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NEVANLINNA'S COEFFICIENTS AND DOUGLAS ALGEBRAS

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Some relations between Douglas algebras and coefficients appearing in Nevanlinna's matrix parametrization of the solutions of the Nevanlinna Pick interpolation problem are studied.

1. Introduction.

Let U denote the analytic functions bounded by one in $\mathbb{D} = \{z : |z| < 1\}$. Given a sequence $\{z_n\} \subset \mathbb{D}$, we consider the classical Nevanlinna Pick interpolation problem

$$\text{(NP)} \qquad f(z_n) = w_n, \qquad n = 1, 2, \dots, \quad f \in U.$$

If this problem has more than one solution, R. Nevanlinna [**4**] found analytic functions P, Q, R and S such that the set of all solutions is given by

$$E = \left\{ \frac{P - Qw}{R - Sw}, \quad w \in U \right\}. \tag{1.1}$$

The functions P, Q, R and S are unique subject to the normalization $S(0) = 0$ and $PS - RQ = \pi$, where

$$\pi(z) = \prod_n \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z}_n z}$$

is the Blaschke product corresponding to $\{z_n\}$.

While the funcions P, Q, R and S arose from classical function theory, it turns out that they are also connected with more recent developments. It is part of Nevanlinna's theory that the functions $P/R, Q/R, S/R$ and $1/R$ belong to U and are linked with π in many ways. (See Lemma 1.)

Suppose (NP) has a solution f_0 satisfying $\sup\{|f_0(z)|, z \in D\} < 1$. Our main result is that then $P/R, Q/R, S/R$ and $1/R$ all belong to a certain subalgebra of H^∞ depending only on π which we shall denote by CDA_π. This algebra is part of the theory of Douglas algebras through the work of S.Y. Chang and D.E. Marshall ([**1**], [**2?**]). Our results in particular answer a problem raised by V. Tolokonnikov in [**11**] where other relations between Douglas algebras and the Nevanlinna Pick problem are studied.

Our methods are based on Nevanlinna's ideas in [**4**] and last but not least on the more recent treatment of the Nevanlinna Pick problem given by J. Garnett in [**2**], where dual extremal methods are used. We also give a new proof of a recent result of Tolokonnikov concerning questions whether (NP) has a unique solution.

Next we introduce some notations and well known results.

Let m denote normalized Lebesgue measure on the unit circle $\mathbb{T} = \{z : |z| = 1\}$. If $1 \leq p \leq \infty$, H^p denote the Hardy space consisting of all $f \in L^p(m)$ whose harmonic extension to D is analytic there. If $p = \infty$, the norm $||f||_p$ in $L^p(m)$ can also be given by

$$||f||_\infty = \sup\{|f(z)| : z \in D\} \qquad f \in H^\infty.$$

For basic properties of H^p, we refer to Garnett's book [**2**].

We recall that $I \in H^\infty$ is called an inner function if $|I(e^{i\alpha})| = 1$ almost everywhere with respect to m. Any Blaschke product is inner, but there are many others ([**2**, p. 75]).

Considering H^∞ as a subalgebra of $L^\infty(m)$, let $D_\pi = [H^\infty, \overline{\pi}]$ be the Douglas algebra generated by H^∞ and the restriction $\overline{\pi}|_{\mathbb{T}}$ of $\overline{\pi}$ to $\mathbb{T}$. Then let $QD_\pi = D_\pi \cap \overline{D_\pi}$ be the maximal C^*-subalgebra of D_π. Define also $QDA_\pi = QD_\pi \cap H^\infty$ and let CDA_π denote the subalgebra of H^∞ generated by all inner functions I invertible in D_π. It is evident that $CDA_\pi \subset QDA_\pi$. For more about these algebras, see [**1**], and [**2**] for example. Let I be an inner function. The property of I being invertible in D_π has a very concrete formulation: If $\{\zeta_n\} \subset D$ and $|\pi(\zeta_n)| \to 1$, then $|I(\zeta_n)| \to 1$.

The special solutions I_α to (NP) given by

$$I_\alpha = \frac{P - Qe^{i\alpha}}{R - Se^{i\alpha}}$$

play an important role in this theory. Nevanlinna showed that each I_α is inner [**4**], and in fact almost all I_α are Blaschke products [**9**]. A Nevanlinna Pick problem is called scaled if it has a solution f_0 satisfying $||f_0||_\infty < 1$.

