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THE GODBILLON-VEY CYCLIC COCYCLE AND LONGITUDINAL DIRAC OPERATORS

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The goal of this paper is to prove the index theorem for the pairing of the Godbillon-Vey cyclic cocycle with the index class of the longitudinal Dirac operator for a codimension one foliation. Let $(X, \mathcal{F})$ be a foliated S^1-bundle over an arbitrary spin manifold M. The Dirac operator on M lifts to a longitudinal elliptic operator D, the longitudinal Dirac operator, on $(X, \mathcal{F})$. The index class of D is an element of the K_0-group of the foliation C^*-algebra $C^*(X, \mathcal{F})$. A densely defined cyclic even-cocycle on $C^*(X, \mathcal{F})$, the Godbillon-Vey cyclic cocycle, is constructed. The main result gives a topological formula for the pairing of the Godbillon-Vey cyclic cocycle with the index class of D. The proof of the main theorem uses a new technique, the pairing with the graph projections.

1. Introduction.

Over the past decade K-theory has come to play significant roles in the study of C^*-algebras. One such role is as a receptor of indices of pseudodifferential operators on foliated manifolds. If P is a longitudinal elliptic operator on a foliated manifold $(X, \mathcal{F})$, then the index of P is an element of the K_0-group of the foliation C^*-algebra $C^*(X, \mathcal{F})$ [**10**]. A transverse invariant measure ν for the foliation generates a trace on the C^*-algebra $C^*(X, \mathcal{F})$. This trace defines an additive map ϕ_ν from the K_0-group into the scalars. Evaluating ϕ_ν on the index of an operator, we obtain a numerical invariant (an analytic index), which depends on the transverse invariant measure ν. The index theorem of A. Connes [**6**] describes the analytic index in terms of the symbol of the operator and the foliation cycle corresponding to the transverse invariant measure.

For many interesting foliations, e.g. Anosov foliations, there does not exist a nontrivial transverse invariant measure. Thus, in order to obtain numerical invariants of operators on such foliations, we need an alternative. A natural candidate is the pairing between K-group and cyclic cohomology. In fact, a trace on a C^*-algebra may be regarded as a densely defined cyclic 0-cocycle. Our aim is to give an index formula for higher dimensional cyclic cocycles. In this direction several authors have obtained results for certain cocycles, see for example [**11**]. Connes and H. Moscovici [**9**] studied the pairing between cyclic cocycles associated with group cocycles and Dirac operators on a Golois covering. In order to compute the pairing they use idempotents constructed by A. Wasserman. Our arguments use graph projections associated with the operators; the advantage is that they provide a direct construction and result in a simple argument.

We focus on a particular cyclic cocycle for a special class of foliations. Let Γ be a discrete group acting freely on a manifold $\widetilde{M}$ so that $\widetilde{M}/\Gamma$ is a closed manifold. Suppose that a Γ-action on the circle S^1, by orientation preserving diffeomorphisms, is given. The S^1-bundle over $\widetilde{M}/\Gamma$ associated with the action is equipped with a foliation $\mathcal{F}$, whose leaves are transverse to the fiber of the bundle. The S^1-bundle X with $\mathcal{F}$ is called a foliated S^1-bundle. When the action satisfies a certain condition (Condition 2.2), the foliation C^*-algebra $C^*(X, \mathcal{F})$ is strongly Morita equivalent to the reduced crossed product $C(S^1) \rtimes \Gamma$. The foliation $\mathcal{F}$ is of codimension one, and transversely orientable. To such a foliation, is assigned a characteristic class, called the Godbillon-Vey class [**13**]. It is a 3-dimensional de Rham cohomology class of X. For foliated S^1-bundles, this characteristic class is interpreted as a group 2-cocycle with values in the space of 1-forms on S^1 [**5**]. Based on this picture, A. Connes studied an analytical interpretation of the Godbillon-Vey class [**8**]. He constructed a densely defined cyclic 2-cocycle τ on the C^*-algebra $C(S^1) \rtimes \Gamma$ and showed that the additive map, induced by τ, coincides with the map, which the Godbillon-Vey class induces on the geometric group $K^0(S^1, \Gamma)$, via the index map $K^0(S^1, \Gamma) \to K_0(C(S^1) \rtimes \Gamma)$.

If P is a longitudinal elliptic operator on a foliated S^1-bundle $(X, \mathcal{F})$, its index $\mathrm{ind}(P)$ is regarded as a class in $K_0(C(S^1) \rtimes \Gamma)$ via the strong Morita equaivalence. We will explicitly compute the value of the additive map mentioned above on the indices of longitudinal Dirac operators. More precisely, we will consider the case where an even-dimensional manifold $\widetilde{M}$ is endowed with a Γ-invariant metric and a Γ-invariant spin structure. We will study the index of the associated Dirac operator D. In order to carry out an explicit computation, the following points have to be taken care of. (1) Since $\mathrm{ind}(D)$ is defined to be a class in the K_0-group of the foliation C^*-algebra, we have to obtain a formula for a densely defined cyclic cocycle on $C^*(X, \mathcal{F})$ (Section 6). The strong Morita equivalence between $C^*(X, \mathcal{F})$ and $C(S^1) \rtimes \Gamma$ yields a homomorphism from $C(S^1) \rtimes \Gamma$ into $C^*(X, \mathcal{F})$. Thus, once we obtain a densely defined cyclic cocycle on $C^*(X, \mathcal{F})$, we can compare this cocycle with Connes's cocycle (Section 9). (2) The index $\mathrm{ind}(D)$ is described in terms of a parametrix of D [**10**], [**9**], and there is not a canonical choice of a parametrix. Thus it seems infeasible to compute the evaluation on such an element. Hence we need a projection "canonically" attached to the operator. The operator extends to a closed operator T; the graph of T is a closed subspace, and the associated orthogonal projection is called the graph projection of T. It will be shown that the graph projection represents $\mathrm{ind}(D)$. A disadvantage of using graph projections is that they lack the regularity which idempotents in [**9**], [**10**] can enjoy. Thus it has to be verified that the graph projection does indeed belong to the domain of the cyclic cocycle.

A use of graph projections in the index problem is a new idea. Once (1) and (2) above are done, the proof of the actual computation of the evaluation (Theorem 8.10) will be straightforward by employing Getzler's symbolic calculus method [**12**].

This work grew out of a study of the K_0-group of the C^*-algebras of Anosov foliations on the unit circle bundle $T_1\Sigma$ of a closed Riemann surface Σ of genus $g > 1$ furnished with a metric of constant negative curvature. Those C^*-algebras are strongly Morita equivalent to crossed product C^*-algebras $C(S^1) \rtimes \pi_1(\Sigma)$, where $\pi_1(\Sigma)$ acts on $C(S^1)$ through linear fractional transformations. Since Anosov foliations on $T_1\Sigma$ have nonzero Godbillon-Vey classes, there must be a class in K_0 on which the cyclic cocycle attains a nonzero value. Our motivation was to describe this class as clearly as possible. This matter will be discussed in Section 10.

2. Foliated Bundles and Its C^*-algebras.

In this section we study the properties of C^*-algebras associated with foliated bundles. On these C^*-algebras we will construct densely defined cyclic cocycles in Section 6.

Let M be a closed Riemannian manifold, and let $\widetilde{M} \to M$ be a Galois covering with deck transformation group Γ. Given a right Γ-action on a closed manifold V by diffeomorphisms, we can construct a fibre bundle $X \to M$ with fibre V. This is the associated bundle

$$p : X = \widetilde{M} \times_\Gamma V \to \widetilde{M}/\Gamma = M,$$

where the right Γ-action on $\widetilde{M} \times V$ is diagonal. The product foliation on $\widetilde{M} \times V$ with leaves $\widetilde{M} \times \{x\}, x \in V$, descends to a foliation $\mathcal{F}$ on X. The projection p restricted to any leaf of $\mathcal{F}$ is a covering map. We call the V-bundle $X \to M$ together with $\mathcal{F}$ a *foliated V-bundle.*

Condition 2.1. Through the paper we assume that a Γ-action on V satisfies the condition: for $g \in \Gamma$, if there exists an open set U in V such that $xg = x$ for all $x \in U$, then g is the identity element of Γ.

