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IRREDUCIBLE NON-DENSE $A_1^{(1)}$-MODULES

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We study the irreducible weight non-dense modules for Affine Lie Algebra $A_1^{(1)}$ and classify all such modules having at least one finite-dimensional weight subspace. We prove that any irreducible non-zero level module with all finite-dimensional weight subspaces is non-dense.

1. Introduction.

Let $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ and $\mathcal{G} = \mathcal{G}(A)$ is the associated Kac-Moody algebra over the complex numbers $\mathbf{C}$ with Cartan subalgebra $H \subset \mathcal{G}$, 1-dimensional center $\mathbf{C}c \subset H$ and root system Δ.

A $\mathcal{G}$-module V is called a *weight* if $V = \bigoplus_{\lambda \in H^*} V_\lambda$, $V_\lambda = \{v \in V \mid hv = \lambda(h)v$ for all $h \in H\}$. If V is an irreducible weight $\mathcal{G}$-module then c acts on V as a scalar. We will call this scalar the *level* of V, For a weight $\mathcal{G}$-module V, set $P(V) = \{\lambda \in H^* \mid V_\lambda \neq 0\}$.

Let $Q = \sum_{\varphi \in \Delta} \mathbf{Z}\varphi$. It is clear that if a weight $\mathcal{G}$-module V is irreducible then $P(V) \subset \lambda + Q$ for some $\lambda \in H^*$. An irreducible weight $\mathcal{G}$-module V is called *dense* if $P(V) = \lambda + Q$ for some $\lambda \in H^*$, and *non-dense* otherwise.

Irreducible dense modules whose weight spaces are all one-dimensional were classified by S. Spirin [**1**] for the algebra $A_1^{(1)}$ and by D. Britten, F. Lemire, F. Zorzitto [**2**] in the general case. It follows from [**2**] that such modules exist only for algebras $A_n^{(1)}$, $C_n^{(1)}$. V. Chari and A. Pressley constructed a family of irreducible integrable dense modules with all infinite-dimensional weight spaces. These modules can be realized as tensor product of standard highest weight modules with so-called loop modules [**3**].

In the present paper we study irreducible non-dense weight $\mathcal{G}$-modules. We use Kac [**4**] as a basic reference for notation, terminology and preliminary results. Our main result is the classification of all irreducible non-dense $\mathcal{G}$-modules having at least one finite-dimensional weight subspace. This includes, in particular, all irreducible highest weight modules. Moreover, we show that this classification includes all irreducible modules of non-zero level whose weight spaces are all finite- dimensional.

The paper is organized as follows. In Section 3 we study generalized Verma modules $M_\alpha^\varepsilon(\lambda,\gamma)$, α is a real root, $\lambda \in H^*$, $\gamma \in \mathbf{C}$, $\varepsilon \in \{+,-\}$ which do not necessarily have a highest weight (cf. [**5**]). By making use of the generalized Casimir operator and generalized Shapovalov form we obtain the criteria of irreducibility for the modules $M_\alpha^\varepsilon(\lambda,\gamma)$ without highest weight (Theorem 3.11).

In Section 4 we classify all irreducible $\mathbf{Z}$-graded modules for the Heisenberg subalgebra $G \subset \mathcal{G}$ with at least one finite-dimensional graded component. Irreducible G- modules with trivial action of c were described earlier in [**6**]. Let $\delta \in \Delta$ such that $\mathbf{Z}\delta - \{0\}$ is the set of all imaginary roots in Δ. Following [**6**] we introduce in Section 5 the category $\tilde{\mathcal{O}}(\alpha)$ of weight $\mathcal{G}$-modules $\tilde{V}$ such that $P(\tilde{V}) \subset \bigcup_{i=1}^{\ell} \{\lambda_i - k\alpha + n\delta \mid k,n \in \mathbf{Z}, k \geq 0\}$ where $\lambda_i \in H^*$, but without any restriction on the action of the center (unlike in [**6**] where the trivial action of the center is required). The irreducible objects in $\tilde{\mathcal{O}}(\alpha)$ are the unique quotients of $\mathcal{G}$-modules $M_\alpha(\lambda, V)$, where $\lambda \in H^*$, V is irreducible $\mathbf{Z}$-graded G-module. Modules $M_\alpha(\lambda,\mathbf{C})$, with $\lambda(c) = 0$ were studied in [**7-9**]. If $\lambda(c) \neq 0$ and at least one graded component of V is finite-dimensional then the module $M_\alpha(\lambda, V)$ is irreducible [**8, 9**]. In Section 6 we classify all irreducible non-dense $\mathcal{G}$-modules with at least one finite-dimensional weight subspace (Theorem 6.2). It turns out that these modules are the quotients of the modules of type $M_\alpha^\varepsilon(\lambda,\gamma)$ or $M_\alpha(\lambda, V)$. Moreover, any irreducible $\mathcal{G}$-module of non-zero level whose weight spaces are all finite- dimensional is the quotient of $M_\alpha^\varepsilon(\lambda,\gamma)$ for some real root α, $\lambda \in H^*$, $\gamma \in \mathbf{C}$, $\varepsilon \in \{+,-\}$ (Theorem 6.3).

2. Preliminaries.

We have the root space decomposition for $\mathcal{G}$: $\mathcal{G} = H \oplus \sum_{\varphi\in\Delta} \mathcal{G}_\varphi$, where dim $\mathcal{G}_\varphi = 1$ for all $\varphi \in \Delta$. Denote by $\mathcal{U}(\mathcal{G})$ the universal enveloping algebra of $\mathcal{G}$, by W the Weyl group and by (,) the standard non-degenerate symmetric bilinear form on $\mathcal{G}$ [**4**, Theorem 3.2]. Let Δ^{re} be the set of real roots in Δ and Δ^{im} be the set of imaginary roots in Δ. Fix $\alpha \in \Delta^{re}$ and consider a subalgebra $\mathcal{G}(\alpha) \subset \mathcal{G}$ generated by $\mathcal{G}_\alpha$ and $\mathcal{G}_{-\alpha}$. Then $\mathcal{G}(\alpha) \simeq sl(2)$ and we fix in $\mathcal{G}(\alpha)$ a standard basis e_α, $e_{-\alpha}$, $h_\alpha = [e_\alpha, e_{-\alpha}]$ where $[h_\alpha, e_{\pm\alpha}] = \pm 2e_{\pm\alpha}$. We will use the following realization of $\mathcal{G}$:

$$\mathcal{G} = \mathcal{G}(\alpha) \otimes \mathbf{C}[t,t^{-1}] \oplus \mathbf{C}c \oplus \mathbf{C}d$$

with $[x\otimes t^n + ac + bd,\, y \otimes t^m + a_1 c + b_1 d] = [x,y]\otimes t^{n+m} + bmy\otimes t^m - b_1 nx \otimes t^n + n\delta_{n,-m}(x,y)c$, for all x, $y \in \mathcal{G}(\alpha)$, a, b, a_1, $b_1 \in \mathbf{C}$. Then $H = \mathbf{C}h_\alpha \oplus \mathbf{C}c \oplus \mathbf{C}d$.

Denote by δ the element of H^* defined by: $\delta(h_\alpha) = \delta(c) = 0$ and $\delta(d) = 1$. Then $\Delta^{im} = \mathbf{Z}\delta - \{0\}$ and $\pi = \{\alpha, \delta - \alpha\}$ is a basis of Δ. Let $\Delta_+ = \Delta_+(\pi)$ be the set of all positive roots with respect to π. The root system Δ can be described in the following way: $\Delta = \{\pm\alpha + n\delta \mid n \in \mathbf{Z}\} \cup \{n\delta \mid n \in \mathbf{Z} - \{0\}\}$. We have $\mathcal{G}_{\pm\alpha+n\delta} = \mathcal{G}_{\pm\alpha} \otimes t^n$, $n \in \mathbf{Z}$, $\mathcal{G}_{n\delta} = \mathbf{C}h_\alpha \otimes t^n$, $n \in \mathbf{Z} - \{0\}$. Set $e_{\alpha+n\delta} = e_\alpha \otimes t^n$, $e_{-\alpha+n\delta} = e_{-\alpha} \otimes t^n$, $n \in \mathbf{Z}$, $e_{m\delta} = h_\alpha \otimes t^m$, $m \in \mathbf{Z} - \{0\}$. Then $[e_{k\delta}, e_{m\delta}] = 2k\delta_{k,-m}c$, $[e_{k\delta}, e_{\pm\alpha+n\delta}] = \pm 2e_{\pm\alpha+(n+k)\delta}$, $[e_{\alpha+k\delta}, e_{-\alpha+m\delta}] = \delta_{k,-m}(h_\alpha + kc) + (1 - \delta_{k,-m})e_{(k+m)\delta}$ for any k, $m \in \mathbf{Z}$.