For general properties of Douglas algebras and more on the Nevanlinna Pick problem, Garnett's book [**2**] is a good reference.

The letter C_i will be used for different absolute constants, while $C(t)$ indicates a constant depending on the parameter t.

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2. Main result.

If (NP) has more than one solution, R. Nevanlinna considered the "Wertevorrat" $\Delta(z) = \{f(z) : f \text{ is a solution of (NP)}\}, z \in \mathbb{D}$. Using (1.1), one can easily check that $\Delta(z)$ is a disc of center $c(z) = (-Q(z)\overline{S(z)}+P(z)\overline{R(z)})(|R(z)|^2 - |S(z)|^2)^{-1}$, and radius $\rho(z) = |\pi(z)|(|R(z)|^2 - |S(z)|^2)^{-1}$.

For later use, we collect some of the properties of Nevanlinna's coefficients.

Lemma 1. *Assume* (NP) *has more than one solution and consider the Nevanlinna's coefficients* P, Q, R, S *appearing in* (1.1). *Then*

(i) P, Q, R, S *have radial limit almost everywhere and* $Q = -\pi\overline{R}$, $P = -\pi\overline{S}$, $|R|^2 - |S|^2 = 1$, $Q\overline{S} - P\overline{R} = 0$, *almost everywhere on the unit circle.*

(ii) $|R(z)|^2 - |S(z)|^2 \geq 1$, $|R(z)|^2 - |P(z)|^2 \geq 1$, $\quad z \in \mathbb{D}$.

(iii) *For any* $e^{i\alpha} \in \partial\mathbb{D}$, $(R - Se^{i\alpha})^{-2}$ *is an exposed point of* H^1.

(iv) *If* $u \in U$ *and* $f = (P - Qu)(R - Su)^{-1}$, *one has*

$$||f||_\infty = \left\| \frac{\overline{S}/\overline{R} - u}{1 - uS/R} \right\|_{L^\infty(\partial\mathbb{D})}.$$

(v) *If* (NP) *is scaled, one has* $\rho(z) \to 1$ *as* $|\pi(z)| \to 1$.

(vi) *If* (NP) *is scaled and* $\gamma = \inf\{||f_0||_\infty : f$ *is a solution of* (NP)$\}$, *then* $R \in H^p$ *for all* $p < \pi(\arcsin(\gamma))^{-1}$.

Proof. (i), (ii), (iii) are well known (see [**8**] and the references there given to [**2**]). Using the relations in (i)

$$\begin{aligned}\left|\frac{P - Qu}{R - Su}(e^{i\theta})\right| &= \left|\frac{Q}{R}(e^{i\theta})\right| \left|\frac{P/Q - u}{1 - uS/R}(e^{i\theta})\right| \\ &= \left|\frac{\overline{S}/\overline{R} - u}{1 - uS/R}(e^{i\theta})\right|, \quad \text{a.e. } e^{i\theta} \in \partial\mathbb{D},\end{aligned}$$

and this is (iv). A proof of (v) can be found in [**10**]. Now, let us prove (vi). Consider $I_\alpha = (P - Qe^{i\alpha})(R - Se^{i\alpha})^{-1}$, for fixed α, $0 \leq \alpha < 2\pi$. Using (i). one can easily check

$$I_\alpha\overline{\pi} = e^{i\alpha}\frac{(R - Se^{i\alpha})^{-2}}{|(R - Se^{i\alpha})^{-2}|}, \qquad \text{a.e. on } \partial\mathbb{D}.$$

Since $\gamma = \operatorname{dist}(I_\alpha\overline{\pi}, H^\infty) < 1$, there exists $g \in H^\infty$ satisfying

$$1 > \gamma = \left\| \frac{(R - Se^{i\alpha})^{-2}}{|(R - Se^{i\alpha})^{-2}|} - g \right\|_\infty .$$

Since $I_\alpha(0) \in \partial\Delta(0)$, one has $\operatorname{dist}(I_\alpha\overline{\pi}, H_0^\infty) = 1$, where $H_0^\infty = \{f \in H^\infty : f(0) = 0\}$. The proof of Lemma 4.3 in ([**2**, p. 386]) shows $|g(z)| \geq 1 - \gamma$, $z \in \mathbb{D}$. Let $\operatorname{Arg}(z)$ be the principal branch of the argument. One has,

$$\left|\operatorname{Arg}\left(g^{-1}(R - Se^{i\alpha})^{-2}\right)\right| \leq \arcsin(\gamma), \quad \text{a.e. on } \partial\mathbb{D}.$$

