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SOBOLEV SPACES ON LIPSCHITZ CURVES

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We study Sobolev spaces on Lipschitz graphs Γ, by means of a square function of a geometric second difference. Given a function in the Sobolev space $W^{1,p}(\Gamma)$ we show that the geometric square function is also in $L^p(\Gamma)$. For $p = 2$ we prove a dyadic analogue of this result, and a partial converse.

1. Introduction.

The Sobolev space on the real line, $W^{1,p}(\mathbf{R})$, is the set of functions in $L^p(\mathbf{R})$ whose distributional derivatives are also functions in $L^p(\mathbf{R})$.

There are several characterizations of these spaces. In the early 80's Dorronsoro (see [**Do**]) gave a mean oscillation characterization of potential spaces, extending earlier results due to R.S. Stritchartz. In the late 80's, Semmes showed that the Sobolev spaces $W^{1,2}(M)$ have many of the properties of $W^{1,2}(\mathbf{R}^n)$ when M is a chord-arc surface (see [**Se**]). Dorronsoro and Semmes used square functions closely related to the square functions we use.

There is a characterization, due to E. Stein (see [**St1**] Ch.V) that involves the second differences of the given function. More precisely, let

$$\Delta_t f(x) = f(x+t) + f(x-t) - 2f(x),$$

and define the square function

$$Sf(x) = \left(\int_0^\infty |\Delta_t f(x)|^2 \frac{dt}{t^3}\right)^{1/2}.$$

Then the following result is true (see [**St1**]):

Theorem A [Stein]. *For $1 < p < \infty$, $f \in W^{1,p}(\boldsymbol{R})$ if and only if f, $Sf \in L^p(\boldsymbol{R})$. Moreover $\|Sf\|_p \sim \|f'\|_p$.*

For $p = 2$ the proof of this theorem is just an application of Plancherel's theorem. In this case $\|Sf\|_2 = \|f'\|_2$.

It is important for applications (eg. boundary problems for PDE's) to obtain similar results when $\mathbf{R}$ is replaced by a curve Γ. Smooth curves can be treated reducing to the case $\Gamma = \mathbf{R}$ after a suitable change of variables.

Difficulties appear when the curve is merely Lipschitz, as it often happens in harmonic analysis (eg. boundedness of the Cauchy integral on Lipschitz curves, see [**Ch**], [**M**], [**CJS**]).

Let Γ be a Lipschitz graph:

$$\Gamma = \{z = x + iA(x) : \ \|A'\|_\infty < \infty\}.$$

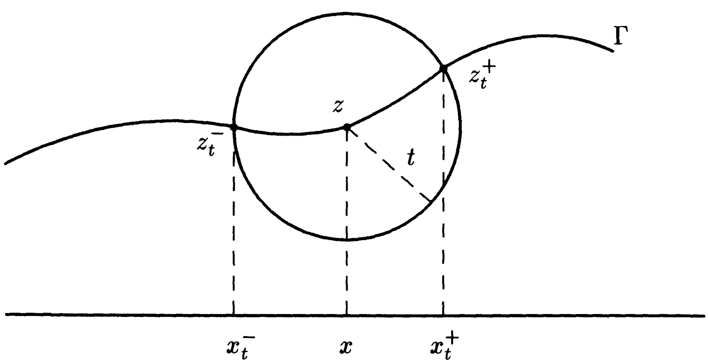
We define the Sobolev space on the curve just pulling back to the line,

$$W^{1,p}(\Gamma) = \{f \in L^p(\Gamma) : f(\tilde{A}) \in W^{1,p}(\mathbf{R}), \quad \tilde{A}(x) = x + iA(x)\}. \tag{1}$$

We introduce a *geometric second difference*, to do it we must restrict our attention to Lipschitz graphs with Lipschitz constant less than one. From now on Γ is always a Lipschitz graph, with $\|A'\|_\infty < 1$. For any $z \in \Gamma$, let

$$\tilde{\Delta}_t f(z) := f(z_t^+) + f(z_t^-) - 2f(z), \tag{2}$$

where $z_t^\pm$ are the *unique* points on Γ at distance t from z. It is clear that one point lies on the right and the other on the left of z, denoted respectively z_t^+ and z_t^-. Let us denote the corresponding x-coordinates x, $x_t^\pm$, see figure below,



We define the *geometric square function*, $\tilde{S}f$, by analogy with Stein's square function Sf; just replacing the second difference by the geometric one,

$$\tilde{S}f(z) = \left(\int_0^\infty |\tilde{\Delta}_t f(z)|^2 \frac{dt}{t^3}\right)^{1/2}, \quad z \in \Gamma.$$

We can prove the following result,

Theorem 1. *Let Γ be a Lipschitz graph with Lipschitz constant less than one. Assume $f \in W^{1,p}(\Gamma)$ then $\tilde{S}f \in L^p(\Gamma)$ for $1 < p < \infty$. Moreover*

$$\|\tilde{S}f\|_{L^p(\Gamma)} \le C\|f'\|_{L^p(\Gamma)}.$$

We can prove dyadic analogues of Theorem 1, and a partial converse. We assume the reader is familiar with the dyadic intervals on the line, and with the Haar basis (see definitions in Section 3).

Let us consider the case $\Gamma = \mathbf{R}$.

Denote by $\mathcal{D}$ the collection of dyadic intervals on the line. Let χ_I denote the characteristic function of the interval I.

Define the *dyadic square function* by:

$$S_d f(x) = \left(\sum_{I \in \mathcal{D}} \frac{|\Delta_I f|^2}{|I|^2} \chi_I(x) \right)^{1/2}, \tag{3}$$

where $\Delta_I f$ denotes the second difference of f associated to the interval $I = [x_I^-, x_I^+]$ centered at x_I, namely:

$$\Delta_I f = f(x_I^+) + f(x_I^-) - 2f(x_I).$$

The square function S_d is a dyadic analogue of the square function defined in the begining of the paper.

In this case, the analogue of Theorem 1 is very simple. The main observation being that the second difference $\Delta_I f$ of an absolutely continuous function f is, up to a scaling factor, the Haar coefficient of the derivative f' corresponding to the interval I. More precisely:

$$\Delta_I f = \langle f', h_I \rangle |I|^{1/2},$$

where the Haar function h_I is the step function supported on I that takes the values $\pm 1/|I|^{1/2}$ on the right and left halves of I, respectively.

The Haar functions indexed on $\mathcal{D}$ form a basis of $L^2(\mathbf{R})$. Hence if $f \in W^{1,2}(\mathbf{R})$, an application of Plancherel's Theorem for orthonormal systems implies:

$$\|f'\|_2^2 = \sum_{I \in \mathcal{D}} |\langle f', h_I \rangle|^2 = \sum_{I \in \mathcal{D}} \frac{|\Delta_I f|^2}{|I|}.$$

The right hand side coincides with the L^2 norm of the dyadic square function, hence:

$$f \in W^{1,2}(\mathbf{R}) \Rightarrow \|S_d f\|_2 = \|f'\|_2.$$

We also get a partial converse.

Define the *dyadic derivative*, Df, of $f \in L^2(\mathbf{R})$, as the L^2 limit (when it exists) of the sequence:

$$D_n f(x) = \frac{f(x_I^+) - f(x_I^-)}{|I|}, \qquad x \in I \in \mathcal{D}_n;$$

where $I = [x_I^-, x_I^+]$, and $\mathcal{D}_n$ denotes the nth-generation of dyadic intervals.

In this case if f and $S_d f$ are in $L^2(\mathbf{R})$, then the limit exists, so Df is in $L^2(\mathbf{R})$. Moreover, $\|Df\|_2 = \|S_d f\|_2$. This is another application of Plancherel's Theorem, once we observe that:

$$Df(x) = \sum_{I \in \mathcal{D}} \frac{\Delta_I f}{|I|^{1/2}} h_I(x).$$

We are ready now to describe the results for Lipschitz curves. We will replace the dyadic square function S_d by a *geometric dyadic square function* $\tilde{S}_d$.

We construct a family $\mathcal{F}$ of intervals related to the geometry of the problem. $\mathcal{F}$ is what we call a *regular dyadic grid.* It preserves the nesting properties of the standard dyadics, but the scaling is more involved. (For the precise definitions see Section 3.2.)

Let Γ be a Lipschitz graph with Lipschitz constant less than one. For a function f on Γ define the *geometric second difference* corresponding to the interval I by:

$$\tilde{\Delta}_I f = f(z_I^+) + f(z_I^-) - 2f(z_I);$$

where $z_I^\pm$ are the points on the curve Γ whose projections coincide with the endpoints, $x_I^\pm$ of I. And z_I is the unique point in Γ which is equidistant to both $z_I^\pm$.

Define now the *geometric dyadic square function:*

$$\tilde{S}_d(z) = \left(\sum_{I \in \mathcal{F}} \frac{|\tilde{\Delta}_I f|^2}{|I|^2} \chi_I(\pi(z)) \right)^{1/2};$$

where $\pi(z)$ is the X-coordinate of z.

We can then prove an analogue of Theorem 1 (for $p = 2$):

Theorem 1′. *Let Γ be a Lipschitz graph with Lipschitz constant smaller than one. Assume $f \in W^{1,2}(\Gamma)$ then $\tilde{S}_d f \in L^2(\Gamma)$. Moreover*

$$\|\tilde{S}_d f\|_2 \leq C\|f'\|_2.$$

We also get a partial converse, which is the main result of this paper.

Define the *dyadic derivative of f associated to the grid $\mathcal{F}$, $D_{\mathcal{F}} f$,* for $f \in L^2(\Gamma)$, as the limit in $L^2(\Gamma)$ (when it exists) of the sequence:

$$D_n f(z) = \frac{f(z_I^+) - f(z_I^-)}{z_I^+ - z_I^-}, \quad \pi(z) \in I \in \mathcal{F}_n,$$

where $\mathcal{F}_n$ is the n$^t h$-generation of $\mathcal{F}$ (see Section 3.2).

Theorem 2. *Let Γ be a Lipschitz graph with Lipschitz constant smaller than one. Assume both f and $\tilde{S}_d f$ are in $L^2(\Gamma)$. Then $D_{\mathcal{F}} f$ exists as a limit in $L^2(\Gamma)$. Moreover, $\|D_{\mathcal{F}} f\|_2 \le C\|\tilde{S}_d\|_2$.*

It should be clear that if we know *a priori* that $f \in W^{1,2}(\Gamma)$, then $f' = D_{\mathcal{F}} f$, and hence $\|f'\|_2 \le C\|\tilde{S}_d f\|_2$.

To prove these theorems we try to mimic the argument described in the case $\Gamma = \mathbf{R}$. We build a Haar basis adjusted to the Lipschitz curve Γ and supported on the grid $\mathcal{F}$ which itself is related to the geometry of the problem. This can be done without great difficulty, we will not get a basis but a frame, exactly as in [**CJS**] for the study of Cauchy integrals on Lipschitz curves.

In this setting the *Haar coefficients* of the derivative will not be exact multiples of $\tilde{\Delta}_I f$. There will be an error that can be controlled by the geometry of the problem.

The proof of Theorem 2 is not as straightforward as in the case of the line. Surprisingly enough it is here where operators like the ones studied in [**P**] appeared first. We will use the techniques developed there. For more details see the introduction to the third section.

The norm $\|\tilde{S}_d f\|_2^2 = \sum_{I\in\mathcal{F}} \frac{|\tilde{\Delta}_I f|^2}{|I|}$ can be regarded as a Riemann sum for

$$\int_R \int_0^\infty |\tilde{\Delta}_t f(z)|^2 \frac{dt}{t^3} dx = \|\tilde{S} f\|_2^2.$$

In the case $\Gamma = \mathbf{R}$ we could use Theorem 2 to prove the full converse of Stein's theorem, averaging over translations and dilations of the dyadic intervals. In the general case it is not clear how to do the averaging, since we no longer have the group structure of the line available. (See [**GJ**] for examples on how to go from dyadic to continuous situations.)

The paper is organized as follows: We will prove Theorem 1 in the next section; we will use a result of Dorronsoro and some Carleson type estimates. This proof, suggested by the referee, greatly simplifies the original proof of the author. In Section 3 we will prove Theorems 1′ and 2, together with all the discrete ingredients (see the introduction to Section 3 for more details).

