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DISTORTION OF BOUNDARY SETS UNDER INNER FUNCTIONS (II)

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We present a study of the metric transformation properties of inner functions of several complex variables. Along the way we obtain fractional dimensional ergodic properties of classical inner functions.

1. Introduction.

An inner function is a bounded holomorphic function from the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ into the unit disk Δ of the complex plane such that the radial boundary values have modulus 1 almost everywhere. If E is a non empty Borel subset of $\partial\Delta$, we denote by $f^{-1}(E)$ the following subset of the unit sphere $\mathbb{S}_n$ of $\mathbb{C}^n$

$$f^{-1}(E) = \left\{\xi \in \mathbb{S}_n : \lim_{r\to 1} f(r\xi) \text{ exist and belongs to } E\right\} .$$

The classical lemma of Löwner, see e.g. [**R**, p. 405], asserts that inner functions f, with $f(0) = 0$, are measure preserving transformations when viewed as mappings from $\mathbb{S}_n$ to $\partial\Delta$, i.e. if E is a Borel subset of $\partial\Delta$ then $|f^{-1}(E)| = |E|$, where in each case $|\cdot|$ means the corresponding normalized Lebesgue measure.

In this paper we extend this result to fractional dimensions as follows:

Theorem 1. *If f is inner in the unit disk Δ, $f(0) = 0$, and E is a Borel subset of $\partial\Delta$, we have:*

$$\operatorname{cap}_\alpha\left(f^{-1}(E)\right) \geq \operatorname{cap}_\alpha(E)\,, \qquad 0 \leq \alpha < 1\,.$$

Moreover, if E is any Borel subset of $\partial\Delta$ with $\operatorname{cap}_\alpha(E) > 0$, equality holds if and only if either f is a rotation or $\operatorname{cap}_\alpha(E) = \operatorname{cap}_\alpha(\partial\Delta)$.

Moreover, it is well known, see [**N**], that if f is not a rotation then f is ergodic, i.e., there are no nontrivial sets A, with $f^{-1}(A) = A$ except for a set of Lebesgue measure zero. This also has a fractional dimensional parallel.

Corollary. *With the hypotheses of Theorem 1, if f is not a rotation and if the symmetric difference between E and $f^{-1}(E)$ has zero α-capacity, then either* $\operatorname{cap}_\alpha(E) = 0$ *or* $\operatorname{cap}_\alpha(E) = \operatorname{cap}_\alpha(\partial\Delta)$.

Theorem 2. *If f is inner in the unit ball of $\mathbb{C}^n$, $f(0) = 0$, and E is a Borel subset of $\partial\Delta$, we have:*

$$\operatorname{cap}_{2n-2+\alpha}\left(f^{-1}(E)\right) \geq K(n,\alpha)^{-1} \operatorname{cap}_\alpha(E)\,, \qquad 0 < \alpha < 1\,,$$

and

$$\frac{1}{\operatorname{cap}_{2n-2}\left(f^{-1}(E)\right)} \leq 1 + (2n-2)\log\frac{1}{\operatorname{cap}_0(E)}\,, \qquad (n > 1)\,.$$

Corollary. *In particular, for any inner function f, we have that*

$$\operatorname{Dim}\left(f^{-1}(E)\right) \geq 2n - 2 + \operatorname{Dim}(E)\,,$$

where Dim *denotes Hausdorff dimension.*

Here $\operatorname{cap}_\alpha$ and cap_0 denote, respectively, α-dimensional Riesz capacity and logarithmic capacity. We refer to [**C**], [**KS**] and [**L**] for definitions and basic background on capacity.

For background and some applications of these results we refer to [**FP**] where it is shown that Theorem 1 holds with some constants depending on α.

The outline of this paper is as follows: In Section 2 we obtain an integral expression for the α-energy that is used in Section 3, where Theorems 1 and 2 are proved. Section 4 contains some further results for the case $n = 1$. In Section 5, we prove an analogous distortion theorem, with Hausdorff measures replacing capacities. Section 6 discusses an open question and some partial results concerning distortion of subsets of the disc.

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2. An integral expression for the α-energy.

In this section we obtain an expression of the α-energy of a signed measure μ in Σ_{N-1} (the unit sphere of $\mathbb{R}^N$) as an L^2-norm of its Poisson extension. This approach is due to Beurling [**B**].

If μ is a signed measure on Σ_{N-1}, and $0 \le \alpha < N-1$, then the α-energy $I_\alpha(\mu)$ of μ is defined as

$$I_\alpha(\mu) = \iint_{\Sigma_{N-1}\times\Sigma_{N-1}} \Phi_\alpha(|x-y|)\, d\mu(x)\, d\mu(y)\,,$$

where

$$\Phi_\alpha(t) = \begin{cases} \log \dfrac{1}{t}\,, & \text{if } \alpha = 0\,, \\ \dfrac{1}{t^\alpha}\,, & \text{if } 0 < \alpha < N-1\,. \end{cases}$$

Recall that if E is a closed subset of Σ_{N-1}, then

$$(\operatorname{cap}_\alpha(E))^{-1} = \inf\{I_\alpha(\mu) :\ \mu \text{ a probability measure supported on } E\}\,,$$

for $0 < \alpha < N-1$,

$$\log \frac{1}{\operatorname{cap}_0(E)} = \inf\{I_0(\mu) :\ \mu \text{ a probability measure supported on } E\}\,,$$

and that the infimum is attained by a unique probability measure μ_e which is called the *equilibrium distribution* of E.

If E is any Borel subset of Σ_{N-1}, then the α-capacity of E is defined as

$$\operatorname{cap}_\alpha(E) = \sup\{\operatorname{cap}_\alpha(K) :\ K \subset E,\ K \text{ compact}\}\,.$$

We recall Choquet's theorem that all Borel sets are *capacitables*, i.e.

$$\operatorname{cap}_\alpha(E) = \inf\{\operatorname{cap}_\alpha(O) : E \subset O,\ O \text{ open}\}\,.$$

As we shall remark later on, for a general Borel set E of Σ_{N-1}, one has

$$\frac{1}{\operatorname{cap}_\alpha(E)} = \inf\{I_\alpha(\mu) :\ \mu \text{ a probability measure},\ \mu(E) = 1\}\,,$$

and analogously for the logarithmic capacity.

We first need to obtain the expansion of the integral kernel Φ_α in terms of the spherical harmonics. We refer to [**SW**, Chap. IV] for details about spherical harmonics; we shall follow its notations.

Let $\mathcal{H}_k$ be the real vector space of the spherical harmonics of degree k in $\mathbb{R}^N$ $(N > 1)$. If a_k is the dimension of $\mathcal{H}_k$, we have

$$a_0 = 1, \quad a_1 = N, \quad a_k = \frac{N+2k-2}{k}\binom{N+k-3}{k-1}. \qquad \text{[\textbf{SW}, p. 145]}$$

If Σ_{N-1} denotes the unit sphere of $\mathbb{R}^N$, the space $L^2(\Sigma_{N-1}, d\xi)$ can be decomposed as

$$L^2(\Sigma_{N-1}, d\xi) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k \,,$$

where $d\xi$ is the usual Lebesgue measure (not normalized).

If ξ, η belongs to Σ_{N-1}, $Z_\eta^k(\xi)$ will denote the zonal harmonic of degree k with pole η, and if $\{Y_1^k, \dots, Y_{a_k}^k\}$ is any orthonormal basis of $\mathcal{H}_k$, we have

$$Z_\eta^k(\xi) = \sum_{m=1}^{a_k} Y_m^k(\xi) Y_m^k(\eta) = Z_\xi^k(\eta). \qquad \text{[\textbf{SW}, p. 143]}$$

The zonal harmonics can be expressed in terms of the ultraspherical (or Gegenbauer) polynomials P_k^λ which are defined by the formula

$$\left(1 - 2rt + r^2\right)^{-\lambda} = \sum_{k=0}^{\infty} P_k^\lambda(t) r^k \,,$$

where $|r| < 1$, $|t| \le 1$ and $\lambda > 0$.

We have [**SW**, p. 149], if $N > 2$,

$$Z_\eta^k(\xi) = C_{k,N} P_k^{(N-2)/2}(\xi \cdot \eta) \,.$$

It is easy to compute the constants $C_{k,N}$. First, if ω_{N-1} denotes the Lebesgue measure of Σ_{N-1}, then

$$\left\| Z_\eta^k \right\|_2^2 = \frac{a_k}{\omega_{N-1}}, \qquad \text{[\textbf{SW}, p. 144]}$$

while, on the other hand,

$$\begin{aligned} \frac{a_k}{\omega_{N-1}} &= C_{k,N}^2 \int_{\Sigma_{N-1}} \left| P_k^{(N-2)/2}(\xi \cdot \eta) \right|^2 \, d\xi \\ &= C_{k,N}^2 \, \omega_{N-2} \int_{-1}^{1} \left| P_k^{(N-2)/2}(t) \right|^2 \left(1 - t^2\right)^{(N-3)/2} \, dt \,. \end{aligned}$$

Now, the polynomials $P_k^{(N-2)/2}(t)$ form an orthogonal basis of

$$L^2 \left([-1,1], \; (1 - t^2)^{(N-3)/2} \, dt \right)$$

[**SW**, p. 151], [**AS**, p. 774], and

$$\left\| P_k^{(N-2)/2} \right\|_2^2 = \frac{\pi \, 2^{4-N} \Gamma(k + N - 2)}{k! \, (2k + N - 2) \Gamma \left(\dfrac{N-2}{2} \right)^2}, \qquad \text{[\textbf{AS}, p. 774]}$$

where $\Gamma(\cdot)$ denotes the Euler's Gamma function, and, therefore

$$C_{k,N}^2 = \frac{a_k}{\omega_{N-1}\,\omega_{N-2}} \left\| P_k^{(N-2)/2} \right\|_2^{-2} = \frac{(N+2k-2)^2}{16\,\pi^N} \Gamma\left(\frac{N-2}{2}\right)^2 .$$

Hence

$$C_{k,N} = \frac{N+2k-2}{4\pi^{N/2}} \Gamma\left(\frac{N-2}{2}\right),$$

and

$$Z_\eta^k(\xi) = \frac{N+2k-2}{4\pi^{N/2}} \Gamma\left(\frac{N-2}{2}\right) P_k^{(N-2)/2}(\xi\cdot\eta) .$$