The Condition 2.1 guarantees that the holonomy groupoid G of $\mathcal{F}$ is a

Hausdorff space, and that

$$G \cong (\widetilde{M} \times \widetilde{M} \times V)/\Gamma,$$

where Γ acts by $(m,n,x)g = (mg,ng,xg)$, $(m,n,x) \in \widetilde{M} \times \widetilde{M} \times V$, $g \in \Gamma$. The groupoid structure of $(\widetilde{M} \times \widetilde{M} \times V)/\Gamma$ is described as follows. Denote by $[m,n,x]$ the class of $(m,n,x) \in \widetilde{M} \times \widetilde{M} \times V$. The source map s and the range map r are given by

$$r([m,n,x]) = [m,x],$$
$$s([m,n,x]) = [n,x].$$

Two elements $[m',n',x']$ and $[m,n,x]$ are composable if and only if there exists a $g \in \Gamma$ such that $n' = mg$, $x' = xg$. In this case,

$$[m',n',x'][m,n,x] = [m'g^{-1},n,x].$$

The lifting to $\widetilde{M}$ of the Riemannian metric on M induces a leafwise Riemannian metric. The latter gives rise to a left Haar system $\{\nu^x\}$ of the groupoid G [**18**].

We recall the definition of foliation C^*-algebras with coefficient [**11**]. Let E be a Hermitian vector bundle over X. Denote by $C_c^\infty(G,E)$ the space of all compactly supported smooth sections of the bundle $(s^*(E))^* \otimes r^*(E)$. So, if $f \in C_c^\infty(G,E)$, then

$$f(\gamma) \in \mathrm{Hom}(E_{s(\gamma)}, E_{r(\gamma)}), \ \gamma \in G.$$

The space $C_c^\infty(G,E)$ has a $*$-algebra stucture:

$$(f_1 * f_2)(\gamma) = \int_{G^{r(\gamma)}} f_1(\gamma')f_2(\gamma'^{-1}\gamma)\, d\nu^{r(\gamma)}(\gamma'),$$
$$f^*(\gamma) = (f(\gamma^{-1}))^*,$$

where $f_1(\gamma')f_2(\gamma'^{-1}\gamma)$ is the composition of maps, and

$$(f(\gamma^{-1}))^* \in \mathrm{Hom}(E_{s(\gamma)}, E_{r(\gamma)})$$

is the adjoint of $f(\gamma^{-1}) \in \mathrm{Hom}(E_{r(\gamma)}, E_{s(\gamma)})$.

Let $\tilde{r}, \tilde{s}$ be the lifting of r, s to $\widetilde{M} \times \widetilde{M} \times V \to \widetilde{M} \times V$, respectively. Thus

$$\tilde{r}(m,n,x) = (m,x) \quad \text{and} \quad \tilde{s}(m,n,x) = (n,x).$$

Denote by $\widetilde{E}$ the lifting to $\widetilde{M} \times V$ of E. It is easy to see that $C_c^\infty(G,E)$ is identified with the space $\mathcal{K}_c$ of those Γ-invariant smooth sections of $(\tilde{s}^*(E))^* \otimes$

$\widetilde{r}^*(E)$ which have Γ-compact supports. Here we say that a subset of $\widetilde{M} \times \widetilde{M} \times V$ is Γ-compact, if its image in $(\widetilde{M} \times \widetilde{M} \times V)/\Gamma$ is compact (Definition 8.3 of [**3**]).

Let $\widetilde{M}_x = \widetilde{M} \times \{x\}$, $x \in V$, and let μ_x be the strictly positive smooth density on $\widetilde{M}_x$ corresponding to the Γ-invariant smooth density on $\widetilde{M}$ through the canonical identification of $\widetilde{M}_x$ and $\widetilde{M}$. Set

$$H_x = L^2\left(\widetilde{E}_x, \mu_x\right),$$

where $\widetilde{E}_x$ is the restriction of $\widetilde{E}$ to $\widetilde{M}_x$. Then the collection $\mathcal{H} = (H_x)_{x\in V}$ together with the space $C_c(\widetilde{E})$, of compactly supported continuous sections of the bundle $\widetilde{E}$ over $\widetilde{M} \times V$, defines a continuous field of Hilbert spaces over V. The Γ-action on $\widetilde{M} \times V$ and $\widetilde{E}$ gives rise to an action on $\mathcal{H}$. We denote this action by $\xi \to g\xi$, for $g \in \Gamma$, and a section ξ of $\mathcal{H}$. The space $\mathrm{End}_\Gamma(\mathcal{H})$ of Γ-equivariant bounded measurable fields of operators $T = (T_x)$, $T_x \in B(H_x)$, is a C^*-algebra, where the norm is given by

$$||T|| = \sup\{||T_x||; x \in V\}.$$

There is a faithful representation $\rho : \mathcal{K}_c \to \mathrm{End}_\Gamma(\mathcal{H})$. For $f \in \mathcal{K}_c$, the operator $\rho(f)$ is defined by

$$[\rho(f)_x\xi](m) = \int_{\widetilde{M}_x} f(m,n,x)\xi(n)\, d\mu_x(n), \tag{2.2}$$

for $\xi \in H_x$. The norm-closure of $\mathcal{K}_c$ with respect to the norm

$$||f|| = ||\rho(f)|| = \sup\{||\rho(f)_x||; x \in V\}, \ f \in \mathcal{K}_c,$$

is, by definition, the C^*-algebra $C^*(X, \mathcal{F}, E)$ of the foliated bundle $(X, \mathcal{F})$ with coefficient E.

Let $C(V) \rtimes \Gamma$ be the reduced crossed product C^*-algebra arising from the (left) Γ-action on $C(V)$ given by

$$(ga)(x) = a(xg), \quad g \in C(V).$$

The C^*-algebra $C(V) \rtimes \Gamma$ is exactly the reduced C^*-algebra associated with the following groupoid. As a topological space this is $V \times \Gamma$. The space of units is V, with $s(x,g) = xg$ and $r(x,g) = x$. Thus $C(V) \rtimes \Gamma$ contains the following dense $*$-subalgebra $C_c(V \times \Gamma)$:

$$(ab)(x,g) = \sum_{h\in\Gamma} a(x,h)b(xh, h^{-1}g),$$
$$a^*(x,g) = \overline{a(xg, g^{-1})}$$

for $a, b \in C_c(V \times \Gamma)$. For each $x \in V$, one has

$$s^{-1}(x) = \{(xg^{-1}, g) \in V \times \Gamma;\ g \in \Gamma\}.$$

Define a $*$-representation L_x of $C_c(V \times \Gamma)$ on $l^2(s^{-1}(x))$ by

$$[L_x(a)\xi](xg^{-1}, g) = \sum_{h \in \Gamma} a(xg^{-1}, h)\xi(xg^{-1}h, h^{-1}g),$$

where $a \in C_c(V \times \Gamma)$ and $\xi \in l^2(s^{-1}(x))$. Then

$$||a|| = \sup\{||L_x(a)||;\ x \in V\} < \infty,$$

and $C(V) \rtimes \Gamma$ is the completion of $C_c(V \times \Gamma)$ with respect to the norm$|| \cdot ||$.

If U_g denotes the characteristic function of $V \times \{g\}$, then U_g belongs to $C_c(V \times \Gamma)$, since V is compact. Any $a \in C_c(V \times \Gamma)$ can be expressed as a finite sum

$$a = \sum_{g \in \Gamma} a_g U_g, \quad a_g \in C(V).$$

The $*$-algebra $C_c(V \times \Gamma)$ is generated by $C(V)$ and $(U_g)_{g \in \Gamma}$, subject to relations: $U_g U_h = U_{gh}$, $g, h \in \Gamma$, $U_g^* = U_{g^{-1}}$, and $U_g a U_g^* = g(a)$, $a \in C(V)$.

Remark 2.3. The collection $\{l^2(s^{-1}(x))\}_{x \in V}$ forms a continuous field of Hilbert spaces, and the correspondence $x \to L_x(a)$ is a continuous field of bounded operators.

Proposition 2.4. *There exists a Hilbert $C(V) \rtimes \Gamma$-module ϵ such that $C^*(X, \mathcal{F}, E)$ is isomorphic to the C^*-algebra $\mathcal{K}(\epsilon)$ of compact operators of ϵ. In particular, $C^*(X, \mathcal{F}, E)$ is strongly Morita equivalent to $C(V) \rtimes \Gamma$.*

Proof. Choose a base point $* \in M$. The image T of $\{*\} \times V$ in X is a complete transversal of $\mathcal{F}$, where $G_T^T = s^{-1}(T) \cap r^{-1}(T)$ and $G_T = s^{-1}(T)$ are identified with $V \times \Gamma$ and $\widetilde{M} \times V$, respectively. Then Proposition 3 of [**14**] implies the assertion. □

We now describe the module ϵ, as we will need the description later. Let $\mathcal{S} = C_c(\widetilde{E})$. A right $C_c(V \times \Gamma)$-action on $\mathcal{S}$ is defined by

$$(\xi f)(m, x) = \sum_{g \in \Gamma} f(xg^{-1}, g)(g^{-1}\xi)(m, x), \quad \xi \in \mathcal{S}, \quad f \in C_c(V \times \Gamma).$$

A $C_c(V \times \Gamma)$-valued inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{S}$ is defined by

$$\langle \xi_1, \xi_2 \rangle(x, g) = \int_{\widetilde{M}_x} (\xi_1(m, x), (g\xi_2)(m, x))_{\widetilde{E}}\, d\mu_x(m),$$

where $(\cdot,\cdot)_{\widetilde{E}}$ is the Hermitian product in $\widetilde{E}$. The module ϵ is the completion of $\mathcal{S}$ with respect to the norm $||\xi|| = ||\langle \xi, \xi\rangle||^{1/2}$.