Throughout this paper C is a constant that might change from line to line. We will use the notation $a \sim b$, for positive numbers a and b, whenever there exists a positive and finite constant C such that $C^{-1}b \le a \le Cb$; we will say, in that case, that a and b are *comparable*.

These results are part of my PhD thesis. I would like to thank my advisor P.W. Jones for suggesting the problem and guiding me through the completion of this work. I extend my warmest thanks to R.R. Coifman and Stephen Semmes for very helpful conversations. Finally, I am grateful to the referee who carefully read this paper, and made a lot of valuable suggestions.

2. Proof of Theorem 1.

We are going to prove in this section the necessity of the boundedness of the geometric square function $\tilde{S}f$ for a function f to be in the Sobolev space of a Lipschitz curve. The idea is to control the geometric square function by Stein's square function. There will be some left overs that can be controlled in turn by Dorronsoro's mixed norm estimate on the approximation of these functions by affine functions. Further errors can be handled by Carleson-type estimates given by the geometry of the curve.

Let us state some geometric lemmas that we will prove at the end of this section.

Recall that $x_t^{\pm}$ are the projections onto the real line of the points on the curve Γ which are at distance t from a given point $z \in \Gamma$ whose projection is x.

Lemma 1. *Let $u_x^+(t) := x_t^+ - x := t_x^+$, for $t > 0$; then $u_x^+ > 0$ is an increasing homeomorphism of t. Moreover it is uniformly bilipschitz on x, i.e.*

$$\frac{1}{C} \leq \frac{d\,u_x^+(t)}{dt} \leq C \quad \forall x, t.$$

Similarly for $\quad u_x^-(t) := x - x_t^- := t_x^- > 0.$

Let us define the following quantities, as they are defined by Peter Jones [**J**] in the Traveling Salesman Problem.

For a point $z \in K$, K a subset of the plane; and $t > 0$, let

$$\beta(z,t) = \inf_L \sup_{w \in K, |w-z| < 2t} t^{-1} \operatorname{dist}(w, L)$$

where L is any line in the plane. This quantity measures how close is the set $K \bigcap \{w : |w - z| < 2t\}$ to a line.

In our case $K = \Gamma$ and, since it is a graph, we will talk indistinctly about $z \in \Gamma$ or its projection $x \in \mathbf{R}$.

In general $t_x^+ \neq t_x^-$. This assymetry is what causes most of the problems. Since the curve is flat enough, we can control the difference

Lemma 2. $\quad |t_x^+ - t_x^-| \leq C\beta(x,t)\, t.$

We will prove Lemmata 1 and 2 at the end.

Recall that μ is a *Carleson measure* on the upper half plane if

$$\int_{\hat{I}} d\mu(x,t) \leq C|I| \quad \forall I \subset \mathbf{R}$$

where I is any interval of the line and $\hat{I}$ is the cube lifted above I.

Finally we can control the β's in the sense that

Lemma 3 (P. Jones' Geometric Lemma). *The measure given by*

$$d\mu(x,t) = \beta^2(x,t)\frac{dt}{t}dx,$$

is a Carleson measure on the upper half plane $\boldsymbol{R}^2_+$.

For a proof of this result see [**J**] and also [**Do**].

We will need the following facts concerning Carleson measures:

Carleson's Lemma. *Given a Carleson measure μ in the upper half plane, and a positive function $F(x,t)$ then*

$$\int_{\boldsymbol{R}}\int_0^\infty [F(x,t)]^p d\mu(x,t) \leq C\int_{\boldsymbol{R}} [F^*(x)]^p dx, \quad 0 < p < \infty$$

where $F^*(x) = \sup_{t>0,|y-x|<t} F(y,t)$.

For a proof of this lemma and the next see [**St2**], Corollary 2.4 in Ch.II.

As an immediate consequence of Carleson's Lemma and the Hardy-Littlewood Maximal Theorem, we conclude that for the case $F(x,t) = |m_{x,t}f|$, where $m_{x,t}f = \frac{1}{2t}\int_{x-t}^{x+t} f(y)dy$ the following inequality is true:

Lemma 4. *Given a Carleson measure μ in the upper half plane, and $f \in L^p(\boldsymbol{R})$ for $1 < p < \infty$, then:*

$$\int_{\boldsymbol{R}}\int_0^\infty |m_{x,t}f|^p d\mu(x,t) \leq C\int_{\boldsymbol{R}} |f(x)|^p dx.$$

We can deduce from this lemma the following mixed norm estimate; here the β's, are the ones given by the geometry, which in particular are bounded by a constant.

Lemma 5. *Given the Carleson measure in the upper half plane,*

$$d\mu(x,t) = \beta^2(x,t)\frac{dt}{t}dx$$

and $f \in L^p(\boldsymbol{R})$ for $1 < p < \infty$, then:

$$\int_{\boldsymbol{R}}\left(\int_0^\infty |m_{x,t}f|^2\beta^2(x,t)\frac{dt}{t}\right)^{p/2} dx \leq C\int_{\boldsymbol{R}} |f(x)|^p dx.$$

We will prove this result at the end of the section.

We are going to use the following result due to Dorronsoro:

Theorem [Dorronsoro]. *Let $f \in W^{1,p}(\boldsymbol{R})$ be given, with $1 < p < \infty$. Then for each $x \in \boldsymbol{R}$ and $t > 0$ there is an affine function $a_{x,t}$ with the following properties:*

$$|a'_{x,t}| \leq Ct^{-1} \int_{x-t}^{x+t} |f'(y)|dy; \tag{4}$$

$$\int_{\mathbf{R}} \left(\int_0^\infty \left(t^{-1} \sup_{|x-y|\leq t} |f(y) - a_{x,t}(y)| \right)^2 \frac{dt}{t} \right)^{p/2} dx \leq C \int_{\mathbf{R}} |f'(x)|^p dx. \tag{5}$$

If we drop the condition (4) this is a special case of Theorem 6 (i) in [**Do**]. The affine function $a_{x,t}$ used by Dorronsoro is the unique one such that:

$$\int_{x-t}^{x+t} [f(y) - a_{x,t}(y)]y^k dy = 0, \qquad k = 0, 1.$$

It can be computed explicitly. It is not hard to see that:

$$|a'_{x,t}| \leq \frac{C}{t} \left[\frac{1}{t} \int_{x-t}^{x+t} |f(y) - m_{x,t}f|dy \right].$$

The following inequality is true for absolutely continuous functions:

$$\frac{1}{t} \int_{x-t}^{x+t} |f(y) - m_{x,t}f|dy \leq C \int_{x-t}^{x+t} |f'(y)|dy;$$

(it is a calculus exercise to check it). Since functions $f \in W^{1,p}(\mathbf{R})$ are absolutely continuous after modifications on a set of measure zero, we see that condition (4) holds in Dorronsoro's Theorem.

Proof of Theorem 1. We want to bound with a constant times the L^p norm of the derivative of a function $f \in W^{1,p}(\mathbf{R})$ the following expression

$$\left(\int_{\mathbf{R}} \left(\int_0^\infty |f(x_t^+) + f(x_t^-) - 2f(x)|^2 \frac{dt}{t^3} \right)^{p/2} dx \right)^{1/p}. \tag{6}$$

Recall that $x_t^+ = x + t_x^+$. To get a symmetric second difference, add and subtract $f(x - t_x^+)$, we can bound (6) by Minkowski's inequality, up to a constant by:

$$(7)\quad \left(\int_{\mathbf{R}}\left(\int_0^\infty |f(x+t_x^+)+f(x-t_x^+)-2f(x)|^2\frac{dt}{t^3}\right)^{p/2}dx\right)^{1/p} + \left(\int_{\mathbf{R}}\left(\int_0^\infty |f(x-t_x^-)-f(x-t_x^+)|^2\frac{dt}{t^3}\right)^{p/2}dx\right)^{1/p}.$$

The first summand can be reduced to the euclidean case. Let us do the change of variable $s = t_x^+ = u_x^+(t)$; by Lemma 1, $s \sim t$, $ds \sim dt$. We can bound the first term by:

$$C\left(\int_{\mathbf{R}}\left(\int_0^\infty |f(x+s)+f(x-s)-2f(x)|^2\frac{ds}{s^3}\right)^{p/2}dx\right)^{1/p},$$

which is bounded by $C\|f'\|_p$ by Theorem A.

We are left with the second integral in (7). This time we will add and subtract $a_{x,t}(x-t_x^-)$ and $a_{x,t}(x-t_x^+)$; where $a_{x,t}$ is the affine function given in Dorronsoro's theorem. Certainly:

$$|f(x-t_x^\pm)-a_{x,t}(x-t_x^\pm)| \le \sup_{|y-x|\le t}|f(y)-a_{x,t}(y)|.$$

We can then bound (7) by a constant times:

$$(8)\quad \left(\int_{\mathbf{R}}\left(\int_0^\infty \left(t^{-1}\sup_{|y-x|\le t}|f(y)-a_{x,t}(y)|\right)^2\frac{dt}{t}\right)^{p/2}dx\right)^{1/p} + \left(\int_{\mathbf{R}}\left(\int_0^\infty |a_{x,t}(x-t_x^-)-a_{x,t}(x-t_x^+)|^2\frac{dt}{t^3}\right)^{p/2}dx\right)^{1/p}$$

The first term is bounded by $C\|f'\|_p$ by Dorronsoro's theorem. The second can be rewritten as:

$$\left(\int_{\mathbf{R}}\left(\int_0^\infty |a'_{x,t}|^2|t_x^+-t_x^-|^2\frac{dt}{t^3}\right)^{p/2}dx\right)^{1/p};$$

and using Dorronsoro's estimate (4) and Lemma 2, we can bound this by

$$\left(\int_{\mathbf{R}}\left(\int_0^\infty \left[\frac{1}{2t}\int_{x-t}^{x+t}|f'|\right]^2\beta^2(x,t)\frac{dt}{t}\right)^{p/2}dx\right)^{1/p};$$

which in turn is bounded by Carleson's mixed norm lemma (Lemma 5), and P. Jones geometric lemma (Lemma 3) by $C\|f'\|_p$.

This finishes the proof of Theorem 1 except for the geometric lemmas, and Carleson's mixed norm lemma. $\square$

Proof of Lemma 1. We want to prove that $u_x^+(t) = t_x^+ = x_t^+ - x$ is an increasing bilipschitz homeomorphism. Clearly u_x^+ is increasing (because Γ is a Lipschitz graph with Lipschitz constant less than one). The inverse of this mapping is given by the distance between the images on the curve Γ of x and $y = x+s$, namely $(u_x^+)^{-1}(s) = |\tilde{A}(x+s) - \tilde{A}(x)|$, where A is the Lipschitz map defining Γ and $\tilde{A}(y) = y + iA(y)$. By hypothesis, $|A(y+h) - A(y)| \le \eta h$ where $\eta < 1$.

Showing that u_x^+ is bilipschitz is equivalent to show that its inverse is bilipschitz. To show this it is enough to show that there exists a constant C such that $\forall x,\ s \ge 0,\ h > 0$

$$\frac{1}{C} \le \frac{(u_x^+)^{-1}(s+h) - (u_x^+)^{-1}(s)}{h} \le C.$$

We can assume without loss of generality that $x = A(x) = 0$. We want to bound $(|\tilde{A}(y+h)| - |\tilde{A}(y)|)/h$, from above and below.

The upper bound is trivial by the triangle inequality and by the fact that the map $\tilde{A}$ is bilipschitz, since

$$h \le |\tilde{A}(y+h) - \tilde{A}(y)| = |h + i(A(y+h) - A(y))| \le h\sqrt{1+\eta^2}.$$

Note that for all z and y,

$$|\tilde{A}(z)|^2 - |\tilde{A}(y)|^2 = z^2 - y^2 + (A^2(z) - A^2(y)).$$

It is not hard to check that for every $0 < y < z$

$$A^2(z) - A^2(y) \ge -\eta^2(z^2 - y^2),$$

therefore $|\tilde{A}(z)|^2 - |\tilde{A}(y)|^2 \ge (1-\eta^2)(z^2 - y^2)$.

Since $|\tilde{A}(z)| \le |z|\sqrt{1+\eta^2}$, then

$$\frac{|z| + |y|}{|\tilde{A}(z)| + |\tilde{A}(y)|} \ge \frac{1}{\sqrt{1+\eta^2}},$$

hence, choosing $z = y + h,\ \ h > 0$ we get that

$$\frac{|\tilde{A}(y+h)| - |\tilde{A}(y)|}{h} \ge \frac{1-\eta^2}{\sqrt{1+\eta^2}},$$

which is certainly larger than zero, since $\eta < 1$.