The case $N = 2$ is slightly different. In this case we can take $P_k^0 \equiv T_k$, the Chebyshev's polynomials defined in $[-1,1]$ by

$$T_k(\cos\theta) = \cos k\theta .$$

It is known that these polynomials form an orthogonal basis of

$$L^2\left([-1,1],\ (1-t^2)^{-1/2}\,dt\right) .$$

In this particular case, if $\xi = e^{i\theta}, \eta = e^{i\psi}$, then $\xi\cdot\eta = \cos(\theta-\psi)$, and

$$\begin{aligned} Z_\eta^k(\xi) &= \frac{1}{\pi}\cos k(\theta-\psi) = \frac{1}{\pi} T_k(\cos(\theta-\psi)) \\ &= \frac{1}{\pi} P_k^0(\xi\cdot\eta), \qquad k = 1,2,\dots, \\ Z_\eta^0(\xi) &= \frac{1}{2\pi} = \frac{1}{2\pi} P_0^0(\xi\cdot\eta) . \end{aligned}$$

Therefore,

$$C_{k,2} = \begin{cases} \dfrac{1}{\pi}, & \text{if } k > 0, \\ \dfrac{1}{2\pi}, & \text{if } k = 0 . \end{cases}$$

We can now write down the expansion of the kernel $\Phi_\alpha(|x-y|)$ in a Fourier series of Gegenbauer's polynomials. Fix, first, α, with $0 < \alpha < N-1$. If we denote by $g(t)$ the function

$$g(t) = \left(\frac{1}{2-2t}\right)^{\alpha/2},$$

then we can express the kernel Φ_α in terms of g as

$$\Phi_\alpha(|\xi-\eta|) = \Phi_\alpha\left(\sqrt{|\xi|^2 - 2\xi\cdot\eta + |\eta|^2}\right) = g(\xi\cdot\eta) .$$

Now, develop $g(t)$ as a Fourier series

$$g(t) = \sum_{k=0}^{\infty} g_k P_k^{(N-2)/2}(t)\,, \quad \text{where} \quad g_k \left\| P_k^{(N-2)/2} \right\|_2^2 = \left\langle g, P_k^{(N-2)/2} \right\rangle ,$$

and conclude

$$\Phi_\alpha(|\xi - \eta|) = g(\xi \cdot \eta) = \sum_{k=0}^{\infty} g^k Z_\eta^k(\xi)\,, \tag{1}$$

where $g^k C_{k,N} = g_k$. Hereafter F will denote the usual hypergeometric function

$$F(a,b;c;t) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{t^m}{m!}\,,$$

where

$$(u)_m = u(u+1)\dots(u+m-1) = \frac{\Gamma(u+m)}{\Gamma(u)}\,.$$

The polynomials $P_k^{(N-2)/2}$ can be expressed in terms of F [**AS**, p. 779].

If $N > 2$,

$$P_k^{(N-2)/2}(t) = \binom{k+N-3}{k} F(-k, k+N-2; (N-1)/2; (1-t)/2)\,.$$

Then,

$$\left\langle g, P_k^{(N-2)/2} \right\rangle = \binom{k+N-3}{k} \int_{-1}^{1} F(-k, k+N-2; (N-1)/2; (1-t)/2) \cdot (2-2t)^{-\alpha/2}(1-t^2)^{(N-3)/2}\, dt\,.$$

Therefore

$$\left\langle g, P_k^{(N-2)/2} \right\rangle = 2^{N-2-\alpha} \binom{k+N-3}{k} \int_0^1 s^{-1+(N-1-\alpha)/2}(1-s)^{-1+(N-1)/2} \cdot F(-k, k+N-2; (N-1)/2; s)\, ds\,.$$

Using the relationship

$$P_k^{(N-2)/2}(-t) = (-1)^k P_k^{(N-2)/2}(t)\,, \qquad \text{[SW, p. 149], [AS, p. 775]}$$

we have

$$\left\langle g, P_k^{(N-2)/2}\right\rangle = 2^{N-2-\alpha}\binom{k+N-3}{k}(-1)^k\int_0^1 s^{-1+(N-1-\alpha)/2}(1-s)^{-1+(N-1)/2} \cdot F(-k,k+N-2;(N-1)/2;1-s)\,ds\,.$$

Term by term integration of the series defining F gives

$$\int_0^1 s^{a-1}(1-s)^{b-1}F(-k,c;b;1-s)\,ds = B(a,b)F(-k,c;a+b;1)\,,$$

where $B(\cdot,\cdot)$ is the Euler's Beta function. Moreover, it is easy to see that ([**AS**, p. 556])

$$F(-k,c;a+b;1) = \frac{\Gamma(a+b)\Gamma(a+b-c+k)}{\Gamma(a+b+k)\Gamma(a+b-c)} = \frac{\Gamma(a+b)}{\Gamma(a+b+k)}(-1)^k\frac{\Gamma(1+c-a-b)}{\Gamma(1+c-a-b-k)}\,,$$

and so

$$(-1)^k\int_0^1 s^{a-1}(1-s)^{b-1}F(-k,c;b;1-s)\,ds = \frac{\Gamma(a)\Gamma(b)\Gamma(1+c-a-b)}{\Gamma(a+b+k)\Gamma(1+c-a-b-k)}\,.$$

This gives

$$\left\langle g, P_k^{(N-2)/2}\right\rangle = 2^{N-2-\alpha}\binom{k+N-3}{k}\frac{\Gamma\left(\frac{N-1-\alpha}{2}\right)\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(k+\frac{\alpha}{2}\right)}{\Gamma\left(N-1-\frac{\alpha}{2}+k\right)\Gamma\left(\frac{\alpha}{2}\right)}\,,$$

and

$$g_k = \frac{\left\langle g, P_k^{(N-2)/2}\right\rangle}{\left\|P_k^{(N-2)/2}\right\|_2^2} = 2^{N-3-\alpha}\frac{N+2k-2}{\sqrt{\pi}}\frac{\Gamma\left(\frac{N-1-\alpha}{2}\right)\Gamma\left(\frac{N}{2}-1\right)\Gamma\left(k+\frac{\alpha}{2}\right)}{\Gamma\left(N-1-\frac{\alpha}{2}+k\right)\Gamma\left(\frac{\alpha}{2}\right)}\,.$$

Therefore,

$$(2) \qquad g^k = g_k C_{k,N}^{-1} = 2^{N-1-\alpha}\pi^{(N-1)/2}\,\frac{\Gamma\left(\frac{N-1-\alpha}{2}\right)\Gamma\left(k+\frac{\alpha}{2}\right)}{\Gamma\left(N-1-\frac{\alpha}{2}+k\right)\Gamma\left(\frac{\alpha}{2}\right)},$$

if $N > 2$. On the other hand, if $N = 2$, the k-th Chebyshev's polynomial is $T_k(t) = F(-k, k; 1/2; (1-t)/2)$, (see [**AS**, p. 779]), and

$$\langle g, P_k^0\rangle = \int_{-1}^{1} (2-2t)^{-\alpha/2} F(-k,k;1/2;(1-t)/2)(1-t^2)^{-1/2}\,dt\,.$$

Using the above computations when $N = 2$, we have that

$$\langle g, P_k^0\rangle = 2^{-\alpha}\pi^{1/2}\,\frac{\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma\left(k+\frac{\alpha}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}+k\right)\Gamma\left(\frac{\alpha}{2}\right)}\,.$$

Moreover it is easy to see, [**AS**, p. 774], that

$$\|P_k^0\|_2^2 = \begin{cases} \frac{\pi}{2}, & \text{if } k > 0\,, \\ \pi\,, & \text{if } k = 0\,, \end{cases}$$

and also that $C_{k,2}^{-1} = 2\,\|P_k^0\|_2^2$.

Then

$$g^k = \frac{\langle g, P_k^0\rangle}{\|P_k^0\|_2^2} C_{k,2}^{-1}\,,$$

and so (2) is also satisfied in this case ($N = 2$). Therefore we have proved the following:

Lemma 1. *For all $N \in \mathbb{N}$, $N > 1$ and $0 < \alpha < N-1$,*

$$\Phi_\alpha(|\xi-\eta|) = \sum_{k=0}^{\infty} g^k Z_\eta^k(\xi)\,,$$

where

$$g^k = 2^{N-1-\alpha}\pi^{(N-1)/2}\,\frac{\Gamma\left(\frac{N-1-\alpha}{2}\right)\Gamma\left(k+\frac{\alpha}{2}\right)}{\Gamma\left(N-1-\frac{\alpha}{2}+k\right)\Gamma\left(\frac{\alpha}{2}\right)}\,.$$

Now we can express the α-energy of a measure μ in terms of its Poisson extension P_μ.

Lemma 2. *If μ is a signed measure supported on Σ_{N-1}, we have:*

(i) *If $0 < \alpha < N-1$, then*

$$I_\alpha(\mu) = C(N,\alpha) \int_0^1 \left\{ \int_{\Sigma_{N-1}} |P_\mu(r\xi)|^2 \, d\xi \right\} r^{\alpha-1}(1-r^2)^{N-2-\alpha} \, dr \, ,$$

with

$$C(N,\alpha) = \frac{4\pi^{N/2}}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{N-\alpha}{2}\right)} \, .$$

(ii) *If $m = \mu(\Sigma_{N-1})$, then*

$$\begin{aligned} I_0(\mu) = {} & \omega_{N-1} \int_0^1 \int_{\Sigma_{N-1}} \left| P_\mu(r\xi) - \frac{m}{\omega_{N-1}} \right|^2 d\xi \, (1-r^2)^{N-2} \frac{dr}{r} \\ & + \frac{m^2}{2} \left[\frac{\Gamma'}{\Gamma}\left(\frac{N}{2}\right) - \frac{\Gamma'}{\Gamma}(N-1) \right] . \end{aligned}$$

In particular, if $N = 2$,

$$I_0(\mu) = 2\pi \int_0^1 \int_0^{2\pi} \left| P_\mu(re^{i\theta}) - \frac{m}{2\pi} \right|^2 d\theta \frac{dr}{r} \, .$$