The representation of $C^*(X,\mathcal{F},E)$ on ϵ given by Proposition 2.4 is:

$$(f * \xi)(\gamma) = \int_{G^{r(\gamma)}} f(\gamma')\xi(\gamma'^{-1}\gamma)\, d\nu^{r(\gamma)}(\gamma'),$$

for $f \in C_c(G,E)$, $\xi \in C_c(G_T, r^*E)$. Through the identification $G \cong (\widetilde{M} \times \widetilde{M} \times V)/\Gamma$, the left $C_c(G,E)$-action is described as

$$(f * \xi)(m,x) = \int_{\widetilde{M}_x} f(m,n,x)\xi(n,x)\, d\mu_x(n),$$

where $(m,x) \in \widetilde{M} \times V \cong G_T$, and f is regarded as a Γ-invariant family of integral kernels on $\widetilde{M} \times \widetilde{M} \times V$. Proposition 3 of [**14**] says that the left $C_c(G,E)$-action extends to a faithful representation of $C^*(X,\mathcal{F},E)$ on ϵ, and that the image of this representation is precisely the space $\mathcal{K}(\epsilon)$ of compact operators of the Hilbert C^*-module ϵ over $C(V) \rtimes \Gamma$.

Let $C_c^{\infty,0}(\widetilde{E})$ be the space of compactly supported sections of $\widetilde{E}$ over $\widetilde{M} \times V$ of class $C^{\infty,0}$ ([**6**]). In an obvious way $C_c^{\infty,0}(\widetilde{E})$ can be regarded as a subspace of sections of the field $\mathcal{H}$. Consider the $*$-algebra of intertwining operators of $\mathcal{H}$ which map $C_c^{\infty,0}(\widetilde{E})$ into itself. Its C^*-closure in $\mathrm{End}_\Gamma(\mathcal{H})$ is denoted by $\mathfrak{B}$.

Proposition 2.5. *There exists a $*$-monomorphism Φ from $\mathfrak{B}$ into the C^*-algebra $\mathcal{L}(\epsilon)$ of all bounded operators of the Hilbert C^*-module ϵ over $C(V) \rtimes \Gamma$.*

Proof. For $\xi \in C_c^{\infty,0}(\widetilde{E})$, denote by ξ_x the restriction of ξ onto $\widetilde{M}_x$. Then $\xi_x \in H_x = L^2(\widetilde{E}_x)$. For $f \in C_c(G_x^T)$, define $S_{\xi,x}(f) \in C_c^{\infty,0}(\widetilde{E})$ by

$$S_{\xi,x}(f)(m) = \sum_{g\in\Gamma} (g^{-1}\xi)(m,x) f(xg^{-1}, g).$$

For $u \in C_c^{\infty,0}(\widetilde{E})$, define $T_{\xi,x}(u) \in C_c(G_x^T)$ by

$$T_{\xi,x}(u)(xg^{-1}, g) = \langle (g^{-1}\xi)_x, u\rangle_x,$$

where $\langle,\rangle_x$ is the inner product of H_x.

We need the following lemma.

Lemma 2.6. *The linear maps $S_{\xi,x}$ and $T_{\xi,x}$ extend to bounded maps*

$$S_{\xi,x} : l^2(G_x^T) \to H_x,$$

and

$$T_{\xi,x} : H_x \to l^2(G_x^T).$$

Moreover,

(1) *$S_{\xi,x}$ is the adjoint of $T_{\xi,x}$;*

(2) *for $\xi, \eta, \zeta \in C_c^{\infty,0}(\widetilde{E})$ and $f \in l^2(G_x^T)$, one has*

$$\begin{aligned} S_{\xi,x}T_{\eta,x}(\zeta_x) &= (\xi\langle\eta,\zeta\rangle)_x, \\ T_{\xi,x}S_{\eta,x}(f) &= L_x(\langle\xi,\eta,\rangle)f; \end{aligned}$$

(3) $||\xi|| = \sup\{||S_{\xi,x}||;\ x \in V\} = \sup\{||T_{\xi,x}||;\ x \in V\}$,
where $||\xi||$ is the norm of ξ in ϵ.

Proof. By a straightforward computation,

$$\langle S_{\xi,x}(f),\ u\rangle_x = \langle f,\ T_{\xi,x}(u)\rangle,$$

where $f \in C_c(G_x^T)$ and $u \in C_c(\widetilde{E})$, and the right-hand side is the inner product in $l^2(G_x^T)$. Let $a \in C_c(V \times \Gamma)$ and $\xi, \eta \in C_c^{\infty,0}(\widetilde{E})$. Then

$$\begin{aligned} S_{\xi,x}(a|_{G_x^T}) &= (\xi a)_x, \\ T_{\xi,x}(\eta_x) &= \langle\xi,\ \eta\rangle|G_x^T. \end{aligned}$$

From this

$$\begin{aligned} S_{\xi,x}T_{\eta,x}(\zeta_x) &= S_{\xi,x}(\langle\eta,\zeta\rangle|_{G_x^T}) \\ &= (\xi\langle\eta,\zeta\rangle)_x. \end{aligned}$$

As for the second equality in the assertion (2), we have

$$\begin{aligned} T_{\xi,x}S_{\eta,x}(f) &= T_{\xi,x}((\eta a)_x) \\ &= \langle\xi,\eta a\rangle|G_x^T \\ &= (\langle\xi,\eta\rangle a)|G_x^T \\ &= L_x(\langle\xi,\eta\rangle)f, \end{aligned}$$

where a is an element of $C_c(V \times \Gamma)$ such that $a|_{G_x^T} = f$. Thus

$$||S_{\xi,x}(f)||^2 = \langle S_{\xi,x}(f), S_{\xi,x}(f)\rangle = \langle f, L_x(\langle\xi,\xi\rangle)f\rangle.$$

From this and facts that $C_c(G_x^T)$ is dense in $l^2(G_x^T)$, and that $L_x(\langle\xi,\xi\rangle)$ is positive, it follows that $S_{\xi,x}$ extends to a bounded linear map, and

$$||S_{\xi,x}|| = ||L_x(\langle\xi,\xi\rangle)||.$$

Consequently, $T_{\xi,x}$ also extends to a bounded linear map, and

$$||T_{\xi,x}|| = ||S_{\xi,x}||.$$

Finally,

$$\begin{aligned} ||\xi||^2 &= ||\langle \xi, \xi\rangle|| = \sup ||L_x(\langle \xi, \xi\rangle)|| \\ &= \sup ||S_{\xi,x}||^2 \\ &= \sup ||T_{\xi,x}||^2. \end{aligned}$$

This completes the proof of the lemma. □

We return to the proof of Proposition 2.5.

Assume that $P = (P_x) \in \mathrm{End}_\Gamma(\mathcal{H})$ and its adjoint $P^* = (P^*_x)$ preserve the space $C_c^{\infty,0}(\widetilde{E})$. Since P is Γ-equivariant, it is readily seen that P defines a $C_c(V \times \Gamma)$-module homomorphism $\widehat{P}$ of $C_c^{\infty,0}(\widetilde{E})$. Furthermore, for $\xi \in C_c^{\infty,0}(\widetilde{E})$, one has

$$\begin{aligned} ||\widehat{P}(\xi)|| &= \sup ||S_{P(\xi),x}|| \\ &= \sup ||P_x S_{\xi,x}|| \\ &\leq \sup ||P_x|| \sup ||S_{\xi,x}|| \\ &= ||P||\ ||\xi||. \end{aligned} \tag{2.7}$$

Thus $\widehat{P}$ is a bounded operator of ϵ. Similarly P^* defines a bounded operator $\widehat{P}^*$ with

$$\langle \widehat{P}\xi, \eta\rangle = \langle \xi, \widehat{P}^*\eta\rangle$$

for $\xi, \eta \in \epsilon$. Therefore $\widehat{P} \in \mathcal{L}(\epsilon)$.