This finishes the proof of the lemma. $\square$

Proof of Lemma 2. We want to show that there exists a constant C independent of x and t such that

$$|t_x^+ - t_x^-| \leq C\, t\beta(x,t),$$

where the β's were defined for $z = \tilde{A}(x)$, by

$$\beta(x,t) = \inf_L \sup_{w\in\Gamma,\ |w-z|<2t} t^{-1} \operatorname{dist}(w, L),$$

and L is any line in the plane.

Notice that the height h of the isosceles triangle drawn through the images on the curve of x, $x_t^+ = x + t_x^+$ and $x_t^- = x - t_x^-$ (which we will denote respectively by z, z_t^+ and z_t^-) is certainly bounded by $t\beta(x,t)$.

Therefore it is enough to show that $|t_x^+ - t_x^-| \leq C\, h$.

Let $\alpha = \alpha(x,t)$ be the common angle in the isosceles triangle. Let $\theta = \theta(x,t)$ be the angle between the horizontal and the chord through z_t^+ and z_t^-. We can assume without loss of generality that $\theta \geq 0$ and that $\arg z \geq \arg z_t^-$. Then high school geometry shows that

$$t_x^- = t\cos(\alpha+\theta), \quad h = t\sin\alpha,$$

$$t\cos\alpha\cos\theta = \frac{x_t^+ - x_t^-}{2} = \tilde{x}_t, \qquad t_x^+ = (x_t^+ - x_t^-) - t_x^-.$$

Therefore $t_x^- = \tilde{x}_t - h\sin\theta$. and $t_x^+ = \tilde{x}_t + h\sin\theta$.

Hence

$$|t_x^+ - t_x^-| = 2h\sin\theta < 2h.$$

We can have a better bound if we notice that $\sin\theta \leq \frac{\eta}{1+\eta^2}$.

This finishes the proof of Lemma 2. $\square$

Proof of Lemma 5. The case $p = 2$ is an immediate consequence of Lemma 4. We will get the inequality for $1 < p < 2$ using the atomic decomposition of the tent spaces T_∞^q for $q \leq 1$ (see [**CMS**]), as suggested by the referee. For $2 < p < \infty$ we will get the result interpolating between a mixed L^2 norm space and the space of Carleson measures.

Case $1 < p < 2$**:** Denote by $\Gamma(x)$ the standard cone whose vertex is x, i.e., $\Gamma(x) = \{(y,t) : |y-x| < t\}$. For a function G on $\mathbf{R}_+^2$, define $A_\infty(G)(x) = \sup_{(y,t)\in\Gamma(x)} |G(y,t)|$.

The *tent space* T^q_∞ consists of exactly those functions G continuous in $\mathbf{R}^2_+$, so that $A_\infty(G) \in L^q(\mathbf{R})$, and for which $G(x,t)$ has non-tangential limits at the boundary almost everywhere. We define $\|G\|_{T^q_\infty} = \|A_\infty(G)\|_q$.

A T^q_∞-atom is a function $a(x,t)$ supported on a tent $\hat{I}$, and such that $\sup_{(x,t)} |a(x,t)| \le 1/|I|^{1/q}$; where I is an interval centered at x_I, and $\hat{I} = \{(x,t) \in \mathbf{R}^2_+ : x \in I, t < |I|/2 - |x - x_I|\}$. Clearly $\|a\|_{T^q_\infty} \le 1$. The atomic decomposition for T^q_∞ when $q \le 1$ given in Proposition 5 on p. 326 of [**CMS**], says that if $G \in T^q_\infty$, $q \le 1$, then $G(x,t) = \sum \lambda_j a_j(x,t)$, where a_j are T^q_∞-atoms. Moreover $\sum |\lambda_j|^q \le \|G\|^q_{T^q_\infty}$.

Let $f \in L^p(\mathbf{R})$ be given and set

$$F(x,t) = |m_{x,t} f|^2.$$

Then F lies in the tent space T^q_∞ of [**CMS**] with $q = p/2 < 1$. Moreover, as an application of the Hardy-Littlewood Theorem, $\|F\|^{p/2}_{T^{p/2}_\infty} \le C\|f\|^p_p$.

It is simple to check for $T^{p/2}_\infty$-atoms, $a(x,t)$, that the quantity:

$$\int_{\mathbf{R}} \left(\int_0^\infty a(x,t)\beta^2(x,t)\frac{dt}{t} \right)^{p/2} dx, \tag{9}$$

is bounded by a constant C independent of the atom a. More precisely, using the support and size properties of the atom we see that (9) is bounded by:

$$\frac{1}{|I|}\int_I \left(\int_0^{|I|/2} \beta^2(x,t)\frac{dt}{t} \right)^{p/2} dx \le \left(\frac{1}{|I|}\int_I \int_0^{|I|/2} \beta^2(x,t)\frac{dt}{t}dx \right)^{p/2} \le C;$$

the first inequality by the Cauchy-Schwartz inequality with $p' = 2/p > 1$, the last one by P. Jones' geometric lemma.

Finally, writing an atomic decomposition for $F(x,t) = \sum \lambda_j a_j(x,t)$, using the above estimate for atoms, and the fact that $p/2 < 1$, we conclude that

$$\int_{\mathbf{R}} \left(\int_0^\infty F(x,t)\beta^2(x,t)\frac{dt}{t} \right)^{p/2} dx \le \sum C\lambda_j^{p/2} \le C\|F\|^{p/2}_{T^{p/2}_\infty} \le C\|f\|^p_p.$$

Case $2 < p < \infty$: Let us introduce the mixed norm spaces, $1 < p < \infty$

$$L^{2,p} = \left\{ f : \mathbf{R}^2_+ \to \mathbf{R}; \|f\|_{2,p} = \left(\int_{\mathbf{R}} \left(\int_0^\infty |f(x,t)|^2 \frac{dt}{t} \right)^{p/2} dx \right)^{1/p} < \infty \right\}.$$

Define the Carleson measure space by:

$$CM = \left\{ g : \mathbf{R}^2_+ \to \mathbf{R}; \|g\|_{CM} = \sup_{I \subset \mathbf{R}} \frac{1}{|I|}\int_I \left(\int_0^{|I|} g^2(x,t)\frac{dt}{t} \right)^{1/2} dx < \infty \right\}.$$

These are Banach spaces with the corresponding norms. We can interpolate between mixed norm spaces and Carleson measure space. In the sense that, given a linear operator T bounded simultaneously from L^2 into $L^{2,2}$, and from L^∞ into CM, it is also bounded from L^p into $L^{2,p}$, for $2 < p < \infty$. See [**CMS**] and [**AM**].

Define the linear operator T for integrable functions by:

$$Tf(x,t) = \beta(x,t)m_{x,t}f.$$

T is bounded from L^2 into $L^{2,2}$, it only remains to check that is bounded from L^∞ into CM. We want to show that:

$$\frac{1}{|I|}\int_I \left(\int_0^{|I|} |m_{x,t}f|^2\beta^2(x,t)\frac{dt}{t}\right)^{1/2} dx < C\|f\|_\infty.$$

Certainly $|m_{x,t}f| \leq \|f\|_\infty$; substituting it into the integral, applying the Cauchy-Schwartz inequality, and using once more P. Jones' geometric lemma we get the desired inequality.

As it was pointed out by the referee, the result for $p > 2$ is related to Remark b on p. 320 of [**CMS**]. This remark addresses essentially the same point, but with integrals in t replaced by integrals over cones.

This finishes the proof of the mixed norm Carleson's lemma. $\square$

3. Dyadic Version.

3.1. Introduction. Let Γ be a Lipschitz graph, $\Gamma = \{z = x + iA(x) : \|A'\|_\infty < \infty\}$. We will assume that $\|A'\|_\infty < 1$, as before.

When $\Gamma = \mathbf{R}$ it is not difficult to see that

$$f \in W^{1,2}(\mathbf{R}) \iff f, \tilde{S}f \in L^2(\mathbf{R}).$$

As we pointed out in the introduction of the paper, in this case this result can be regarded as a continuous version of Plancherel's theorem for the Haar basis. The key observation being that the Haar coefficients of the derivative f' of an absolutely continuous function f are, up to a scaling factor, the second difference of f at the corresponding interval.

We will take advantage of this natural *dyadic* interpretation in order to develop a discrete approach to the problem.

In Section 3.2 we will introduce the *regular dyadic grids* (substitutes for an ordinary dyadic grid). We will construct some Haar systems associated to these grids and to a *nice* complex measure $d\sigma$ (by nice we mean absolutely continuous with respect to Lebesgue measure, and such that $|\sigma(I)| \sim |I|$ for all intervals I in the grid, where $\sigma(I) = \int_I d\sigma$).

In Section 3.3 we will construct a regular dyadic grid $\mathcal{F}$ adjusted to the geometry of the problem and the corresponding Haar system $\{h_I^\sigma\}_{I\in\mathcal{F}}$, associated to the measure $d\sigma = (1+iA'(x))dx$ (this measure is certainly *nice*). We will show that this particular Haar system is a *frame*, i.e. it behaves almost like an orthonormal basis (see [**CJS**].) The deviation from the standard basis is controlled by a geometric quantity estimated in a *Geometric Lemma* (dyadic version of P. Jones Geometric Lemma 3, which in this case is very easy to prove; see [**J**]), and a discrete version of Carleson's Lemma.

Define the *geometric second difference* associated to the interval $I = (x_I^-, x_I^+)$ by

$$\tilde{\Delta}_I f = f(z_I^+) + f(z_I^-) - 2f(z_I),$$

where $z_I^\pm = x_I^\pm + iA(x_I^\pm)$, and $z_I \in \Gamma$ and is equidistant to $z_I^\pm$.

Define the *geometric dyadic square function*

$$\tilde{S}_d f(z) = \left(\sum_{I\in\mathcal{F}} \frac{|\tilde{\Delta}_I f|^2}{|I|^2}\chi_I(\pi(z))\right)^{1/2};$$

where $\pi(z)$ is the X-coordinate of z.

We can prove the dyadic analogue of Theorem 1, for $p=2$,

Theorem 1′. *Given $f \in W^{1,2}(\Gamma)$ then*

$$\|\tilde{S}_d f\|^2_{L^2(\Gamma)} = \sum_{I\in\mathcal{F}} \frac{|\tilde{\Delta}_I f|^2}{|I|} \le C\|f'\|^2_{L^2(\Gamma)}.$$

If we do not know *a priori* that $f \in W^{1,2}(\Gamma)$ we can still show a partial converse. Let $\mathcal{F}_n$ denotes the nth generation of the grid $\mathcal{F}$. Define the *dyadic derivative of f associated to the grid $\mathcal{F}$, $D_\mathcal{F} f$*, as the limit in $L^2(\Gamma)$, when it exists, of the sequence:

$$D_k f(z) = \frac{f(z_I^+) - f(z_I^-)}{z_I^+ - z_I^-}; \qquad \pi(z) \in I \in \mathcal{F}_k(J).$$

Theorem 2. *Assume that $f \in L^2(\Gamma)$ and that*

$$\|\tilde{S}_d f\|^2_{L^2(\Gamma)} = \sum_{I\in\mathcal{F}} \frac{|\tilde{\Delta}_I f|^2}{|I|} < \infty.$$

Then $D_\mathcal{F} f$ exists and is in $L^2(\Gamma)$. Moreover

$$\|D_\mathcal{F} f\|^2_{L^2(\Gamma)} \le C\|\tilde{S}_d f\|^2_{L^2(\Gamma)}.$$

It will be enough to prove local versions of these theorems. By this we mean to replace $\mathbf{R}$ by an interval J and prove the corresponding statements uniformly on J.

In Section 3.4 we will prove a local version of Theorem 1′. To do this we will use the orthogonality of the *Haar system* constructed and Carleson's Lemma for regular dyadic grids.

In Section 3.5 we will prove a local version of Theorem 2. We will reduce the problem to the boundedness of an operator, $P_{b,\sigma}$, that formally looks like the operator defined in [**P**] by,

$$P_b g = \sum_{n=0}^{\infty} \Delta_n g \prod_{j=n+1}^{\infty} (1 + \Delta_j b);$$

where g is a square integrable function, b comes from the geometry and is in the space of bounded mean oscillation functions (BMO), and $\Delta_n f$ is the projection onto the subspace generated by the Haar functions corresponding to the nth generation of the dyadics.