Proof. Let $\left\{\mu_j^k\right\}$, $k \geq 0$, $1 \leq j \leq a_k$, be the Fourier coefficientes of μ, i.e.,

$$\mu \sim \sum_{k=0}^{\infty} \sum_{j=1}^{a_k} \mu_j^k Y_j^k \, .$$

Recall that P_μ is defined by

$$P_\mu(r\xi) = \int_{\Sigma_{N-1}} p(\eta, r\xi) \, d\mu(\eta) \, ,$$

where $p(\eta, r\xi)$ is the classical (normalized) Poisson kernel

$$p(\eta, r\xi) = \frac{1}{\omega_{N-1}} \frac{1-r^2}{|\eta - r\xi|^N} \, .$$

We have [**SW**, p. 145]

$$p(\eta, r\xi) = \sum_{k=0}^{\infty} r^k Z_\eta^k(\xi) = \sum_{k,j} r^k Y_j^k(\eta) Y_j^k(\xi) \, .$$

Now, Plancherel's theorem gives that

$$P_\mu(r\xi) = \sum_{k,j} r^k \mu_j^k Y_j^k(\xi)\,.$$

Using again Plancherel's theorem we obtain that

$$\int_{\Sigma_{N-1}} |P_\mu(r\xi)|^2\, d\xi = \sum_{k,j} r^{2k} \left|\mu_j^k\right|^2 ,$$

and so if we denote by Λ the right hand side in (i), we have that

$$\Lambda = C(N,\alpha) \sum_{k,j} \left|\mu_j^k\right|^2 \int_0^1 r^{2k+\alpha-1}(1-r^2)^{N-2-\alpha}\, dr\,,$$

and, substituting $r^2 = t$, we get that

$$\Lambda = \frac{C(N,\alpha)}{2} \sum_{k,j} \frac{\Gamma\left(k+\frac{\alpha}{2}\right)\Gamma(N-1-\alpha)}{\Gamma\left(k+N-1-\frac{\alpha}{2}\right)} \left|\mu_j^k\right|^2 = \sum_{j,k} g^k \left|\mu_j^k\right|^2 .$$

Note that we have used the known duplication formula for the Gamma function in the last equality.

On the other hand, by (1),

$$\Phi_\alpha(|\xi-\eta|) = \sum_{k=0}^{\infty} g^k Z_\eta^k(\xi) = \sum_{k,j} g^k Y_j^k(\eta) Y_j^k(\xi)\,,$$

and using Plancherel's theorem we obtain that

$$\int_{\Sigma_{N-1}} \Phi_\alpha(|\xi-\eta|)\, d\mu(\eta) = \sum_{k,j} g^k \mu_j^k Y_j^k(\xi)\,,$$

$$I_\alpha(\mu) = \sum_{k,j} g^k \left|\mu_j^k\right|^2 = \Lambda\,.$$

This finishes the proof of (i).

In order to prove (ii) observe that

$$\int_{\Sigma_{N-1}} \left|P_\mu(r\xi) - \frac{m}{\omega_{N-1}}\right|^2 d\xi + \frac{m^2}{\omega_{N-1}} = \int_{\Sigma_{N-1}} |P_\mu(r\xi)|^2\, d\xi\,.$$

Integrating this equality we have that

$$\begin{aligned} I_\alpha(\mu) = {} & C(N,\alpha) \int_0^1 \int_{\Sigma_{N-1}} \left|P_\mu(r\xi) - \frac{m}{\omega_{N-1}}\right|^2 d\xi\, r^{\alpha-1}(1-r^2)^{N-2-\alpha}\, dr \\ & + m^2\, U(\alpha)\,, \end{aligned}$$

where

$$U(\alpha) = \frac{\Gamma(N/2)\Gamma(N-1-\alpha)}{\Gamma\left((N-\alpha)/2\right)\Gamma(N-1-\alpha/2)},$$

and hence

$$\lim_{\alpha \to 0} \frac{I_\alpha(\mu) - m^2\, U(\alpha)}{\alpha} \\ = \omega_{N-1} \int_0^1 \int_{\Sigma_{N-1}} \left| P_\mu(r\xi) - \frac{m}{\omega_{N-1}} \right|^2 d\xi\, (1-r^2)^{N-2} \frac{dr}{r}.$$

On the other hand,

$$\begin{aligned} \lim_{\alpha \to 0} \frac{I_\alpha(\mu) - m^2\, U(\alpha)}{\alpha} &= \lim_{\alpha \to 0} \frac{I_\alpha(\mu) - m^2}{\alpha} - m^2 \lim_{\alpha \to 0} \frac{U(\alpha) - 1}{\alpha} \\ &= I_0(\mu) - m^2 U'(0), \end{aligned}$$

and

$$U'(0) = \frac{1}{2}\left[\frac{\Gamma'}{\Gamma}\left(\frac{N}{2}\right) - \frac{\Gamma'}{\Gamma}(N-1)\right].$$

This finishes the proof of Lemma 2. □

3. Distortion of α-capacity.

We need the following lemmas.

Lemma 3. *Let μ be a finite positive measure in $\partial\Delta$, and let f be an inner function. Then, there exists a unique positive measure $\widetilde{\nu}$ in $\mathbb{S}_n$ such that $P_\mu \circ f = P_{\widetilde{\nu}}$ and*

$$\widetilde{\nu}\left(f^{-1}(\text{support } \mu)\right) = \widetilde{\nu}(\mathbb{S}_n).$$

Moreover, if $f(0) = 0$, then

$$\frac{1}{\omega_{2n-1}} \widetilde{\nu}(\mathbb{S}_n) = \frac{1}{2\pi} \mu(\partial\Delta).$$

Proof. It is essentially the same proof as that of Lemma 1 of [**FP**], but see Lemma 10 below for further details.

A different normalization is useful; choosing $\nu = (2\pi/\omega_{2n-1})\widetilde{\nu}$, one obtai

$$P_\nu = \frac{2\pi}{\omega_{2n-1}} P_\mu \circ f \qquad \text{and} \qquad \nu(\mathbb{S}_n) = \mu(\partial\Delta).$$

The following is well known

Lemma 4. (Subordination principle). *Let $f : \mathbb{B}_n \longrightarrow \Delta$ be a holomorphic function such that $f(0) = 0$, and let $v : \Delta \longrightarrow \mathbb{R}$ be a subharmonic function. Then*

$$\frac{1}{\omega_{2n-1}} \int_{\mathbb{S}_n} v(f(r\xi))\, d\xi \leq \frac{1}{2\pi} \int_0^{2\pi} v(re^{i\theta})\, d\theta\, .$$

It will be relevant later on to recall the well known fact that, in the case $n = 1$, equality in Lemma 4 holds for a given r, $0 < r < 1$, if and only if either v is harmonic in $\Delta_r = \{|z| < r\}$ or f is a rotation. Note also that there is no such equality statement when $n > 1$ since in higher dimensions the extremal functions in Schwarz's lemma are not so clearly determined (see e.g. [**R**, p. 164]).

Lemma 5. *Let μ be a signed measure on $\partial\Delta$, f an inner function with $f(0) = 0$, and ν a signed measure on $\mathbb{S}_n$ such that*

$$P_\nu = (2\pi/\omega_{2n-1}) P_\mu \circ f\, .$$

Then

(i) *If $n = 1$ and $0 \leq \alpha < 1$, then*

$$I_\alpha(\nu) \leq I_\alpha(\mu)\, .$$

(ii) *If $n > 1$ and $0 < \alpha < 1$, then*

$$I_{2n-2+\alpha}(\nu) \leq K(n,\alpha) I_\alpha(\mu)\, ,$$

where

$$K(n,\alpha) = \frac{(n-1)!\, \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(n-1+\frac{\alpha}{2}\right)}\, .$$

If $\alpha = 0$ and $m = \mu(\partial\Delta) = \nu(\mathbb{S}_n)$, we have

$$I_{2n-2}(\nu) \leq (2n-2) I_0(\mu) + m^2\, .$$

The measure ν is obtained from Lemma 3 by splitting μ into its positive and negative parts. Note that for fixed α,

$$K(n,\alpha) \sim n^{1-\alpha/2}\, \Gamma\left(\frac{\alpha}{2}\right), \qquad \text{as } n \to \infty\, ,$$

while for fixed $n > 1$

$$K(n,\alpha) \sim \frac{C_n}{\alpha}, \qquad \text{as } \alpha \to 0\, .$$

Let us observe also that $K(n,\alpha)$ takes the value 1 for $n = 1$.

Proof. Since $\left|P_\mu - \frac{m}{2\pi}\right|^2$ and $\left|P_\mu\right|^2$ are subharmonic, we obtain by subordination, Lemma 4, that if $n = 1$ and $\alpha = 0$

$$\int_0^{2\pi} \left|P_\nu - \frac{m}{2\pi}\right|^2 d\theta = \int_0^{2\pi} \left|P_\mu(f) - \frac{m}{2\pi}\right|^2 d\theta \le \int_0^{2\pi} \left|P_\mu - \frac{m}{2\pi}\right|^2 d\theta\,,$$

and if $n \ge 1$, $0 < \alpha < 1$, that

$$(3) \qquad \int_{\mathbb{S}_n} |P_\nu|^2\, d\xi = \left(\frac{2\pi}{\omega_{2n-1}}\right)^2 \int_{\mathbb{S}_n} |P_\mu(f)|^2\, d\xi \le \frac{2\pi}{\omega_{2n-1}} \int_0^{2\pi} |P_\mu|^2 d\theta\,.$$

In the first case, we obtain

$$I_0(\nu) \le I_0(\mu)$$

by integrating with respect to $2\pi\, dr/r$ and applying Lemma 2, part (ii).