So, the same is true on $\mathbb{D}$ and using a result in ([**2**, p. 114]), one gets

$$\left(g^{-1}(R - Se^{i\alpha})^{-2}\right)^{-1} \in H^p, \quad p < \frac{\pi}{2\arcsin(\gamma)}.$$

Hence $(R - Se^{i\alpha})^2 \in H^p$, for $p < \pi(2\arcsin(\gamma))^{-1}$ and it follows $R \in H^p$, for $p < \pi(\arcsin(\gamma))^{-1}$. This finishes the proof of Lemma 1. □

Let (NP) be an scaled Nevanlinna problem, V. Tolokonnikov proved that the extremal solutions I_α are invertible in D_π [**11**]. Our next result is an extension of this.

Proposition . *Let* (NP) *be a scaled Nevanlinna Pick problem and* I_α *one of its extremal solutions,* $0 \leq \alpha < 2\pi$. *Then* $D_{I_\alpha} = D_\pi$.

Proof. As mentioned before, it is known that I_α is invertible in D_π. We present another proof of it. From (v) of Lemma 1, $\rho(z) \to 1$ whenever $|\pi(z)| \to 1$. Since $I_\alpha(z) \in \partial\Delta(z)$, one gets $|I_\alpha(z)| \to 1$. Hence, I_α is invertible in D_π and $D_{I_\alpha} \subset D_\pi$.

For the converse assume

$$|I_\alpha(z_n)| \to 1.$$

Since the Nevanlinna Pick problem (NP) is scaled, the "Wertevorrat" $\Delta(z_n)$ must meet a fixed disc inside the unit disc. Actually, $f_0(z_n), I_\alpha(z_n) \in \Delta(z_n)$, where f_0 is a solution to (NP) with $||f_0||_\infty < 1$. Hence, for large n,

$$|\pi(z_n)| \geq \rho(z_n) \geq \frac{1}{4}(1 - ||f_0||_\infty) > 0$$

and one deduces that π is invertible in D_{I_α}.

The Proposition can also be immediately deduced from the proof of Theorem 2.1 in [**1**]. □

Remark. The hypothesis on the scaling of the Nevanlinna Pick problem is essential. In fact, there exist non scaled Nevanlinna Pick problems and points $\beta_n \in \mathbb{D}$ such that

$$\sup\{|w| : w \in \Delta(\beta_n)\} \underset{n\to\infty}{\longrightarrow} 0, \qquad |\pi(\beta_n)| \underset{n\to\infty}{\longrightarrow} 1$$

see [**5**]. Then, $I_\alpha(\beta_n) \to 0$, $0 \le \alpha < 2\pi$, and no I_α is invertible in D_π.

The following result is known although we have not found it in the literature. We thank the referee for pointing out it to us.

Lemma 2. *Given $u, |u| = 1$ and $z, |z| \le 1$, one has that*

$$z = \int_0^{2\pi} \frac{z - ue^{i\alpha}}{1 - \bar{z}ue^{i\alpha}} \frac{d\alpha}{2\pi}$$

can be uniformly approximated by its Riemann sums.

Proof. Multiplying by $\bar{u}$ if necessary, one may assume $u = 1$. For $w = e^{2\pi i n^{-1}}$, one has

$$z - \frac{1}{n}\sum_{k=1}^{n} \frac{z - w^k}{1 - w^k\bar{z}} = \bar{z}^{n-1}\frac{1 - |z|^2}{1 - \bar{z}^n}, \qquad |z| < 1.$$

This can be shown expanding in a series and using

$$\sum_{k=1}^{n} w^{pk} = 0$$

unless $p \equiv 0 \bmod n$. By continuity the same holds if $\bar{z}^n \neq 1$. Now, the inequalities

$$\begin{aligned}\left| z - \frac{1}{n}\sum_{k=1}^{n} \frac{z - w^k}{1 - w^k\bar{z}} \right| &\le \frac{|z|^{n-1}(1 + |z|)(1 - |z|)}{1 - |z|^n} \\ &= \frac{1 + |z|}{1 + |z|^{-1} + \cdots + |z|^{-(n-1)}} \le \frac{2}{n}\end{aligned}$$

finish the proof. □

Assume (NP) is scaled. In [**11**] it is proved that the functions P/R, $\pi R^{-2}(S/R)^k$, $k \ge 0$, belong to CDA_π and it is asked if $R^{-1} \in CDA_\pi$. Next, we complete these results.

