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THE POSITIVE DIMENSIONAL FIBRES OF THE PRYM MAP

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The fibres of positive dimension of the Prym map are characterized.

Let C be an irreducible complex smooth curve of genus g. Let $\pi : \tilde{C} \longrightarrow C$ be a connected unramified double covering of C.

The *Prym variety* associated to the covering is, by definition, the component of the origin of the Kernel of the norm map

$$P(\tilde{C}, C) = \operatorname{Ker}(Nm_\pi)^0 \subset J\tilde{C}.$$

It is a principally polarized abelian variety (p.p.a.v.) of dimension $g(\tilde{C}) - g = g - 1$.

One defines the Prym map

$$\begin{array}{cccc} P_g : & \mathcal{R}_g & \longrightarrow & \mathcal{A}_{g-1} \\ & (\tilde{C} \xrightarrow{\pi} C) & \longmapsto & P(\tilde{C}, C), \end{array}$$

where $\mathcal{R}_g$ is the coarse moduli space of the coverings π as above and $\mathcal{A}_{g-1}$ stands for the coarse moduli space of p.p.a.v.'s of dimension $g-1$.

It is well-known that this map is generically injective for $g \geq 7$ (Friedman-Smith, Kanev). On the other hand this map is never injective; this is a consequence of the tetragonal construction due to Donagi (see [**Do1**] for a description of the construction). This fact is already implicit in the results of Mumford ([**M**]):

The coarse moduli space $\mathcal{RH}_g$ of unramified double coverings of smooth hyperelliptic curves of genus g has $[\frac{g-1}{2}] + 1$ irreducible components $\mathcal{RH}_{g,t}$, $t = 0, ..., [\frac{g-1}{2}]$. For an element $(\tilde{C}, C) \in \mathcal{RH}_{g,t}$ there exist two hyperelliptic curves

$$p_1 : C_1 \longrightarrow \mathbb{P}^1, \quad p_2 : C_2 \longrightarrow \mathbb{P}^1$$

of genus $g(C_1) = t \leq g - t - 1 = g(C_2)$ such that

a) $\tilde{C} = C_1 \times_{\mathbb{P}^1} C_2$ and

b) $C = \tilde{C}/(\sigma_1 \circ \sigma_2)$, where σ_1 (resp. σ_2) is the involution on $\tilde{C}$ attached to the branched covering $\tilde{C} \longrightarrow C_1$ (resp. $\tilde{C} \longrightarrow C_2$).

Mumford proves (loc. cit. p. 346) that one has an isomorphism of p.p.a.v.

$$P(\tilde{C}, C) \cong JC_1 \times JC_2.$$

Consequently the fibres of the restriction of P_g to $\mathcal{RH}_g$ have positive dimension. In fact $P_g(\mathcal{RH}_{g,t})$ is contained in the product $\mathcal{JH}_t \times \mathcal{JH}_{g-t-1}$, where $\mathcal{JH}_s$ stands for the locus of Jacobians of hyperelliptic curves of genus s. Thus

$$\dim \mathcal{RH}_{g,t} = 2g - 1 > \dim \mathcal{JH}_t \times \mathcal{JH}_{g-t-1} = \begin{cases} 2g-4 & \text{if } t \neq 0 \\ 2g-3 & \text{if } t = 0. \end{cases}$$

On the other hand positive dimensional fibres also appear for some coverings of bi-elliptic curves (a curve is called bi-elliptic if it can be represented as a ramified double covering of an elliptic curve).

In this note we characterize the fibres of positive dimension of the Prym map. To state our theorem we need some notation: let $\mathcal{RB}_g$ be the coarse moduli space of the unramified double coverings $\pi : \tilde{C} \longrightarrow C$ such that C is a smooth bi-elliptic curve of genus g. This variety has $[\frac{g-1}{2}] + 2$ irreducible components

$$\mathcal{RB}_g = \left(\bigcup_{t=0}^{[\frac{g-1}{2}]} \mathcal{RB}_{g,t} \right) \cup \mathcal{RB}'_g$$

(see [**N**] for more details).

We obtain:

Theorem. *Assume $g \geq 13$. A fibre of P_g is positive dimensional at $(\tilde{C}, C)$ if and only if C is either hyperelliptic or*

$$(\tilde{C}, C) \in \bigcup_{t \geq 1} \mathcal{RB}_{g,t}.$$

Proof. If C is hyperelliptic we apply the results of Mumford. On the other hand, all the irreducible components of the fibres of $P_{g|\mathcal{RB}_{g,t}}$ are positive dimensional for $t \geq 1$ (see [**N**, §20]). This finishes one implication.