In the second case, using Lemma 2, part (i), and Lemma 4 with $v = |P_\mu|^2$, we have that

$$\begin{aligned} I_{2n-2+\alpha}(\nu) &= C(2n, 2n-2+\alpha) \int_0^1 \left\{ \int_{\mathbb{S}_n} |P_\nu(r\xi)|^2\, d\xi \right\} r^{2n-2+\alpha-1} \frac{dr}{(1-r^2)^\alpha} \\ &\le \frac{C(2n, 2n-2+\alpha)}{C(2,\alpha)}\, C(2,\alpha) \\ &\quad \cdot \frac{2\pi}{\omega_{2n-1}} \int_0^1 \left\{ \int_0^{2\pi} \left|P_\mu(re^{i\theta})\right|^2\, d\theta \right\} r^{\alpha-1} \frac{dr}{(1-r^2)^\alpha} \\ &= K(n,\alpha) I_\alpha(\mu)\,, \end{aligned}$$

where

$$K(n,\alpha) = \frac{(n-1)!\,\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(n-1+\frac{\alpha}{2}\right)}\,.$$

Finally, since $\nu(\mathbb{S}_n) = m$,

$$\int_{\mathbb{S}_n} \left|P_\nu(r\xi) - \frac{m}{\omega_{2n-1}}\right|^2\, d\xi = \int_{\mathbb{S}_n} |P_\nu(r\xi)|^2\, d\xi - \frac{m^2}{\omega_{2n-1}}\,,$$

and so, Lemma 2 gives, if $n > 1$, that

$$I_{2n-2}(\nu) = m^2 + \frac{4\pi^n}{(n-2)!} \int_0^1 \int_{\mathbb{S}_n} \left|P_\nu(r\xi) - \frac{m}{\omega_{2n-1}}\right|^2\, d\xi\, r^{2n-3}\, dr\,.$$

By Lemmas 3 and 4, we get that

$$\begin{aligned}\int_{\mathbb{S}_n} \left| P_\nu(r\xi) - \frac{m}{\omega_{2n-1}} \right|^2 d\xi &= \int_{\mathbb{S}_n} \left| \frac{2\pi}{\omega_{2n-1}} P_\mu(f(r\xi)) - \frac{m}{\omega_{2n-1}} \right|^2 d\xi \\ &= \left(\frac{2\pi}{\omega_{2n-1}} \right)^2 \int_{\mathbb{S}_n} \left| P_\mu(f(r\xi)) - \frac{m}{2\pi} \right|^2 d\xi \\ &\leq \frac{2\pi}{\omega_{2n-1}} \int_0^{2\pi} \left| P_\mu(re^{i\theta}) - \frac{m}{2\pi} \right|^2 d\theta \,.\end{aligned}$$

Therefore

$$\begin{aligned}I_{2n-2}(\nu) &\leq m^2 + \frac{4\pi^n}{(n-2)!} \int_0^1 \frac{2\pi}{\omega_{2n-1}} \int_0^{2\pi} \left| P_\mu(re^{i\theta}) - \frac{m}{2\pi} \right|^2 d\theta \, \frac{dr}{r} \\ &= m^2 + \frac{4\pi^n}{(n-2)!} \frac{1}{\omega_{2n-1}} I_0(\mu) \\ &= m^2 + (2n-2) I_0(\mu) \,.\end{aligned}$$

The proof of Lemma 5 is finished. □

Finally, we can prove

Theorem 1. *If f is inner in the unit disk Δ, $f(0) = 0$, and E is a Borel subset of $\partial\Delta$, we have:*

$$\operatorname{cap}_\alpha \left(f^{-1}(E) \right) \geq \operatorname{cap}_\alpha(E) \,, \qquad 0 \leq \alpha < 1 \,.$$

Moreover, if E is any Borel subset of $\partial\Delta$ with $\operatorname{cap}_\alpha(E) > 0$, equality holds if and only if either f is a rotation or $\operatorname{cap}_\alpha(E) = \operatorname{cap}_\alpha(\partial\Delta)$.

Notice the following consequence concerning invariant sets. It is well known that an inner function f with $f(0) = 0$, which is not a rotation, is ergodic with respect to Lebesgue measure, see e.g. [**P**]. As a consequence of the above, it is also ergodic with respect to α-capacity. More precisely,

Corollary. *With the hypotheses of Theorem 1, if f is not a rotation and if the symmetric difference between E and $f^{-1}(E)$ has zero α-capacity, then either $\operatorname{cap}_\alpha(E) = 0$ or $\operatorname{cap}_\alpha(E) = \operatorname{cap}_\alpha(\partial\Delta)$.*

In higher dimensions we have

Theorem 2. *If f is inner in the unit ball of $\mathbb{C}^n$, $f(0) = 0$, and E is a Borel subset of $\partial\Delta$, we have:*

$$\operatorname{cap}_{2n-2+\alpha} \left(f^{-1}(E) \right) \geq K(n,\alpha)^{-1} \operatorname{cap}_\alpha(E) \,, \qquad 0 < \alpha < 1 \,,$$

and

$$\frac{1}{\operatorname{cap}_{2n-2}\left(f^{-1}(E)\right)} \leq 1+(2n-2)\log\frac{1}{\operatorname{cap}_0(E)}, \qquad (n>1).$$

Proof of Theorems 1 *and* 2. To prove the inequalities in the theorems we may assume that E is closed. Assume first that $n=1$, $0<\alpha<1$. Let us denote by μ_e the α-equilibrium probability distribution of E, and let ν be the probability measure such that $P_\nu = P_{\mu_e}\circ f$. By Lemma 5,

$$I_\alpha(\nu) \leq I_\alpha(\mu_e) = \left(\operatorname{cap}_\alpha(E)\right)^{-1}. \tag{4}$$

But, from Lemma 3, $\nu\left(f^{-1}(E)\right)=1$, and so

$$I_\alpha(\nu) = \iint_{f^{-1}(E)\times f^{-1}(E)} \Phi_\alpha(|z-w|)\,d\nu(z)\,d\nu(w).$$

Now, let $\{K_n\}$ be an increasing sequence of compacts subsets in $\partial\Delta$, $K_n \subset f^{-1}(E)$, such that $\nu(K_n)\nearrow 1$. Then, for each $n\geq 1$,

$$\begin{aligned} I_\alpha(\nu) &= \iint_{f^{-1}(E)\times f^{-1}(E)} \Phi_\alpha(|z-w|)\,d\nu(z)\,d\nu(w) \\ &\geq \nu(K_n)^2 \iint_{K_n\times K_n} \Phi_\alpha(|z-w|)\,\frac{d\nu(z)}{\nu(K_n)}\,\frac{d\nu(w)}{\nu(K_n)} \\ &\geq \nu(K_n)^2\left(\operatorname{cap}_\alpha(K_n)\right)^{-1} \\ &\geq \nu(K_n)^2\left(\operatorname{cap}_\alpha\left(f^{-1}(E)\right)\right)^{-1}, \end{aligned}$$

and consequently

$$I_\alpha(\nu) \geq \left(\operatorname{cap}_\alpha\left(f^{-1}(E)\right)\right)^{-1}. \tag{5}$$

The inequality in Theorem 1 follows now from (4) and (5).

The cases $n>1$ (Theorem 2) and $n=1$, $\alpha=0$ are completely analogous.

Proof of the equality statement of Theorem 1. First we prove it assuming that E is closed, to show the ideas that we will use to demonstrate the general case.

Suppose that $0<\alpha<1$. We have seen that

$$\frac{1}{\operatorname{cap}_\alpha\left(f^{-1}(E)\right)} \leq I_\alpha(\nu) \leq I_\alpha(\mu_e) = \frac{1}{\operatorname{cap}_\alpha(E)}.$$

Therefore, if E and $f^{-1}(E)$ have the same α-capacity, then

$$I_\alpha(\nu) = I_\alpha(\mu_e),$$

and this is possible only if for all $r \in (0,1)$,

$$\int_0^{2\pi} \left|P_{\mu_e}\left(re^{i\theta}\right)\right|^2 d\theta = \int_0^{2\pi} \left|P_{\mu_e}\left(f\left(re^{i\theta}\right)\right)\right|^2 d\theta .$$

This can occur only if either f is a rotation or $|P_{\mu_e}|^2$ is harmonic. In the latter case, we obtain that μ_e is normalized Lebesgue measure, or equivalently that $\mathrm{cap}_\alpha(E) = \mathrm{cap}_\alpha(\partial\Delta)$. Since E is closed, it follows that $E = \partial\Delta$.

In order to prove the general case we need a characterization of the α-capacity of E when E is not closed (see Lemma 6 below). We begin by recalling some facts about convergence of measures.

We will say that a sequence of signed measures $\{\sigma_n\}$ with supports contained in a compact set K converges w^* to a signed measure σ if

$$\int h(x)\, d\sigma_n(x) \underset{n\to\infty}{\longrightarrow} \int h(x)\, d\sigma(x) , \qquad \text{for all } \ h \in C(K) .$$

Here, the w^*-convergence refers to the duality between the space of signed measures on K and the space $C(K)$ of continuous functions with support contained in K.

In this Section, we will denote by $\mathcal{M}_\alpha(K)$ $(0 \le \alpha < 1)$ the vector space of all signed measures whose support is contained in the set K and whose α-energy is finite. $\mathcal{M}_\alpha(\mathbb{C})$ or $\mathcal{M}_\alpha(\overline{\Delta})$ is denoted simply by $\mathcal{M}_\alpha$, and $\mathcal{M}_\alpha^+$ denotes the corresponding cone of positive measures.

The positivity properties of I_α [**L**, p. 79-80] allow us to define an inner product in $\mathcal{M}_\alpha$ (for $0 < \alpha < 1$) and e.g. in $\mathcal{M}_0(\{|z| = 1/2\})$ (for $\alpha = 0$) as follows

$$\langle \sigma, \gamma \rangle = \iint \Phi_\alpha(|x-y|)\, d\sigma(x) d\gamma(y) .$$

Observe that the associated norm verifies

$$\|\sigma\|^2 = I_\alpha(\sigma) .$$

In the next lemma we collect some useful information concerning the above inner product.

Lemma 6.

(i) *If* $0 < \alpha < 1$, *K is a compact subset of* $\mathbb{C}$, $\{\sigma_n\}$ *is a Cauchy sequence (with respect to the inner product) in* $\mathcal{M}_\alpha^+(K)$ *and* $\sigma_n \overset{w^*}{\longrightarrow} \sigma$, *then*

$$\|\sigma_n - \sigma\| \longrightarrow 0 , \qquad \text{as } \ n \to \infty .$$

(ii) *If E is any Borel subset of K, then*

$$\frac{1}{\mathrm{cap}_\alpha(E)} = \inf\left\{ I_\alpha(\mu) : \ \ \mu \ \text{ a probability measure, } \ \mu(E) = 1 \right\} ,$$

and there exists a probability measure μ_e supported on $\overline{E}$ such that

$$\frac{1}{\mathrm{cap}_\alpha(E)} = I_\alpha(\mu_e)\,.$$

In fact, if K_n is an increasing sequence of compact subsets of E such that

$$\mathrm{cap}_\alpha(K_n) \nearrow \mathrm{cap}_\alpha(E)\,,$$

and if μ_n is the equilibrium distribution of K_n, then

$$\mu_n \xrightarrow{w^*} \mu_e \qquad \textit{and} \qquad \|\mu_n - \mu_e\| \longrightarrow 0\,,$$

as $n \to \infty$.

These statements remain true in the case $\alpha = 0$, if K is a compact subset of Δ.

