(x,y)) + (\Upsilon(x,y) - p), f(x,y,p)\}, & \text{for } p \geq \Upsilon(x,y) \\ f(x,y,p), & \text{otherwise} \end{cases}$$

and

$$m(x,y,p) = l(x,y,\pi(p,-N,N)).$$

Thus α and β are lower and upper solutions for

$$y'' = m(x,y,y') \text{ for almost all } x \in [0,1]. \tag{3.11}$$

It is easy to see that m satisfies the conditions of Theorem 1 and thus there is a solution y of problem (3.11) and (1.2) satisfying $\alpha \leq y \leq \beta$. To show that this is a solution of our problem it suffices to show that $\Phi(x,y) \leq y' \leq \Upsilon(x,y)$. From the boundary conditions there are no solutions for $y'(0) \notin [\Phi(0,y(0)), \Upsilon(0,y(0))]$. Suppose that $y'(t) < \Phi(t,y(t))$ for some $t \in (0,1]$. By continuity and the definition of N we may choose t and $u \in (0,t)$ such that $-N < y'(x) < \Phi(x,y(x))$ for all $x \in (u,t]$ and $y'(u) = \Phi(u,y(u))$. Now

$$\begin{aligned}(y'(x) - \Phi(x,y(x)))' &= m(x,y(x),y'(x)) \\ &\qquad - \Phi_x(x,y(x)) - \Phi_y(x,y(x))\Phi(x,y(x)) \\ &> f(x,y(x),\Phi(x,y(x))) \\ &\qquad - \Phi_x(x,y(x)) - \Phi_y(x,y(x))\Phi(x,y(x)) \geq 0,\end{aligned}$$

a contradiction. Thus $\Phi(x,y) \leq y'$. Similarly the $y' \leq \Upsilon(x,y)$ and the result follows. □

Remark 10. The conditions $G((C,D);(E,F)) = 0$ only if

$$E \in [\Phi(0,C), \Upsilon(0,C)],$$

(3.9) and (3.10) guarentee the solution y satisfies

$$\Phi(x, y(x)) \leq y'(x) \leq \Upsilon(x, y(x)).$$

There are other ways to guarentee this as for example in the case of periodic boundary conditions where we may replace the inequality signs in (3.9) and (3.10) by not equals to signs. See for example Schmitt [**25**].

4. Applications.

To show that the boundary conditions (1.2) are compatible we must show that (2.11) holds. Usually this follows easily from the properties of Brouwer degree (see, for example, Lloyd [**18**]) however the following lemma often suffices.

Lemma 11. *Let $M = (M^0, M^1) \in C(\bar{\Delta}; \mathbb{R}^2)$ satisfy*

$$M^0(\alpha(0), D) \leq 0, M^0(\beta(0), D) \geq 0$$

and

$$M^1(C, \alpha(1)) \leq 0, M^1(C, \beta(1)) \geq 0$$

for all $(C, D) \in \bar{\Delta}$ and

$$M(C, D) \neq 0$$

for all $(C, D) \in \partial\Delta$, then $d(M, \Delta, 0) \neq 0$.

This follows since $\mathcal{S} = \theta M + (1-\theta)(I-p)$ is a homotopy of M with $I-p$ where I is the identity on $\mathbb{R}^2$ and $p \in \Delta$ is any point.

Problems of the form (1.1) and (1.2), usually for the case f is continuous, have been considered by many authors. Shooting methods have been used combined variously with the maximum principle, with the Jordan separation theorem, the Kneser–Hukuhara continuum theorem and/or the Ważewski retraction theorem. Often these have been refined in the process. See Baxley [**4**], Baxley and Brown [**3**], Bernfeld and Palamides [**8**], Jackson and Klassen [**14**], Jackson and Palamides [**15**], Palamides [**23, 24**] and their references. In [**24**] Palamides used an extension of Ważewski's retraction principle involving the Kneser–Hukuhara continuum theorem and the maximum principle to prove the following existence result.