Theorem 1. *Let* (NP) *be a scaled Nevanlinna Pick problem, E the set of its solutions and*

$$E = \left\{\frac{P - Qw}{R - Sw} : w \in U\right\}$$

its Nevanlinna's parametrization. Let D_π be the Douglas algebra generated by H^∞ and $\overline{\pi}|_{\mathbb{T}}$. Then, the functions $P/R, Q/R, S/R, 1/R$ belong to the algebra CDA_π.

Proof. Since $|S/R(e^{i\theta})| \le 1$, Lemma 2 shows

$$\frac{1}{2\pi}\int_0^{2\pi} I_\alpha(e^{i\theta})\,d\alpha = P/R(e^{i\theta}), \quad \text{a.e. } e^{i\theta}\in\mathbb{T},$$

and the integral can be uniformly approximated by its Riemann sums. Since I_α are inner functions invertible in D_π, one gets $P/R \in CDA_\pi$.

Since Q/R is an inner function, one only has to show that Q/R is invertible in D_π. If $|\pi(z)| \to 1$, by (v) of Lemma 1, the disc $\Delta(z)$ tends to the unit disc, that is to say,

$$\rho(z) = \frac{|Q/R(z) - P/R(z)S/R(z)|}{1-|S/R(z)|^2} \to 1$$
$$c(z) = \frac{P/R(z) - Q/R(z)\overline{S/R(z)}}{1-|S/R(z)|^2} \to 0.$$

Hence,

$$\begin{aligned} 0 \leftarrow \frac{P/R(z)S/R(z) - Q/R(z)}{1-|S/R(z)|^2} + Q/R(z) \\ = \frac{P/R(z)S/R(z) - Q/R(z)|S/R(z)|^2}{1-|S/R(z)|^2}\end{aligned}$$

and one gets $|Q/R(z)| \to 1$. Therefore $Q/R \in CDA_\pi$.

Since by (i) of Lemma 1 $Q\overline{S} = P\overline{R}$ a.e. on the unit circle, one has $S/R = \overline{(P/R)}(Q/R) \in CD_\pi$ and since it is analytic, $S/R \in CDA_\pi$.

Using $R = \overline{Q}\pi$ a.e. on the unit circle, one gets $\overline{(1/R)}Q/R = \pi/R \in H^\infty$. Then, for $0<\delta<1$,

$$\delta\frac{1}{R} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\frac{Q}{R}e^{i\alpha} + \frac{1}{R}\delta}{1+e^{i\alpha}\overline{(\delta/R)}Q/R}\,d\alpha$$

uniformly on the unit circle. Since Q/R is an inner function invertible in D_π, so is

$$\frac{Q/Re^{i\alpha} + \delta/R}{1+e^{i\alpha}\overline{(\delta/R)}Q/R}, \qquad e^{i\alpha}\in\partial\mathbb{D},$$

and one gets $R^{-1}\in CDA_\pi$. ☐

3. An example.

The results of last section may suggest that if one takes $w \in CDA_\pi$, $w \in U$ in Nevanlinna's formula, the resulting function $f = (P - Qw)(R - Sw)^{-1}$ may also belong to CDA_π. This is of course the case if $||w||_\infty < 1$, because of the relation

$$f = (P/R - wQ/R)\sum_{n=0}^{\infty}(wS/R)^n.$$

It has been surprising to us that for general $w \in U \cap CDA_\pi$, the function f may not belong to CDA_π. In fact, f may not belong to the bigger algebra QA_π, which consists of the holomorphic functions in the unit disc which belong to $D_\pi \cap \overline{D_\pi}$. To show this, we need to construct a scaled Nevanlinna Pick problem such that the corresponding function R is not bounded. We will do the construction in the upper half plane.