Lemma 6 is contained in [**L**, p. 82, 89, 145] if $0 < \alpha < 1$. The case $\alpha = 0$ is similar, though we need the restriction $K \subset \Delta$ so that $\|\cdot\|$ is a norm [**L**, p. 80].

Now we are ready to finish the proof of Theorem 1. Let E be a Borel subset of $\partial\Delta$ such that

$$\mathrm{cap}_\alpha\left(f^{-1}(E)\right) = \mathrm{cap}_\alpha(E) > 0\,. \tag{6}$$

We choose an increasing sequence of compact sets $K_n \subset E$ such that $\mathrm{cap}_\alpha(K_n) \nearrow \mathrm{cap}_\alpha(E)$. Let μ_n be the α-equilibrium measure of K_n and let μ_e be the probability measure supported on $\overline{E}$ given by Lemma 6. We have

$$\mu_n \xrightarrow{w^*} \mu_e \qquad \text{and} \qquad I_\alpha(\mu_n) \searrow I_\alpha(\mu_e)\,,$$

as $n \to \infty$. In fact,

$$\|\mu_n - \mu_e\| \to 0\,, \qquad \text{as } n \to \infty\,.$$

Let ν_n be the probability measure, with $\nu_n(f^{-1}(K_n)) = 1$, such that $P_{\nu_n} = P_{\mu_n} \circ f$ (see Lemma 3). We can suppose after extracting a subsequence if necessary, that ν_n converges w^* to a probability measure ν on $\overline{f^{-1}(E)}$. Since the Poisson kernel is continuous in Δ we obtain, by using the w^*-convergence, that

$$P_{\mu_n} \to P_{\mu_e} \quad \text{and} \quad P_{\nu_n} \to P_\nu\,, \qquad \text{as } n \to \infty\,,$$

pointwise. Therefore $P_\nu = P_{\mu_e} \circ f$, which in particular shows that ν is a probability measure supported on $f^{-1}(\overline{E})$.

Claim. *$I_\alpha(\nu_n) \to I_\alpha(\nu)$ as $n \to \infty$.*

Since ν_n is a probability measure on $f^{-1}(E)$, Lemma 6 guarantees that

$$\frac{1}{\operatorname{cap}_\alpha(f^{-1}(E))} \le I_\alpha(\nu_n),$$

and so, by letting $n \to \infty$, and using that $P_\nu = P_{\mu_e} \circ f$ (by Lemma 5) we obtain that

$$\frac{1}{\operatorname{cap}_\alpha(f^{-1}(E))} \le I_\alpha(\nu) \le I_\alpha(\mu_e) = \frac{1}{\operatorname{cap}_\alpha(E)}.$$

From (6), we deduce that $I_\alpha(\nu) = I_\alpha(\mu_e)$. Finally, we can reason as in the case of E being closed and conclude that either f is a rotation or μ_e is normalized Lebesgue measure, i.e., $\operatorname{cap}_\alpha(E) = \operatorname{cap}_\alpha(\partial\Delta)$.

Proof of the Claim. Consider first the case $0 < \alpha < 1$. Since $P_{\nu_p - \nu_n} = P_{\mu_p - \mu_n} \circ f$, by Lemma 5 we obtain that

$$\|\nu_p - \nu_n\|^2 = I_\alpha(\nu_p - \nu_n) \le I_\alpha(\mu_p - \mu_n) = \|\mu_p - \mu_n\|^2 \underset{p,n\to\infty}{\longrightarrow} 0.$$

Therefore $\{\nu_n\}$ is a Cauchy sequence in the norm and so, by Lemma 6, we have that

$$\|\nu_n - \nu\| \to 0 \qquad \text{and} \qquad I_\alpha(\nu_n) \to I_\alpha(\nu)$$

as $n \to \infty$.

For $\lambda > 0$, and $A \subset \mathbb{C}$, we will denote by λA the set $\lambda A = \{\lambda z : z \in A\}$.

If E is a Borel subset of $\partial\Delta$, then $\frac{1}{2}E$ is a Borel subset of $\{|z| = 1/2\}$. Also, if σ is a probability measure in $\partial\Delta$, we will denote by σ^* the probability measure in $\{|z| = 1/2\}$ defined by

$$\sigma(A) = \sigma^*\left(\frac{1}{2}A\right), \tag{7}$$

for A a Borel subset of $\partial\Delta$. It is clear that

$$I_0(\sigma^*) = I_0(\sigma) + \log 2. \tag{8}$$

Now, in order to prove the case $\alpha = 0$, let μ_n^* and ν_n^* be the measures defined from μ_n and ν_n by (7). Then using again Lemma 5 and (8) we have that

$$\begin{aligned}\|\nu_p^* - \nu_n^*\|^2 &= I_0(\nu_p^* - \nu_n^*) = I_0(\nu_p - \nu_n) + \log 2 \\ &\le I_0(\mu_p - \mu_n) + \log 2 = \left\|\mu_p^* - \mu_n^*\right\|^2 \underset{p,n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Therefore $\{\nu_n^*\}$ is a Cauchy sequence in the norm and again by Lemma 6, we obtain that

$$\|\nu_n^* - \nu^*\| \to 0 \qquad \text{and} \qquad I_0(\nu_n^*) \to I_0(\nu^*)$$

as $n \to \infty$. It follows, from (8) that

$$I_0(\nu_n) \to I_0(\nu), \qquad \text{as } n \to \infty.$$

□

4. Some further results on distortion of capacity in the case n = 1.

First we show that Theorem 1 is sharp. In what follows $|\cdot|$ will denote not normalized Lebesgue measure in $\partial\Delta$ (i.e. $|\partial\Delta| = 2\pi$).

Proposition 1. $\text{cap}_\alpha(f^{-1}(E))$ *can take any value between* $\text{cap}_\alpha(E)$ *and* $\text{cap}_\alpha(\partial\Delta)$. *More precisely, given* $0 < s \leq t < \text{cap}_\alpha(\partial\Delta)$ *there exist a Borel subset* E *of* $\partial\Delta$ *and an inner function* f *with* $f(0) = 0$ *such that* $\text{cap}_\alpha(E) = s$ *and* $\text{cap}_\alpha(f^{-1}(E)) = t$.

In order to prove this, we need the following lemma whose proof will given later.

Lemma 7. *Let* I *be any closed interval in* $\partial\Delta$ *with* $|I| > 0$, *and let* B *be a finite union of closed intervals in* $\partial\Delta$ *such that* $|B| = |I|$. *Then there exists an inner function* f *such that*

$$f(0) = 0 \qquad \textit{and} \qquad f^{-1}(I) = B.$$

In fact, if $0 < |I| < 2\pi$, *then* f *is unique.*

Remark. It is natural to wonder if this lemma holds in higher dimensions, more precisely: Is it true that given an interval I in $\partial\Delta$ and a Borel subset B of $\mathbb{S}_n$ such that

$$\frac{|B|}{\omega_{2n-1}} = \frac{|I|}{2\pi},$$

there is an inner function $f : \mathbb{B}_n \longrightarrow \Delta$ such that $f^{-1}(I) \overset{\circ}{=} B$?

It is not possible to construct such f by using the Ryll-Wojtaszczyk polynomials (see [**R1**]), since in that case the following stronger result would be true too:

Given E, I subsets of $\partial\Delta$ with $|E| = |I|$ and $N \in \mathbb{N}$, there exists an inner function $f : \Delta \longrightarrow \Delta$ such that

$$E = f^{-1}(I)\,, \qquad \text{and} \qquad f^{(j)}(0) = 0\,, \quad \text{if } j \le N\,.$$

But it is easy to see, as a consequence of Lemma 8, that in general this is not possible.

Proof of Proposition 1. Let I be a closed interval in $\partial\Delta$ centered at 1 and such that $\operatorname{cap}_\alpha(I) = s$. Consider the function $g(z) = z^2$. Then (see e.g. [**FP**] or Proposition 3 below),

$$s = \operatorname{cap}_\alpha(I) < \operatorname{cap}_\alpha\left(g^{-1}(I)\right) < \cdots < \operatorname{cap}_\alpha\left(g^{-k}(I)\right) \underset{k\to\infty}{\longrightarrow} \operatorname{cap}_\alpha(\partial\Delta)\,.$$

Therefore, if $t = \operatorname{cap}_\alpha\left(g^{-k}(I)\right)$ for some k, we are done.

Note that $g^{-k}(I)$ consists of 2^k closed intervals of length $|I|/2^k$ and centered at the points $z_{j,k} = e^{2\pi j i/2^k}$ $(j = 1, \dots, 2^k)$.

If $\operatorname{cap}_\alpha\left(g^{-(k-1)}(I)\right) < t < \operatorname{cap}_\alpha\left(g^{-k}(I)\right)$ a simple continuity argument shows that there exist a finite union B of 2^k closed intervals in $\partial\Delta$ of total length $|I|$ with $\operatorname{cap}_\alpha(B) = t$.

Finally, applying Lemma 7 to the pair I, B we obtain an inner function f with $f(0) = 0$ and $f^{-1}(I) = B$. □

Proof of Lemma 7. Let u be the Poisson integral of the characteristic function of B, and let $\tilde{u}$ be its conjugate harmonic function chosen such that $\tilde{u}(0) = 0$. Since $u(0) = |B|/2\pi$ the holomorphic function $F = u + i\tilde{u}$ transforms Δ into the strip $S = \{\omega : 0 < \operatorname{Re}\omega < 1\}$. Notice that F has radial boundary values except for a finite number of points, and F applies the interior of B into $\{\omega : \operatorname{Re}\omega = 1\}$ and $\partial\Delta \setminus B$ into $\{\omega : \operatorname{Re}\omega = 0\}$.

Now, let G be the Riemann mapping of S chosen such that

$$G(|B|/2\pi) = 0\,.$$

G transforms $\{\omega : \operatorname{Re}\omega = 1\}$ onto an interval J of $\partial\Delta$. On the other hand, the function $h = G \circ F$ is clearly an inner function, $h(0) = 0$ and $h^{-1}(I) = B$. By composing h with an appropriate rotation we finish the proof of the existence statement.

To show the uniqueness of f, it is sufficient to prove the following

Lemma 8. *If A is any Borel subset of $\partial\Delta$, such that $\int_A e^{-i\theta}d\theta \neq 0$, and f, g are inner functions with $f(0) = g(0) = 0$ such that*

$$f^{-1}(A) \overset{\circ}{=} g^{-1}(A)\,,$$

then $f \equiv g$.