Theorem 2.1 of [**24**]. *Let f satisfy the Carathéodory conditions and for each fixed $p \in \mathbb{R}$ and almost all $x \in [0,1]$ let $f(x, \cdot, p)$ be nondecreasing for $y \in [\alpha(x), \beta(x)]$, where $\alpha(x) = -m + \gamma(x)$ and $\beta(x) = m - \gamma(x)$, $\gamma(x) = (1 - [1 + K\nu m^\nu x]^{(\nu-1)/\nu})/(K(\nu-1)m^{\nu-1})$ and $K > 0, m > 0, \nu = 2k+1,\ k =$*

$1, 2, \ldots$, and $\hat{m} = -\gamma'(1)$ are such that: For each $\rho > 0$ there exists a nondecreasing real-valued function $M(\rho)$ such that

$$|f(x, y, p_2) - f(x, y, p_1)| \leq M(\rho)|p_2 - p_1|, \text{ for } |p_2 - p_1| \leq \rho$$
$$|f(x, 0, p)| \leq Kp^{\nu+1}, \text{ for } |p| \geq \hat{m}$$

for almost all $x \in [0,1]$ and all $y \in [\alpha(x), \beta(x)]$, where $M(\rho) \leq K\rho^\nu$, for $\rho \geq \hat{m}$. Let f satisfy the Knobloch–Nagumo–Schmitt condition. Assume that there exists $\theta_0 \in (\pi/4, \pi/2)$ such that $G((C, D); (P, Q))$ is continuous for $(C, P, D, Q) \in E \times \hat{E}$, where

$$E = \left\{(C, C\tan\theta) : -m \leq C \leq m \text{ and } \frac{\pi}{4} \leq \theta \leq \theta_0\right\} \text{ and}$$
$$\hat{E} = [\alpha(1), \beta(1)] \times [q_m, q_M]$$

where $q_m = \min\{\Phi(1, y) : \alpha(1) \leq y \leq \beta(1)\}$ and $q_M = \max\{\Upsilon(1, y) : \alpha(1) \leq y \leq \beta(1)\}$. Moreover assume that:

1. *For each fixed $C \in [-m, m], \phi \in [\pi/4, \theta_0]$ and $(D, Q) \in \hat{E}$*

$$g^1((y, \alpha(1)); (y\tan\phi, -\hat{m})), \quad g^1((C, z); (C\tan\phi, Q))$$

and $g^1(y, D); (C\tan\phi, q)$ are nondecreasing functions with respect to the corresponding variables y, z or q but $g^1((C, z); (C\tan\phi, q))$ is (strictly) increasing with respect to both variables z and q and furthermore for each $y \in [-m, m]$ and $\theta \in [\pi/4, \theta_0]$ we have

$$g^1((y, \alpha(1)); (y\tan\theta, -\hat{m})) \leq 0 \leq g^1((y, \beta(1)); (y\tan\theta, \hat{m})). \tag{4.1}$$

2. *For each point $y \in [-m, m]$ we have*

$$\Phi(0, y) \leq y, y\tan\theta_0 \leq \Upsilon(0, y).$$

3. *For each pair of points (y_1, z_1, q_1) and $(y_2, z_2, q_2) \in [-m, m] \times \hat{E}$ we have*

$$g^0((y_1, z_1); (y_1, q_1))g^0((y_2, z_2); (y_2\tan\theta_0, q_2)) \leq 0. \tag{4.2}$$

Then problem (1.1) and (1.2) has a solution y such that for all $x \in [0,1]$ $\alpha(x) \leq y(x) \leq \beta(x)$ and $\Phi(x, y(x)) \leq y'(x) \leq \Upsilon(x, y(x))$.

We indicate how this result can be generalised after first showing how it follows from our Theorem 2.