In Section 3.6 the operator $P_{b,\sigma}$ is analized. The strategy is the same as in [**P**]. We can rewrite the paraseries P_b in terms of the weight $\omega = \prod_{j=0}^{\infty}(1 + \Delta_j b)$ (see p. 581). The necessary and sufficient conditions for the boundedness of the operator P_b in L^2 are described in [**P**], and they reduce to a reverse Hölder condition on the weight. In our case the grid will be the regular dyadic grid $\mathcal{F}$; the Haar functions will not be the standard ones either. Nevertheless, we can mimic what we did in [**P**]. As we could expect, the boundedness of the operator will depend upon the boundedness of a weighted maximal operator, and this will be so provided the weight ω satisfies a Reverse Hölder condition on the grid. The proof in this case is simpler than in [**P**]; after a minute of reflexion we see that both the weight and the grid come from the geometry and some of the difficulties are cancelled out.

3.2. Dyadic grids and Haar functions. Consider a fix interval J. A *dyadic grid associated to* J is a collection of nested intervals $\mathcal{F}(J)$ such that $\mathcal{F}(J) = \bigcup_{n=0}^{\infty} \mathcal{F}_n(J)$. The *generations* $\mathcal{F}_n$ are defined inductively by $\mathcal{F}_{n+1}(J) = \bigcup_{I \in \mathcal{F}_n(J)} \mathcal{F}_1(I)$, and given any interval I, its first generation $\mathcal{F}_1(I) = \{I_l, I_r\}$ is a partition of I into two disjoint intervals that we will call the *children* of I.

A *regular dyadic grid associated to* J is a dyadic grid such that there is a constant $\frac{1}{2} \leq C < 1$, such that given any interval $\tilde{I} \in \mathcal{F}(J)$ and I a child of $\tilde{I}$ then

$$(1 - C)|\tilde{I}| \leq |I| \leq C|\tilde{I}|.$$

If $C = \frac{1}{2}$ we get the ordinary dyadic decomposition of J. In this case given any $I \in \mathcal{F}_n(J)$, $|I| = 2^{-n}|J|$.

If $C > \frac{1}{2}$ then we can only say that for any $I \in \mathcal{F}_n(J)$

$$(1 - C)^n|J| \leq |I| \leq C^n|J|.$$

This implies that given any point $x \in J$, if I_n is the unique interval in the nth generation that contains x then

$$\bigcap_{n=0}^{\infty} I_n = \{x\}; \quad \lim_{n\to\infty} |I_n| = 0.$$

It also implies that intervals of a given generation are comparable, but the comparison bounds are not independent of the generation.

We say that $\mathcal{F}$ is a *dyadic grid on* $\boldsymbol{R}$ if there exists a sequence of intervals $\{J_n\}_{n\geq 0}$ such that:

(i) $J_n \in \mathcal{F}_1(J_{n+1})$,

(ii) $\mathbf{R} = \cup_{n\geq 0} J_n$;

in that case $\mathcal{F} = \cup_{n\geq 0}\mathcal{F}(J_n)$. The generations can be defined by:

$$\mathcal{F}_k = \begin{cases} \cup_{n\geq 0}\mathcal{F}_{n+k}(J_n) & \text{for } k \geq 0 \\ \cup_{n\geq -k}\mathcal{F}_{n+k}(J_n) & \text{for } k < 0. \end{cases}$$

$\mathcal{F}$ is a *regular* dyadic grid on $\mathbf{R}$ if there exists a constant $1/2 < C < 1$ such that $(1 - C)|\tilde{I}| \leq |I| \leq C|\tilde{I}|$, for all $I \in \mathcal{F}$, $\tilde{I}$ parent of I.

Given any regular dyadic grid associated to an interval J, $\mathcal{F}(J)$, and an absolutely continuous measure σ, such that $|\sigma(I)| \sim |I|$, for all $I \in \mathcal{F}(J)$; there is a *Haar system* associated to them. More precisely for each $I \in \mathcal{F}(J)$, let I_r, I_l be the right and left children of I respectively, define

$$h_I^\sigma(x) = \left(\frac{\sigma(I_r)\sigma(I_l)}{\sigma(I)}\right)^{1/2} \left(\frac{1}{\sigma(I_r)}\chi_{I_r}(x) - \frac{1}{\sigma(I_l)}\chi_{I_l}(x)\right); \tag{10}$$

and

$$h_o^\sigma(x) = \frac{1}{\sigma^{1/2}(J)}\chi_J(x), \tag{11}$$

where χ_I is the characteristic function of I.

Clearly each h_I^σ is supported on I and is constant on each child. Moreover its mean value with respect to $d\sigma$ is zero. Therefore, if we denote by $\langle .,. \rangle_\sigma$ the bilinear operation $\langle f, g\rangle_\sigma = \int fg\, d\sigma$ (notice that there is no conjugation), the system $\{h_I^\sigma\}_{I\in\mathcal{F}(J)}$ behaves like an orthonormal system with respect to this pseudo inner product, i.e. $\langle h_I^\sigma, h_{I'}^\sigma\rangle_\sigma$ is zero if $I \neq I'$, and one if $I = I'$. The function h_o^σ is certainly "orthogonal" with respect to the bilinear form $\langle .,.\rangle_\sigma$ to all the h_I^σ's and $\langle h_o^\sigma, h_o^\sigma\rangle_\sigma = 1$. Let us state this result as the first part of the next lemma.

Lemma 6. *The* Haar system associated *to the regular dyadic grid $\mathcal{F}$ and the measure σ as defined above satisfies the following properties:*

- *"orthonormality" with respect to the bilinear form $\langle .,.\rangle_\sigma$.*
- *"reconstruction formula" for functions $f \in L^2_{\mathrm{Loc}}(J, d\sigma)$:*

$$f(x) = \sum_{I\in\mathcal{F}'(J)} \langle f, h_I^\sigma\rangle_\sigma h_I^\sigma(x), \qquad \sigma - a.e.x \tag{12}$$

where $\mathcal{F}'(J)$ is the grid $\mathcal{F}(J)$ with a second copy J_o of J and we agree that $h_{J_o}^\sigma := h_o^\sigma$.

The proof of this lemma is an standard application of Lebesgue's Differentiation Theorem (see for example [**P**] p. 631), replacing by the corresponding expectation and difference operators as defined next.

Define E_n^σ the *expectation operator* with respect to $d\sigma$, associated to the grid, by

$$E_n^\sigma f(x) = \frac{1}{\sigma(I)} \int_I f(y)d\sigma(y) \qquad x \in I \in \mathcal{F}_n(J). \tag{13}$$

Define the *difference operator*,

$$\Delta_n^\sigma f = E_{n+1}^\sigma f - E_n^\sigma f. \tag{14}$$

Observe that $E_o^\sigma f(x) = \langle f, h_o^\sigma\rangle_\sigma h_o^\sigma(x)$, and for $n > 0$,

$$\Delta_n^\sigma f(x) = \sum_{I\in\mathcal{F}_n(J)} \langle f, h_I^\sigma\rangle_\sigma h_I^\sigma(x). \tag{15}$$

We can use Plancherel's Theorem for orthogonal systems if the measure $d\sigma$ is positive (in that case we have an honest inner product); to get that

$$\|f\|^2_{L^2(J,d\sigma)} = \sum_{I\in\mathcal{F}'(J)} |\langle f, h_I^\sigma\rangle_\sigma|^2.$$

In particular, if $d\sigma = dx$ we have the *standard Haar basis* associated to the grid $\mathcal{F}$, that we will denote by $\{h_I\}_{I\in\mathcal{F}}$. for the record, note that,

$$h_I(x) = \left(\frac{|I_r||I_l|}{|I|}\right)^{1/2} \left(\frac{1}{|I_r|}\chi_{I_r}(x) - \frac{1}{|I_l|}\chi_{I_l}(x)\right). \tag{16}$$

We want to deal with complex measures, and we want to say something about the function being in ordinary $L^2(J)$. That is we would like to know under which conditions the system $\{h_I^\sigma\}_{I\in\mathcal{F}(J)}$ is a *frame* in $L^2(J)$. By this we mean that we can reconstruct the functions as in (12), and we can also recover the L^2 norm. More precisely, there exists a constant $C > 0$ such that

$$\frac{1}{C}\sum_{I\in\mathcal{F}'(J)} |\langle f, h_I^\sigma\rangle_\sigma|^2 \le \|f\|^2_{L^2(J)} \le C\sum_{I\in\mathcal{F}'(J)} |\langle f, h_I^\sigma\rangle_\sigma|^2. \tag{17}$$

In [**CJS**] a Haar system adjusted to a Lipschitz curve is built. There the grid is the ordinary dyadic grid and the measure involved is $d\sigma = z'(x)dx$, where z is the arclength parametrization. It turns out that in this case the system is a frame.

In the next section we will construct a Haar system associated to a regular dyadic grid $\mathcal{F}$ and to a measure $d\sigma$ related to the given Lipschitz curve. We will show that this particular system is a *frame.*

Carleson's lemma is still valid in this context. A *Carleson sequence with respect to* $\mathcal{F}(J)$ is a sequence of complex numbers $\{b_I\}_{I\in\mathcal{F}(J)}$ such that there exists a constant C (*Carleson's constant*) such that

$$\sum_{I\in\mathcal{F}(I_o)} |b_I| \le C|I_o|, \quad \forall I_o \in \mathcal{F}(J).$$

Lemma 7 (Carleson's Lemma). *Given* $\{b_I\}$ *a Carleson sequence with respect to* $\mathcal{F}(J)$ *and any sequence of positive numbers* $\{\lambda_I\}$ *then*

$$\sum_{I\in\mathcal{F}(J)} \lambda_I|b_I| \le C\int_J \lambda^*(x)dx,$$

where C *is the Carleson constant of the* $\{b_I\}$ *and* $\lambda^*(x) = \sup_{x\in I\in\mathcal{F}(J)} \lambda_I$.

A proof for the standard dyadic grid can be found in [**M**] p. 273. The proof for regular dyadic grids is essentially the same.

3.3. Our Grid. Given the Lipschitz graph $\Gamma = \{z = x + iA(x); \|A'\|_\infty$ $\eta < \infty\}$. We assume, as before, that $\eta < 1$.

Fix an interval J, let $\Gamma_J = \tilde{A}(J)$, i.e. the piece of the graph Γ whose projection is J.

We will construct a Haar system, adjusted to the Lipschitz graph Γ_J, but also to the geometry of our problem. In general the supporting dyadic grid will not be the ordinary dyadics (except in the trivial case when Γ_J is a line) but it will be a regular dyadic grid. The measure will be

$$d\sigma = (1 + iA'(x))dx. \tag{18}$$

To define the grid it is enough to indicate how to produce the children of a given interval. Let I be any interval, let us denote its left and right endpoints by x_I^- and x_I^+ respectively. Let $z_I^+ = x_I^+ + iA(x_I^+)$ and similarly z_I^-. Let z_I be the point on the curve Γ which is equidistant from z_I^+ and z_I^- (it is well defined because $\|A'\|_\infty < 1$). Let x_I be the point in I such that $z_I = x_I + iA(x_I)$. The children of I will then be

$$I_l = (x_I^-, x_I), \quad I_r = (x_I, x_I^+).$$

Lemma 8. *The grid $\mathcal{F}(J)$ defined by this procedure is a regular dyadic grid.*

Proof. Clearly, the vector $z_I^+ - z_I^- = \int_I d\sigma(x) = \sigma(I)$.