Here $\overset{\circ}{=}$ *denotes equality up to a set of zero Lebesgue measure.*

Proof. Let $F : \Delta \longrightarrow \{\omega : \ 0 < \operatorname{Re}\omega < 1\}$ be the holomorphic function given by

$$F(z) = \frac{1}{2\pi}\int_A \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\,.$$

F is univalent in a neighbourhood of 0, because

$$F'(0) = \frac{1}{\pi}\int_A e^{-i\theta}d\theta \neq 0\,.$$

Now, observe that $\operatorname{Re}(F \circ f) = \operatorname{Re}(F \circ g)$ almost everywhere on $\partial\Delta$. Since $\operatorname{Re}(F \circ f)$ and $\operatorname{Re}(F \circ g)$ are bounded harmonic functions it follows that $F \circ f = F \circ g + ic$ in Δ, where c is a real constant. Since $f(0) = g(0)$, we deduce that $F \circ f = F \circ g$ which proves the lemma because F is univalent in a neighbourhood of 0. □

Observe that, in particular, the condition $\int_A e^{-i\theta}d\theta \neq 0$ is satisfied e.g. if A is any interval in $\partial\Delta$ with $0 < |A| < 2\pi$.

The condition $\int_A e^{-i\theta}d\theta \neq 0$ is not only a technicality. If A is k-symmetrical (i.e., there exists a subset $A_0 \subset A$, with $A_0 \subset [0, 2\pi/k]$, such that $A \overset{\circ}{=} A_0 \cup (A_0 + 2\pi/k) \cup (A_0 + 4\pi/k) \cup \cdots \cup (A_0 + 2\pi(k-1)/k)$), and $\int_A e^{-ik\theta}d\theta \neq 0$, then $f = \omega g$, where ω is a k-th root of unity. To see this, one can use Lemma 8 with the functions $h \circ f$, $h \circ g$ and the set $h(A)$, where $h(z) = z^k$.

Also, note that if A is the union of two intervals in $\partial\Delta$, then $f = \pm g$, because $\int_A e^{-i\theta}d\theta = 0$ implies that A is 2-symmetrical.

Notice that if the function g in Lemma 8 were the identity, and $0 < |A| < 2\pi$, then, by ergodicity, we would have that f is a rotation of rational angle. This, together with the above remark, could suggest that perhaps the following statement was true:

If A is any Borel subset of $\partial\Delta$, such that $0 < |A| < 2\pi$, and f, g are inner functions with $f(0) = g(0) = 0$ such that

$$f^{-1}(A) \overset{\circ}{=} g^{-1}(A)\,,$$

then $f \equiv \lambda g$ with $|\lambda| = 1$.

But this is false as the next example shows: Let B be the following Blaschke product

$$B(z) = z\,\frac{2z-1}{2-z}\,.$$

By applying a theorem of Stephenson [**S**, Theorem 3] to the pair $B, -B$, one obtains two inner functions f and g with $f(0) = g(0) = 0$, such that

$$B \circ f = -B \circ g\,.$$

But then $(B(f))^2 = (B(g))^2$, and so, if we had $f = \lambda g$, we could conclude that $B(z) = -B(\lambda z)$. But, since $B'(0) \neq 0$, we had $\lambda = -1$, i.e., $B(z) = -B(-z)$, a contradiction.

The following is well known, at least for $\alpha = 0$, see for instance [**A**, p. 35-36] where it is credited to Beurling.

Proposition 2. *Let $0 \leq \alpha < 1$. If I is any interval in $\partial\Delta$, then I has the minimum α-capacity between all the Borel subsets of $\partial\Delta$ with the same Lebesgue measure than I.*

Proof. Let E be a Borel set such that $|E| = |I|$. A standard approximation argument shows that for all $\varepsilon > 0$ there exists a finite union B_ε of closed intervals such that

$$\big|\,|E| - |B_\varepsilon|\,\big| < \varepsilon \qquad \text{and} \qquad |\operatorname{cap}_\alpha(E) - \operatorname{cap}_\alpha(B_\varepsilon)\,| < \varepsilon\,.$$

Let I_ε be a closed interval with the same center than I and such that $|I_\varepsilon| = |B_\varepsilon|$. By Lemma 7, we can find an inner function f_ε such that

$$f_\varepsilon(0) = 0 \qquad \text{and} \qquad f_\varepsilon^{-1}(I_\varepsilon) = B_\varepsilon\,.$$

Therefore, by Theorem 1,

$$\operatorname{cap}_\alpha(E) + \varepsilon \geq \operatorname{cap}_\alpha(B_\varepsilon) \geq \operatorname{cap}_\alpha(I_\varepsilon)\,,$$

but $\operatorname{cap}_\alpha(I_\varepsilon) \to \operatorname{cap}_\alpha(I)$ as $\varepsilon \to 0$. □

The following proposition is not unexpected since ergodic theory says that $f^{-k}(E)$ is well spread on $\partial\Delta$. Hereafter $f^k = f \circ \cdots \circ f$ denotes the k-iterate of f and $f^{-k} = (f^k)^{-1}$.

Proposition 3. *If $f : \Delta \longrightarrow \Delta$ is inner but not a rotation, $f(0) = 0$, $0 \leq \alpha < 1$ and E is a Borel subset of $\partial\Delta$ with $\operatorname{cap}_\alpha(E) > 0$, then*

$$\operatorname{cap}_\alpha\left(f^{-k}(E)\right) \to \operatorname{cap}_\alpha(\partial\Delta) \qquad \textit{as} \;\; k \to \infty\,.$$

The proof of this result is an easy consequence of the following lemma.

Lemma 9. *With the hypotheses of Proposition 3, if μ is any probability measure on E with finite α-energy and if ν_k is the probability measure in $f^{-k}(E)$ such that $P_{\nu_k} = P_\mu \circ f^k$, then*

$$I_\alpha(\nu_k) \longrightarrow I_\alpha\left(\frac{|\cdot|}{2\pi}\right) \qquad \textit{as} \;\; k \to \infty\,.$$

With this, we have

$$\frac{1}{\operatorname{cap}_\alpha\left(f^{-k}(E)\right)} \leq I_\alpha(\nu_k) \longrightarrow I_\alpha\left(\frac{|\cdot|}{2\pi}\right) = \frac{1}{\operatorname{cap}_\alpha(\partial\Delta)}$$

giving us the conclusion of Proposition 3.

Proof of Lemma 9. We will prove it for $0 < \alpha < 1$; the case $\alpha = 0$ being similar.

By Lemma 2 (i), we have with an appropriate function g_α that

$$I_\alpha(\sigma) = \int_0^1 \int_0^{2\pi} \left|P_\sigma(re^{i\theta})\right|^2 d\theta\, g_\alpha(r)\, dr$$

for any probability measure σ on $\partial\Delta$.

Using (3) we have for all $r \in (0,1)$ that

$$\int_0^{2\pi} \left|P_{\nu_k}(re^{i\theta})\right|^2 d\theta \leq \int_0^{2\pi} \left|P_\mu(re^{i\theta})\right|^2 d\theta\,.$$

Since μ has finite α-energy, the right hand side in the last inequality, as a function of r, belongs to $L^1(g_\alpha(r)\, dr)$. Therefore, by using the Lebesgue's dominated convergence theorem, we would be done if we show that

$$\int_0^{2\pi} \left|P_{\nu_k}(re^{i\theta})\right|^2 d\theta \longrightarrow \frac{1}{2\pi} \qquad \text{as } k \to \infty\,, \tag{9}$$

for each r with $0 < r < 1$. But, by Schwarz's lemma, and since f is not a rotation, $\left|f^k\left(re^{i\theta}\right)\right| \longrightarrow 0$ as $k \to \infty$, uniformly on θ for r fixed. Therefore, for each r, $P_{\nu_k}\left(re^{i\theta}\right) = P_\mu\left(f^k\left(re^{i\theta}\right)\right) \longrightarrow 1/2\pi$, as $k \to \infty$, uniformly on θ, and this implies (9). ☐

Even in the case when $\operatorname{cap}_\alpha(E) = 0$, the sets $f^{-k}(E)$ are well spread on $\partial\Delta$.

Proposition 4. *If $f : \Delta \longrightarrow \Delta$ is an inner function (but not a rotation) with $f(0) = 0$, E is any non empty Borel subset of $\partial\Delta$, and μ is any probability measure on E, then for some absolute constant C and a positive constant A that only depends on $|f'(0)|$, we have that*

$$\left|\nu_k(I) - \frac{|I|}{2\pi}\right| < C\, e^{-Ak}\,,$$

for each interval $I \subset \partial\Delta$. In particular,

$$\nu_k \longrightarrow \frac{|\cdot|}{2\pi}$$

in the usual weak-$*$ topology.

Here ν_k is the probability measure concentrated in $f^{-k}(E)$ such that $P_{\nu_k} = P_\mu \circ f^k$.

Proof. The proof is similar to that of Lemma 3 in [**P**], but using here the fact that $P_{\nu_k} = P_\mu \circ f^k$ instead of Lemma 1 in [**P**]. □

Proposition 5. *If $f : \mathbb{B}_n \longrightarrow \Delta$ is inner, then f assumes in $\partial\mathbb{B}_n$ all the values in $\partial\Delta$.*

Proof. Let $f : \mathbb{B}_n \longrightarrow \Delta$ be an inner function. It is enough to prove that $f^{-1}\{1\} \neq \varnothing$. But,

$$u := \operatorname{Re}\left(\frac{1+f}{1-f}\right) = \frac{1-|f|^2}{|1-f|^2} > 0, \qquad \text{in } \mathbb{B}_n\,. \tag{10}$$

Therefore, u is harmonic and positive in $\mathbb{B}_n$ and so there exists a positive measure in $\mathbb{S}_n$ such that

$$\operatorname{Re}\left(\frac{1+f}{1-f}\right) = P_\mu\,.$$

By (10) P_μ tends radially to 0 a.e. with respect to Lebesgue measure, since f is inner and (by Privalov's theorem, (see e.g., [**R**, Theorem 5.5.9])) f can assume the value 1 at most in a set of zero Lebesgue measure. Then, the Radon-Nikodym derivative of μ with respect to Lebesgue measure is zero a.e., and so μ is a singular measure.

By Lemma 11 it follows that $P_\mu \to +\infty$ in a set of full μ-measure. But this is the same to say that $f(re^{i\theta}) \to 1$ in that set. □

When the inner function f has order $k \geq 1$ at 0, we can improve Theorem 1 in the case α=0.