For a Lie algebra $\mathcal{A}$, $S(\mathcal{A})$ will denote the corresponding symmetric algebra. We will identify the algebra $\mathcal{U}(H) = S(H)$ with the ring of polynomials $\mathbf{C}[H^*]$ and denote by σ the involutive antiautomorphism on $\mathcal{U}(\mathcal{G})$ such that $\sigma(e_\alpha) = e_{-\alpha}$, $\sigma(e_{\delta-\alpha}) = e_{\alpha-\delta}$. Set $\mathcal{N}_+ = \sum_{\varphi \in \Delta_+} \mathcal{G}_\varphi$, $\mathcal{N}_- = \sum_{\varphi \in \Delta_+} \mathcal{G}_{-\varphi}$.

3. Generalized Verma modules.

The center of $\mathcal{U}(\mathcal{G}(\alpha))$ is generated by the Casimir element $z_\alpha = (h_\alpha + 1)^2 + 4e_{-\alpha}e_\alpha$. Denote

$$\mathcal{N}_\alpha^+ = \sum_{\varphi \in \Delta_+ - \{\alpha\}} \mathcal{G}_\varphi, \qquad \mathcal{N}_\alpha^- = \sum_{\varphi \in \Delta_+ - \{\alpha\}} \mathcal{G}_{-\varphi},$$
$$T_\alpha = S(H) \otimes \mathbf{C}[z_\alpha], \qquad E_\alpha^\varepsilon = (H + \mathcal{G}(\alpha)) \oplus \mathcal{N}_\alpha^\varepsilon,\ \varepsilon \in \{+, -\}.$$

Let $\lambda \in H^*, \gamma \in \mathbf{C}$. Consider the 1-dimensional T_α-module $\mathbf{C}v_\lambda$ with the action $(h \otimes z_\alpha^n)v_\lambda = h(\lambda)\gamma^n v_\lambda$ for any $h \in S(H)$, and construct an $H + \mathcal{G}(\alpha)$-module

$$V(\lambda, \gamma) = \mathcal{U}(\mathcal{G}(\alpha) + H) \bigotimes_{\cdot T_\alpha} \mathbf{C}v_\lambda.$$

It is clear that the module $V(\lambda, \gamma)$ has a unique irreducible quotient $V_{\lambda,\gamma}$.

Proposition 3.1.

(i) *If V is an irreducible weight $H + \mathcal{G}(\alpha)$-module then $V \simeq V_{\lambda,\gamma}$ for some $\lambda \in H^*$, $\gamma \in \mathbf{C}$.*

(ii) *$V_{\lambda,\gamma} \simeq V_{\lambda',\gamma'}$ if and only if $\gamma = \gamma'$, $\lambda' = \lambda + n\alpha$, $n \in \mathbf{Z}$, $\gamma \neq (\lambda(h_\alpha) + 2\ell + 1)^2$ for all integers ℓ, $0 \leq \ell < n$ if $n \geq 0$ or for all integers ℓ, $n \leq \ell < 0$ if $n < 0$.*

Proof. This is essentially the classification of irreducible weight $sl(2)$-modules. □

Let $\lambda \in H^*$, $\gamma \in \mathbf{C}$, $\varepsilon \in \{+, -\}$. Consider $V_{\lambda,\gamma}$ as E_α^ε-module with trivial action of $\mathcal{N}_\alpha^\varepsilon$ and construct the $\mathcal{G}$-module

$$M_\alpha^\varepsilon(\lambda, \gamma) = \mathcal{U}(\mathcal{G}) \bigotimes_{\mathcal{U}(E_\alpha^\varepsilon)} V_{\lambda,\gamma}$$

associated with $\alpha, \lambda, \gamma, \varepsilon$.

The module $M_\alpha^\varepsilon(\lambda, \gamma)$ is called a generalized Verma module. Notice that $V_{\lambda,\gamma}$ does not have to be finite-dimensional.

Proposition 3.2.

(i) *$M_\alpha^\varepsilon(\lambda, \gamma)$ is a free $\sigma(\mathcal{U}(\mathcal{N}_\alpha^\varepsilon))$- module with all finite-dimensional weight subspaces.*

(ii) *$M_\alpha^\varepsilon(\lambda, \gamma)$ has a unique irreducible quotient, $L_\alpha^\varepsilon(\lambda, \gamma)$.*

(iii) *$M_\alpha^\varepsilon(\lambda, \gamma) \simeq M_{\pm\alpha}^{\varepsilon'}(\lambda', \gamma')$ if and only if $\varepsilon = \varepsilon'$, $\gamma = \gamma'$, $\lambda' = \lambda + n\alpha, n \in \mathbf{Z}$ and $\gamma \neq (\lambda(h_\alpha) + 2\ell + 1)^2$ for all $\ell \in \mathbf{Z}$, $0 \leq \ell < n$ if $n \geq 0$ or for all $\ell \in \mathbf{Z}$, $n \leq \ell < 0$ if $n < 0$.*

Proof. Follows from the construction of $\mathcal{G}$- module $M_\alpha^\varepsilon(\lambda, \gamma)$ and Proposition 3.1. □

Let $R_\lambda = \{(\lambda(h_\alpha) + 2\ell + 1)^2 \mid \ell \in \mathbf{Z}\}$. Recall that V is called a highest weight module with respect to $\mathcal{N}_+$ and with highest weight $\lambda \in H^*$ if $V = \mathcal{U}(\mathcal{G})v$, $v \in V_\lambda$ and $V_{\lambda+\varphi} = 0$ for all $\varphi \in \Delta_+(\pi)$. Proposition 3.2, (iii) implies that $M_\alpha^\varepsilon(\lambda, \gamma)$ and $L_\alpha^\varepsilon(\lambda, \gamma)$ are highest weight modules with respect to some choice of basis of Δ and, therefore, are the quotients of Verma modules [**4**], if and only if $\gamma \in R_\lambda$. The theory of highest weight modules was developed in [**4**, **10**].

Corollary 3.3.

(i) *Let V be an irreducible weight $\mathcal{G}$-module, $0 \neq v \in V_\lambda$ and $\mathcal{N}_\alpha^\varepsilon v = 0$. Then $V \simeq L_\alpha^\varepsilon(\lambda, \gamma)$ for some $\gamma \in \mathbf{C}$.*

(ii) *Let $\lambda \notin R_\lambda$. $L_\alpha^\varepsilon(\lambda, \gamma) \simeq L_{\alpha'}^{\varepsilon'}(\lambda', \gamma')$ if and only if $\varepsilon = \varepsilon'$, $\alpha' = \alpha$ or $\alpha' = -\alpha$, $\gamma = \gamma'$, $\lambda' = \lambda + n\alpha$, $n \in \mathbf{Z}$ and $\gamma \neq (\lambda(h_\alpha) + 2\ell + 1)^2$ for all $\ell \in \mathbf{Z}$, $0 \leq \ell < n$ if $n \geq 0$ or for all $\ell \in \mathbf{Z}$, $n \leq \ell < 0$ if $n < 0$.*

Proof. Since V is irreducible $\mathcal{G}$- module, $V' = \mathcal{U}(\mathcal{G}(\alpha))v$ is an irreducible $\mathcal{G}(\alpha)$-module and $V \simeq \sigma(\mathcal{U}(\mathcal{N}_\alpha^\varepsilon))V'$. Then V is a homomorphic image of $M_\alpha^\varepsilon(\lambda, \gamma)$ for some $\gamma \in \mathbf{C}$ and, thus, $V \simeq L_\alpha^\varepsilon(\lambda, \gamma)$ which proves (i). (ii) follows from Proposition 3.2, (iii). □

From now on we will consider the modules $M_\alpha^+(\lambda, \gamma)(= M(\lambda, \gamma))$. All the results for the modules $M_\alpha^-(\lambda, \gamma)$ can be proved analogously. Set $z = z_\alpha$. For $\lambda \in H^*$, $\gamma \in \mathbf{C}$ and integer $n \geq 0$ we denote by $z(n)$ the restriction of z to the subspace $M(\lambda, \gamma)_{\lambda - n(\delta - \alpha)}$.

Proposition 3.4. *If $\gamma \neq (\lambda(h_\alpha) + 2\ell + 1)^2$ for all $0 \leq \ell < 2n$ then* $\operatorname{Spec} z(n) = \{(2k \pm \sqrt{\gamma})^2 \mid k \in \mathbf{Z}, 0 \leq k \leq n\}$.