Outline. As in [**24**], $\alpha < \beta$ are lower and upper solutions for (1.1). There are two cases to consider. The first case is $g^0((C, D); (C, Q))$ not identically 0 on $[\alpha(0), \beta(0)] \times \hat{E}$. We modify G without changing its zero set and also denote the modification by G. Then we extend G to $\bar{\Delta}\times\mathbb{R}^2$ without changing its zero set when $(D,Q)\in\hat{E}$ so that the extension is compatible with α and β. Then solutions of the new problem are solutions.

Replace $g^0((C,D);(P,Q))$ by $Cg^0((C,D);(P,Q))$. Let $s_i:[\alpha(0),\beta(0)]\to\mathbb{R}$, $i=1,2$ be defined by

$$-s_1(-C)=s_2(C)=\begin{cases} C\tan\theta_0, & \text{for } C\geq 0\\ C, & \text{for } C<0.\end{cases}$$

By (4.2), replacing g^0 by $-g^0$, if necessary, we may assume that

$$g^0((C,D);(s_2(C),Q))\leq 0$$

and

$$g^0((C,D);(s_1(C),Q))\geq 0$$

for all $(D,Q)\in\hat{E}$. We extend g^0 as a continuous function to $\bar{\Delta}\times\mathbb{R}^2$ satisfying $g^0((C,D);(P,Q))<0$ for all $P>s_2(C)$ and $g^0((C,D);(P,Q))>0$ for all $P<s_1(C)$ for all $(D,Q)\in[\alpha(1),\beta(1)]\times\mathbb{R}$. By (4.1), and monotonicity, $g^1((C,\alpha(1));(P,Q))\leq 0$ for all $q_m\leq Q\leq\alpha'(1)$ and $g^1((C,\beta(1));(P,Q))\geq 0$ for all $q_M\geq Q\geq\beta'(1)$ and all $(C,P)\in E$; it is not difficult to show from the definition of lower and upper solutions and the Knobloch–Nagumo–Schmitt condition that $q_m<\alpha'(1)$ and $\beta'(1)<q_M$. Extend g^1 to a continuous function on $\bar{\Delta}\times\mathbb{R}^2$ so that $g^1((C,\alpha(1));(P,Q))\leq 0$ for all $Q\leq\alpha'(1)$ and $g^1((C,\beta(1));(P,Q))\geq 0$ for all $Q\geq\beta'(1)$. It is easy to see that G now has the required properties. We show that $\alpha'(x)\geq\Phi(x,\alpha(x))$ and $\Upsilon(x,\beta(x))\geq\beta'(x)$ almost everywhere so that Theorem 2 applies and a solution exists.

Now $\alpha'(0)=-m=\alpha(0)$ so $\alpha'(0)\geq\Phi(0,\alpha(0))$ by assumption (2). Suppose that $z(t)=\alpha'(t)-\Phi(t,\alpha(t))<0$ for some $t\in(0,1]$. By continuity we may choose $u\in[0,t)$ such that $z(u)=0$ and $z<0$ on $(u,t]$. By (2.3), (2.8) and the lipschitz condition on f there is a constant k such that

$$\begin{aligned} z'(x) &\geq f(x,\alpha(x),\alpha'(x))-\Phi_x(x,\alpha(x))-\Phi_y(x,\alpha(x))\alpha'(x)\\ &\geq f(x,\alpha(x),\alpha'(x))-f(x,\alpha(x),\Phi(x,\alpha(x)))\\ &\qquad -\Phi_y(x,\alpha(x))(\alpha'(x)-\Phi(x,\alpha(x)))\\ &\geq kz(x)\end{aligned}$$

for almost all $x\in[u,t]$, a contradiction. Thus $\alpha'(x)\geq\Phi(x,\alpha(x))$ almost everywhere. Similarly $\Upsilon(x,\beta(x))\geq\beta'(x)$ almost everywhere.