Theorem 3. *If $f : \Delta \longrightarrow \Delta$ is inner,*

$$f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0\,, \qquad f^{(k)}(0) \neq 0\,, \qquad (k \geq 1)\,,$$

and E is a Borel subset of $\partial\Delta$, then

$$\operatorname{cap}_0\left(f^{-1}(E)\right) \geq \left(\operatorname{cap}_0(E)\right)^{1/k}\,. \tag{11}$$

Moreover, if $\operatorname{cap}_0(E) > 0$, equality holds if and only if either $f(z) = \lambda z^k$, with $|\lambda| = 1$, or $\operatorname{cap}_0(E) = \operatorname{cap}_0(\partial\Delta)$.

Proof. For such a function f, Schwarz's lemma says us that $|f(z)| \leq |z|^k$, with equality only if $f(z) = \lambda z^k$ with $|\lambda| = 1$. With this in mind, the subordination principle says now (see e.g. [**HH**]) that if v is a subharmonic function in Δ, then

$$\int_0^{2\pi} v\left(f\left(re^{i\theta}\right)\right)\,d\theta \le \int_0^{2\pi} v\left(r^k e^{i\theta}\right)\,d\theta\,,$$

with equality for a given r only if v is harmonic in $\{|z| < r\}$ or f is a rotation of z^k.

Now, in order to prove (11), we can assume that E is closed. If μ_e is the equilibrium probability distribution of E and ν is the probability measure in $f^{-1}(E)$ such that $P_\nu = P_\mu \circ f$, then

$$\begin{aligned} I_0(\nu) &= 2\pi \int_0^1 \int_0^{2\pi} \left| P_{\mu_e}\left(f\left(re^{i\theta}\right)\right) - \frac{1}{2\pi}\right|^2 d\theta\,\frac{dr}{r} \\ &\le 2\pi \int_0^1 \int_0^{2\pi} \left| P_{\mu_e}\left(r^k e^{i\theta}\right) - \frac{1}{2\pi}\right|^2 d\theta\,\frac{dr}{r}\,. \end{aligned}$$

Substituting $r^k = t$, we obtain that

$$I_0(\nu) \le \frac{1}{k}\, I_0(\mu_e)\,.$$

This finishes the proof of (11). The equality statement can be proved in the same way as that of Theorem 1. □

Remark. For other α's $(0 < \alpha < 1)$ we can show

$$\frac{1}{\operatorname{cap}_\alpha\left(f^{-1}(E)\right)} - \frac{1}{\operatorname{cap}_\alpha(\partial\Delta)} \le \frac{C_\alpha}{k^{1-\alpha}}\left(\frac{1}{\operatorname{cap}_\alpha(E)} - \frac{1}{\operatorname{cap}_\alpha(\partial\Delta)}\right)$$

where C_α is a constant depending only on α.

We expect $C_\alpha = 1$, but we have not been able to show this.

5. Distortion of α-content.

The following is an extension of Löwner's lemma.

Theorem 4. *If $f : \mathbb{B}_n \longrightarrow \Delta$ is inner, $f(0) = 0$ and E is a Borel subset of $\partial\Delta$, then, for $0 < \alpha \le 1$,*

(i) $$M_{2n-2+\alpha}\left(f^{-1}(E)\right) \ge C_{n,\alpha}\, M_\alpha(E)$$

and

(ii) $$\mathcal{M}_{2(n-1+\alpha)}\left(f^{-1}(E)\right) \ge C'_{n,\alpha}\, M_\alpha(E)\,.$$

Here M_β and $\mathcal{M}_\beta$ denote, respectively, β-dimensional content with respect to the euclidean metric and with respect to the metric in $\mathbb{S}_n$ given by

$$d(a,b) = |1 - \langle a, b\rangle|^{1/2},$$

where $\langle a, b\rangle = \sum a_j \bar{b}_j$ is the inner product in $\mathbb{C}^n$. This metric is equivalent to the Carnot-Carathèodory metric in the Heisenberg group model for $\mathbb{S}_n$. We refer to [**R**] for details about this metric.

Recall that in a general metric space (X, d) the α-content of a set $E \subset X$ is defined as

$$M_\alpha(E) = \inf \left\{ \sum_i r_i^\alpha : \ E \subset \bigcup_i B_d(x_i, r_i) \right\}.$$

Observe that, as a consequence of Theorem 4, one obtains

Corollary. *If $f : \mathbb{B}_n \longrightarrow \Delta$ is inner and E is a Borel subset of $\partial\Delta$, then*

$$\operatorname{Dim}(f^{-1}(E)) \geq 2n - 2 + \operatorname{Dim}(E)$$

and

$$\mathcal{D}im\,(f^{-1}(E)) \geq 2n - 2 + 2\operatorname{Dim}(E)$$

where Dim *and $\mathcal{D}im$ denote, respectively, Hausdorff dimension with respect to the euclidean metric and the metric d.*

In order to prove Theorem 4 we will prove a lemma about Poisson integrals. We need to consider the classical Poisson kernel (not normalized)

$$P(\xi, z) = \frac{1 - |z|^2}{|\xi - z|^{2n}} \qquad (z \in \mathbb{B}_n\,,\ \xi \in \mathbb{S}_n)\,,$$

and the invariant Poisson kernel

$$Q(\xi, z) = \frac{(1 - |z|^2)^n}{|1 - \langle z, \xi\rangle|^{2n}} \qquad (z \in \mathbb{B}_n\,,\ \xi \in \mathbb{S}_n)\,.$$

Of course, they coincide if $n = 1$. In this section if ν is a positive measure in $\mathbb{S}_n$, we will denote by P_ν the function

$$P_\nu(z) = \int_{\mathbb{S}_n} P(\xi, z)\, d\nu(\xi)$$

and by Q_ν the invariant Poisson extension of ν

$$Q_\nu(z) = \int_{\mathbb{S}_n} Q(\xi, z)\, d\nu(\xi)\,.$$

Lemma 10. *Let μ be a finite positive measure in $\partial\Delta$, and let $f : \mathbb{B}_n \longrightarrow \Delta$ be an inner function. Then, there exists a finite measure $\nu \geq 0$ in $\mathbb{S}_n$ such that $P_\mu \circ f = P_\nu$, and if ν has singular part σ and continuous part γ, and we denote by A the set*

$$A = \{\xi \in \mathbb{S}_n : \ P_\sigma(r\xi) \to +\infty\,, \ \textit{as } r \to 1\}$$

and by B the set

$$B = \left\{\xi \in \mathbb{S}_n : \ \exists \lim_{r\to 1} f(r\xi) = f(\xi), \ |f(\xi)| = 1 \ \ \textit{and} \ \ \lim_{r\to 1} P_\gamma(r\xi) > 0\right\},$$

then A has full σ-measure, B has full γ-measure and

$$A \cup B \subset f^{-1}(\text{support}\,\mu)$$

and so

$$\nu\left(f^{-1}(\text{support}\,\mu)\right) = \|\nu\|\,.$$

The same is true if we replace P_ν by $Q_{\nu'}$ $(P_\mu \circ f = Q_{\nu'})$ and A, B by the following sets

$$A' = \{\xi \in \mathbb{S}_n : \ Q_{\sigma'}(r\xi) \to +\infty\,, \ \textit{as } r \to 1\}\,,$$

and

$$B' = \left\{\xi \in \mathbb{S}_n : \ \exists \lim_{r\to 1} f(r\xi) = f(\xi), \ |f(\xi)| = 1 \ \ \textit{and} \ \ \lim_{r\to 1} Q_{\gamma'}(r\xi) > 0\right\},$$

where σ' and γ' denote, respectively, the singular and the continuous part of ν'.

Proof. We will prove the lemma only for the measure ν', since the proof of the result for ν is similar and standard.

Let $U : \Delta \longrightarrow \mathbb{C}$ be a holomorphic function such that $\operatorname{Re} U = P_\mu$. Then $U \circ f$ is also holomorphic and so $\operatorname{Re}(U \circ f) = P_\mu \circ f$ is pluriharmonic, i.e. harmonic and $\mathcal{M}$-harmonic (see e.g. [**R**, Theorem 4.4.9]). Therefore there exist finite positive measures ν and ν' in $\mathbb{S}_n$ such that

$$P_\mu \circ f = P_\nu\,, \qquad P_\mu \circ f = Q_{\nu'}\,.$$

Let us denote by E the support of μ. If $\xi \in A'$, then $|f(r\xi)| \to 1$ as $r \to 1$. The curve $\{f(r\xi) : \ 0 \leq r < 1\}$ in Δ must end on a unique point $e^{i\psi} = f(\xi) \in \Delta$, since otherwise we would have $P_\mu \equiv +\infty$ on a set of positive Lebesgue measure. Now, $e^{i\psi} \in E$, since otherwise P_μ vanishes continuously at $e^{i\psi}$. Therefore $A' \subset f^{-1}(E)$. Similarly one sees that $B' \subset f^{-1}(E)$.

The set A' has full σ'-measure since by the inequality (14), that we will prove later,

$$\left\{\xi \in \mathbb{S}_n : \ \underline{\mathrm{D}}\,\sigma'(\xi) = \infty\right\} \subset A',$$

where

$$\underline{\mathrm{D}}\,\sigma'(\xi) = \liminf_{r\to 0} \frac{\sigma'(B_d(\xi,r))}{|B_d(\xi,r)|},$$

and the set $\{\xi : \ \underline{\mathrm{D}}\,\sigma'(\xi) = \infty\}$ has full σ'-measure (see Lemma 11 below). Let us observe that ([**R**, p. 67])

$$|B_d(\xi,r)| \sim r^{2n}.$$

The set B' has full γ'-measure, since as $r \to 1$

$$Q_{\gamma'}(r\xi) \longrightarrow \frac{d\gamma'}{dL} \qquad \text{a.e.}$$

with respect to Lebesgue measure L (see, e.g., [**R**, Theorem 5.4.9]) and $\left\{\frac{d\gamma'}{dL} > 0\right\}$ has full γ'-measure. □

Lemma 11. *Suppose that μ is a singular positive Borel measure (with respect to Lebesgue measure) in $\mathbb{S}_n$. Then*

$$\underline{\mathrm{D}}\,\mu(x) = \infty \qquad a.e. \ \ \mu.$$

Proof. Let $\mathcal{A}$ be a Borel set such that $|\mathcal{A}| = 0$, and μ is concentrated on $\mathcal{A}$. Define for $\alpha > 0$

$$\mathcal{A}_\alpha = \left\{x \in \mathcal{A} : \ \underline{\mathrm{D}}\,\mu(x) < \alpha\right\}.$$

It is enough to prove that $\mu(\mathcal{A}_\alpha) = 0$, and by regularity that $\mu(K) = 0$ for all K compact subset of $\mathcal{A}_\alpha$.