Proof. Denote $V_n = M(\lambda, \gamma)_{\lambda - n(\delta - \alpha)}$, $n > 0$. One can easily show that $V_n = e_{\alpha-\delta}V_{n-1} + e_{-\delta}e_\alpha V_{n-1} + e_{-\alpha-\delta}e_\alpha^2 V_{n-1}$. Let $V_{n-1} = \oplus V_{n-1}(\tau), \tau \in \mathbf{C}$, where $V_{n-1}(\tau) = \{v \in V_{n-1} \mid \exists N : (z(n-1)-\tau)^N v = 0\}$. Then the subspace $e_{\alpha-\delta}V_{n-1}(\tau) + e_{-\delta}e_\alpha V_{n-1}(\tau) + e_{-\alpha-\delta}e_\alpha^2 V_{n-1}(\tau) \subset V_n$ is $z(n)$- invariant and $z(n)$ has on it the eigenvalues τ and $(2 \pm \sqrt{\tau})^2$, thanks to the condition $\gamma \neq (\lambda(h_\alpha) + 2\ell + 1)^2$, $0 \leq \ell < 2n$, which implies that $z(n)$ has eigenvalues $(2k \pm \sqrt{\gamma})^2$, $0 \leq k \leq n$. □

Corollary 3.5. *If $\gamma \notin R_\lambda$ then e_α and $e_{-\alpha}$ act injectively on $M(\lambda, \gamma)$.*

Proof. If $\gamma \notin R_\lambda$ then $\operatorname{Spec} z(n) \bigcap R_{\lambda - n\beta} = \emptyset$ for all integer $n \geq 0$ by Proposition 3.4 and, therefore, e_α and $e_{-\alpha}$ act injectively on $M(\lambda, \gamma)$. □

Fix $\rho \in H^*$ such that $(\rho, \alpha) = 1$, $(\rho, \delta) = 2$. Since $M(\lambda, \gamma)$ is a restricted module, i.e. for every $v \in M(\lambda, \gamma)$, $\mathcal{G}_\varphi v = 0$ for all but a finite number of positive roots φ, we have well-defined action of a generalized Casimir operator Ω on $M(\lambda, \gamma)$ [**4**]:

$$\Omega v = (\mu + 2\rho, \mu)v + 2 \sum_{\varphi \in \Delta_+} \bar{e}_{-\varphi} e_\varphi v, \; v \in M(\lambda, \gamma)_\mu,$$

where $\bar{e}_{-\varphi} \in \mathcal{G}_{-\varphi}$, $(\bar{e}_{-\varphi}, e_\varphi) = 1$, $\varphi \in \Delta_+$. Set $\tilde{\Omega} = 2\Omega + id$.

Let $s_\alpha \in W$, $s_\alpha(\mu) = \mu - (\mu, \alpha)\alpha$, $\mu \in H^*$.

Lemma 3.6. *For a $\mathcal{G}$-module $M(\lambda, \gamma)$*

$$\tilde{\Omega} = [(\lambda + 2\rho + s_\alpha(\lambda + 2\rho), \lambda) + \gamma]id.$$

Proof. Follows from [**4**, Th.2.6] and definition of $\tilde{\Omega}$. □

Lemma 3.7. *Let $n > 0, \beta = \delta - \alpha$, $0 \neq v \in M(\lambda, \gamma)_{\lambda - n\beta}$, $\gamma \neq (\lambda(h_\alpha) + 2\ell + 1)^2$ for all $0 \leq \ell < 2n$ and $\mathcal{N}_\alpha^+ v = 0$. Then $k^2\gamma = (n(\lambda(c) + 2) - k^2)^2$ for some $k \in \mathbf{Z}$, $0 \leq k \leq n$.*

Proof. It follows from Lemma 3.6 that $z(n)v = \gamma' v$ and

$$(\lambda - n\beta + 2\rho + s_\alpha(\lambda - n\beta + 2\rho), \lambda - n\beta) + \gamma' = (\lambda + 2\rho + s_\alpha(\lambda + 2\rho), \lambda) + \gamma$$

which implies

$$\gamma' = \gamma + 4n(\lambda(c) + 2).$$

But, $\gamma' = (2k \pm \sqrt{\gamma})^2$ for some $k \in \mathbf{Z}$, $0 \leq k \leq n$ by Proposition 3.4. Therefore, $k^2\gamma = (n(\lambda(c) + 2) - k^2)^2$ which completes the proof. □

Corollary 3.8. *Let $\lambda \in H^*$, $\gamma \in \mathbf{C} - R_\lambda$. If $k^2\gamma \neq (n(\lambda(c) + 2) - k^2)^2$ for all $n, k \in \mathbf{Z}$, $n > 0$, $0 \leq k \leq n$ then $\mathcal{G}$-module $M(\lambda, \gamma)$ irreducible.*

Proof. If the $\mathcal{G}$-module $M(\lambda, \gamma)$ has a non-trivial submodule M, then M contains a non-zero vector v of weight $\lambda - n(\delta - \alpha)$, $n > 0$, such that $\mathcal{N}_\alpha^+ v = 0$. Now, the statement follows from Lemma 3.7. □

Consider the following decomposition of $\mathcal{U}(\mathcal{G})$:

$$\mathcal{U}(\mathcal{G}) = (\mathcal{N}_\alpha^- \mathcal{U}(\mathcal{G}) + \mathcal{U}(\mathcal{G})\mathcal{N}_\alpha^+) \oplus T_\alpha \mathbf{C}[e_\alpha]e_\alpha \oplus T_\alpha \mathbf{C}[e_{-\alpha}]e_{-\alpha} \oplus T_\alpha.$$

Let j be the projection of $\mathcal{U}(\mathcal{G})$ to T_α. Introduce the generalized Shapovalov form F, a symmetric bilinear form on $\mathcal{U}(\mathcal{G})$ with values in T_α, as follows (cf. [**11**]): $F(x,y) = j(\sigma(x)y)$, $x, y \in \mathcal{U}(\mathcal{G})$. The algebra $\mathcal{U}(\mathcal{G})$ is Q-graded: $\mathcal{U}(\mathcal{G}) = \bigoplus_{\eta \in Q} \mathcal{U}(\mathcal{G})_\eta$. It is clear that $F(\mathcal{U}(\mathcal{G})_{\eta_1}, \mathcal{U}(\mathcal{G})_{\eta_2}) = 0$ if $\eta_1 \neq \eta_2$. Denote $\mathcal{U}(\mathcal{N}_-)_{-\eta} = \mathcal{U}(\mathcal{N}_-) \bigcap \mathcal{U}(\mathcal{G})_{-\eta}$ and let F_η be a restriction of F to $\mathcal{U}(\mathcal{N}_-)_{-\eta}$.

For $\lambda \in H^*$, $\gamma \in \mathbf{C}$, consider the linear map $\theta_{\lambda,\gamma} : T_\alpha \to \mathbf{C}$ defined by $\theta_{\lambda,\gamma}(h \otimes z^n) = h(\lambda)\gamma^n$ for any $h \in S(H)$, $n \in \mathbf{Z}_+$.

Set $\lambda_k = \lambda + k\alpha$, $k \in \mathbf{Z}$. Let $\mu = \lambda - n(\delta - \alpha) \in P(M(\lambda,\gamma))$, $n \in \mathbf{Z}_+$ and $\gamma \neq (\lambda(h_\alpha) + 2s + 1)^2$ for all integer s, $0 \leq s < 2n$. Then $\lambda_{2n} \in P(M(\lambda,\gamma))$, $M(\lambda,\gamma)_{\lambda_{2n}} = \mathbf{C}v_n$ and $M(\lambda,\gamma)_\mu = \mathcal{U}(\mathcal{N}_-)_{-n(\alpha+\delta)}v_n$. Set $F^{(n)} = F_{n(\alpha+\delta)}$. We define a a bilinear $\mathbf{C}$-valued form F_μ^0 on $M(\lambda,\gamma)_\mu$ as follows:

$$F_\mu^0(u_1 v_n,\, u_2 v_n) = \theta_{\lambda_{2n},\gamma}\left(F^{(n)}(u_1,\, u_2)\right),\; u_1,\, u_2 \in \mathcal{U}(\mathcal{N}_-)_{-n(\alpha+\delta)}.$$

One can see that dim $L(\lambda,\gamma)_\mu = \text{rank } F_\mu^0$.

Lemma 3.9. *Let $\lambda \in H^*$, $\gamma \in \mathbf{C} - R_\lambda$. The following conditions are equivalent:*

(i) *$M(\lambda,\gamma)$ is irreducible.*

(ii) *$F^0_{\lambda - n(\delta-\alpha)}$ is non-degenerate for all integers $n > 0$.*

(iii) *$\theta_{\lambda_{2n},\gamma}\left(\det F^{(n)}\right) \neq 0$ for all integers $n > 0$.*

Proof. Follows from the Corollary 3.5. □

Consider in T_α the following polynomials: $f_{m,k} = k^2 z - (m(c+2) - k^2)^2$, $g_s = z - (h_\alpha + 2s + 1)^2$, $s, m, k \in \mathbf{Z}$, $0 \leq k \leq m$. Lemma 3.7 implies that if $\theta_{\lambda,\gamma}(g_s) \neq 0$ for all $s \in \mathbf{Z}$, $0 \leq s < 2n$ and $\theta_{\lambda_{2m},\gamma}(f_{m,k}) \neq 0$ for all $m, k \in \mathbf{Z}$, $0 < m \leq n$, $0 \leq k \leq m$, then $M(\lambda,\gamma)_{\lambda-n(\delta-\alpha)} = L(\lambda,\gamma)_{\lambda-n(\delta-\alpha)}$ and $\theta_{\lambda_{2n},\gamma}\left(\det F^{(n)}\right) \neq 0$. We conclude that the polynomial det $F^{(n)}$ is not identically equal to zero and has its zeros in the union of zeros of polynomials $f_{m,k}$, $0 < m \leq n$, $0 \leq k \leq m$, g_s, $0 \leq s \leq 2n$. Therefore, det $F^{(n)}$ is a product of factors of type $f_{m,k}$ and g_s.