We show that the correspondence $P \to \widehat{P}$ is injective. From the inequality (2.7),

$$\|\widehat{P}\|_{\mathcal{L}(\epsilon)} \leq \|P\|.$$

Assume that $\widehat{P} = 0$. Let $P = \lim_{j\to\infty} P^{(j)}$ in norm in $\mathrm{End}_\Gamma(\mathcal{H})$ where we have that $P^{(j)}$ preserves $C_c^{\infty,0}(\widetilde{E})$. Then, for $\xi \in C_c^{\infty,0}(\widetilde{E})$, we have

$$\lim_{j\to\infty} \|\widehat{P}^{(j)}(\xi)\| = 0.$$

Notice that any $\xi \in C_c^{\infty,0}(\widetilde{E})$ is written as $\xi = \alpha\langle \beta, \gamma\rangle$ for some $\alpha, \beta, \gamma \in C_c^{\infty,0}(\widetilde{E})$. Then

$$P_x^{(j)}\xi_x = P_x^{(j)}(\alpha\langle\beta,\gamma\rangle)_x = P_x^{(j)} S_{\alpha,x} T_{\beta,x}(\gamma_x).$$

Therefore

$$\begin{aligned}\sup\|P_x^{(j)}\xi_x\| &\le \sup\|P_x^{(j)}S_{\alpha,x}\|\sup\|T_{\beta,x}(\gamma_x)\| \\ &\le C\|\widehat{P}^{(j)}(\alpha)\|\end{aligned}$$

for some $C > 0$. Thus $\sup\|P_x^{(j)}\xi_x\| \to 0$ as $j \to \infty$. Hence $P_x\xi_x = 0$ for all $x \in V$. This means that $P = 0$ in $\mathrm{End}_\Gamma(\mathcal{H})$. Thus $P \to \widehat{P}$ is an injective $*$-homomorphism, and in particular,

$$\|\widehat{P}\|_{\mathcal{L}(\epsilon)} = \|P\| = \sup\|P_x\|.$$

This ends the proof of Proposition 2.5. □

Remark 2.8. The foliation C^*-algebra $C^*(X,\mathcal{F},E)$ is a subalgebra of $\mathfrak{B}$, and the restriction to $C^*(X,\mathcal{F},E)$ of the embedding of $\mathfrak{B}$ into $\mathcal{L}(\epsilon)$ is exactly the isomorphism

$$C^*(X,\mathcal{F},E) \to \mathcal{K}(\epsilon)$$

given in Proposition 2.4.

Remark 2.9. When the Γ-action on V does not satisfy the Condition 2.1, the structure of the holonomy groupoid is more complex, and $C^*(X,\mathcal{F},E)$ is not strongly Morita equivalent to $C(V)\rtimes\Gamma$. Thus the arguments above do not apply to this case. However, if one uses the C^*-algebra of the fundamental groupoid, in place of the holonomy groupoid, then the results in this paper remain valid.

3. Algebra of Pseudodifferential Operators.

For a given foliated bundle $(X,\mathcal{F})$, the C^*-algebra $C^*(X,\mathcal{F},E)$ defined in the preceding section contains pseudodifferential operators. In this section we will introduce a dense Banach subalgebra $\mathfrak{A}$ of $C^*(X,\mathcal{F},E)$ and will show that $\mathfrak{A}$ is holomorphically closed.

Let $\widetilde{E}^0$ and $\widetilde{E}^1$ be Γ-equivariant Hermitian vector bundles over $\widetilde{M}\times V$. Let $P : C_c^{\infty,0}(\widetilde{E}^0) \to C^{\infty,0}(\widetilde{E}^1)$ be a continuous linear map. We say that P is a Γ-equivariant family of pseudodifferential operators of order r if

(1) P is Γ-equivariant,

(2) for each $x \in V$, the operator P restricts to $\widetilde{M}_x$ to give a pseudodifferential operator of order r

$$P_x : C_c^\infty(\widetilde{E}_x^0) \to C^{\infty,0}(\widetilde{E}_x^1),$$

(3) the distributional kernel of P has Γ-compact support.

Conditions (1) and (2) imply that the distributional kernel is regarded as a distribution on $\widetilde{M} \times \widetilde{M} \times V$ and is Γ-invariant.

Denote by $\Psi^r_\Gamma\left(\widetilde{E}^0, \widetilde{E}^1\right)$ the space of all Γ-equivariant families of pseudodifferential operators of order $\leq r$ from $\widetilde{E}^0$ to $\widetilde{E}^1$. When $\widetilde{E}^0 = \widetilde{E}^1 = \widetilde{E}$, we use $\Psi^r_\Gamma(\widetilde{E})$ instead of $\Psi^r_\Gamma\left(\widetilde{E}^0, \widetilde{E}^1\right)$. A basic fact is that if $P \in \Psi^r_\Gamma\left(\widetilde{E}^0, \widetilde{E}^1\right)$, $Q \in \Psi^s_\Gamma\left(\widetilde{E}^1, \widetilde{E}^2\right)$, then $QP \in \Psi^{r+s}_\Gamma\left(\widetilde{E}^0, \widetilde{E}^2\right)$. If $P \in \Psi^r_\Gamma\left(\widetilde{E}^0, \widetilde{E}^1\right)$, then its formal adjoint P^* belongs to $\Psi^r_\Gamma\left(\widetilde{E}^1, \widetilde{E}^0\right)$. So, in particular, $\Psi^0_\Gamma(\widetilde{E})$ is a $*$-algebra.

Recall [**11**] that by a tangential operator we mean a continuous linear operator $D : C^{\infty,0}_c(\widetilde{E}^0) \to C^{\infty,0}(\widetilde{E}^1)$ such that D is Γ-equivariant and that for each $x \in V, D$ restricts to $\widetilde{M}_x$ to give a continuous linear operator $D_x : C^\infty_c(\widetilde{E}^0_x) \to C^\infty(\widetilde{E}^1_x)$.

Let Δ_x be the Laplacian on $\widetilde{M}_x$ twisted by $\widetilde{E}_x$. Then Δ_x acts on the sections of $\widetilde{E}_x$. Denote by $W^s_x(\widetilde{E})$ the completion of $C^\infty_c(\widetilde{E}_x)$ with respect to the Sobolev s-norm:

$$\|f\|^2_{s,x} = \langle f, (I + \Delta_x)^s f\rangle_x,$$

where $\langle \cdot, \cdot\rangle_x$ is the inner product of $H_x = L^2(\widetilde{E}_x)$. We obtain a continuous field $W^s_\tau(\widetilde{E}) = \left(W^s_x(\widetilde{E})\right)_{x\in V}$ of Hilbert spaces over V, which we shall call a *tangential Sobolev field* [**15**, p. 78].

A tangential operator D is *smoothing* if D induces a bounded operator

$$W^s_\tau(\widetilde{E}) \to W^t_\tau(\widetilde{E})$$

for all $s, t \in \mathbb{R}$. A smoothing operator is *compactly smoothing* if its distributional kernel has Γ-compact support.

For a tangential operator P, and $s, t \in \mathbb{R}$, set

$$\|P_x\|_{s,t} = \sup\left\{(\|P_x\xi\|_{s,x})/\|\xi\|_{t,x}; \quad \xi \in C^\infty_c(\widetilde{E}_x)\right\},$$

and

$$\|P\|_{s,t} = \sup\left\{\|P_x\|_{s,t}; \quad x \in V\right\}.$$

Of course, $\|P_x\|_{s,t}$, $\|P\|_{s,t}$ might be infinite. However it is true that if $P \in \Psi^r_\Gamma(\widetilde{E})$, then

$$\|P\|_{s-r,s} < \infty,$$

for any s. In particular, P extends to an intertwining operator

$$W^s_\tau(\widetilde{E}) \to W^{s-r}_\tau(\widetilde{E}).$$

If P belongs to $\Psi_\Gamma^{-\infty}(\widetilde{E}) = \bigcap_r \Psi_\Gamma^r(\widetilde{E})$, then P is a compactly smoothing operator. Moreover, one can see that $\Psi_\Gamma^{-\infty}(\widetilde{E})$ is contained in $C^*(X, \mathcal{F}, E)$, here $\widetilde{E}$ is the lifting of E to $\widetilde{M} \times V$.

Let $S^*\mathcal{F}$ be the unit cosphere bundle of $\mathcal{F}$, and let π be the canonical projection $S^*\mathcal{F} \longrightarrow X$. Let E^0, E^1 be Hermitian bundles over X, and let $\widetilde{E}^0, \widetilde{E}^1$ be the liftings to $\widetilde{M} \times V$ of E^0, E^1, respectively. The principal symbol map is $\sigma_r : \Psi_\Gamma^r\left(\widetilde{E}^0, \widetilde{E}^1\right) \to C^{\infty,0}(S^*\mathcal{F}, \mathrm{Hom}(\pi^*E^0, \pi^*E^1))$. We say that $P \in \Psi_\Gamma^r\left(\widetilde{E}^0, \widetilde{E}^1\right)$ is elliptic if $\sigma_r(P)$ is invertible.