Consider $z_n = iy_n$, where $y_{n+1} < cy_n$, for some fixed $0 < c < 1$ and $z_n^* = x_n + iy_n$, where $x_n > 0$ is a decreasing sequence, $\sup x_n y_n^{-1}$ is a small number to be chosen later, $x_n y_n^{-1} \to 0$ as $n \to \infty$, but

$$\sum_n (x_n y_n^{-1})^2 = +\infty. \tag{3.1}$$

Let B and B^* be the Blaschke products in the upper half plane with zeros $\{z_n\}$ and $\{z_n^*\}$ and B_1, B_1^* the Blaschke products with zeros $\{\varphi(z_n)\}, \{\varphi(z_n^*)\}$, where φ is a conformal map from the upper half plane to the unit disc.

Lemma 3. *With the notations above, the Nevanlinna Pick problem*

$$f(\varphi(z_n)) = B_1^*(\varphi(z_n)), \qquad n = 1, 2, \dots, \quad f \in U \tag{$*$}$$

is scaled. Moreover, if

$$\{f \in H^\infty : f \ \ \textit{solves} \ \ (*)\} = \left\{\frac{P - Qw}{R - Sw} : w \in U\right\}$$

is Nevanlinna's parametrization of the set of its solutions, one has

$$\lim_{\theta \to 0} |R(e^{i\theta})| = +\infty.$$

Proof. We will prove the Lemma in the upper half plane. Let $x \in \mathbb{R}$, as in ([**2**, p. 432]), one can compute

(3.2)

$$\operatorname{Arg}\frac{B^*(x)}{B(x)} = \sum_n \operatorname{Arg}\left(\frac{x - z_n}{x - \overline{z_n}}\right) - \operatorname{Arg}\left(\frac{x - z_n^*}{x - \overline{z_n^*}}\right) = 2\int_0^{x_n} \frac{y_n}{(x-t)^2 + y_n^2}\, dt.$$

Now, if $F \in H^1$, one has

$$\left|\int_{\mathbb{R}} F(x) \operatorname{Arg} \frac{B^*(x)}{B(x)}\, dx\right| = 2\left|\sum_n \int_0^{x_n} \int_{\mathbb{R}} \frac{y_n}{(x-t)^2+y_n^2} F(x)\, dx\, dt\right|$$
$$\leq 2\sum_n \int_0^{x_n} |F(t+iy_n)|\, dt \leq 2K \sup(x_n y_n^{-1}) ||F||_1$$

because the linear measure σ on $\bigcup_n [iy_n, x_n + iy_n]$ is a Carleson measure, with $\sigma(Q) \leq \sup_n (x_n y_n^{-1}) l(Q)$ where Q is a square lying on the real line and $l(Q)$ is the length of its side. So, given $\varepsilon > 0$, if $\sup_n x_n y_n^{-1}$ is sufficiently small, one gets $||\operatorname{Arg}(B^*/B)||_{BMO} < \varepsilon$, and hence

$$\text{(3.3)} \qquad \operatorname{Arg}(B^*/B) = u + \widetilde{v}, \quad ||u||_\infty \leq C\varepsilon, \quad ||v||_\infty \leq C\varepsilon,$$

where $\widetilde{v}$ is the conjugate function of v and C is an absolute constant ([**2**, p. 248]).

Now,

$$||B^*/B - e^{v+i\widetilde{v}}||_\infty \leq 2C\varepsilon,$$

hence

$$\text{(3.4)} \qquad \operatorname{dist}(B^*/B, H^\infty) \leq 2C\varepsilon < 1$$

and $(*)$ is scaled.

On other hand,

$$||B/B^* - e^{-v-i\widetilde{v}}||_\infty \leq 2C\varepsilon,$$

so

$$\text{(3.5)} \qquad \operatorname{dist}(B/B^*, H^\infty) \leq 2C\varepsilon < 1.$$

Now, (3.4) and (3.5) give that B^* is an extremal solution of $(*)$, that is to say, there exists $0 \leq \alpha < 2\pi$,

$$B^* = \frac{P - Qe^{i\alpha}}{R - Se^{i\alpha}}.$$

Thus, applying (3.3) and (i) of Lemma 1,

$$\exp(i(u+\widetilde{v})) = B^*/B = \frac{(R-Se^{i\alpha})^{-2}}{|(R-Se^{i\alpha})^{-2}|}.$$