Lemma 3.10. *Let $n, m \in \mathbf{Z}$, $n > 0$, $0 < m \leq n$. Then $f_{m,k}$ is a factor of* det $F^{(n)}$ *if and only if k is a divisor of m or $k = 0$.*

Proof. Assume that k is a divisor of m or $k = 0$. Set $r = 2n + 2m + k$. Consider $\lambda \in H^*$ and $\gamma \in \mathbf{C} - \mathbf{Z}$ such that $\theta_{\lambda,\gamma}(f_{m,k}) = \theta_{\lambda,\gamma}(g_r) = 0$. For integer $s \geq 0$

set $\nu_s = \lambda_{-s} = \lambda - s\alpha$. Then $\theta_{\nu_s,\gamma}(f_{m,k}) = \theta_{\nu_s,\gamma}(g_{r+s}) = 0$ and $\nu_s(h_\alpha) \notin \mathbf{Z}$, which implies that $\theta_{\nu_s,\gamma}(g_\ell) \neq 0$ for all $\ell \in \mathbf{Z}$, $\ell < r+s$. Thus, the form $F^0_{\nu_s - i\beta}$, $\beta = \delta - \alpha$ is defined for all $s \geq 0$, $0 < i \leq n$ and $M(\nu_s, \gamma) \simeq M(\lambda_r)$, $s \geq 0$ by Proposition 3.2, (iii), where $M(\lambda_r)$ is the Verma module with highest weight $\lambda_r = \lambda + r\alpha$. Therefore, $M(\nu_s,\gamma)_{\nu_s - i\beta} \simeq M(\lambda_r)_{\nu_s - i\beta}$, $0 < i \leq n$ as T_α-modules. The operator $z(m)$ has eigenvectors w_s^+, $w_s^- \in M(\lambda_r)_{\nu_s - m\beta}$ with eigenvalues $\gamma^+ = (\lambda(h_\alpha)+4(n+m+k)+1)^2$ and $\gamma^- = (\lambda(h_\alpha)+4(n+m)+1)^2$ respectively. Since $\theta_{\nu_s,\gamma}(f_{m,k}) = 0$, then

$$\gamma^* = \gamma + 4m(\lambda(c) + 2) \in \{\gamma^+, \gamma^-\}$$

and

$$(\nu_s+2\rho+s_\alpha(\nu_s+2\rho),\nu_s)+\gamma = (\nu_s-m\beta+2\rho+s_\alpha(\nu_s-m\beta+2\rho),\nu_s-m\beta)+\gamma^*.$$

Let $w_s^* \in \{w_s^+, w_s^-\}$ and $z(m)w_s^* = \gamma^* w_s^*$. Then

$$\tilde{\Omega} w_s^* = [(\nu_s - m\beta + 2\rho + s_\alpha(\nu_s - m\beta + 2\rho), \nu_s - m\beta) + \gamma^*] w_s^*$$

by Lemma 3.6. But, $w_s^* \in M(\lambda_r)$ and

$$\tilde{\Omega}\, w_s^* = (2(\lambda_r + 2\rho, \lambda_r) + 1) w_s^*$$

by Corollary 2.6 in [**4**]. Hence

$$2(\lambda_r + 2\rho, \lambda_r) + 1 = (\nu_s - m\beta + 2\rho + s_\alpha(\nu_s - m\beta + 2\rho), \nu_s - m\beta) + \gamma^*$$

and

$$(\lambda_r + 2\rho, \lambda_r) = (\lambda_r + 2\rho - \tau^*, \lambda_r - \tau^*)$$

where $\tau^* = m\delta - k\alpha$ if $\gamma^* = \gamma^+$ and $\tau^* = m\delta + k\alpha$ if $\gamma^* = \gamma^-$. If k divides m or $k = 0$ then τ^* is a quasiroot and $D = Hom_{\mathcal{G}}(M(\lambda_r - \tau^*), M(\lambda_r)) \neq 0$ [**10**, Prop. 4.1].

Let $0 \neq \chi \in D$. Then $\chi(M(\lambda_r - \tau^*)) \cap M(\lambda_r)_{\nu_s - n\beta} \neq 0$ and therefore, $\theta_{\lambda_{2n-s},\gamma}(\det F^{(n)}) = 0$ for any integer $s \geq 0$. It implies that if $\lambda \in H^*$, $\gamma \in \mathbf{C} - \mathbf{Z}$ and $\theta_{\lambda,\gamma}(f_{m,k}) = 0$ then $\theta_{\lambda,\gamma}(\det F^{(n)}) = 0$. Thus, $f_{m,k}$ is a factor of $\det F^{(n)}$. Conversely, suppose that $f_{n,k}$ is a factor of $\det F^{(n)}$, $k \neq 0$ and k is not a divisor of n. Let $r = 4n+k$. Consider a pair $(\lambda,\gamma) \in H^* \times (\mathbf{C} - \mathbf{Z})$ such that $\theta_{\lambda,\gamma}(f_{n,k}) = \theta_{\lambda,\gamma}(g_r) = 0$ but $\theta_{\lambda,\gamma}(f_{p,q}) \neq 0$ for all $0 < p < n$, $0 \leq q \leq p$ (such λ and γ always exist). Then $\theta_{\lambda,\gamma}(\det F^{(n)}) = 0$ and the Verma module $M(\lambda_r)$ has an irreducible subquotient with highest weight $\lambda_r - \tau^*$, where τ^* is one of $n\delta + k\alpha$, $n\delta - k\alpha$. But, this contradicts the Theorem 2 in [**10**]. Therefore, $f_{n,k}$ can not be a factor of $\det F^{(n)}$ if $k \neq 0$ and k is not a divisor of n.

Let now $0 < m < n$, $0 < k < m$, k is not a divisor of m and $f_{m,k}$ is a factor of det $F^{(n)}$. Consider a pair $(\lambda, \gamma) \in H^* \times \mathbf{C}$ such that $\theta_{\lambda,\gamma}(f_{m,k}) = 0$, $\theta_{\lambda,\gamma}(f_{p,q}) \neq 0$ for all $p, q \in \mathbf{Z}$, $0 < p \leq n$, $0 \leq q \leq p$, $(p,q) \neq (m,k)$ and $\theta_{\lambda,\gamma}(g_s) \neq 0$ for all $s \in \mathbf{Z}$. As it was shown above $f_{m,k}$ is not a factor of det $F^{(m)}$ which implies that $\theta_{\lambda_{2m},\gamma}(\det F^{(m)}) \neq 0$. Now it follows from Lemma 3.7 that $M(\lambda,\gamma)_{\lambda-n\beta} = L(\lambda,\gamma)_{\lambda-n\beta}$ and $\theta_{\lambda_{2n},\gamma}(\det F^{(n)}) \neq 0$. But, this contradicts the assumption that $f_{m,k}$ is a factor of det $F^{(n)}$. The Lemma is proved. □

For $n \in \mathbf{Z}$, $n > 0$ denote $X_n = \{0\} \cup \{k \in \mathbf{Z}_+ \mid \frac{n}{k} \in \mathbf{Z}\}$.

Theorem 3.11. *Let $\lambda \in H^*$, $\gamma \in \mathbf{C} - R_\lambda$. $\mathcal{G}$-module $M(\lambda,\gamma)$ is irreducible if and only if $k^2\gamma \neq (n(\lambda(c)+2) - k^2)^2$ for all $n \in \mathbf{Z}$, $n > 0$, $k \in X_n$.*

Proof. Follows from Lemmas 3.9 and 3.10. □

4. Irreducible representations of the Heisenberg subalgebra.

Consider the Heisenberg subalgebra $G = \mathbf{C}c \oplus \sum_{k \in \mathbf{Z}-\{0\}} \mathcal{G}_{k\delta} \subset \mathcal{G}$. It is a $\mathbf{Z}$-graded algebra with $deg\, c = 0$, $deg\, e_{k\delta} = k$. This gradation induces a $\mathbf{Z}$-gradation on the universal enveloping algebra $\mathcal{U}(G) : \mathcal{U}(G) = \bigoplus_{i \in \mathbf{Z}} \mathcal{U}_i$.

In this section we study the irreducible $\mathbf{Z}$-graded G- modules. The central element c acts as a scalar on each such module. In general, we say that a G-module V is a module of level $a \in \mathbf{C}$ if c acts on V as a multiplication by a.

4.1. G-Modules of non-zero level. Let $G_+ = \sum_{k>0} \mathcal{G}_{k\delta}$, $G_- = \sum_{k<0} \mathcal{G}_{k\delta}$. For $a \in \mathbf{C}^* = \mathbf{C} - \{0\}$, let $\mathbf{C}v_a$ be the 1- dimensional $G_\varepsilon \oplus \mathbf{C}c$-module for which $G_\varepsilon v_a = 0$, $cv_a = av_a$, $\varepsilon \in \{+,-\}$. Consider the G-module

$$M^\varepsilon(a) = \mathcal{U}(G) \bigotimes_{\mathcal{U}(G_\varepsilon \oplus \mathbf{C}c)} \mathbf{C}v_a$$

associated with a and ε.