Proposition 3.1. ([**15**, Prop. 7.12], [**6**, p. 128]). *Let $P \in \Psi_\Gamma^r\left(\widetilde{E}^0, \widetilde{E}^1\right)$ be elliptic. Then there exists $Q \in \Psi_\Gamma^{-r}\left(\widetilde{E}^1, \widetilde{E}^0\right)$ such that $I - PQ$ and $I - QP$ are compactly smoothing.*

The operator Q given by Proposition 3.1 is called a *parametrix* of P.

Every $P \in \Psi_\Gamma^0(\widetilde{E})$ is regarded as an intertwining operator in $\mathrm{End}_\Gamma\left(W_\tau^0(\widetilde{E})\right)$. Thus $\Psi_\Gamma^0(\widetilde{E}) \subseteq \mathfrak{B}$. Let $\wp_0$ denote the C^*-closure of $\Psi_\Gamma^0(\widetilde{E})$ in $\mathrm{End}_\Gamma\left(W_\tau^0(\widetilde{E})\right)$. The principal symbol map σ_0 extends to a $*$-homomorphism

$$\sigma : \wp_0 \longrightarrow C(S^*\mathcal{F}, \mathrm{End}(\pi^*E)),$$

and the sequence

$$0 \to C^*(X, \mathcal{F}, E) \to \wp_0 \xrightarrow{\sigma} C(S^*\mathcal{F}, \mathrm{End}(\pi^*E)) \to 0$$

is exact.

Fix an $N > \dim M$. For $P \in \Psi_\Gamma^{-1} = \Psi_\Gamma^{-1}(\widetilde{E})$, set

$$|||P||| = \max\left(||P||_{1-N,-N}, ||p||_{N,N-1}\right).$$

Then by the interpolation method of Calderon, for all $-N \le s \le N-1$, one has

$$||P||_{s,s} \le ||P||_{s+1,s} \le |||P|||.$$

Certainly, $|||\cdot|||$ is a norm on Ψ_Γ^{-1}. A staightforward computation shows that

$$|||PQ||| \le |||P|||\ |||Q|||$$

for $P, Q \in \Psi_\Gamma^{-1}$.

Let $\mathfrak{A}$ be the Banach algebra completion of Ψ_Γ^{-1} with respect to $|||\cdot|||$.

Lemma 3.2. *There exists an injective homomorphism $\alpha : \mathfrak{A} \to \wp_0$.*

Proof. Since $||P||_{0,0} \le |||P|||$, there exists a homomorphism $\alpha : \mathfrak{A} \to \wp_0$. We prove the injectivity of α. Let $\{P_j\}$ be a Cauchy sequence in Ψ_Γ^{-1} with respect to $|||\cdot|||$. It suffices to show that if $\alpha(P_j)\to 0$ in $\wp_0$, then $P_j\to 0$ in $\mathfrak{A}$. Since $\{P_j\}$ is a Cauchy sequence in $\mathfrak{A}$, it is a Cauchy sequence also with respect to $||\cdot||_{s+1,s}, -N\le s\le N-1$. Therefore, there exist intertwining operators $P^{(s)}$, of fields of Hilbert spaces $W^s_\tau(\widetilde{E})\to W^{s+1}_\tau(\widetilde{E})$ such that

$$||P_j-P^{(s)}||_{s+1,s}\to 0 \quad \text{as} \quad j\to\infty.$$

Recall that $C^{\infty,0}_c(\widetilde{E})$ is a total subspace of $W^s_\tau(\widetilde{E})$. For $\xi\in C^{\infty,0}_c(\widetilde{E})$ and for $s\ge -1$, we have

$$\begin{aligned}||P^{(s)}\xi||_0 &\le ||(P_j-P^{(s)})\xi||_0+||P_j\xi||_0\\ &\le ||(P_j-P^{(s)})\xi||_{s+1}+||P_j||_{0,0}||\xi||_0\\ &\le ||(P_j-P^{(s)})||_{s+1,s}||\xi||_s+||P_j||_{0,0}||\xi||_0\to 0\end{aligned}$$

as $j\to\infty$.

Hence $P^{(s)}\xi=0$ for all $\xi\in C^{\infty,0}_c(\widetilde{E})$. Consequently $P^{(s)}=0$.

Assume, now, that $s+1<0$. Then

$$\begin{aligned}||P^{(s)}\xi||_{s+1} &\le ||(P_j-P^{(s)})\xi||_{s+1}+||P_j\xi||_{s+1}\\ &\le ||(P_j-P^{(s)})\xi||_{s+1}+||P_j\xi||_0\\ &\le ||(P_j-P^{(s)})||_{s+1,s}||\xi||_s+||P_j||_{0,0}||\xi||_0\to 0\end{aligned}$$

as $j\to\infty$.

Hence $P^{(s)}\xi=0$ for all $\xi\in C^{\infty,0}_c(\widetilde{E})$. Thus $P_j\to 0$ in $\mathfrak{A}$. □

From now on, we regard $\mathfrak{A}$ as a subalgebra of $\wp_0$. In particular, an element $P\in\mathfrak{A}$ is interpreted as a collection of operators $P=(P_s)$ such that $P_s: W^s_\tau(\widetilde{E})\to W^{s+1}_\tau(\widetilde{E})$ is bounded for $-N\le s\le N-1$, and such that

$$P_s|W^s_\tau(\widetilde{E})=P_t \quad \text{if} \quad s<t.$$

Let $\mathfrak{A}^+$ be $\mathfrak{A}$ with unit adjointed. As an algebra, $\mathfrak{A}^+$ is identified with the algebra generated by $\mathfrak{A}$ and the identity I of $\wp_0$. Then a sequence $\{\lambda_i I+P_i\}$ in $\mathfrak{A}^+$ converges to $\lambda I+P$ in $\mathfrak{A}^+$ if and only if

$$\lambda_i\to\lambda \qquad \text{in} \quad \mathbb{C},$$

and

$$P_i\to P \qquad \text{in} \quad \mathfrak{A}.$$

Theorem 3.3. *The dense subalgebra $\mathfrak{A}^+$ of $C^*(X,\mathcal{F},E)^+$ is holomorphically closed.*

In order to prove Theorem 3.3 we need the two lemmata below.

Lemma 3.4. *If $P+I$, $P \in \mathfrak{A}$, is invertible in $C^*(X,\mathcal{F},E)^+$, then $(I+P)_s : W_\tau^s(\widetilde{E}) \to W_\tau^{s+1}(\widetilde{E})$ is invertible for $|s| \leq N$.*

Proof. Let $0 \leq s \leq 1$. Obviously, $(I+P)_s : W_\tau^s(\widetilde{E}) \to W_\tau^s(\widetilde{E})$ is injective. By the Open Mapping Theorem, if $(I+P)_s$ is surjective, then it is invertible. Let $\eta \in W_\tau^0(\widetilde{E})$. Since $(I+P)_0 : W_\tau^0(\widetilde{E}) \to W_\tau^0(\widetilde{E})$ is invertible, there exists a $\xi \in W_\tau^0(\widetilde{E})$ such that $(I+P)\xi = \eta$. Then $\xi = \eta - P\xi \in W_\tau^1(\widetilde{E}) + W_\tau^s(\widetilde{E}) \subseteq W_\tau^s(\widetilde{E})$, since $P\xi \in W_\tau^1(\widetilde{E})$. Thus $(I+P)_s : W_\tau^s(\widetilde{E}) \to W_\tau^s(\widetilde{E})$ is injective.

By an induction, using the fact that P maps $W_\tau^{N-1}(\widetilde{E})$ into $W_\tau^N(\widetilde{E})$, we can show that $(I+P)_s$ is invertible for $0 \leq s \leq N$.

As for $-N \leq s \leq 0$, use the nondegenerate pairing

$$W_\tau^{-s}(\widetilde{E}) \times W_\tau^s(\widetilde{E}) \to \mathbb{C}$$

and the fact that $(\xi, (I+P)_s\eta) = ((I+P^*)_{-s}\xi, \eta)$ to deduce the conclusion. $\square$

By Lemma 3.4, we know that when $I+P$ is invertible, it induces invertible operators at each level $W_\tau^s(\widetilde{E}) \to W_\tau^s(\widetilde{E})$.

Lemma 3.5. *Let $I+P$, $P \in \Psi_\Gamma^{-1}$, be invertible in $\wp_0$. Then $(I+P)^{-1} \in \mathfrak{A}^+$.*

Sublemma. *If $Q \in \Psi_\Gamma^0$ is invertible in $\wp_0$, then there exists a sequence $\{A_i\}$ in Ψ_Γ^0 such that $I - A_iQ$ is compactly smoothing, and that*

$$||I - A_iQ||_{s,t} \to 0 \quad \text{as} \quad i \to \infty \quad \text{for all} \quad s,t.$$

Proof of Sublemma. Since Q is invertible in $\wp_0$, its principal symbol $\sigma(P)$ is invertible, i.e. Q is elliptic. Then there exists $R \in \Psi_\Gamma^0$ such that $I - QR$, $I - RQ$ are compactly smoothing.