Let $\theta_I = \arg \sigma(I)$. Notice that by construction, $|\sigma(I_l)| = |\sigma(I_r)| := t_I$. Therefore $\alpha_I := \theta_{I_l} - \theta_I = \theta_I - \theta_{I_r}$ (here α_I is the common angle in the isosceles triangle defined by z_I, z_I^+ and z_I^-). Since the curve is a Lipschitz graph, then certainly both θ_I and α_I are bounded in absolute value by $\theta := \arctan \|A'\|_\infty < \pi/4$. In particular, since $|I| = |\sigma(I)| \cos \theta_I$ and by construction $|\sigma(\tilde{I})| = 2|\sigma(I)| \cos \alpha_{\tilde{I}}$ (where I is a kid of $\tilde{I}$) then

$$(1-C)|\tilde{I}| \leq |I| \leq C|\tilde{I}|; \quad \text{for } C = \frac{1+\eta^2}{2}.$$

Since $0 \leq \eta < 1$ clearly $\frac{1}{2} \leq C < 1$. □

The *Haar system* associated to $\mathcal{F}(J)$ and to $d\sigma = (1 + iA'(x))dx$ is, as we can see by (10) and the fact that $\sigma(I_r)/\sigma(I_l) = e^{2i\alpha_I}$, given by:

$$h_I^\sigma(x) = \frac{1}{\sigma^{1/2}(I)} \left(e^{i\alpha_I} \chi_{I_r}(x) - e^{-i\alpha_I} \chi_{I_l}(x) \right). \tag{19}$$

Proposition 1. *The Haar system defined above is a frame on $L^2(J)$.*

Proof. The proof is essentially the same as the one in [**CJS**].

Let us compare the standard Haar basis, $\{h_I\}_{I \in \mathcal{F}(J)}$, associated to the grid $\mathcal{F}(J)$ (see (16)), and the new system. It is not hard to see that

$$h_I^\sigma(x) = c_I h_I(x) + d_I |I|^{1/2} \sin \alpha_I \frac{\chi_{I_l}(x)}{|I_l|},$$

where $|c_I| \sim 1$, $|d_I| \sim 1$, uniformly on I.

Therefore,

$$\langle f, h_I^\sigma \rangle_\sigma = c_I \int f h_I d\sigma + d_I |I|^{1/2} \sin \alpha_I \frac{1}{|I_l|} \int_{I_l} f d\sigma.$$

Recall that $d\sigma = (1 + iA'(x))dx$ and let us denote the mean value with respect to the Lebesgue measure by $m_I g = \frac{1}{|I|}\int_I g\,dx$, and recall that $\langle .,. \rangle$ denotes the ordinary inner product in L^2. Then we can rewrite the right hand side in the last equality as

$$c_I \langle f(1+iA'), h_I\rangle + d_I |I|^{1/2} \sin\alpha_I \, m_{I_l} f(1+iA').$$

Also notice that,

$$\langle f, h_o^\sigma\rangle_\sigma = c_J \langle f(1+iA'), h_o\rangle,$$

where $|c_J| \sim 1$ as well.

Since $|c_I| \sim 1$ and $|d_I| \sim 1$ independently of I; then

$$\begin{aligned} (20) \qquad \sum_{I\in\mathcal{F}'(J)} |\langle f, h_I^\sigma\rangle|^2 \le C \sum_{I\in\mathcal{F}'(J)} |\langle f(1+iA'), h_I\rangle|^2 \\ + C \sum_{I\in\mathcal{F}(J)} |I| \sin^2\alpha_I \, |m_{I_l} f(1+iA')|^2. \end{aligned}$$

The first term on the right hand side of this inequality is clearly bounded by a multiple of $\|f\|_{L^2(J)}$, since $\{h_I\}_{I\in\mathcal{F}'(J)}$ is a basis on $L^2(J)$ and $\|1+iA'\|_\infty < 2$.

The second term can be controlled by Carleson's Lemma on regular dyadic grids, provided we can show that

Lemma 9 (Geometric Lemma). *The sequence $b_I = |I|\sin^2\alpha_I$, $I \in \mathcal{F}(J)$ satisfies Carleson's condition with Carleson's constant independent of the base interval J.*

We will prove this lemma at the end of the section. Assume it is true, and let $\lambda_I = |m_{I_l} f(1+iA')|^2$. Clearly

$$\lambda^*(x) \le CM^2|f|,$$

where M is the ordinary Hardy-Littlewood maximal operator.

By Carleson's Lemma and the boundedness on L^2 of M, we get that

$$\sum_{I\in\mathcal{F}(J)} |I|\sin^2\alpha_I |m_{I_l} f(1+iA')|^2 \le C\|f\|^2_{L^2(J)}.$$

Therefore, for all $f \in L^2(J)$

$$(21) \qquad \sum_{I\in\mathcal{F}'(J)} |\langle f, h_I^\sigma\rangle_\sigma|^2 \le C\|f\|^2_{L^2(J)}.$$

The converse now follows from a standard polarization argument (see [**CJS**]).

This finishes the proof of the proposition. □

We will say that a locally integrable function b is in $\mathrm{BMO}(\mathcal{F},\sigma,J)$ if there exists a constant C such that

$$(22) \qquad \sum_{I\in\mathcal{F}_k(I_o)} |\langle b, h_I^\sigma\rangle_\sigma|^2 \le C|I_o| \quad \forall I_o\in\mathcal{F}(J).$$

Remark. Since the square of the absolute value of the sequence $b_I = i\sigma^{1/2}(I)\sin\alpha_I$ is a Carleson sequence with respect to $d\sigma$ and $\mathcal{F}(J)$ (Geometric Lemma 9), the function

$$b(x) = \sum_{I\in\mathcal{F}(J)} b_I h_I^\sigma(x)$$

is a well defined $L^2(J)$ function and is in $\mathrm{BMO}(\sigma,\mathcal{F},J)$; moreover, there exists constant $0<\epsilon<1$ such that for all I, $|b_I h_I^\sigma(x)|\le 1-\epsilon$.

Proof of Lemma 9. This proof is the same as the proof of the Lipschitz case in the Travelling Salesman Problem (see [**J**].)

Since $|I|\sim|\sigma(I)|$, it is enough to show that the sequence $|\sigma(I)|\sin^2\alpha_I$ satisfies Carleson's condition.

Denote by Γ_{I_o} the image curve of the interval I_o.

Certainly the arclength of Γ_{I_o} is comparable to $|I_o|$. We can compute this length $l(\Gamma_{I_o})$, by successive polygonal approximations to Γ_{I_o}.

Let $\sigma_I = \{z = z_I^- + t\sigma(I) : 0\le t\le 1\}$, be the chord built joining the images of the endpoints of I on Γ. Clearly, $|\sigma_I| = |\sigma(I)|$.

Let $\Gamma_o = \sigma_{I_o}$ and define for $n>0$

$$\Gamma_n = \bigcup_{I\in\mathcal{F}_n(I_o)} \sigma_I.$$

Clearly $\Gamma_n \longrightarrow \Gamma_{I_o}$ and $l(\Gamma_n)\longrightarrow l(\Gamma_{I_o})$.

Therefore

$$l(\Gamma_{I_o}) - l(\Gamma_o) = \sum_{n=0}^{\infty} l(\Gamma_{n+1}) - l(\Gamma_n).$$

It is easy to compare the lengths of two succesive polygonals,

$$l(\Gamma_{n+1}) - l(\Gamma_n) = \sum_{I\in\mathcal{F}_n(I_o)} \left(|\sigma(I_r)| + |\sigma(I_l)| - |\sigma(I)|\right).$$

By definition of the grid, $|\sigma(\tilde{I})| = 2|\sigma(I)|\cos\alpha_{\tilde{I}}$, for $\tilde{I}$ parent of I; hence, since Γ is Lipschitz, $|\sigma(I_r)| + |\sigma(I_l)| - |\sigma(I)| \sim |\sigma(I)|\sin^2\alpha_I$.

Therefore
$$l(\Gamma_{I_o}) - l(\Gamma_o) \sim \sum_{I\in\mathcal{F}_n(I_o)} |\sigma(I)| \sin^2 \alpha_I.$$

Finally since $l(\Gamma_o) = |\sigma(I_o)| \sim |I_o|$ and $l(\Gamma_{I_o}) \sim |I_o|$ we see that for all $I_o \in \mathcal{F}(J)$
$$\sum_{I\in\mathcal{F}_n(I_o)} |\sigma(I)| \sin^2 \alpha_I \leq C|I_o|.$$

This finishes the proof of Lemma 9. □

The bilinear form $\langle .,. \rangle_\sigma$ is not an honest inner product. We would like to study the boundedness in L^2 of certain operators and their *adjoints* with respect to the bilinear form. Let us state here a lemma that we will use later. The proof of the lemma is an exercise in functional analysis left to the reader.

Lemma 10. *Given T and T^* linear operators in $L^2(J)$ such that*
$$\langle Tf, g\rangle_\sigma = \langle f, T^*g\rangle_\sigma, \quad \forall f, g \in L^2(J),$$
then T is bounded in $L^2(J)$ if an only if T^* is bounded in $L^2(J)$.

3.4. Proof of Theorem 1′. Suppose $f \in W^{1,2}(\Gamma)$, where $\Gamma = \{x + iA(x) : \|A'\|_\infty = \eta < 1\}$. Let $\tilde{A}(x) = x + iA(x)$.

By definition of the Sobolev space on the curve, $f(\tilde{A})$ and $(f(\tilde{A}))'$ are in $L^2(\mathbf{R})$. We can assume that $f(\tilde{A})$ is absolutely continuous.

Let $d\sigma = (1 + iA'(x))dx$, be the measure used in the previous section. There we showed that given an interval I then (see (19))
$$h_I^\sigma(x) = \frac{1}{\sigma^{1/2}(I)} \left(e^{i\alpha_I}\chi_{I_r}(x) - e^{-i\alpha_I}\chi_{I_l}(x)\right).$$

Clearly $(f(\tilde{A}))' = f'(\tilde{A})(1 + iA')$, and by the fundamental theorem of calculus,
$$\langle f'(\tilde{A}), h_I^\sigma\rangle_\sigma = \frac{1}{\sigma^{1/2}(I)} \left[e^{i\alpha_I} f(z_I^+) + e^{-i\alpha_I} f(z_I^-) - 2\cos\alpha_I f(z_I)\right],$$

where $I_r = [x_I, x_I^+]$, $I_l = [x_I^-, x_I]$ and $z_I^\pm = \tilde{A}(x_I^\pm)$.

The right hand side is almost the geometric second difference that we associated to I, namely $\tilde{\Delta}_I f = f(z_I^+) + f(z_I^-) - 2f(z_I)$.

Let us introduce an *adjusted geometric second difference*

$$\Delta_I f = e^{i\alpha_I} f(z_I^+) + e^{-i\alpha_I} f(z_I^-) - 2\cos\alpha_I f(z_I). \tag{23}$$

Observe that when $\Gamma = \mathbf{R}$, the two differences $\Delta_I f$ and $\tilde{\Delta}_I f$ coincide with the ordinary second difference.

Remark. This *adjusted* second difference is, in some sense, a better behaved object. If we define

$$\Delta_t f(z) = e^{i\alpha(z,t)} f(z_t^+) + e^{-i\alpha(z,t)} f(z_t^-) - 2\cos\alpha(z,t) f(z);$$

then Δ_t will annihilate linear holomorphic functions. This is something that an ordinary second difference does but ours does not!! The nonlinearity introduced in the construction of $z_t^{\pm}$ is compensated in Δ_t by the introduction of the correction factors $e^{\pm i\alpha(z,t)}$ and $\cos\alpha(z,t)$.

Fix an interval J. We just showed that if $f \in W^{1,2}(\Gamma)$ then for all $I \in \mathcal{F}(J)$,

$$\langle f'(\tilde{A}), h_I^\sigma \rangle_\sigma = \frac{\Delta_I f}{\sigma^{1/2}(I)}.$$

Also recall that

$$\langle f'(\tilde{A}), h_o^\sigma \rangle_\sigma = \frac{1}{\sigma^{1/2}(J)} \int_J f'(\tilde{A}) d\sigma = \frac{f(z_J^+) - f(z_J^-)}{\sigma^{1/2}(J)}.$$

Let $\Gamma_J = \tilde{A}(J)$. Notice that $\|f'\|_{L^2(\Gamma_J, dz)} = \|(f(\tilde{A}))'\|_{L^2(J)} \sim \|f'(\tilde{A})\|_{L^2(J)}$, therefore by Proposition 1 it follows that

$$\|f'\|^2_{L^2(\Gamma_J)} \sim \sum_{I \in \mathcal{F}(J)} \frac{|\Delta_I f|^2}{|\sigma(I)|} + \frac{|f(z_J^+) - f(z_J^-)|^2}{|\sigma(J)|}. \tag{24}$$

If we replace Δ_I by $\tilde{\Delta}_I$ we can still show a local version of Theorem 1$'$. Since $|I| \sim |\sigma(I)|$, we can use either of them in the estimates.