The module $M^\varepsilon(a)$ is a $\mathbf{Z}$-graded: $M^\varepsilon(a) = \sum_{i \in \mathbf{Z}} M^\varepsilon(a)_i$ where

$$M^\varepsilon(a)_i = (\sigma(\mathcal{U}(G_\varepsilon)) \cap \mathcal{U}_i) \otimes v_a.$$

Proposition 4.1.

(i) *The G-module $M^\varepsilon(a)$ is irreducible.*

(ii) *$M^\varepsilon(a)$ is a $\sigma(\mathcal{U}(G_\varepsilon))$-free module.*

(iii) $\dim M^{\varepsilon}(a)_i = P(|\, i\,|)$ *where* $P(n)$ *is a partition function.*

Proof. (ii) and (iii) follow directly from the definition of $M^{\varepsilon}(a)$. Since $a \neq 0$ one can easily show that for any non-zero $u \in \sigma(\mathcal{U}(G_{\varepsilon}))$ there exists $u' \in \mathcal{U}(G_{\varepsilon})$ such that $0 \neq u'uv_a \in M^{\varepsilon}(a)_0$ which implies (i) and completes the proof. □

Lemma 4.2. *If* V *is a* $\mathbf{Z}$*-graded* G*-module of level* $a \in \mathbf{C}^*$ *and* $\dim V_i < \infty$ *for at least one* $i \in \mathbf{Z}$ *then*

$$\text{Spec } e_{\delta}e_{-\delta} \mid_V \subset \{2ma \mid m \in \mathbf{Z}\}.$$

Proof. Let $v \in V_j$ be a non-zero eigenvector of $e_{\delta}e_{-\delta}$ with eigenvalue b and $b \neq 2ma$ for all $m \in \mathbf{Z}$. Since $a \neq 0$, if $e_{n\delta}v = 0$ then $e_{-n\delta}v \neq 0$, $n \in \mathbf{Z} - \{0\}$. Denote $Y = \{n \in \mathbf{Z} - \{0,1\} \mid e_{n\delta}v \neq 0\}$. We may assume without lost of generality that $j = i$ and $|\, Y \cap \mathbf{Z}_+ \,| = \infty$. Elements e_{δ} and $e_{-\delta}$ act injectively on the subspace spanned by $e_{\delta}^k v$, $e_{-\delta}^k v$, $k \in \mathbf{Z}$. Then, for each $k \in Y \cap \mathbf{Z}_+$, $e_{\delta}e_{-\delta}(e_{k\delta}v) = be_{k\delta}v$ and $0 \neq e_{-\delta}^k e_{k\delta}v \in V_i$. Set $w_k = e_{-\delta}^k e_{k\delta}v$. Then $e_{\delta}e_{-\delta}w_k = (b + 2ka)w_k$, $k \in Y \cap \mathbf{Z}_+$. This contradicts the assumption that $\dim V_i < \infty$. Therefore, $b = 2ma$ for some $m \in \mathbf{Z}$. □

For a $\mathbf{Z}$-graded G-module V and $j \geq 0$ denote by $V^{[j]}$ the $\mathbf{Z}$-graded G-module with $(V^{[j]})_i = V_{i-j}$, $i \in \mathbf{Z}$.

We describe now all irreducible $\mathbf{Z}$-graded G-modules of non-zero level with finite-dimensional components.

Proposition 4.3.

(i) *Let* V *be an irreducible* $\mathbf{Z}$*-graded* G*-module of level* $a \in \mathbf{C}^*$ *such that* $\dim V_i < \infty$ *for at least one* $i \in \mathbf{Z}$. *Then* $V^{[j]} \simeq M^{\varepsilon}(a)$ *for some* $\varepsilon \in \{+,-\}$, $j \in \mathbf{Z}$.

(ii) $\text{Ext}^1((M^{\varepsilon}(a))^{[j]}, M^{\varepsilon'}(a)) = 0$ *for any* $j \in \mathbf{Z}$, $\varepsilon, \varepsilon' \in \{+,-\}$.

Proof. (i) By Lemma 4.2 Spec $X \mid_V \subset \{2ma \mid m \in \mathbf{Z}\}$ where X stands for $e_{\delta}e_{-\delta}$. Let $V_i \neq 0$, n be an integer with maximal absolute value such that $2na \in$ Spec $X \mid_{V_i}$ and let $0 \neq v \in V_i$, $Xv = 2nav$. Assume that $n > 0$. Then $e_{k\delta}v = 0$ for all $k > 1$. Indeed, if $e_{k\delta}v \neq 0$ for some $k > 1$ then $X(e_{k\delta}v) = e_{k\delta}Xv = 2na\, e_{k\delta}v$ and $2(n+k)a$ is an eigenvalue of X on V_i which contradicts the assumption. Therefore, $e_{k\delta}v = 0$ for all $k > 1$. Consider the element $\tilde{v} = e_{\delta}^{n-1}v \neq 0$. Then $e_{-\delta}e_{\delta}\tilde{v} = e_{k\delta}\tilde{v} = 0$, $k > 1$. If $e_{\delta}\tilde{v} \neq 0$ then $v_p = e_{\delta}^p\tilde{v} \neq 0$, $e_{k\delta}v_p = 0$ and, hence $e_{-k\delta}v_p \neq 0$ for all $p > 0$, $k > 1$. This would imply that $\dim V_i = \infty$. Therefore, $e_{\delta}\tilde{v} = 0$ and $V = \mathcal{U}(G)\tilde{v} \simeq M^+(a)$ up to a shifting of gradation. If $n \leq 0$ then, clearly,

$V \simeq M^-(a)$ up to a shifting of gradation. Suppose that $V_i = 0$ but, for example, $V_{i-1} \neq 0$. Then $e_{k\delta}v = 0$ for any non-zero $v \in V_{i-1}$ for all $k > 0$ and thus $V = \mathcal{U}(G)v \simeq M^+(a)$ up to a shifting of gradation. This completes the proof of (i).
(ii) Follows from the proof of (i) and Proposition 4.1, (ii). □

Lemma 4.4. *Every finitely-generated* $\mathbf{Z}$*-graded* G*-module* V *of level* $a \in \mathbf{C}^*$ *such that* dim $V_i < \infty$ *for at least one* $i \in \mathbf{Z}$ *has a finite length.*

Proof. If $V_i = 0$ then statement follows from Proposition 4.3. Let $V_i \neq 0$, n be an integer with maximal absolute value such that $2na \in$ Spec $e_\delta e_{-\delta}\,|_{V_i}$ and v be a corresponding eigenvector. It follows from the proof of Proposition 4.3, (i) that $V' = \mathcal{U}(G)v \simeq M^\varepsilon(a)$ up to a shifting of gradation. Consider a G-module $\tilde{V} = V/V'$. Then dim $\tilde{V}_i <$ dim V_i and we can complete the proof by induction on dim V_i. □

Now we are in the position to establish the completely reducibility for for finitely-generated G-modules of non-zero level with finite-dimensional components.

Proposition 4.5. *Every finitely-generated* $\mathbf{Z}$*-graded* G*-module* V *of a non-zero level such that* dim $V_i < \infty$ *for at least one* $i \in \mathbf{Z}$ *is completely reducible.*

Proof. Follows from Lemma 4.4 and Proposition 4.3. □

4.2. ***G*-modules of level zero.** The irreducible G-modules of level zero are classified by V. Chari [**6**]. We recall this classification.

Let $\tilde{G} = \mathcal{U}(G)/\mathcal{U}(G)c$ and let $g : \mathcal{U}(G) \to \tilde{G}$ be the canonical homomorphism. For $r > 0$ consider a $\mathbf{Z}$-graded ring $L_r = \mathbf{C}[t^r, t^{-r}]$, $deg\, t = 1$ and denote by P_r the set of graded ring epimorphisms $\Lambda : \tilde{G} \to L_r$ with $\Lambda(1) = 1$. Let $L_0 = \mathbf{C}$ and $\Lambda_0 : \tilde{G} \to \mathbf{C}$ is a trivial homomorphism such that $\Lambda_0(1) = 1$, $\Lambda_0(g(e_{k\delta})) = 0$ for all $k \in \mathbf{Z} - \{0\}$. Set $P_0 = \{\Lambda_0\}$.

Given $\Lambda \in P_r$, $r \geq 0$ define a G-module structure on L_r by:

$$e_{k\delta}t^{rs} = \Lambda(g(e_{k\delta}))t^{rs}, \quad k \in \mathbf{Z} - \{0\}\,, \quad ct^{rs} = 0, s \in \mathbf{Z}.$$

Denote this G-module by $L_{r,\Lambda}$.

Proposition 4.6.

(i) *Let* V *be an irreducibe* $\mathbf{Z}$*-graded* G*-module of level zero. Then* $V \simeq L_{r,\Lambda}$ *for some* $r \geq 0$, $\Lambda \in P_r$ *up to a shifting of gradation.*

(ii) $L_{r,\Lambda} \simeq L_{r',\Lambda'}$ *if and only if* $r = r'$ *and there exists* $b \in \mathbf{C}^*$ *such that* $\Lambda(g(e_{k\delta})) = b^k\Lambda'(g(e_{k\delta}))$, $k \in \mathbf{Z} - \{0\}$.