Since Q is invertible in $\wp_0$, there exists a sequence $\{B_i\}$ in Ψ_Γ^0 such that

$$||Q^{-1} - B_i||_{0,0} \to 0 \quad \text{as} \quad i \to \infty.$$

Put $A_i = 2R + B_i - RQB_i - B_iQR - RQR + RQB_iQR$. Then $A_i \in \Psi_\Gamma^0$. We have

$$I - A_iQ = (I - RQ)(I - B_iQ)(I - RQ).$$

Since $S = I - RQ$ is compactly smoothing,

$$\begin{aligned} ||I - A_iQ||_{s,t} &= ||S(Q^{-1} - B_i)QS||_{s,t} \\ &\leq ||S||_{s,0}||Q^{-1} - B_i||_{0,0}||Q||_{0,0}||S||_{0,t} \to 0 \quad \text{as} \quad i \to \infty, \end{aligned}$$

and the sublemma is proved. □

Proof of Lemma 3.5. By the sublemma, there exists a sequence $\{A_i\}$ of order zero ψDO's such that $I - A_i(I+P)$ is compactly smoothing, and such that

$$||I - A_i(I+P)||_{s,t} \to 0 \quad \text{as} \quad i \to 0 \quad \text{for all} \quad s,t.$$

Notice that $I - A_i = A_iP + (I - A_i(I+P))$ belongs to Ψ_Γ^{-1}. Thus $A_i = I + B_i$ with $B_i \in \Psi_\Gamma^{-1}$. Set

$$T_i = I - (I + B_i)(I+P) \in \Psi_\Gamma^{-\infty}.$$

We have $(I+P)^{-1} - I = B_i + T_i(I+P)^{-1}$. The operator $(I+P)^{-1} - I$ maps $W_\tau^s(\widetilde{E})$ into $W_\tau^{s+1}(\widetilde{E})$ for $-N \le s \le N-1$. Therefore

$$||\,((I+P)^{-1} - I) - B_i||_{s+1,s} \quad \text{is finite,}$$

and

$$\begin{aligned} ||\,((I+P)^{-1} - I) - B_i||_{s+1,s} &= ||T_i(I+P)^{-1}||_{s+1,s} \\ &\le ||T_i||_{s+1,s}||(I+P)^{-1}||_{s,s} \to 0 \quad \text{as} \quad i \to \infty. \end{aligned}$$

This means that $(I+P)^{-1} = I+Q$, with $Q \in \mathfrak{A}$. Thus $(I+P)^{-1} \in \mathfrak{A}^+$. □

Proof of Theorem 3.3. The proof uses the well-known fact that an algebra is holomorphically closed if and only if the resolvents are contained in the algebra itself. Since $\mathfrak{A}$ is an ideal of $C^*(X,\mathcal{F},E)^+$, no elements of $\mathfrak{A}$ are invertible in $C^*(X,\mathcal{F},E)^+$. So it is sufficient to consider elements of the form $I+P$, $P \in \mathfrak{A}$. Since $P \in \mathfrak{A}$, there exists a sequence $\{P_i\}$ in Ψ_Γ^{-1} such that

$$|||P_i - P||| \to 0 \quad \text{as} \quad i \to \infty.$$

Then, in particular, $||(I+P) - (I+P_i)||_{0,0} \to 0$ as $i \to \infty$. As $I+P$ is invertible in $C^*(X,\mathcal{F},E)^+$, one may assume that $I+P_i$ is also invertible in $C^*(X,\mathcal{F},E)^+$ for all i. From

$$(I+P)^{-1} - I = (I+P)^{-1}(I - (I+P)) = -(I+P)^{-1}P,$$

it follows that $(I+P)^{-1} - I$ maps $W_\tau^s(\tilde{E})$ into $W_\tau^{s+1}(\tilde{E})$ for $-N \le s \le N-1$. As bounded operators on $W_\tau^s(\tilde{E})$, one has that

$$\begin{aligned} &||(I+P_i)^{-1} - (I+P)^{-1}||_{s,s} \\ &\qquad \le \Big(||(I+P) - (I+P_i)||_{s,s}||(I+P)^{-1}||_{s,s}^2\Big) \\ &\qquad\qquad / \Big(1 + ||(I+P_i) - (I+P)||_{s,s}||(I+P)^{-1}||_{s,s}^2\Big) \end{aligned}$$

From this, it follows that $\sup\{||(I+P_i)^{-1}||_{s,s};\, i\} < \infty$. Moreover, one can see that $\sup\{||(I+P_i)^{-1}||_{s,s};\, i,\ |s| \leq N\} < \infty$. Then we have

$$\begin{aligned}
||((I+P)^{-1} - I) - ((I+P_i)^{-1} - I)||_{s+1,s} \\
&\leq ||(I+P)^{-1} - (I+P_i)^{-1}||_{s+1,s} \\
&\leq ||(I+P)^{-1}[(I+P_i) - (I+P)](I+P_i)^{-1}||_{s+1,s} \\
&\leq ||(I+P)^{-1}||_{s+1,s+1}||P_i - P||_{s+1,s}||(I+P_i)^{-1}||_{s,s}.
\end{aligned}$$

Since $||(I+P_i)^{-1}||_{s,s}$ is uniformly bounded, as $i \to \infty$ one has

$$||((I+P)^{-1} - I) - ((I+P_i)^{-1} - I)||_{s+1,s} \to 0.$$

This means that $|||(I+P)^{-1} - (I+P_i)^{-1}||| \to 0$. By Lemma 3.5, $(I+P_i)^{-1} \in \mathfrak{A}^+$. Consequently $(I+P)^{-1} \in \mathfrak{A}^+$. $\square$

Applying Theorem 3.3 to the bundle E^k, one obtain that $M_k(\mathfrak{A})^+$ is holomorphically closed in $M_k(C^*(X,\mathcal{F},E))^+$. From this we get the following (see [**3**]).

Proposition 3.6. *The canonical inclusion $\mathfrak{A} \subseteq C^*(X,\mathcal{F},E)$ induces an isomorphism*

$$K_0[\mathfrak{A}] \xrightarrow{\cong} K_0[C^*(X,\mathcal{F},E)].$$

4. Modular Automorphism Groups.

A volume form on the fibre of the foliated bundle $(X,\mathcal{F})$ gives rise to a weight on $C^*(X,\mathcal{F},E)$. We will show that the modular automorphism group, associated with the weight, leaves the Banach algebra $\mathfrak{A}$ invariant, and induces a one-parameter group of automorphisms.

Throughout the rest of the paper, assume that V is oriented, and Γ acts on V by orientation preserving diffeomorphisms. Let ω_V be a volume form on V. For $g \in \Gamma$, a positive real-valued function λ_g on V is determined by

$$\lambda_g \omega_V = g(\omega_V).$$

The correspondence $g \to \lambda_g$ satisfies the cocycle condition:

$$\lambda_{gh} = g(\lambda_h)\lambda_g, \qquad g,h \in \Gamma. \tag{4.1}$$

Let ϕ be the state on $C(V) \rtimes \Gamma$ associated with the volume form ω_V. Then

$$\phi(f) = \int_V f_e \omega_V$$

if

$$f = \sum f_g U_g \in C_c(V \times \Gamma).$$

The modular automorphism group (σ_t) of ϕ leaves $C(V) \rtimes \Gamma$ invariant. We have

$$\sigma_t(f) = \sum \lambda_g^{-it} f_g U_g$$

for $f = \sum f_g U_g$. Actually, σ_t is implemented by the following unitary Δ^{it} on $L^2(V) \otimes l^2(\Gamma)$ defined by

$$\left[\Delta^{it}\xi\right](x,g) = \lambda_g^{-it}(x)\xi(xg).$$

Let $\widetilde{\omega}$ be a Γ-invariant volume form on $\widetilde{M}$. Choose an orientation on X so that for a Γ-invariant volume form ω on $\widetilde{M} \times V$, there exists a positive smooth function ψ on $\widetilde{M} \times V$ such that

$$\widetilde{\omega} \wedge \omega_V = \psi\omega.$$

As above, let $\widetilde{E}$ be a Γ-equivariant Hermitian bundle over $\widetilde{M} \times V$. Recall $\mathcal{S} = C_c(\widetilde{E})$. Define a linear operator Δ^{it} $(t \in \mathbb{R})$ on $\mathcal{S}$ by

$$\Delta^{it}(\xi) = \psi^{-it}\xi, \qquad \xi \in \mathcal{S}. \tag{4.2}$$

Lemma 4.3. *The linear operator Δ^{it} extends to a bounded operator Δ^{it} : $\epsilon \to \epsilon$ which satisfies:*