The second case is $g^0((C,D);(C,Q))$ identically 0 on $[\alpha(0),\beta(0)]\times\hat{E}$. In this case we replace E by the set $E_1=\{(C,C):C\in[\alpha(0),\beta(0)]\}$ and $s_1(C)=s_2(C)=C$. The rest of the proof remains unchanged. $\square$

Remark 12. In the second case of the proof above the boundary condition g^0 admits the solution $y(0) = y'(0)$ and in our proof we have effectively replaced g^0 by the simpler $y(0) - y'(0)$ and our solution satisfies this.

In the above proof the monotonicity assumptions on g^1 are used only to guarentee that $g^1((C, \alpha(1)); (P, Q)) \leq 0$ for all $q_m \leq Q \leq \alpha'(1)$ and $g^1((C, \beta(1)); (P, Q)) \geq 0$ for all $q_M \geq Q \geq \beta'(1)$ and all $(C, P) \in E$ and hence can be relaxed. In Palamides's proof they are required in a shooting argument and it is not clear how they can be weakened.

We do not need either the local lipschitz or monotonicity conditions on f required in Palamides's proof for application of the maximum principle in a shooting argument. We used the local lipschitz condition only along $(x, \alpha(x), \alpha'(x))$ and $(x, \beta(x), \beta'(x))$ and only to show that $\Upsilon(x, \beta(x)) \geq \beta'(x)$ and $\alpha'(x) \geq \Phi(x, \alpha(x))$ almost everywhere. We used the monotonicity condition on f only in the construction of the lower and upper solutions. Palamides also used the monotonicity condition on f in the construction of the lower and upper solutions.

Moreover the other results of [**24**] also follow from our Theorems 1 and 2; in the statement of Theorem 2.2 of [**24**] conditions on g have been omitted although the intended conditions are clear.

We illustrate the improvement our results represent over [**24**] by modifying the example presented there.

Example. Let

$$f(x, y, p) = -\sin x - (\cos x - y^2) 2 \sin y - p^5 \text{ for } x \in [0, 1],$$

$$g^0((y(0), y(1)); (y'(0), y'(1))) = -y'(0) + y(0) + (y^2(1) - y'^2(1))/10 \text{ and}$$

$$g^1((y(0), y(1)); (y'(0), y'(1))) = y(0) + y'(0) + 6y(1) + y'(1) + \sin(y(0) - y'(1)).$$

Thus we have translated the x interval so it is now $[0, 1]$ and modified f in the y variable so that it is no longer monotonic with respect to y. In view of our remark above Palamides's example has a solution with $y(0) = y'(0)$ so we have modified g^0 to avoid this. Also we modified g^1 to avoid monotonicity.

To see that there is a solution we apply Theorem 2 to a modified problem. We let $\beta(x) = \pi/2 = -\alpha(x)$ and $\Upsilon(x, y) = 2 = -\Phi(x, y)$. It is easy to check that the $\alpha < \beta$ are lower and upper solutions and that the Knobloch–Nagumo–Schmitt condition is satisfied. As above we set $\hat{E} = [-\pi/2, \pi/2] \times [-2, 2]$ but replace E by $E_2 = \hat{E}$. It is easy to check that if Ψ is a strongly inwardly pointing vector field with $|\Psi| \le 2$ then conditions (2.10) and (2.11) are satisfied. We modify G for $(y'(0), y'(1)) \notin [-2,2]^2$ by projecting $(y'(0), y'(1))$ in the obvious way so that the modified G is strongly compatible with α and β. Thus, by Theorem 2 there is a solution y of the modified problem. However given the bounds on y and on y' the solution lies in the region where G was not modified. Thus y is the required solution.

Notice that from our analysis of Theorem 2.1 of [**24**] and the above it is clear we could have obtained existence of solutions for the example of [**24**] from our Theorem 2 using constant α, β, Υ and Φ.

5. Boundary Set Conditions.

In this section we consider problem (1.1) and (1.3) again assuming that there exist lower and upper solutions $\alpha \le \beta$, respectively, and look for solutions y lying between α and β.