Theorem 1$'$ (Local version). *Given $f \in W^{1,2}(\Gamma)$, then for every interval $J \in \mathcal{F}$*

$$\sum_{I \in \mathcal{F}(J)} \frac{|\tilde{\Delta}_I f|^2}{|\sigma(I)|} + \frac{|f(z_J^+) - f(z_J^-)|^2}{|\sigma(J)|} \leq C \|f'\|^2_{L^2(\Gamma_J)},$$

uniformly on J.

Remark. This local version implies Theorem 1$'$. Since it holds uniformly on J, and clearly $f \in W^{1,2}(\Gamma)$ implies that $\frac{|f(z_J^+) - f(z_J^-)|^2}{|\sigma(J)|} \leq \|f'\|^2_{L^2(\Gamma_J)}$ (more is actually true: $f \in W^{1,2}(\Gamma)$, f absolutely continuous, implies that $\frac{|f(z_J^+) - f(z_J^-)|^2}{|\sigma(J)|} \to 0$, as $J \to \mathbf{R}$, this is a consequence of the elementary fact that for any function $f \in L^2(\mathbf{R})$, $\frac{1}{|I|}(\int_I f)^2 \to 0$ as $|I| \to \infty$.) Denote by $\mathcal{F} = \cup_{n\geq 0}\mathcal{F}(J_n)$ where $J_n \in \mathcal{F}_1(J_{n+1})$ and $\mathbf{R} = \cup_{n\geq 0} J_n$, then certainly

$$\sum_{I\in\mathcal{F}} \frac{|\tilde{\Delta}_I f|^2}{|\sigma(I)|} \leq C\|f'\|^2_{L^2(\Gamma)};$$

which is the conclusion we were seeking.

Proof of Theorem 1′ (Local version). After observation (24), we see that it is enough to compare $\sum_{I\in\mathcal{F}(J)} |\tilde{\Delta}_I f|^2/|\sigma(I)|$ and $\sum_{I\in\mathcal{F}(J)} |\Delta_I f|^2/|\sigma(I)|$.

In particular

$$\Delta_I f = \cos\alpha_I \tilde{\Delta}_I f + i\sin\alpha_I \left(f(z_I^+) - f(z_I^-)\right). \tag{25}$$

Since $f \in W^{1,2}(\Gamma)$, we can assume that $f(\tilde{A})$ is absolutely continuous; i.e. $f(\tilde{A})(b) - f(\tilde{A})(a) = \int_a^b (f(\tilde{A}))'(x)dx = \int_a^b f'(\tilde{A})d\sigma$. Hence if we denote the mean value of g with respect to σ on I by $m_I^\sigma g$, then

$$\frac{f(z_I^+) - f(z_I^-)}{\sigma(I)} = m_I^\sigma f'(\tilde{A}).$$

Therefore

$$\sum_{I\in\mathcal{F}(J)} \frac{|\tilde{\Delta}_I f|^2}{|\sigma(I)|} \leq C \sum_{I\in\mathcal{F}(J)} \frac{|\Delta_I f|^2}{|\sigma(I)|} + C \sum_{I\in\mathcal{F}(J)} |\sigma(I)| \sin^2\alpha_I |m_I^\sigma f'(\tilde{A})|^2.$$

The second summand on the right hand side is bounded by $\|f'(\tilde{A})\|_2^2 \sim \|f'\|^2_{L^2(\Gamma)}$ by Carleson's Lemma and the same argument with the maximal function that we used at the end of Proposition 1.

This finishes the proof of the local version of Theorem 1′. □

3.5. Proof of Theorem 2. If we do not know *a priori* that $f \in W^{1,2}(\Gamma)$ but only that $f \in L^2(\Gamma)$ and that for a fixed interval J,

$$\frac{|f(z_J^+) - f(z_J^-)|^2}{|\sigma(J)|} + \sum_{I\in\mathcal{F}(J)} \frac{|\tilde{\Delta}_I f|^2}{|\sigma(I)|} < \infty, \tag{26}$$

we can still say something. Certainly (26) does not carry enough information about the smoothness of f. for instance it only considers the values of f at a countable number of points which is negligible. Nevertheless, if (26) is true the sequence

$$D_k^J f(z) = \frac{f(z_I^+) - f(z_I^-)}{\sigma(I)}, \qquad \pi(z) \in I \in \mathcal{F}_k(J)$$

will converge to a function $D^J f$ in $L^2(\Gamma_J)$, that we will call the *dyadic derivative of f on J with respect to the grid $\mathcal{F}(J)$* (clearly if we start with a differentiable function f then the sequence converges pointwise to f' in J). More precisely, we can prove the following:

Theorem 2 (Local version). *Let $f \in L^2(\Gamma_J)$ and assume* (26). *Then, the sequence $D_k^J f$ defined above converges to a function $D^J f \in L^2(\Gamma_J)$. Moreover,*

$$\|D^J f\|_{L^2(\Gamma_J)}^2 \leq C \left(\frac{|f(z_J^+) - f(z_J^-)|^2}{|\sigma(J)|} + \sum_{I \in \mathcal{F}(J)} \frac{|\tilde{\Delta}_I f|^2}{|\sigma(I)|} \right),$$

where the constant C is independent of the base interval J.

Remark. To get the global estimate, denote by $\mathcal{F} = \cup_{n \geq 0} \mathcal{F}_1(J_n)$ where $J_n \in \mathcal{F}(J_{n+1})$ and $\mathbf{R} = \cup_{n \geq 0} J_n$, as in the remark right after the local version of Theorem 1′. Clearly, $\mathcal{F}(J_n) \subset \mathcal{F}(J_{n+1}) \subset \ldots \subset \mathcal{F}$, assume that $\sum_{I \in \mathcal{F}} \frac{|\tilde{\Delta}_I f|^2}{|\sigma(I)|} < \infty$. This implies that (26) holds uniformly on J_n (since $|f(z_J^+) - f(z_J^-)|^2/|\sigma(J)| \leq c \sum_{J \subset I \in \mathcal{F}} |\tilde{\Delta}_I f|^2/|\sigma(I)|$). Given $f \in L^2(\Gamma)$, we will get a sequence of functions $D^n f$ defined by $D^{J_n} f$ on Γ_{J_n} and zero otherwise, uniformly bounded in L^2. By construction $D^{n+1} f|_{J_n} = D^n f|_{J_n}$, hence $D^n f \to D_{\mathcal{F}} f$ in the L^2 sense as $n \to \infty$, and

$$\|D_{\mathcal{F}} f\|_{L^2(\Gamma)}^2 \leq C \sum_{I \in \mathcal{F}} \frac{|\tilde{\Delta}_I f|^2}{|\sigma(I)|}.$$

Hence, Theorem 2 is proved, up to the local version.

Proof of Theorem 2 (Local version). Fix an interval J. Let us drop the superscripts J in the notation for dyadic derivative (i.e. D_k and D will be used instead of D_k^J and D^J).

We do not know *a priori* that f' exists, so we cannot use Carleson's Lemma straight away as we did in the previous section.

Nevertheless, notice that for every $x_I \in I \in \mathcal{F}_k(J)$ we can write by (25)

$$\Delta_I f = \cos \alpha_I \tilde{\Delta}_I f + i\sigma(I) \sin \alpha_I D_k f(x_I), \tag{27}$$

by an abuse of language, we are identifying $D_k f$ with $D_k f(\tilde{A})$, and we are writing $D_k f(x)$ instead of $D_k f(\tilde{A}(x))$.

It is not hard to see that

$$D_{k+1} f(x) - D_k f(x) = \frac{\Delta_I f}{\sigma^{1/2}(I)} h_I^\sigma(x), \quad x \in I \in \mathcal{F}_k(J).$$

Therefore, multiplying (27) by $h_I^\sigma/\sigma^{1/2}(I)$ and using the last equality we get for every $x \in I \in \mathcal{F}_k(J)$

(28)
$$D_{k+1}f(x) = \cos\alpha_I \frac{\tilde{\Delta}_I f}{\sigma^{1/2}(I)} h_I^\sigma(x) + (1 + i\sigma^{1/2}(I)\sin\alpha_I h_I^\sigma(x))D_k f(x).$$

By hypothesis and Proposition 1, the function

$$g(x) = \sum_{I\in\mathcal{F}(J)} \cos\alpha_I \frac{\tilde{\Delta}_I f}{\sigma^{1/2}(I)} h_I^\sigma(x), \tag{29}$$

is in $L_o^2(J)$.

Let $b(x) = \sum_{I\in\mathcal{F}(J)} b_I h_I^\sigma(x)$, where $b_I = i\sigma^{1/2}(I)\sin\alpha_I$. By the remark on p. 573, b is in $\mathrm{BMO}(\mathcal{F},\sigma,J)$.

Moreover, with the notation of Section 3.2 p. 569,

$$\Delta_k^\sigma g(x) = \sum_{I\in\mathcal{F}_k(J)} \cos\alpha_I \frac{\tilde{\Delta}_I f}{\sigma^{1/2}(I)} h_I^\sigma(x), \quad E_o^\sigma g = 0,$$

and similarly for $\Delta_k^\sigma b(x)$.

With this notation we can rewrite (28) for all $k \geq 0$ as

$$D_{k+1}f(x) = \Delta_k^\sigma g(x) + (1 + \Delta_k^\sigma b(x))\, D_k f(x). \tag{30}$$

This is the recurrence equation that we solved in [**P**] under some conditions on b.

Let us replace $D_k f$ by the corresponding sum and continue down until we reach $k = 0$. We get

$$D_{k+1}f = \Delta_k^\sigma g + \sum_{n=0}^{k-1} \Delta_n^\sigma g \prod_{j=n+1}^{k} (1 + \Delta_j^\sigma b) + D_o f \prod_{j=0}^{k}(1 + \Delta_j^\sigma b). \tag{31}$$

The last summand on the right hand side of this equation is a multiple of $D_o f = \frac{f(z_J^+) - f(z_J^-)}{\sigma(J)}$ which is not necessarily zero.

Lemma 11. *The sequence $\omega_k = \prod_{j=0}^{k}(1 + \Delta_j^\sigma b)$ converges in $L^2(J)$ and a.e. to the function $\omega = \prod_{j=0}^{\infty}(1 + \Delta_j^\sigma b)$. Moreover $\|\omega\|_{L^\infty(J)} \leq 1$.*

We will prove this lemma at the end of the section. These products had been studied in [**FKP**].

The first two summands in the right hand side of (31) look formally like the finite sum operator P_b^k in [**P**]. The only differences are that here the supporting grid is not the standard dyadic grid and the measure is $d\sigma$ instead of the Lebesgue measure. The function b comes from the geometry, just as the measure $d\sigma$ and the grid do. All the algebra is still valid, including the algebra to pass to the corresponding finite *paraseries*.

Let us define the analogous finite sum operators, for $b \in \mathrm{BMO}(\sigma, \mathcal{F}, J)$ and $g \in L^2_o(J, d\sigma)$ (the space of functions in $L^2(J)$ with mean value zero on J with respect to $d\sigma$)

$$P^k_{b,\sigma} g := \sum_{n=0}^{k-1} \Delta^\sigma_n g \prod_{j=n+1}^{k} (1 + \Delta^\sigma_j b) + \Delta^\sigma_k g(x). \tag{32}$$

Proposition 2. *The operators $P^k_{b,\sigma}$ converge to a bounded operator in $L^2(J)$.*

To show the convergence of the martingale $D_k f$ (see (31)), it is enough to show that $P^k_{b,\sigma} g$ converges to a function in $L^2(J)$ since the other term converges to $\omega D_o f$, a multiple of $\omega \in L^2(J)$ (by Lemma 11), where $D_o f = \frac{f(z_J^+) - f(z_J^-)}{\sigma(J)}$. As a consequence of Proposition 2,

$$\|P^k_{b,\sigma} g\|^2_{L^2(J)} \leq C \|g\|^2_{L^2(J)} \leq C \sum_{I \in \mathcal{F}(J)} \frac{|\Delta_I f|^2}{|\sigma(I)|}.$$

It is clear that $\|\omega D_o f\|^2_{L^2(J)} \leq C \frac{|f(z_J^+) - f(z_J^-)|^2}{|\sigma(J)|}$, because by Lemma 11, $|\omega| \leq 1$. Therefore, in the limit, the function $Df = \lim_{k\to\infty} D_k f$, will be in $L^2(J)$ and moreover,

$$\|Df\|^2_{L^2(J)} \leq C \sum_{I \in \mathcal{F}(J)} \frac{|\Delta_I f|^2}{|\sigma(I)|} + C \frac{|f(z_J^+) - f(z_J^-)|^2}{|\sigma(J)|},$$

where C is a constant independent of J. The local version of Theorem 2 is proved up to the study of the operators $P^k_{b,\sigma}$, and the weight ω (Lemma 11). $\square$

3.6. Convergence of the operators $P^k_{b,\sigma}$. Since formally the operators $P^k_{b,\sigma}$ look exactly like the ones treated in [**P**], we want to analize them in a similar way.