(1) $\langle \Delta^{it}(\xi), \Delta^{it}(\eta)\rangle = \sigma_t(\langle \xi, \eta\rangle), \qquad \xi, \eta \in \epsilon,$

(2) $\Delta^{it}(\xi a) = (\Delta^{it}(\xi))\sigma_t(a), \qquad \xi \in \epsilon, \quad a \in C(V) \rtimes \Gamma,$

(3) $\Delta^{is}\Delta^{it}(\xi) = \Delta^{i(s+t)}(\xi), \qquad t, s \in \mathbb{R},\ \xi \in E.$

Proof. (1) By the definition of $C(V) \rtimes \Gamma$-valued inner product and (4.2), the equality holds for $\xi, \eta \in \mathcal{S}$, $t \in \mathbb{R}$. Then

$$\begin{aligned}\sup &\{||\Delta^{it}(\xi)||/||\xi||;\ \xi \in \mathcal{S},\ \xi \neq 0\} \\ &= \sup\left\{||\sigma_t(\langle\xi,\xi\rangle)||^{1/2}/||\langle\xi,\xi\rangle||^{1/2};\ \xi \in \mathcal{S},\ \xi \neq 0\right\} = 1.\end{aligned}$$

Hence Δ^{it} extends to a bounded operator on a Banach space ϵ, and the equality holds for all $\xi \in \epsilon$.

(2) A straightforward computation shows that the equality (2) is true for $\xi \in \epsilon$, $a \in C(V) \rtimes \Gamma$. By continuity, the equality holds for all $\xi \in \epsilon$ and $a \in C(V) \rtimes \Gamma$.

(3) From the definition of Δ^{it} and continuity, the conclusion follows. ∎

Statement (2) of Lemma 4.3 means that Δ^{it} is not $C(V) \rtimes \Gamma$-linear.

Lemma 4.4. (1) *If $P \in \mathcal{L}(\epsilon)$, then $\Delta^{it} P \Delta^{-it} \in \mathcal{L}(\epsilon)$, and $||\Delta^{it} P \Delta^{-it}|| = ||P||$.*

(2) *We have $\Delta^{it} \mathcal{K}(\epsilon) \Delta^{-it} \subseteq \mathcal{K}(\epsilon)$.*

Proof. (1) Let $\xi \in \epsilon$, and let $a \in C(V) \rtimes \Gamma$. By Lemma 4.3,

$$\begin{aligned}\left(\Delta^{it} P \Delta^{-it}\right)(\xi a) &= \Delta^{it} P\left(\left(\Delta^{-it}(\xi)\right) \sigma_{-t}(a)\right) \\ &= \Delta^{it}\left(P\left(\Delta^{-it}(\xi)\right) \sigma_{-t}(a)\right) \\ &= \left(\Delta^{it} P \Delta^{-it}(\xi)\right) a.\end{aligned}$$

We have also that

$$\begin{aligned}\langle\left(\Delta^{it} P^* \Delta^{-it}\right)(\xi), \eta\rangle &= \sigma_t\langle P^* \Delta^{-it}(\xi), \Delta^{-it}(\eta)\rangle \\ &= \sigma_t\langle \Delta^{-it}(\xi), P\Delta^{-it}(\eta)\rangle \\ &= \langle \xi, \Delta^{it} P \Delta^{-it}(\eta)\rangle.\end{aligned}$$

This means that $(\Delta^{it} P \Delta^{-it})^* = \Delta^{it} P^* \Delta^{-it}$. Obviously, $\Delta^{it} P \Delta^{-it}$, $\Delta^{it} P^* \Delta^{-it}$ are bounded. Thus

$$\Delta^{it} P \Delta^{-it} \in \mathcal{L}(\epsilon).$$

Since $\Delta^{it} : \epsilon \to \epsilon$ is a surjective isometry,

$$||\Delta^{it} P \Delta^{-it}|| = ||P||.$$

(2) Let $\xi, \eta \in \epsilon$. By the definition of rank one operators $\theta_{\xi,\eta}$ and Lemma 4.4,

$$\Delta^{it} \theta_{\xi,\eta} \Delta^{-it} = \theta_{\Delta^{it}\xi, \Delta^{it}\eta}.$$

Therefore $\Delta^{it} \mathcal{K}(\epsilon) \Delta^{-it} \subseteq \mathcal{K}(\epsilon)$. ☐

Definition 4.5. For $P \in \mathcal{L}(\epsilon)$, set

$$\widehat{\sigma}_t(P) = \Delta^{it} P \Delta^{-it} \in \mathcal{L}(\epsilon).$$

Proposition 4.6. *The operator $\{\widehat{\sigma}_t\}_{t \in \mathbb{R}}$ on $\mathcal{L}(\epsilon)$ amounts to a one-parameter group of automorphisms of the C^*-algebra $\mathcal{L}(\epsilon)$. Moreover, $\{\widehat{\sigma}_t\}$ preserves $\mathcal{K}(\epsilon)$.*

Proof. It is easy to see that $t \to \widehat{\sigma}_t$ is strongly continuous. The conclusion follows from Lemma 4.4. ☐

Notice that Δ^{it} preserves $C_c^{\infty,0}(\widetilde{E})$.

Lemma 4.7. *If $P \in \Psi_\Gamma^r(\widetilde{E})$, then $\Delta^{it}P\Delta^{-it} \in \Psi_\Gamma^r(\widetilde{E})$.*

Proof. We only have to show that $\Delta^{it}P\Delta^{-it}$ is Γ-equivariant. (Other properties of elements of $\Psi_\Gamma^r(\widetilde{E})$ are obvious.) For $g \in \Gamma$, $\xi \in C_c^{\infty,0}(\widetilde{E})$, we have

$$g(\Delta^{it}(\xi)) = g(\psi^{-it}\xi) = g(\psi)^{-it}g(\xi) = \lambda_g^{-it}\psi^{-it}g(\xi) = \lambda_g^{-it}\Delta^{it}(\xi).$$

Hence

$$\begin{aligned} g(\Delta^{it}P\Delta^{-it}) = \lambda_g^{-it}\Delta^{it}gP\Delta^{-it} &= \lambda_g^{-it}\Delta^{it}Pg\Delta^{-it} \\ &= \lambda_g^{-it}\Delta^{it}P\lambda_g^{it}\Delta^{-it} = (\Delta^{it}P\Delta^{-it})g, \end{aligned}$$

because the multiplication by $\lambda_g^{it} \in C^\infty(V)$ commutes with operators Δ^{it} and P. $\square$

Lemma 4.8. *The linear operator Δ^{it} extends to a bounded operator on $W_\tau^s(\widetilde{E})$ for all s.*

Proof. Recall that the L^2-inner product induces a well-defined pairing

$$\langle \cdot, \cdot \rangle_x : W_x^s \times W_x^{-s} \to \mathbb{C}$$

such that $|\langle \xi, \eta \rangle_x| \le ||\xi_x||_s ||\eta_x||_{-s}$. Let $s \ge 0$. Set $Q = \psi^{it}\Lambda^{2s}\psi^{-it}$. Thanks to Lemma 4.7, $Q \in \Psi_\Gamma^{2s}(\widetilde{E})$. We have

$$\begin{aligned} ||\Delta^{it}(\xi)_x||_s^2 &= \langle \Delta^{it}(\xi)_x, \Lambda^{2s}\Delta^{it}(\xi)_x \rangle \\ &= \langle \xi_x, (Q\xi)_x \rangle \\ &\le ||\xi_x||_s ||Q_x\xi_x||_{-s} \\ &\le ||\xi_x||_s ||Q_x||_{-s,s} ||\xi_x||_s. \end{aligned}$$

Therefore $\Delta^{it} : W_\tau^s(\widetilde{E}) \to W_\tau^s(\widetilde{E})$ is bounded for $s \ge 0$. Then by nondegeneracy of the pairing $W_x^s \times W_x^{-s} \to \mathbb{C}$, we see that $\Delta^{it} : W_\tau^s(\widetilde{E}) \to W_\tau^s(\widetilde{E})$ is bounded for all s. $\square$

By Lemma 4.8, there exists a constant $C > 0$ such that

$$|||\Delta^{it}P\Delta^{-it}||| \le C|||P||| \quad \text{for} \quad P \in \Psi_\Gamma^{-1}(\widetilde{E}).$$

By continuity, $\widehat{\sigma}_t(P) = \Delta^{it}P\Delta^{-it}$, $P \in \mathfrak{A}$, gives rise to an $\mathbb{R}$-action on the Banach algebra $\mathfrak{A}$. Denote by δ the generator of $(\widehat{\sigma}_t)$, i.e.

$$\delta(P) = \lim_{t\to 0} i(\widehat{\sigma}_t(P) - P)/t, \quad P \in \mathfrak{A}.$$

Then δ is a closed derivation of $\mathfrak{A}$, whose domain contains $\Psi_\Gamma^{-1}(\widetilde{E})$. Set $\varphi = \log \psi$. Then by a straightforward computation we obtain that

$$\delta(P) = [\varphi, P] = \varphi P - P\varphi, \quad P \in \Psi_\Gamma^{-1}(\widetilde{E}),$$

where φ is regarded as pointwise multiplication operator.