Proof. (i) is essentially Lemma 3.6 in [**6**]; (ii) follows from [**6**, Prop. 3.8]. □

Remark 4.7. All the results of Section 4, except Proposition 4.1 (iii), are hold for the Heisenberg subalgebra of an arbitrary Affine Lie Algebra.

5. The category $\tilde{\mathcal{O}}(\alpha)$.

Let $\alpha \in \pi$. Following [**6**] we define category $\tilde{\mathcal{O}}(\alpha)$ to be the category of weight $\mathcal{G}$-modules M satisfying the condition that there exist finitely many elements $\lambda_1, ..., \lambda_r \in H^*$ such that $P(M) \subseteq \bigcup_{i=1}^{r} D(\lambda_i)$ where

$$D(\lambda_i) = \{\lambda_i + k\alpha + n\delta \mid k, n \in \mathbf{Z},\ k \leq 0\}.$$

Notice that the trivial action of c, as in [**6**], is no longer required. It is clear that $\tilde{\mathcal{O}}(\alpha)$ is closed under the operations of taking submodules, quotients and finite direct sums.

Denote $B_\alpha = \sum_{n \in \mathbf{Z}} \mathcal{G}_{\alpha + n\delta}$. Then $\mathcal{G} = B_{-\alpha} \oplus (H + G) \oplus B_\alpha$.

Let V be an irreducible $\mathbf{Z}$-graded G-module of level $a \in \mathbf{C}$ and let $\lambda \in H^*$, $\lambda(c) = a$. Then we can define a $B = (H + G) \oplus B_\alpha$-module structure on V by setting: $hv_i = (\lambda + i\delta)(h)v_i$, $B_\alpha v_i = 0$ for all $h \in H$, $v_i \in V_i$, $i \in \mathbf{Z}$.

Consider the $\mathcal{G}$-module

$$M_\alpha(\lambda, V) = \mathcal{U}(\mathcal{G}) \bigotimes_{\mathcal{U}(B)} V$$

associated with α, λ, V.

Proposition 5.1.

(i) *The $\mathcal{G}$-module $M_\alpha(\lambda, V)$ is $S(B_{-\alpha})$- free.*

(ii) *$M_\alpha(\lambda, V)$ has a unique irreducible quotient $L_\alpha(\lambda, V)$.*

(iii) *$P(M_\alpha(\lambda, V)) = (D(\lambda) - \{\lambda + n\delta \mid n \in \mathbf{Z}\}) \cup P(V) \subset D(\lambda)$.*

(iv) *$M_\alpha(\lambda, V) \simeq M_{\alpha'}(\lambda', V')$ if and only if $\alpha' \in \{\alpha + n\delta \mid n \in \mathbf{Z}\}$ and there exists $i \in \mathbf{Z}$ such that $\lambda = \lambda' + i\delta$ and $V^{[i]} \simeq V'$ as graded G-modules.*

Proof. Follows from the construction of $\mathcal{G}$- module $M_\alpha(\lambda, V)$. □

Now we describe the classes of isomorphisms of irreducible modules in $\tilde{\mathcal{O}}(\alpha)$.

Proposition 5.2.

(i) *Let $\tilde{V}$ be an irreducible object in $\tilde{\mathcal{O}}(\alpha)$. Then there exist $\lambda \in H^*$ and an irreducible G- module V such that $\tilde{V} \simeq L_\alpha(\lambda, V)$.*

(ii) $L_\alpha(\lambda, V) \simeq L_\alpha(\lambda', V')$ *if and only if there exists* $i \in \mathbf{Z}$ *such that* $\lambda = \lambda' + i\delta$ *and* $V^{[i]} \simeq V'$ *as graded G-modules.*

Proof. One can see that $\tilde{V}$ contains a non-zero element $v \in \tilde{V}_\lambda$ such that $B_\alpha v = 0$. Then $V = \mathcal{U}(G)v$ is an irreducible $\mathbf{Z}$-graded G- module and $\tilde{V} \simeq \mathcal{U}(B_{-\alpha})V$. This implies that $\tilde{V}$ is a homomorphic image of $M_\alpha(\lambda, V)$ and, therefore, is isomorphic to $L_\alpha(\lambda, V)$, which proves (i). Part (ii) follows from Proposition 5.1, (iv). □

Lemma 5.3. *If* $0 < \dim L_\alpha(\lambda, V)_\mu < \infty$ *for some* $\mu \in H^*$ *then* $\dim V_i < \infty$ *for all* $i \in \mathbf{Z}$.

Proof. If $\lambda(c) = 0$ then $V^{[j]} \simeq L_{r,\Lambda}$ for some $r \geq 0$, $\Lambda \in P_r$, $j \in \mathbf{Z}$ by Proposition 4.6 and, hence $\dim V_i \leq 1$ for all $i \in \mathbf{Z}$. Let $\lambda(c) = a \in \mathbf{C}^*$ and $V^{[j]} \simeq M^\varepsilon(a)$, for any $j \in \mathbf{Z}$, $\varepsilon \in \{+,-\}$. By Proposition 4.3, (i), $\dim V_i = \infty$ for all i. If $a \in \mathbf{Q}_+$ ($a \notin \mathbf{Q}_+$ respectively) then $\lambda(h_\alpha) - na \notin \mathbf{Z}_+$ for all integer $n \geq n_0$ ($n \leq n_0$ respectively) and for some $n_0 \in \mathbf{Z}$. Thus, $e_{\alpha-n\delta}e_{-\alpha+n\delta}$ acts injectively on $L_\alpha(\lambda, V)$ for all $n \geq n_0$ ($n \leq n_0$ respectively) which implies that $\dim L_\alpha(\lambda, V)_\mu = \infty$. But, this contradicts the assumption. We conclude that $V^{[j]} \simeq M^\varepsilon(a)$ for some $j \in \mathbf{Z}$, $\varepsilon \in \{+,-\}$ and $\dim V_i < \infty$ for all $i \in \mathbf{Z}$. □

Theorem 5.4. *Let* $\tilde{V} \in \tilde{\mathcal{O}}(\alpha)$ *be an irreducible.*

(i) [**6**] *If* $\tilde{V}$ *is of level zero then* $\tilde{V} \simeq L_\alpha(\lambda, L_{r,\Lambda})$ *for some* $\lambda \in H^*$, $\lambda(c) = 0$, $r \geq 0$, $\Lambda \in P_r$.

(ii) *If* $\tilde{V}$ *is of level* $a \in \mathbf{C}^*$ *and* $\dim \tilde{V}_\mu < \infty$ *for at least one* $\mu \in P(\tilde{V})$ *then* $\tilde{V} \simeq L_\alpha(\lambda, M^\varepsilon(a))$ *for some* $\lambda \in H^*$, $\lambda(c) = a$, $\varepsilon \in \{+,-\}$.

Proof. (i) follows from Propositions 5.2 and 4.6, while (ii) follows from Lemma 5.3, Propositions 5.2 and 4.3. □

In some cases we can describe the structure of modules $L_\alpha(\lambda, V)$.

Let $\lambda(c) = 0$, $r = 0$, $\Lambda = \Lambda_0$, $L_{0,\Lambda_0} \simeq \mathbf{C}$. Set $\tilde{M}(\lambda) = M_\alpha(\lambda, \mathbf{C})$. Notice that $\tilde{M}(\lambda) \simeq S(B_{-\alpha})$ as vector spaces and, therefore, $P(\tilde{M}(\lambda)) = \{\lambda - n\alpha + k\delta \mid k, n \in \mathbf{Z},\, n > 0\} \cup \{\lambda\}$ and

$$\dim \tilde{M}(\lambda)_{\lambda - n\alpha + k\delta} = \infty, n > 1, \dim \tilde{M}(\lambda)_\lambda = \dim \tilde{M}(\lambda)_{\lambda-\alpha+k\delta} = 1, k \in \mathbf{Z}.$$

Proposition 5.5.

(i) $L_\alpha(\lambda, \mathbf{C}) \simeq \tilde{M}(\lambda)$ *if and only if* $\lambda(h_\alpha) \neq 0$.

(ii) *If* $\lambda(h_\alpha) = 0$ *then* $L_\alpha(\lambda, \mathbf{C})$ *is a trivial one-dimensional module.*

Proof. Proposition follows from [**7**, Proposition 6.2] and is also proved in [**8**]. □

Let $\lambda(c) = a \in \mathbf{C}^*$. Set $M^\varepsilon(\lambda, a) = M_\alpha(\lambda, M^\varepsilon(a))$. We have

$$P(M^\varepsilon(\lambda, a)) = \{\lambda - k\alpha + n\delta \mid k, n \in \mathbf{Z},\ k > 0\} \cup \{\lambda - \varepsilon n\delta \mid n \in \mathbf{Z}_+\}$$

and

$$\dim M^\varepsilon(\lambda, a)_{\lambda - k\alpha + n\delta} = \infty,\ k > 0, n \in \mathbf{Z}, \dim M^\varepsilon(\lambda, a)_{\lambda - \varepsilon n\delta} = P(n),\ n \in \mathbf{Z}_+.$$

Proposition 5.6. [**8**, **9**] $L_\alpha(\lambda, M^\varepsilon(a)) \simeq M^\varepsilon(\lambda, a)$.