In this setting we can define the *paraproduct*

$$\Pi^\sigma_b g = \sum_{j=0}^{\infty} E^\sigma_j g \Delta^\sigma_j b \tag{33}$$

and its *adjoint* with respect to $\langle .,. \rangle_\sigma$

$$(34) \qquad (\Pi_b^\sigma)^* g = \sum_{j=0}^{\infty} \Delta_j^\sigma g \Delta_j^\sigma b.$$

(It is easy to check that $\langle \Pi_b^\sigma g, f\rangle_\sigma = \langle g, (\Pi_b^\sigma)^* f\rangle_\sigma$.)

For $b \in \mathrm{BMO}(\mathcal{F}, \sigma, J)$ the paraproduct is bounded in $L^2(J)$ by Carleson's Lemma and so is its adjoint by Lemma 10.

The basic product and composition rules for the expectation and difference operators are true (see Definitions (13), (14), and see [**P**], and [**Ga**]), namely

$$E_n^\sigma \Delta_j^\sigma = \begin{cases} \Delta_j^\sigma & \text{if } n > j \\ 0 & \text{otherwise} \end{cases};$$

$$\Delta_n^\sigma f \times \Delta_j^\sigma g = \Delta_n^\sigma (f \times \Delta_j^\sigma g) \text{ when } n > j.$$

Therefore for all $i_1 < i_2 < \ldots < i_M$ and $n \leq i_M$

$$(35) \qquad E_n^\sigma (\Delta_{i_1}^\sigma f_1 \times \ldots \times \Delta_{i_M}^\sigma f_M) = 0;$$

and for all $M \geq n$

$$(36) \qquad E_n^\sigma \left(\sum_{k \geq M} \Delta_k^\sigma \right) = 0.$$

Let $b = \sum_{I \in \mathcal{F}(J)} b_I h_I^\sigma$, where $b_I = i\sigma^{1/2}(I) \sin \alpha_I$. By the remark on p. 573 $b \in \mathrm{BMO}(\mathcal{F}, \sigma, J)$.

We can now reproduce word by word what we did in [**P**], except for

Proposition 3. *The operator*

$$(37) \qquad P_b^\sigma g = \sum_{n=0}^{\infty} \Delta_n^\sigma g \prod_{j=n+1}^{\infty} (1 + \Delta_j^\sigma b),$$

is well defined and is bounded on $L^2(J)$.

Nevertheless we can do similar computations to the ones done in [**P**] to prove the analogous result. Let us assume that it is true for a moment, and let us go back to our problem. We want to study the convergence of $P_{b,\sigma}^k g$ as $k \to \infty$. Let $b_k = \sum_{n=0}^k \Delta_n^\sigma b$.

Then clearly

$$P_{b_k}^\sigma g = P_{b,\sigma}^k g + (g - g_k).$$

Therefore $P_{b,\sigma}^k g$ will converge simultaneously with $P_{b_k}^\sigma g$ (since $(g - g_k) \to 0$). But reproducing the proof of the corresponding theorem in [**P**], we see that $P^{\sigma}_{b_k} g$ converges to $P^{\sigma}_b g = (I - \Pi^{\sigma}_b)^{-1} g$. Therefore $P^k_{b,\sigma} g$ converges to $P^{\sigma}_b g$, which is a function in L^2, by Proposition 3.

Proof of Proposition 3. As in the proof of the analogous result in [**P**], the weight ω (see Lemma 11) can be used to rewrite the operator so that it will now look like the operators P_ω treated in [**P**].

Recall that

$$\omega(x) = \prod_{n=0}^{\infty} (1 + \Delta^{\sigma}_n b(x)). \tag{38}$$

As a byproduct of the proof of Lemma 11 we will get (see (56)) that

$$E^{\sigma}_n \omega = \prod_{j=0}^{n-1} (1 + \Delta^{\sigma}_j b), \tag{39}$$

which is equivalent to

$$m^{\sigma}_I \omega = \prod_{I' \supset I} (1 + b_{I'} h^{\sigma}_{I'}(x_I)), \quad x_I \in I. \tag{40}$$

With this in mind we can rewrite the operator P^{σ}_b as

$$\begin{aligned} P^{\sigma}_b g(x) &= \sum_{n=0}^{\infty} \frac{\omega(x) \Delta^{\sigma}_n g(x)}{E^{\sigma}_{n+1} \omega(x)} \\ &= \sum_{I \in \mathcal{F}(J)} \frac{\omega(x) \langle g, h^{\sigma}_I \rangle_{\sigma} h^{\sigma}_I(x)}{m^{\sigma}_I \omega (1 + b_I h^{\sigma}_I(x))}. \end{aligned} \tag{41}$$

Written in this way the operator looks formally like what we called P_ω in [**P**]. The main step over there was to study the boundedness of the adjoint operator.

Let

$$(P^{\sigma}_b)^* g(x) = \sum_{I \in \mathcal{F}(J)} \left[\frac{1}{m^{\sigma}_I \omega} \int \frac{\omega g h^{\sigma}_I}{1 + b_I h^{\sigma}_I} d\sigma \right] h^{\sigma}_I(x). \tag{42}$$

It is easy to check that for all $f, g \in L^2(J)$

$$\langle P^{\sigma}_b f, g \rangle_{\sigma} = \langle f, (P^{\sigma}_b)^* g \rangle_{\sigma}.$$

Therefore by Lemma 10 it is enough to show the boundedness of the operator $(P^{\sigma}_b)^*$. Since $\{h^{\sigma}_I\}$ is a frame, it is enough to show that there exists a constant C such that for every $g \in L^2(J)$

$$\sum_{I \in \mathcal{F}(J)} \left| \frac{1}{m^{\sigma}_I \omega} \int \frac{\omega g h^{\sigma}_I}{1 + b_I h^{\sigma}_I} d\sigma \right|^2 \leq C \|g\|_2^2. \tag{43}$$

We can rewrite the operator in a simpler form.

Let $I \in \mathcal{F}(J)$ be a child of $\tilde{I}$. Then by (40), for any $x_I \in I$,

$$m_I^\sigma \omega = m_{\tilde{I}}^\sigma \omega (1 + b_{\tilde{I}} h_{\tilde{I}}^\sigma (x_I)). \tag{44}$$

Therefore, recalling that $\sigma(I_l) = e^{i\alpha_I}\sigma(I)/2\cos\alpha_I$ and $\sigma(I_r) = e^{-2i\alpha_I}\sigma(I_l)$, we get

$$\begin{aligned}\frac{1}{m_I^\sigma \omega}\int \frac{\omega g h_I^\sigma}{1 + b_I h_I^\sigma} d\sigma &= \frac{1}{\sigma^{1/2}(I)}\left[\frac{e^{i\alpha_I}\int_{I_r}\omega g\, d\sigma}{m_I^\sigma \omega (1 + b_I h_I^\sigma (x_{I_r}))} - \frac{e^{-i\alpha_I}\int_{I_l}\omega g\, d\sigma}{m_I^\sigma \omega (1 + b_I h_I^\sigma (x_{I_l}))}\right] \\ &= 2\cos\alpha_I \sigma^{1/2}(I)\left[\frac{m_{I_r}^\sigma \omega g}{m_{I_r}^\sigma \omega} - \frac{m_{I_l}^\sigma \omega g}{m_{I_l}^\sigma \omega}\right].\end{aligned}$$

Let $d\mu = \omega\, d\sigma$. With this notation (43) is equivalent to

$$\sum_{I \in \mathcal{F}(J)} |\sigma(I)||m_{I_r}^\mu g - m_{I_l}^\mu g|^2 \le C\|g\|_2^2, \tag{45}$$

where $m_I^\mu g$ denotes the mean value of g on I with respect to μ.

Remark. The left hand side of (45) resembles the $L^2(d\sigma)$ norm of the standard dyadic square function $Sf(x) = (\sum_{x\in I}(m_{I_r}f - m_{I_l}f)^2)^{1/2}$, namely,

$$\|Sf\|_{L^2(d\sigma)} = \sum_{I\in\mathcal{D}} \sigma(I)|m_{I_r}f - m_{I_l}f|^2.$$

It is known that such an operator is bounded in $L^2(d\sigma)$ for $d\sigma = vdx$ if and only if the weight v is in the Muckenhoup class A_2 (see [**GC-Rf**] for the general weight theory). There is a very nice proof of this result in [**B**]. Our proof follows the ideas in that paper.

Lemma 12. *The measure μ restores dyadicity to $\mathcal{F}$. More precisely, for every $I \in \mathcal{F}(J)$, I child of $\tilde{I}$, $\mu(\tilde{I}) = 2\mu(I)$.*

Proof. By definition and using (44) for any $x_I \in I$

$$\frac{\mu(I)}{\mu(\tilde{I})} = \frac{\sigma(I)}{\sigma(\tilde{I})}(1 + b_{\tilde{I}} h_{\tilde{I}}^\sigma (x_I)).$$

It is not hard to see that for $x \in I$

$$1 + b_{\tilde{I}} h_{\tilde{I}}^\sigma (x) = \begin{cases} e^{i\alpha_{\tilde{I}}}\cos\alpha_{\tilde{I}} & x \in \tilde{I}_r \\ e^{-i\alpha_{\tilde{I}}}\cos\alpha_{\tilde{I}} & x \in \tilde{I}_l \end{cases}. \tag{46}$$

Also recall that $\sigma(I) = e^{i\theta_I}|\sigma(I)|$, $\theta_{I_r} = \theta_I - \alpha_I$, $\theta_{I_l} = \theta_I + \alpha_I$ and $|\sigma(I_r)| = |\sigma(I_l)| = |\sigma(I)|/2\cos\alpha_I$. Therefore for $x \in I$

$$\frac{\sigma(I)}{\sigma(\tilde{I})} = \frac{1}{2(1 + b_{\tilde{I}} h^{\sigma}_{\tilde{I}}(x_I))}.$$

This finishes the proof of the lemma. $\square$

It is not hard to see, after the last lemma, that

$$\frac{m^{\mu}_{I_l} g + m^{\mu}_{I_r} g}{2} = m^{\mu}_I g.$$

We recall that for all complex numbers z, w the following identity holds,

$$\frac{|z-w|^2}{2} = \left(|z|^2 - \left|\frac{z+w}{2}\right|^2\right) + \left(|w|^2 - \left|\frac{z+w}{2}\right|^2\right).$$

Let $z = m^{\mu}_{I_l} g$, $w = m^{\mu}_{I_r} g$ and $(z+w)/2 = m^{\mu}_I g$. Then (45) is equal, up to a constant, to

$$\sum_{I \in \mathcal{F}(J)} |\sigma(\tilde{I})| \left(|m^{\mu}_I g|^2 - |m^{\mu}_{\tilde{I}} g|^2\right). \tag{47}$$

Adding and subtracting $2|\sigma(I)||m^{\mu}_I g|^2$ we get

(48)
$$\sum_{I \in \mathcal{F}(J)} \left(|\sigma(\tilde{I})| - 2|\sigma(I)|\right) |m^{\mu}_I g|^2 + \sum_{I \in \mathcal{F}(J)} \left(2|\sigma(I)||m^{\mu}_I g|^2 - |\sigma(\tilde{I})||m^{\mu}_{\tilde{I}} g|^2\right).$$

The first summand in the last expression can be bounded by

$$C \sum_{I \in \mathcal{F}(J)} \sin^2 \alpha_I |\sigma(I)||m^{\mu}_I g|^2, \tag{49}$$

because $\left||\sigma(\tilde{I})| - 2|\sigma(I)|\right| = 2|\cos\alpha_I - 1||\sigma(I)| \leq C \sin^2 \alpha_I |\sigma(I)|$.