Fix $\varepsilon > 0$. Since $K \subset \mathcal{A}_\alpha \subset \mathcal{A}$, $|K| = 0$ and so there exists an open set V with $K \subset V$ and $|V| < \varepsilon$ ($|\cdot|$ denotes Lebesgue measure).

Now, for each $x \in K$, we can find $r_x > 0$ such that

$$\frac{\mu(B_d(x,r_x))}{|B_d(x,r_x)|} < \alpha \qquad \text{and} \qquad B_d(x, r_x/3) \subset V.$$

The family $\{B_d(x, r_x/3) : \ x \in K\}$ covers K, hence we can extract a finite subcollection Φ that also covers K. Now, using a Vitaly-type lemma (see, e.g., [**R**, Lemma 5.2.3]), we can find a disjoint subcollection Γ of Φ such that

$$K \subset \bigcup_{\Gamma} B_d(x_i, r_{x_i}).$$

Note that as a consequence of Proposition 5.1.4 in [**R**] we have that

$$\Theta_d := \sup_{\delta} \frac{|B_d(x, r_x)|}{|B_d(x, r_x/3)|} < \infty .$$

Therefore

$$\begin{aligned}\mu(K) &\le \sum_{\Gamma} \mu\left(B_d(x_i, r_{x_i})\right) < \alpha \sum_{\Gamma} |B_d(x_i, r_{x_i})| \\ &< \Theta_d\, \alpha \sum_{\Gamma} |B_d(x_i, r_{x_i}/3)| \le \Theta_d\, \alpha\, |V| < \Theta_d\, \alpha\, \varepsilon .\end{aligned}$$

□

Proof of Theorem 4. We will prove only (ii), since (i) is obtained in a similar way.

Assume, as we may, that E is a closed subset of $\partial\Delta$ and $M_\alpha(E) > 0$. Then, see e.g. [**T**, p. 64], there exists a positive mass distribution on E of finite total mass, such that: (a) $\mu(E) = M_\alpha(E)$, (b) $\mu(I) \le C_\alpha |I|^\alpha$ for any open interval I, where C_α is a constant independent of E. A standard estimate shows that

$$P_\mu(z) \le \frac{C_\alpha}{(1-|z|)^{1-\alpha}} , \qquad (z \in \Delta) , \tag{12}$$

with C_α a new constant. Let $\nu' \ge 0$ be a measure in $\mathbb{S}_n$ such that $P_\mu \circ f = Q_{\nu'}$. Schwarz's lemma (see e.g. [**R**, Theorem 8.1.2]) and (12) give the corresponding inequality for ν':

$$Q_{\nu'}(z) \le \frac{C_\alpha}{(1-\|z\|)^{1-\alpha}} , \qquad (z \in \mathbb{B}_n) . \tag{13}$$

We claim that for each $z \in \mathbb{B}_n$

$$Q_{\nu'}(z) \ge C_n \, \frac{\nu'(B_d(\xi, (2(1-\|z\|))^{1/2}))}{(1-\|z\|)^n} , \qquad (z \in \mathbb{B}_n) , \tag{14}$$

where $\xi = z/\|z\|$ and $B_d(\xi, R)$ denotes the d-ball with center ξ and radius R.

Assuming (14) for the moment and using (13), we obtain that

$$\nu'(B_d(\xi, R)) \le C_{n,\alpha}\, R^{2(n-1+\alpha)} , \qquad (\xi \in \mathbb{S}_n , R > 0) . \tag{15}$$

If we cover the set $A' \cup B'$ (see Lemma 14) with d-balls of radii R_i, we see by (15) that

$$\nu'(A' \cup B') \le C_{n,\alpha} \sum_i R_i^{2(n-1+\alpha)}$$

and so

$$\begin{aligned}\|\nu'\| = \nu'(A' \cup B') &\le C_{n,\alpha}\, \mathcal{M}_{2(n-1+\alpha)}(A' \cup B') \\ &\le C_{n,\alpha}\, \mathcal{M}_{2(n-1+\alpha)}\left(f^{-1}(E)\right) .\end{aligned}$$

So, since $f(0) = 0$,

$$M_\alpha(E) = \|\mu\| = \|\nu'\| \le C_{n,\alpha}\, \mathcal{M}_{2(n-1+\alpha)}\left(f^{-1}(E)\right) .$$

Therefore, in order to finish the proof, it remains only to prove (14). Observe first that we can assume that $\xi = e_1 = (1, 0, \dots, 0)$ since d is invariant under the unitary transformations of $\mathbb{S}_n$ for the inner product $\langle \cdot, \cdot \rangle$. Now, if $z = re_1$, write $\delta^2 = 2(1-r)$. If $\eta \in B_d(e_1, \delta)$, then

$$|1 - r\eta_1| \le |1 - \eta_1| + |\eta_1|(1-r) \le 3(1-r) .$$

Hence, if $\eta \in B_d(e_1, \delta)$

$$Q(\eta, z) = \left(\frac{1-r^2}{|1-r\eta_1|^2}\right)^n \ge \frac{9^{-n}}{(1-r)^n} .$$

Since Q is invariant under the action of the unitary group for the inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{S}_n$, we obtain that if $z = r\xi$ and $\eta \in B_d(\xi, \delta)$, then

$$Q(\eta, z) \ge \frac{9^{-n}}{(1-r)^n} .$$

Finally,

$$Q_{\nu'}(z) \ge \int_{B_d(\xi,\delta)} Q(\eta, z)\, d\nu'(\eta) \ge 9^{-n} \frac{\nu'(B_d(\xi, \delta))}{(1-r)^n} .$$

□

6. Distortion of subsets of the disc.

We have discussed how inner functions distort boundary sets. There are some results on how they distort subsets of Δ. On the one hand Hamilton [**H**] has shown that

Theorem H. *For all Borel subsets E of Δ,*

$$H_\alpha\left(f^{-1}(E)\right) \ge H_\alpha(E) , \qquad 0 < \alpha \le 1 ,$$

where H_α denotes α-Hausdorff measure.

One naturally expects the following to be true:

If $f : \Delta \longrightarrow \Delta$ is inner, $f(0) = 0$ and E is a Borel subset of Δ, then

$$\operatorname{cap}_\alpha\left(f^{-1}(E)\right) \geq \operatorname{cap}_\alpha(E)\,.$$

This we can prove only if $\alpha = 0$. The idea comes from [**P1**, p. 336].

Theorem 5. *Let $f : \Delta \longrightarrow \Delta$ be an inner function. If for some $k \geq 1$*

$$f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0\,, \qquad f^{(k)}(0) \neq 0\,,$$

then,

$$\operatorname{cap}_0\left(f^{-1}(E)\right) \geq \left(\operatorname{cap}_0(E)\right)^{1/k}\,,$$

for all Borel subsets of Δ. Moreover, this inequality is sharp.

Sketch of proof. By approximation, it is enough to prove it if E is closed and f is a finite Blaschke product. Let f be

$$f(z) = z^k \prod_{j=1}^{d} e^{i\nu_j} \frac{z - a_j}{1 - \bar{a}_j z}\,.$$

Denote by g_E, g_F the Green's functions of the unbounded connected component of $\hat{\mathbb{C}} \setminus E$ and $\hat{\mathbb{C}} \setminus F$ (here $F = f^{-1}(E)$) with pole at ∞. Therefore,

$$g_E(z) - \log|z| = \log\frac{1}{\operatorname{cap}_0(E)} + O\left(|z|^{-1}\right)\,,$$
$$g_F(z) - \log|z| = \log\frac{1}{\operatorname{cap}_0(F)} + O\left(|z|^{-1}\right)\,,$$

as $|z| \to \infty$. Moreover, since $k \geq 1$

$$g_E(f(z)) - k\log|z| + \log\prod_{j=1}^{d}|a_j| = \log\frac{1}{\operatorname{cap}_0(E)} + O\left(|z|^{-1}\right)\,,$$

as $|z| \to \infty$. It is easy to see that

$$g_E(f(z)) - \sum_{j=1}^{d} g_F\left(z, \bar{a}_j^{-1}\right)$$

is harmonic in the unbounded connected component of $\mathbb{C}\setminus\left(F \cup \left(\cup_{j=1}^{d}\left\{\bar{a}_j^{-1}\right\}\right)\right)$ and it is bounded at the points $\bar{a}_j^{-1}$ (here $g_F(z, \bar{a}_j^{-1})$ denotes the Green's function of the unbounded connected component of $\hat{\mathbb{C}} \setminus F$ with pole at $\bar{a}_j^{-1}$). Therefore, the function

$$G(z) = \frac{1}{k} g_E(f(z)) - g_F(z) - \frac{1}{k} \sum_{j=1}^{d} g_F\left(z, \overline{a}_j^{-1}\right) \tag{16}$$

is harmonic and bounded in the unbounded connected component of $\hat{\mathbb{C}} \setminus F$. Since $G = 0$ on the outer boundary of F, it follows that $G \equiv 0$.

Now, by using the symmetry of Green's function, we have that

$$g_F\left(z, \bar{a}_j^{-1}\right) \longrightarrow g_F\left(\bar{a}_j^{-1}\right), \qquad \text{as } |z| \to \infty,$$

and so, from (16),

$$\log \frac{1}{\mathrm{cap}_0(E)} - \log \prod_{j=1}^{d} |a_j| - k \log \frac{1}{\mathrm{cap}_0(F)} - \sum_{j=1}^{d} g_F\left(\overline{a}_j^{-1}\right) = 0. \tag{17}$$

On the other hand, since $F \subset \Delta$, the maximum principle says that

$$g_F(z) \geq g_\Delta(z) = \log |z|, \qquad |z| > 1.$$

Hence, from (17), we obtain that

$$\log \frac{1}{\mathrm{cap}_0(E)} - \log \prod_{j=1}^{d} |a_j| - k \log \frac{1}{\mathrm{cap}_0(F)} \geq \sum_{j=1}^{d} \log |a_j|^{-1},$$

and the inequality in the theorem follows.

Finally, to show that the inequality is sharp one simply has to consider the function $f(z) = z^k$. □

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