Recall, that $\mathcal{G}$-module $\tilde{V}$ is called *integrable* if $e_{\pm\alpha}$ and $e_{\pm(\delta-\alpha)}$ act locally nilpotently on $\tilde{V}$. All irreducible integrable $\mathcal{G}$- modules in $\tilde{\mathcal{O}}(\alpha)$ of level zero were classified in [**6**]. In fact, they are the only integrable modules in $\tilde{\mathcal{O}}(\alpha)$.

Corollary 5.7. *If $\tilde{V}$ is irreducible integrable $\mathcal{G}$-module in $\tilde{\mathcal{O}}(\alpha)$ then $\tilde{V}$ is of level zero.*

Proof. Suppose $\tilde{V}$ is of level $a \neq 0$. Since $\tilde{V}$ is integrable, it follows from Proposition 5.6 that $\tilde{V} \neq L_\alpha(\lambda, M^\varepsilon(a))$, $\varepsilon \in \{+, -\}$. Then $\tilde{V} \simeq L_\alpha(\lambda, V)$ and for any $k \in \mathbf{Z}_+$ there exist $i > k$, $j < -k$ such that $V_i \neq 0$, $V_j \neq 0$. Now the same arguments as in the proof of Lemma 5.3 show that $e_{-\alpha}$ and $e_{\delta-\alpha}$ are not locally nilpotent on such module and, therefore, $\tilde{V}$ has a zero level. □

Remark. (i) The structure of modules $L_\alpha(\lambda, L_{r,\Lambda})$, $r > 0$ is unclear is general. Some examples were considered in [**1**, **12**].
(ii) Most of the results of Section 5 can be generalized for an arbitrary Affine Lie Algebra [**6**, **7**, **12**].

6. Non-dense $\mathcal{G}$-modules.

Definition. An irreducible weight $\mathcal{G}$-module V is called dense if $P(V) = \lambda + Q$ for some $\lambda \in H^*$ and non-dense otherwise.

In this section we classify all irreducible non-dense $\mathcal{G}$- modules with at least one finite-dimensional weight subspace. Our main result is the following Theorem.

Theorem 6.2. *If $\tilde{V}$ is an irreducible non-dense $\mathcal{G}$-module with at least one finite-dimensional weight subspace then $\tilde{V}$ belongs to one of the following disjoint classes:*

(i) *highest weight modules with respect to some choice of π;*

(ii) $L^\varepsilon_\alpha(\lambda, \gamma)$, $\alpha \in \Delta^{re}$, $\lambda \in H^*$, $\gamma \in \mathbf{C} - R_\lambda$, $\varepsilon \in \{+, -\}$;

(iii) $L_\alpha(\lambda, L_{r,\Lambda})$, $\alpha \in \Delta^{re}$, $\lambda \in H^*$, $\lambda(c) = 0$, $r \geq 0$, $\Lambda \in P_r$.

(iv) $L_\alpha(\lambda, M^\varepsilon(a))$, $\alpha \in \Delta^{re}$, $\lambda \in H^*$, $a \in \mathbf{C}^*$, $\lambda(c) = a$, $\varepsilon \in \{+, -\}$.

Moreover, we can describe the irreducible $\mathcal{G}$-modules of non-zero level with finite-dimensional weight subspaces.

Theorem 6.3. *Let $\tilde{V}$ be an irreducible $\mathcal{G}$-module of level $a \neq 0$ with all finite-dimensional weight subspaces. Then $\tilde{V} \simeq L^\varepsilon_\alpha(\lambda, \gamma)$ for some $\alpha \in \Delta^{re}$, $\lambda \in H^*$, $\lambda(c) = a$, $\gamma \in \mathbf{C}$, $\varepsilon \in \{+, -\}$.*

Remark 6.4. Theorems 6.2, 6.3 imply that in order to complete the classification of all weight irreducible $\mathcal{G}$-modules one has to study the following classes:

- (i) Modules of type $L_\alpha(\lambda, V)$ where V is a graded irreducible G-module of non-zero level with all infinite- dimensional components.
- (ii) Dense $\mathcal{G}$-modules of zero level.
- (iii) Dense $\mathcal{G}$-modules of non-zero level with an infinite-dimensional weight subspace.

These classification problems are still open.

The proof of Theorem 6.2 is based on some preliminary results. We start with the following Definition.

Definition 6.5. A subset $P \subset \Delta$ is called closed if $\beta_1, \beta_2 \in P$, $\beta_1 + \beta_2 \in \Delta$ imply $\beta_1 + \beta_2 \in P$. A closed subset $P \subset \Delta$ is called a partition if $P \cap -P = \emptyset$, $P \cup -P = \Delta$.

Lemma 6.6. *Let P be a partition, $P \ni \delta$, $P^{re} = P \cap \Delta^{re}$, $\beta \in \Delta^{re}$.*

- (i) *If $| P^{re} \cap \{\beta + k\delta \mid k \in \mathbf{Z}_+\} | < \infty$ or $| P^{re} \cap \{-\beta + k\delta \mid k \in \mathbf{Z}\} | < \infty$ then $P^{re} = \{\varphi + n\delta \mid n \in \mathbf{Z}\}$ for some $\varphi \in \Delta^{re}$.*
- (ii) *If $| P^{re} \cap \{\beta + k\delta \mid k \in \mathbf{Z}\} | = | P^{re} \cap \{-\beta + k\delta \mid k \in \mathbf{Z}_+\} | = \infty$ then $P = \Delta_+(\tilde{\pi})$ for some basis $\tilde{\pi}$ of Δ.*

Proof. Recall that $\Delta = \{\pm\beta + k\delta \mid k \in \mathbf{Z}\} \cup \{n\delta \mid n \in \mathbf{Z} - \{0\}\}$. It follows from [**7**] that there exist $w \in W$ and $\beta' \in \Delta^{re}$ such that

$$wP = \{\beta' + k\delta \mid k \in \mathbf{Z}\} \cup \{k\delta \mid k > 0\}$$

or

$$wP = \{\beta' + n\delta, -\beta' + k\delta \mid n \geq 0,\, k > 0\} \cup \{k\delta \mid k > 0\} = \Delta_+(\pi')$$

where $\pi' = \{\beta', \delta - \beta'\}$. Then

$$P = \{w^{-1}\beta' + k\delta \mid k \in \mathbf{Z}\} \cup \{k\delta \mid k > 0\}$$

or $P = \Delta_+(w^{-1}\pi')$. This implies the statement of Lemma. $\square$

Definition 6.7. A non-zero element v of a $\mathcal{G}$-module V is called admissible if $\mathcal{N}^{\varepsilon}_{\varphi}v = 0$ or $B_{\varphi}v = 0$, for some $\varphi \in \Delta^{re}$, $\varepsilon \in \{+,-\}$.

Lemma 6.8. *If the $\mathcal{G}$-module V contains a non-zero vector $v \in V_\lambda$ such that $e_\varphi v = 0$ and $\lambda + k\delta \notin P(V)$ for some $\varphi \in \Delta^{re}$, $k \in \mathbf{Z} - \{0\}$ then V contains an admissible vector.*

Proof. We will assume that $k > 0$. The case $k < 0$ can be considered analogously. We prove the Lemma by the induction on k. Let $k = 1$. Then we have $e_{\varphi+m\delta}v = e_\delta v = 0$ for all $m \geq 0$. If $e_{\varphi-i\delta}v = 0$ for all $i > 0$ then $B_\varphi v = 0$ and v is admissible. Let $e_{\varphi-n\delta}v \neq 0$ for some $n > 0$ and $e_{\varphi-i\delta}v = 0$, $0 \leq i < n$. Set $\tilde{v} = e_{\varphi-n\delta}v \neq 0$. Then $e_{\varphi-i\delta}\tilde{v} = e_\delta\tilde{v} = e_{-\varphi+(n+1)\delta}\tilde{v} = 0$, $i < n$ and, thus, $e_\psi\tilde{v} = 0$ for any $\psi \in \tilde{P} = \{\varphi - i\delta, -\varphi + (n+j+1)\delta, (j+1)\delta \mid i < n, j \geq 0\}$. One can see that $\tilde{P} \cup \{-\varphi + n\delta\}$ is a partition and $\tilde{P} = \Delta_+(\tilde{\pi}) - \{\varphi'\}$ for some $\varphi' \in \Delta^{re}$, $\tilde{\pi} = \{\varphi', \delta - \varphi'\}$, by Lemma 6.6. Hence, $\mathcal{N}^+_{\varphi'}\tilde{v} = 0$ which proves the Lemma for $k = 1$.