Problems of the form (1.1) and (1.3) for the case f is continuous have been considered by many authors. Shooting methods have been used combined with with the Jordan separation theorem (see Bebernes and Fraker [**7**] and Bebernes and Wilhelmsen [**5**, **6**] and their references).

We show that analogues of the results of Bebernes and Fraker [**7**] for the case f is continuous can be derived from our results.

In order to state our results we need some notation (see Bebernes and Fraker [**7**]). For $x \in [0,1]$ let $C(x) = \{(x,y,p) \in \bar{\omega} \times \mathbb{R}\}$, $S_\alpha(x) = \{(x,y,p) \in C(x) : y = \alpha(x)\}$, and $S_\beta(x) = \{(x,y,p) \in C(x) : y = \beta(x)\}$. For the convenience of the reader and the sake of completeness we recall the definition of compatibility of boundary sets (see [**29**]).

Definition 13. We say the pair of sets $\{\mathcal{J}(0), \mathcal{J}(1)\} \subset \mathbb{R}^2$ is strongly compatible, respectively compatible, for (1.1), α and β if there exists $G \in C(\bar{\Delta} \times \mathbb{R}^2; \mathbb{R}^2)$ which is strongly compatible, respectively compatible, for (1.1), α and β and such that $G((C,D);(E,F)) \neq 0$ for all $(C,E,D,F) \notin \mathcal{J}(0) \times \mathcal{J}(1)$.

Definition 14. Let $\mathcal{J}(i) \subset [\alpha(i), \beta(i)] \times \mathbb{R}$, $i = 0$ or 1 be a closed connected set. We say it is of compatible type 1 if there is $(\alpha(i), u(i)) \in \mathcal{J}(i)$, where $(-1)^i(\alpha'(i) - u(i)) \ge 0$, and there is $(\beta(i), u(i)) \in \mathcal{J}(i)$, where $(-1)^i(u(i) - \beta'(i)) \ge 0$. We say it is of compatible type 2 if for every $p \in \mathbb{R}$ there is $y \in [\alpha(i), \beta(i)]$ such that $(y,p) \in \mathcal{J}(i)$. If it is of compatible type 1 or 2 we say simply it is of compatible type.

Theorem 3 [**29**, Theorem 4]. *Let the sets $\mathcal{J}(i) \subset \mathbb{R}^2$, $i = 0,1$ be of compatible type, then the pair $\{\mathcal{J}(0), \mathcal{J}(1)\}$ is compatible for (1.1), α and β.*

The next result is an immediate consequence of Theorems 1 and 3.

Theorem 4. *Assume that there exist lower and upper solutions, $\alpha \leq \beta$, respectively, for* (1.1), *that f satisfies a Bernstein–Nagumo–Zwirner condition and and that the sets $\mathcal{J}(i)$, $i = 0, 1$ are of compatible type. Then there is a solution of* (1.1) *and* (1.3) *lying between α and β.*

We now state the analogue for measurable f of [**7**, Theorem 1].

Theorem 5. *Assume that there exist lower and upper solutions $\alpha \leq \beta$ for* (1.1) *and that for any $B_2 \geq 0$ and $t_0 \in (0, 1]$ there is $N(B_2) > 0$ such that any solution of* (1.1) *with $|y'(0)| \leq B_2$ and $\alpha(x) \leq y(x) \leq \beta(x)$ for all $x \in [0, t_0]$ satisfies $|y'(x)| \leq N(B_2)$ for all $x \in [0, t_0]$. If $\mathcal{J}(0)$ is compact and of compatible type 1 and $\mathcal{J}(1)$ is of compatible type 2, then problem* (1.1) *and* (1.3) *has a solution lying between α and β.*