Consider $H = \exp(iu - \widetilde{u} + v + i\widetilde{v}) \in H^1$ and hence

$$\frac{H}{|H|} = \frac{(R-Se^{i\alpha})^{-2}}{|(R-Se^{i\alpha})^{-2}|}.$$

By (iii) of Lemma 1 $(R - Se^{i\alpha})^{-2}$ is an exposed point of H^1, so

$$H = M(R - Se^{i\alpha})^{-2}, \qquad M \in \mathbb{C},$$

and $|M(R - Se^{i\alpha})^{-2}(x)| = \exp(v(x) - \widetilde{u}(x))$. Now, by (3.3),

$$\begin{aligned} v(x) - \widetilde{u}(x) = -\widetilde{\mathrm{Arg}}(B^*/B)(x) &= \frac{-2}{\pi}\sum_n \int_0^{x_n} \frac{x-t}{(x-t)^2 + y_n^2}\,dt \\ &= \frac{1}{\pi}\sum_n \ln\left(\frac{(x-x_n)^2 + y_n^2}{x^2 + y_n^2}\right). \end{aligned}$$

Now, let $x > 0$. Using the inequality $\ln(t^{-1}) \le c(\delta)(1-t)$ for $\delta \le t \le 1$, one gets

$$\begin{aligned} \left|\sum_{x_n : |x_n - x| < x} \ln\left(\frac{(x-x_n)^2 + y_n^2}{x^2+y_n^2}\right)\right| &= \sum_{x_n : |x_n - x| < x} \ln\left(\frac{(x-x_n)^2 + y_n^2}{x^2+y_n^2}\right)^{-1} \\ &\le C \sum_{x_n : |x_n - x| < x} \frac{2x_n x - x_n^2}{x^2 + y_n^2} \\ &\le \frac{2C}{x} \sum_{x_n : |x_n - x| < x} x_n \le C_1. \end{aligned}$$

On the other hand, considering k with $x_k > 2x > x_{k+1}$ one has

$$\begin{aligned} \sum_{x_n : |x_n - x| \ge x} \ln\left(\frac{(x-x_n)^2 + y_n^2}{x^2+y_n^2}\right) &= \sum_{n=1}^{k} \ln\left(\frac{(x-x_n)^2 + y_n^2}{x^2+y_n^2}\right) \\ &\ge C\sum_{n=1}^{k} \frac{x_n(x_n - 2x)}{x^2 + y_n^2} \ge C_2 \sum_{n=1}^{k-1} x_n^2 y_n^{-2}. \end{aligned}$$

Also, if $x < 0$, $v(x) - \widetilde{u}(x) \ge -C_3 + v(-x) - \widetilde{u}(-x)$. So, (3.1) gives

$$\lim_{x\to 0} \left|(R - Se^{i\alpha})^{-2}(x)\right| = +\infty,$$

and thus $\lim_{x\to 0} |S/R(x)| = 1$. So, by (i) of Lemma 1, $\lim_{x\to 0} |R(x)| = +\infty$ and this finishes the proof of Lemma 3. □

Now, consider the Nevanlinna Pick problem $(*)$ given by Lemma 3 and

$$\gamma = \inf\{\|f\|_\infty : f \text{ is solution of } (*)\}.$$

For $1 > t > \gamma$, Proposition of last section gives that there exists an inner function J, $tJ = (P - Qw_0)(R - Sw_0)^{-1} \in CDA_\pi$. Using Theorem 1 one can see that $w_0 \in CDA_\pi$. Now consider an interpolating sequence $\{\alpha_n\}$ approaching to 1, with $|\pi(\alpha_n)| \to 1$ as $n \to \infty$, where $\pi = B_1$, and let I be the Blaschke product with zeros $\{\alpha_n\}$. Then, by Lemma 3, $R^{-2}I$ is continuous up to the circle. Also (iv) of Lemma 1 gives

$$\left\| \frac{\overline{S}/\overline{R} - w_0}{1 - w_0 S/R} \right\|_{L^\infty(\partial\mathbb{D})} = t \tag{3.6}$$

and then $|w_0(e^{i\theta})| \leq |S/R(e^{i\theta})| + c(1 - |S/R(e^{i\theta})|)$, $0 \leq \theta < 2\pi$, for some fixed $c = c(t) < 1$. Therefore $w_1 = w_0 + (1-c)R^{-2}I \in U \cap CDA_\pi$.