Assume now that the Lemma is proved for all $0 < k' < k$ and consider two cases:

(i) There exists $n \in \mathbf{Z}$, $0 < n < k$ such that $e_{\varphi+i\delta}v = 0$ for all $0 \leq i < n$ but $e_{\varphi+n\delta}v \neq 0$. Then $e_{\varphi+i\delta}\tilde{v} = e_{-\varphi+(k-n)\delta}\tilde{v} = 0$, $0 \leq i < n$ where $\tilde{v} = e_{\varphi+n\delta}v$ and $e_{-\varphi+(k-n)\delta}\tilde{v} \in V_{\lambda+k\delta} = 0$. If $k - n = 1$ or $k - n > 1$ and $e_{-\varphi+\delta}\tilde{v} = 0$ then $\mathcal{N}_+v = 0$ and $\tilde{v}$ is admissible. Let $k - n > 1$ and $v' = e_{-\varphi+\delta}\tilde{v} \neq 0$. Then $v' \in V_{\lambda'}$, $e_{\varphi'}v' = 0$, $\lambda' + (k-n-1)\delta \notin P(V)$ where $\lambda' = \lambda + (n+1)\delta$, $\varphi' = -\varphi + (k-n)\delta$ and V has an admissible element by the induction hypotheses.

(ii) Let $e_{\varphi+i\delta}v = 0$ for all $0 \leq i \leq k$. Since $e_{k\delta}v = 0$ we have $e_{\varphi+i\delta}v = 0$ for all $i \geq 0$. If $\tilde{v}_m = e_{m\delta}v \neq 0$ for some $0 < m < k$ then $\tilde{v}_m \in V_{\lambda'}$, $\lambda' = \lambda + m\delta$, $e_\varphi\tilde{v}_m = 0$, $\lambda' + (k-m)\delta \notin P(V)$ and we can apply induction. Assume that $\tilde{v}_m = 0$ for all $0 < m < k$. Then we have $e_{\varphi+i\delta}v = e_{m\delta}v = 0$, $i \geq 0$, $0 < m \leq k$. If $e_{\varphi-j\delta}v = 0$ for all $j > 0$ then $B_\varphi v = 0$ and v is admissible. Otherwise, let n be a minimal positive integer such that $\tilde{v} = e_{\varphi-n\delta}v \neq 0$. Then $e_{\varphi-j\delta}\tilde{v} = e_{-\varphi+(n+k)\delta}\tilde{v} = e_{i\delta}\tilde{v} = 0$, $i \geq 0$, $j < n$. Assume that $e_{-\varphi+(n+1)\delta}\tilde{v} = 0$. We have $e_\psi\tilde{v} = 0$ for any $\psi \in \tilde{P} = \{\varphi - j\delta, -\varphi + (n+m)\delta, m\delta \mid j < n, m > 0\}$. The set $\tilde{P} \cup \{-\varphi + n\delta\}$ is a partition, $|\ \tilde{P}^{re} \cap \{\varphi + i\delta \mid i \geq 0\}\ | = |\ \tilde{P}^{re} \cap \{-\varphi + i\delta \mid i > 0\}\ | = \infty$ and, therefore, $\tilde{P} = \Delta_+(\tilde{\pi}) - \{\varphi'\}$ for some $\varphi' \in \Delta^{re}$, $\tilde{\pi} = \{\varphi', \delta - \varphi'\}$ by Lemma 6.6. We conclude that $\mathcal{N}^+_{\varphi'}\tilde{v} = 0$ and $\tilde{v}$ is admissible. Finally, suppose that $v' = e_{-\varphi+(n+1)\delta}\tilde{v} \neq 0$. Then $v' \in V_{\lambda'}$, $e_\varphi v' = 0$, $\lambda' + (k-1)\delta \notin P(V)$ where λ' stands for $\lambda + \delta$ and, thus V has an admissible element by the assumption of induction. This completes the proof of Lemma. $\square$

Proposition 6.9. *Let V be an irreducible non-dense $\mathcal{G}$-module. Then V contains an admissible element.*

Proof. Let $\lambda \in P(V)$ and $\lambda + \varphi \notin P(V)$ for some $\varphi \in \Delta$. We can assume that $\varphi \in \Delta^{re}$. Indeed, let $\varphi = \delta$. If $e_\alpha v = e_{\delta-\alpha} v = 0$ for some $0 \neq v \in V_\lambda$, $\alpha \in \Delta^{re}$ then V is a highest weight module with respect to $\{\alpha, \delta - \alpha\}$ and v is admissible. If, for example, $e_\alpha v \neq 0$ then $\lambda' = \lambda + \alpha \in P(V)$ and $\lambda' + (\delta - \alpha) \notin P(V)$. Hence, we can assume that $\lambda + \varphi \notin P(V)$, $\varphi \in \Delta^{re}$. Let $0 \neq v \in V_\lambda$. If $v' = e_{\varphi - n\delta} v \neq 0$ for some $n \in \mathbf{Z} - \{0\}$ then $e_\varphi v' = 0$, $v' \in V_{\tilde{\lambda}}$, $\tilde{\lambda} = \lambda + \varphi - n\delta$, $\tilde{\lambda} + n\delta \notin P(V)$ and Proposition follows from Lemma 6.8. If $e_{\varphi - n\delta} v = 0$ for all $n \in \mathbf{Z}$ then $B_\varphi v = 0$ and v is admissible. □

Corollary 6.10. *If $\tilde{V}$ is an irreducible non-dense $\mathcal{G}$-module then either $\tilde{V} \simeq L^\varepsilon_\alpha(\lambda, \gamma)$ or $\tilde{V} \simeq L_\alpha(\lambda, V)$ for some $\alpha \in \Delta^{re}$, $\lambda \in H^*$, $\gamma \in \mathbf{C}$, $\varepsilon \in \{+, -\}$ and irreducible G- module V.*

Proof. Follows from Proposition 6.9, Corollary 3.3 (i) and Proposition 5.2. □

Now Theorem 6.2 follows from Corollary 6.6 and Theorem 5.4.

Proof of Theorem 6.3. Let $\mu \in P(\tilde{V})$. Consider the $\mathcal{G}$-submodule $V = \mathcal{U}(G)\tilde{V}_\mu \subset \tilde{V}$. Then it follows from Proposition 4.5 that V is completely reducible and moreover each irreducible component is isomorphic to $M^\varepsilon(a)$, $\varepsilon \in \{+, -\}$ up to a shifting of gradation by Proposition 4.3, (i). Denote by V^+ the sum of all irreducible components of V isomorphic to $M^+(a)$ and assume that $V^+ \neq 0$. Let $0 \neq v \in V^+ \cap \tilde{V}_\chi$, $\chi \in P(\tilde{V})$ and $V^+ \cap \tilde{V}_{\chi+\delta} = 0$. We will show that for any $\alpha \in \Delta^{re}$ there exists $m_\alpha \in \mathbf{Z}_+$ such that $e_{\alpha + m\delta} v = 0$ for all $m \geq m_\alpha$. Indeed, let $v_0 = e_\alpha v \neq 0$. Consider the G-module $\mathcal{U}(G)v_0$ which is again completely reducible by Proposition 4.5. If $e_{k\delta} v \neq 0$ for all $k > 0$ then $v_k = e^k_\delta v_0 \neq 0$ for all $k > 0$. But, for big enough k, v_k will belong to the direct sum of irreducible components of $\mathcal{U}(G)v_0$ each of which is isomorphic to $M^-(a)$ up to a shifting of gradation. This contradicts Proposition 4.1, (ii), since $e^2_\delta v_k = 2^{k+2} e_{\alpha+(k+2)\delta} v = 2e_{2\delta} v_k$. Thus, there exists $m_\alpha \geq 0$ such that $e_{\alpha + m_\alpha \delta} v = 0$ and, therefore, $e_{\alpha + m\delta} v = 0$ for any $m \geq m_\alpha$.

Suppose that $\chi + \delta \in P(\tilde{V})$. Since $\tilde{V}$ is irreducible there exists $0 \neq u \in \mathcal{U}(\mathcal{G})$ such that $0 \neq uv \in \tilde{V}_{\chi+\delta}$. It follows from the discussion above that $e_{n\delta} uv = 0$ for big enough $n \in \mathbf{Z}_+$. The G-submodule $V' = \mathcal{U}(G)uv$ is completely reducible by Proposition 4.5 and since $V^+ \cap \tilde{V}_{\chi+\delta} = 0$, any irreducible component $L \subset V'$ such that $L \cap \tilde{V}_{\chi+\delta} \neq 0$ is isomorphic to $M^-(a)$ up to a shifting of gradation. Hence, $e_{n\delta} \tilde{v} \neq 0$ for any non-zero $\tilde{v} \in V' \cap \tilde{V}_{\chi+\delta}$ by Proposition 4.1, (ii) and $e_{n\delta} uv \neq 0$ in particular. This contradiction implies that $\chi + \delta \notin P(\tilde{V})$ and therefore $\tilde{V}$ is a non-dense

$\mathcal{G}$-module. Applying Theorem 6.2 we conclude that $\tilde{V} \simeq L_\alpha^\varepsilon(\lambda, \gamma)$ for some $\alpha \in \Delta^{re}$, $\lambda \in H^*$, $\lambda(c) = a$, $\gamma \in \mathbf{C}$, $\varepsilon \in \{+, -\}$ which completes the proof. □

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