Proof. By compactness there is $B_2 > 0$ such that $(y(0), y'(0)) \in \mathcal{J}(0)$ implies that $|y'(0)| \leq B_2$. By assumption we may choose N such that $|y'| < N$ for all solutions y of (1.1) with $(x, y) \in \bar{\omega}$ on $[0, 1]$ and L such that $L > \max\{|\alpha'(x)|, |\beta'(x)|, N : x \in [0, 1]\}$. Let

$$j(x, y, p) = f(x, y, \pi(p; -L, L))$$

for all $(x, y, p) \in [0, 1] \times \mathbb{R}^2$. Consider

$$y'' = j(x, y, y') \qquad \text{for all } x \in [0, 1] \tag{5.1}$$

together with (1.3). Now α and β are lower and upper solutions for (5.1) so by Theorem 4 there is a solution y for this problem between α and β. Assume that $|y'| \geq L$ for some $x \in [0, 1]$. Set $t = \min\{x \in [0, 1] : |y'(x)| \leq L\}$. By continuity and the choice of L, we have $t > 0$ and $|y'(x)| \leq L$ for all $x \in [0, t]$. Thus y is a solution of (1.1) on $[0, t]$ lying between α and β but $|y'(t)| > N$ a contradiction. Thus $|y'| < L$ on $[0, 1]$ and y is the required solution. $\square$

The above result can be sharpened as follows.

Definition 15. Set

$$S_0 = \{(y, -L) : \alpha(0) \leq y \leq \beta(0)\} \cup \{(\alpha(0), p) : \alpha'(0) \geq p \geq -L\}, \tag{5.2}$$

$$S_2 = \{(y, L) : \alpha(0) \leq y \leq \beta(0)\} \cup \{(\beta(0), p) : \beta'(0) \leq p \leq L\}, \tag{5.3}$$

$$S_1 = \{(y, -L) : \alpha(1) \leq y \leq \beta(1)\} \cup \{(\beta(1), p) : \beta'(1) \geq p \geq -L\} \text{ and} \tag{5.4}$$

$$S_3 = \{(y, L) : \alpha(1) \leq y \leq \beta(1)\} \cup \{(\alpha(1), p) : \alpha'(1) \leq p \leq L\}. \tag{5.5}$$

We can now state the analogue for measurable f of [**7**, Theorem 3].

Theorem 6. *Assume that there exist lower and upper solutions $\alpha \leq \beta$ for* (1.1) *and that f satisfies a Bernstein–Nagumo–Zwirner condition. Further* *assume that* $\mathcal{J}(i) \subseteq C(i)$, $i = 0, 1$ *are closed connected sets satisfying* $\mathcal{J}(0) \cap \{S_0 \cup S_2\} \neq \emptyset$ *and* $\mathcal{J}(1) \cap \{S_1 \cup S_3\} \neq \emptyset$, *where the* S_i *are given by* (5.2) *to* (5.5) *and* $L > \max\{|\alpha'(x)|, |\beta'(x)| : x \in [0,1]\}$ *satisfies* (2.8). *Then there is a solution* y *lying between* α *and* β.

Proof. This follows since either $(\beta(0), u) \in \mathcal{J}(0)$ for some $u \geq \beta'(0)$ or $(y, L) \in \mathcal{J}(0)$ for some $y \in [\alpha(0), \beta(0))$ and we add the straight line segment joining (y, L) to $(\beta(0), L)$ to $\mathcal{J}(0)$. Similarly, either $(\alpha(0), u) \in \mathcal{J}(0)$ for some $u \leq \alpha'(0)$ or $(y, -L) \in \mathcal{J}(0)$ for some $y \in (\alpha(0), \beta(0)]$ and we add the straight line segment joining $(y, -L)$ to $(\alpha(0), -L)$ to $\mathcal{J}(0)$. Similarly we modify $\mathcal{J}(1)$ as above. Thus the modified $\mathcal{J}(i)$ are of compatible type and, by Theorem 3, there exists a solution for (1.1) and (1.3). This solution satisfies $|y'| < L$ and hence is the required solution. □

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Received February 12, 1993. The author would like to thank CMA at the Australian National University, and in particular Professor N.S. Trudinger for their hospitality and support.

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