Now, assume $f = (P - Qw_1)(R - Sw_1)^{-1} \in QA_\pi$. Thus,

$$f - tJ = \pi(w_1 - w_0)(R - Sw_0)^{-1}(R - Sw_1)^{-1} \in QA_\pi.$$

Let σ denote the pseudohyperbolic metric, $\sigma(z, w) = |z - w|\,|1 - \overline{w}z|^{-1}$. Since $|\pi(\alpha_n)| \to 1$ as $n \to \infty$, writing $g = (w_1 - w_0)(R - Sw_0)^{-1}(R - Sw_1)^{-1}$, from the fact that $\pi g \in QA_\pi$ one can deduce

$$\max_{\sigma(z,\alpha_n)\leq r} |g(z)| - \min_{\sigma(z,\alpha_n)\leq r} |g(z)| \to 0, \qquad \text{as } n \to \infty,$$

for any $r < 1$, because otherwise, taking a subsequence of $\{\alpha_n\}$, for some fixed $r < 1$, there would exist $\delta > 0$ and $z_n, \sigma(\alpha_n, z_n) \leq r$, such that

$$(1 - |z_n|)\,|g'(z_n)| \geq \delta.$$

Then, by subharmonicity, for $m < 1$, it would follow

$$\int_{D_n} |g'(w)|^2\, dm(w) \geq C_1(m)\delta$$

where D_n is the disc of center z_n and radius $m(1 - |z_n|)$. So,

$$\int_{D_n} |g'(w)|^2 (1 - |w|)\, dm(w) \geq C_2(m)\delta(1 - |z_n|)$$

and using a result in [**2**, p. 381], this would contradict the fact $\pi g \in QA_\pi$.

Since $g(\alpha_n) = 0$, one gets

$$\max_{\sigma(z,\alpha_n)\leq r} |g(z)| \to 0, \qquad \text{as } n \to \infty. \tag{3.7}$$

But, (3.6) and (v) of Lemma 1 give

$$|1 - S/R(z)w_i(z)| \leq C_1(t)\,(1 - |S/R(z)|^2) \leq C_1(t)|R(z)|^{-2}, \quad i = 0, 1,\ z \in \mathbb{D},$$

so,

$$\max_{\sigma(z,\alpha_n)\leq r} |g(z)| \geq \frac{1 - c}{1 - C_1} \max_{\sigma(z,\alpha_n)\leq r} |I(z)|.$$

Since $\{\alpha_n\}$ is an interpolating sequence, this contradicts (3.7). Therefore $f \notin QA_\pi$.

4. A question about uniqueness.

The question whether (NP) has a unique solution is in general delicate. A necessary condition for uniqueness is of course that $||f||_\infty = 1$ for any solution f to (NP). If there is $f_0 \in H^\infty$ with $||f_0||_\infty < 1$ solving the reduced problem $f(z_n) = w_n$, $n \geq N$ for some $N \geq 2$, we shall call (NP) semiscaled. In [**11**], Tolokonnikov obtained the following nice result

Theorem 2. (Tolokonnikov). *If a Nevanlinna Pick problem is semiscaled, but not scaled, then any solution is inner and hence must be unique.*

It should be observed that previous results due to T. Nakazi [**3**] an K.O. Oyma [**7**] easily follow from Theorem 2.

Proof. Let us use the notation from the introduction and assume that the Nevanlinna Pick problem (NP) is scaled. One can assume $N = 1$. If $\{z_0, w_0\}$ is an extra pair of points consider the extended problem

$$(*) \qquad f(z_n) = w_n, \qquad n = 0, 1, 2, \dots, \quad f \in U.$$

One can assume $z_0 = 0$. The sets $F = \{f \in H^\infty : ||f||_\infty \leq 1,\ f(z_n) = w_n,\ n \geq 1\}$ and $B = \{f(0) : f \in F,\ ||f||_\infty < 1\}$ are convex. Suppose B is non-empty and that the only functions in F with $f(0) = w_0$ have norm 1. We will show that such f are inner. Since the average of two inner functions is not inner, this will also prove uniqueness.

If $||f||_\infty \leq 1$, $||g||_\infty < 1$ and $0 < \epsilon < 1$, then $||\epsilon g + (1-\epsilon) f||_\infty < 1$, and hence $\overline{B} = \{f(0) : f \in F\}$. The assumptions mean that $w_0 \in \overline{B} \setminus B$. The proof in [**2**, p. 152] works verbatim, and shows that any $f \in F$ with $f(z_0) \in \partial B$ must be inner. □

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