This last expression can be bounded in turn using Carleson's Lemma by

$$C \int_J |M^{\mu} g(x)|^2 dx,$$

where

$$M^{\mu} g(x) = \sup_{x \in I \in \mathcal{F}(J)} |m^{\mu}_I g|. \tag{50}$$

Let

$$a_m = \sum_{I\in\mathcal{F}_m(J)} 2|\sigma(I)||m_I^\mu g|^2$$
$$= \sum_{I\in\mathcal{F}_{m+1}(J)} |\sigma(\tilde{I})||m_{\tilde{I}}^\mu g|^2.$$

Clearly the second term in (48) is a telescopic sum for this sequence, hence it equals to $\sum_{n=1}^{\infty}(a_m - a_{m-1}) = \lim_{m\to\infty} a_m - a_0$.

But

$$a_m \leq C\int_J g_m^2(x)\,dx;$$

where

$$g_m(x) = \sum_{I\in\mathcal{F}_m(J)} |m_I^\mu g|\chi_I(x).$$

Clearly for all m

$$g_m(x) \leq M^\mu g(x),$$

and therefore $a_m \leq \|M^\mu g\|^2_{L^2(J)}$.

Finally we can bound (47) by a constant times the L^2 norm of $M^\mu g$, and we will be done as soon as we can show that this maximal function is bounded on $L^2(J)$.

Lemma 13. *The maximal operator M^μ is bounded on $L^2(J)$.*

Proof. By definition

$$m_I^\mu g = \frac{\int_I \omega g\,d\sigma}{\int_I \omega d\sigma}.$$

It is enough to show that ω satisfies a *weighted Reverse Hölder* $(2+\epsilon)$ *condition*; namely, that there exists $\epsilon > 0$ such that for all $I \in \mathcal{F}(J)$

$$\left(\frac{1}{|I|}\int_I |\omega|^{2+\epsilon}dx\right)^{1/2+\epsilon} \leq C\frac{1}{|I|}\left|\int_I \omega d\sigma\right|. \tag{51}$$

Let us assume that (51) is true. for $I \in \mathcal{F}(J)$, $g \in L^2(J)$, and by Hölder's inequality with $p = 2+\epsilon$, $q = \frac{2+\epsilon}{1+\epsilon}$ we get

$$|m_I^\mu g|^2 \leq \frac{1}{|\frac{1}{|I|}\int_I \omega d\sigma|^2}\left(\frac{1}{|I|}\int_I |\omega|^p|d\sigma|\right)^{2/p}\left(\frac{1}{|I|}\int_I |g|^q|d\sigma|\right)^{2/q}.$$

Since $|d\sigma| \sim dx$ and by (51) we can bound this by

$$C\left[\frac{1}{|I|}\int_I |g(x)|^q dx\right]^{2/q} \leq C(M\,|g|^q(y))^{2/q}, \quad y \in I$$

where M is now the ordinary Hardy-Littlewood maximal operator which is bounded in L^s for all $s > 1$. In particular, since $g \in L^2$ then $|g|^q \in L^{2/q}$, where $2/q > 1$ by hypothesis, and therefore,

$$\int_J |M^\mu g|^2 dx \leq C \int_J |M(|g|^q)|^{2/q} dx \leq C \int_J |g|^2 dx.$$

This proves the lemma; the only missing step is (51). □

It is enough to show that ω satisfies (51) for $\epsilon = 0$. This resembles the classical result of Gehring (see [**Ge**]), that says that if a weight satisfies a Reverse Hölder condition of order p, it does satisfy a condition of order $p+\epsilon$ for some positive ϵ.

Lemma 14. *There exists a constant C such that*

$$\frac{1}{|I|} \int_I |\omega|^2 dx \leq C |m_I^\sigma \omega|^2, \quad \forall I \in \mathcal{F}(J).$$

We will prove this lemma at the end, and as a corollary of it and of the precise description of ω, we will conclude that,

Lemma 15. *There exist $\epsilon > 0$ such that* (51) *is true for all $I \in \mathcal{F}(J)$.*

Proof of Lemma 11: Let

$$\omega_k(x) = \prod_{n=0}^{k} \prod_{I \in \mathcal{F}_n(J)} (1 + b_I h_I^\sigma(x)). \tag{52}$$

Notice that by (46)

$$\omega_k(x) = e^{i \sum_{n=0}^{k} s_n(x) \alpha_n(x)} \prod_{n=0}^{k} \cos \alpha_n(x), \tag{53}$$

where for $x \in I \in \mathcal{F}_n(J)$ we define $\alpha_n(x) = \alpha_I$, $\theta_n(x) = \theta_I$, and $s_n(x) = s_I(x) = \begin{cases} 1 & x \in I_r \\ -1 & x \in I_l \end{cases}$. Recall that $\theta_{I_r} = \theta_I - \alpha_I$, $\theta_{I_l} = \theta_I + \alpha_I$; therefore $\theta_{n+1}(x) = \theta_n(x) - s_n(x)\alpha_n(x)$ and $\sum_{n=0}^{k} s_n \alpha_n = \theta_0 - \theta_{k+1}$.

Hence

$$\omega_k(x) = e^{i(\theta_J - \theta_{k+1}(x))} \prod_{n=0}^{k} \cos \alpha_n(x). \tag{54}$$

Clearly $|\omega_k(x)| \leq 1$, therefore $\omega_k \in L^2(J)$ and $\|\omega_k\|_{L^2(J)} \leq C|J|^{1/2}$, $\forall k$. Moreover, $|\omega_{k+1}| \leq |\omega_k|$, hence it is a decreasing sequence. Therefore there is a subsequence convergent to a function $\omega \in L^2(J)$.

We can also say something about a.e. convergence. Since Γ is a Lipschitz graph parametrized by A, then A is differentiable a.e. Let $x \in J$ be a point where $A'(x)$ exists. Clearly $\theta_k(x) \to \arctan A'(x)$. On the other hand, the infinite product $\prod_{n=0}^{\infty} \cos\alpha_n(x)$ converges for each fixed x simultaneously with $\sum_{n=0}^{\infty}(1-\cos\alpha_n(x)) \sim \sum_{n=0}^{\infty}\sin^2\alpha_n(x)$.

But

$$\int_J \left(\sum_{n=0}^{\infty}\sin^2\alpha_n(x)\right)dx = \sum_{n=0}^{\infty}\int_J\left(\sum_{I\in\mathcal{F}_n(J)}\sin^2\alpha_I\,\chi_I(x)\right)dx$$
$$= \sum_{I\in\mathcal{F}(J)}|I|\sin^2\alpha_I;$$

this last expression is bounded by $C|J|$ by the geometric lemma (Lemma 9). Therefore $\sum_{n=0}^{\infty}\sin^2\alpha_n(x) < \infty$ for a.e. $x \in J$.

Hence for a.e $x \in J$

$$\lim_{k\to\infty}\omega_k(x) = e^{i(\theta_J - \arctan A'(x))}\prod_{n=0}^{\infty}\cos\alpha_n(x).$$

In conclusion, ω is well defined as the L^2 limit of the ω_k and also as a pointwise limit; for a.e. x ,

$$\omega(x) = e^{i\theta_J}(1+iA'(x))^{-1}\prod_{n=0}^{\infty}\cos\alpha_n(x). \tag{55}$$

This finishes the proof of the lemma. □

We can safely write

$$\omega(x) = \prod_{n=0}^{\infty}(1+\Delta_n^{\sigma}b(x)).$$

It is not hard to see that

$$E_j^{\sigma}\omega(x) = \prod_{n=0}^{j-1}(1+\Delta_n^{\sigma}b(x)), \tag{56}$$

which is equivalent for $x_I \in I$ to

$$m_I^{\sigma}\omega = \prod_{I'\supset I}(1+b_{I'}h_{I'}^{\sigma}(x_I)). \tag{57}$$

To prove this last statement, observe that $\omega = \prod_{n=0}^{j-1}(1+\Delta_n^\sigma b)\prod_{n=j}^{\infty}(1+\Delta_n^\sigma b)$. The first factor is constant for all $x \in I \in \mathcal{F}_j(J)$ and the second factor looks like 1+ sums of products of $\Delta_k^\sigma b$ where $k \geq j$. When we compute the mean value on intervals $I \in \mathcal{F}_j(J)$ we pick the value of the first factor at a point $x_I \in I$ times the mean value of just the function $f(x) = 1$, because all the other summands have mean value zero by (35).

Now (56) implies that $1 + \Delta_n^\sigma b = E_{n+1}^\sigma \omega / E_n^\sigma \omega$, which in turn implies that

$$\Delta_n^\sigma b = \frac{E_{n+1}^\sigma \omega - E_n^\sigma \omega}{E_n^\sigma \omega} = \frac{\Delta_n^\sigma \omega}{E_n^\sigma \omega}.$$

Therefore

$$b_I = \langle b, h_I^\sigma \rangle_\sigma = \frac{\langle \omega, h_I^\sigma \rangle_\sigma}{m_I^\sigma \omega}. \tag{58}$$

Proof of Lemma 14. Because the system $\{h_I^\sigma\}_{I \in \mathcal{F}'(I_o)}$ is a frame for $L^2(I_o)$ and $\omega \in L^2(I_o)$ for all $I_o \in \mathcal{F}(J)$ then

$$\int_{I_o} |\omega|^2 dx \sim \sum_{I \in \mathcal{F}(I_o)} |\langle \omega, h_I^\sigma \rangle_\sigma|^2 + |m_{I_o}^\sigma \omega|^2 |\sigma(I_o)|.$$

But by (58), $\langle \omega, h_I^\sigma \rangle_\sigma = b_I m_I^\sigma \omega$.

Therefore to prove the lemma, it is enough to check that for every $I_o \in \mathcal{F}(J)$

$$\sum_{I \in \mathcal{F}(I_o)} |b_I|^2 |m_I^\sigma \omega|^2 \leq C |m_{I_o}^\sigma \omega|^2 |I_o|.$$

But for $I, I' \in \mathcal{F}(I_o)$ and $x_I \in I$, by (57), and (54)

$$m_I^\sigma \omega = m_{I_o}^\sigma \omega \prod_{I \subset I' \subseteq I_o} (1 + b_{I'} h_{I'}(x_I)) = m_{I_o}^\sigma \omega \, e^{i(\theta_{I_o} - \theta_I)} \prod_{I \subset I' \subseteq I_o} \cos \alpha_{I'}.$$

Hence $|m_I^\sigma \omega| \leq |m_{I_o}^\sigma \omega|$ and since $b_I = i\sigma^{1/2}(I) \sin \alpha_I$ then

$$\sum_{I \in \mathcal{F}(I_o)} |b_I|^2 |m_I^\sigma \omega|^2 \leq |m_{I_o}^\sigma \omega|^2 \sum_{I \in \mathcal{F}(I_o)} |\sigma(I)| \sin^2 \alpha_I.$$

But the second factor on the right hand side is bounded by $C|I_o|$ by the geometric lemma (Lemma 9.)

Notice that the constants involved are independent of the base interval J. This finishes the proof of the lemma. □

Proof of (51) (Lemma 15). We conclude immediately from Lemma 14 that for all $I \in \mathcal{F}(J)$,

$$|m_I^\sigma \omega| \sim m_I |\omega|. \tag{59}$$

This observation and Lemma 14 imply that the weight $|\omega|$ satisfies a Reverse Hölder condition of order two on the intervals of the grid. Namely, for all $I \in \mathcal{F}(J)$,

$$\left(\frac{1}{|I|}\int_I |\omega|^2 dx\right)^{1/2} \leq C m_I|\omega|. \tag{60}$$

This is enough to ensure that the weight $|\omega|$ satisfies a Reverse Hölder condition of order $2+\epsilon$, for some $\epsilon > 0$. Namely, for $I \in \mathcal{F}(J)$,

$$\left(\frac{1}{|I|}\int_I |\omega|^{2+\epsilon} dx\right)^{1/2+\epsilon} \leq C m_I|\omega|. \tag{61}$$

Since $|m_I^\sigma \omega| \sim m_I|\omega|$ (see (59)), we then get the desired result.

That condition (60) implies condition (61) for some $\epsilon > 0$ is Gehring's Theorem. One can follow word by word the proof in [G] p. 260; you need the RH_2 condition to be true on a lot of subintervals of the starting interval J, enough so that a Calderon-Zygmund decomposition argument can be used. Usually the intervals used are those that come from a standard dyadic decomposition of J, but it is straightforward to check that it can also be done if the intervals are given by a regular dyadic grid associated to J.

This finishes the proof of (51). □

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