The first step to see the opposite implication is to prove that the curve C is tetragonal (i.e. there exists a g^1_4 on C).

Let $\eta \in JC$ be the two-torsion point characterizing the covering and denote by L the line bundle $\omega_C \otimes \eta$. It is easy to check that L is very ample if C is non-tetragonal. Let Φ_L be the projective embedding of C defined by L.

As in Beauville ([**B**, p. 379]), we replace $\mathcal{R}_g$ and $\mathcal{A}_{g-1}$ by the corresponding functors. Then, the Prym map defines a morphism of functors Pr_g. Our hypothesis on the fibre of P_g implies that the cotangent map to Pr_g at $(\tilde{C}, C)$ is not surjective. By loc. cit. Prop. (7.5), this map can be shown as the cup-product map

$$S^2H^0(C, L) \longrightarrow H^0(C, L^{\otimes 2})$$

followed by the isomorphism induced in cohomology by $L^{\otimes 2} \cong \omega^{\otimes 2}$. Hence, the non-surjectivity implies that $\Phi_L(C)$ is not a projectively normal curve.

We recall Theorem 1 in [**G-L**]: If L is very ample and

$$\deg(L) \geq 2g + 1 - 2h^1(L) - \mathrm{Cliff}(C),$$

then $\Phi_L(C)$ is projectively normal (where Cliff(C) is the Clifford index of C).

Since $h^1(L) = 0$ and $\deg(L) = 2g - 2$ one obtains Cliff$(C) \leq 2$. By using Clifford's Theorem and [**Ma**, Propositions 7 and 8], it follows that the curve either possess a g^1_4 or is plane curve of degree six. The second case contradicts $g \geq 13$.

Thus C is tetragonal. Since $g \geq 13$ the results in [**De**] can be applied: either the fibre is finite (generically, three elements) or we are in one of the following three possibilities: C is either hyperelliptic or bi-elliptic or trigonal.

Assume that C is bi-elliptic. Theorems (9.4), (10.9) and (10.10) in [**N**] states that $P_g^{-1}(P(\tilde{C}, C))$ consists of two points for every $(\tilde{C}, C) \in \mathcal{RB}_{g,0} \cup \mathcal{RB}'_g$, hence

$$(\tilde{C}, C) \in \bigcup_{t \geq 1} \mathcal{RB}_{g,t}.$$

To finish the proof we have to rule out the case: C trigonal. In [**R**], Recillas (cf. also [**Do2**]) establishes an isomorphism

$$\tau : \mathcal{RT}_g \xrightarrow{\cong} \mathcal{M}^{tet,0}_{g-1},$$

where $\mathcal{RT}_g$ is the coarse moduli space of unramified double coverings of trigonal curves and $\mathcal{M}^{tet,0}_{g-1}$ is the moduli space of pairs (X, g^1_4) of tetragonal curves X and a base-point-free tetragonal linear series on X not containing divisors of the form $2x + 2y$. This map satisfies that

$$\tau(\tilde{C}, C) = (X, g^1_4) \Longrightarrow P(\tilde{C}, C) \cong JX \quad \text{(as p.p.a.v.)}.$$

Let us fix $(\tilde{C}, C)$ as above and let $(\tilde{D}, D) \in \mathcal{R}_g$ such that $P(\tilde{D}, D) \cong P(\tilde{C}, C) \cong JX$. Since C is not hyperelliptic, then the singular locus of the theta divisor of $P(\tilde{D}, D)$ has codimension 3 by [**M**, p. 344]. In loc. cit. a list of the Prym varieties with such property appear. We obtain that D is either trigonal or bi-elliptic. Since $P(\tilde{D}, D)$ is the Jacobian of a curve the bi-elliptic case contradicts [**S**].

Hence it suffices to prove that all the fibres of the restriction of P_g to $\mathcal{RT}_g$ are zero dimensional. This follows from the bijection τ. Indeed, a curve X of genus $g \geq 12$ has at most one base-point-free g_4^1 without divisors of the form $2x+2y$; otherwise there exists a map $f : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and then either the genus is ≤ 9 or X is bi-elliptic. By [**T**, Lemma (4.3)] the linear series of degree 4 and dimension 1 on a bi-elliptic curve come from g_2^1 linear series on the elliptic curve, thus divisors of the forbidden form appear.

Now the classical Torelli Theorem says that

$$\begin{aligned} \mathcal{M}_{g-1}^{tet,0} &\longrightarrow \mathcal{A}_{g-1} \\ (X, g_4^1) &\longmapsto JX \end{aligned}$$

is injective. Composing with τ we are done. □

Remark. Note that if one drops the hypothesis on the genus, at least one gets that the Clifford index of C is ≤ 2.

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