Proposition 4.9. *If $P \in \Psi_\Gamma^{-1}(\widetilde{E})$, then $\delta(P) \in \Psi_\Gamma^{-2}(\widetilde{E})$.*

Proof. Recall first the definition of ψ, i.e.

$$\widetilde{\omega} \wedge \omega_V = \psi\omega.$$

From this, $g(\widetilde{\omega}) \wedge g(\omega_V) = g(\psi)g(\omega)$, $g \in \Gamma$. Since $\widetilde{\omega}$ and ω are Γ-invariant, and $g(\omega_V) = \lambda_g\omega_V$, we have

$$\lambda_g\widetilde{\omega} \wedge \omega_V = g(\psi)\omega.$$

Therefore we have

$$\lambda_g\psi = g(\psi), \quad g \in \Gamma, \tag{4.10}$$

and

$$\log \lambda_g + \varphi = g(\varphi). \tag{4.11}$$

Since $\varphi \in C^\infty(\widetilde{M} \times V)$, both φP and $P\varphi$ are continuous linear operators $C_c^{\infty,0}(\widetilde{E}) \to C_c^{\infty,0}(\widetilde{E})$, and $(\varphi P)_x = \varphi_x P_x$, $(P\varphi)_x = P_x\varphi_x$ are ψDO's on $\widetilde{M}_x$ for every $x \in V$. By asymptotic expansion of the symbols, we can see that $\varphi_x P_x - P_x\varphi_x$ is a ψDO of order -2. Hence we only have to show that $[\varphi, P]$ is Γ-invariant. We have

$$\begin{aligned} g(\varphi P - P\varphi) &= g(\varphi)Pg - Pg(\varphi)g \\ &= (g(\varphi)P - Pg(\varphi))g \\ &= (\varphi P - P\varphi)g + (\log\lambda_g P - P\log\lambda_g) \qquad \text{by (4.11)} \\ &= (\varphi P - P\varphi)g, \end{aligned}$$

because $\log \lambda_g$ commutes with P.

5. Godbillon-Vey Classes.

Throughout this section, V denotes the circle S^1 with the canonical volume form dx. The foliation $\mathcal{F}$ on $X = \widetilde{M} \times_\Gamma V$ is transversely orientable and codimension one. To such a foliation, a characteristic class $gv(\mathcal{F})$, called the Godbillon-Vey class, is assigned. In this section we will give a description of $gv(\mathcal{F})$ in terms of function ψ introduced in the preceding section. We will use this description in Section 8.

Let θ be an arbitrary 1-form on X defining $\mathcal{F}$. By integrability, there exists a 1-form η such that $d\theta = \eta \wedge \theta$. The Godbillon-Vey class then given by $[\eta \wedge d\eta] \in H^3_{DR}(X)$ ([**13**]).

Let $\widetilde{\theta}, \widetilde{\eta}$ be the lifting of θ, η respectively to $\widetilde{M} \times V$. Let Ω be the pullback of $\widetilde{\omega}$ by $\widetilde{M} \times V \to \widetilde{M}$. Then $\omega = \Omega \wedge \widetilde{\theta}$ is a Γ-invariant volume form on $\widetilde{M} \times V$.

Since $\widetilde{\theta}$ and $\omega_V = dx$ define the same foliation on $\widetilde{M} \times V$, there exists a nowhere vanishing smooth function f on $\widetilde{M} \times V$ such that $\widetilde{\theta} = f\omega_V$. Then

$$\omega = \Omega \wedge \widetilde{\theta} = f\Omega \wedge \omega_V = f\psi\omega.$$

So $f = 1/\psi$. Consequently, $\widetilde{\theta} = (1/\psi)\omega_V$. From this

$$d\widetilde{\theta} = \widetilde{\eta} \wedge \widetilde{\theta} = (1/\psi)\widetilde{\eta} \wedge \omega_V.$$

On the other hand

$$d\widetilde{\theta} = d(1/\psi\omega_V) = d(1/\psi) \wedge \omega_V,$$

for ω_V is closed. From these,

$$(1/\psi)\widetilde{\eta} \wedge \omega_V = d(1/\psi) \wedge \omega_V. \tag{5.1}$$

Recall that $\varphi = \log\psi$ and $\widetilde{\omega} \wedge \omega_V = \psi\omega$. Thus $-d\varphi \wedge \omega_V = \eta \wedge \omega_V$.

The tangent bundle T of $\widetilde{M} \times V$ has a splitting

$$T = T' \oplus T'',$$

where T' (resp. T'') consists of vectors tangential to $\widetilde{M}_x$, $x \in V$ (resp. $\{a\} \times V$, $a \in \widetilde{M}$). Set

$$\Omega^{n,m} = C^\infty(\Lambda^n(T')^* \otimes \Lambda^m(T'')^*).$$

The exterior derivative d splits as

$$d = d' + (-1)^n d'' \quad \text{on} \quad \Omega^{n,m},$$

where d' and d'' are exterior derivatives in the direction of $\widetilde{M}$ and V, respectively.

The $(1,0)$-component and $(0,1)$-component of the 1-form $\widetilde{\eta}$ are denoted $\widetilde{\eta}'$ and $\widetilde{\eta}''$, respectively. Since the wedge product with ω_V induces an injection

$$\Omega^{1,0} \to \Omega^{1,1},$$

it follows from (5.1) that

$$-d'\varphi = -(d\varphi)' = \widetilde{\eta}'.$$

Then

$$\begin{aligned} \widetilde{\eta} \wedge d\widetilde{\eta} &= (\widetilde{\eta}' + \widetilde{\eta}'') \wedge d(\widetilde{\eta}' + \widetilde{\eta}'') \\ &= -\widetilde{\eta}' \wedge d''\widetilde{\eta}' + \widetilde{\eta}' \wedge d'\widetilde{\eta}'', \end{aligned} \tag{5.2}$$

because $d\widetilde{\eta}' = -d'd'\varphi = 0$ and $\Omega^{n,m} = 0$ for $m > 1$. We have

$$\begin{aligned} d(\widetilde{\eta}' \wedge \widetilde{\eta}'') &= (d'\widetilde{\eta}' - d''\widetilde{\eta}') \wedge \widetilde{\eta}'' - \widetilde{\eta}' \wedge (d'\widetilde{\eta}'' + d''\widetilde{\eta}'') \\ &= -\widetilde{\eta}' \wedge d'\widetilde{\eta}''. \end{aligned} \tag{5.3}$$

Notice that $\widetilde{\eta}' \wedge \widetilde{\eta}''$ is Γ-invariant, sine the Γ-action on $\widetilde{M} \times V$ preserves the decomposition $T = T' \oplus T''$.

Proposition 5.4. *The Godbillon-Vey class of $\mathcal{F}$ is given by the cohomology class*

$$[-d'\varphi \wedge d''d'\varphi] \in H^3_{DR}(X).$$

Proof. By (5.2) and (5.3),

$$\widetilde{\eta} \wedge d\widetilde{\eta} = -d'\varphi \wedge d''d'\varphi - d(\widetilde{\eta}' \wedge \widetilde{\eta}'').$$

Since $\widetilde{\eta} \wedge d\widetilde{\eta}$ and $\widetilde{\eta}' \wedge \widetilde{\eta}''$ are Γ-invariant, so is $d'\varphi \wedge d''d'\varphi$. Therefore $-d'\varphi \wedge d''d'\varphi$ defines a 3-form on X, and

$$[\eta \wedge d\eta] = [-d'\varphi \wedge d''d'\varphi] \in H^3_{DR}(X).$$

□

Remark 5.5. Equality (4.11) together with the fact that $\log \lambda_g$ on $\widetilde{M} \times V$ is constant in the direction of $\widetilde{M}$ implies that $d'\varphi$ is Γ-invariant.

6. Cyclic Cocycles.

In this section we will construct a densely defined cyclic cocycle on the algebra $\mathfrak{A}$. This cocycle can be interpreted as an analytical variant of $gv(\mathcal{F})$.

As in the preceding sections, let $\widetilde{E}$ be a given Γ-equivariant bundle over $\widetilde{M} \times S^1$. Define a new right Γ-action by

$$v \cdot g = \lambda_g(x)^{-1} vg, \qquad v \in \widetilde{E}_{(m,x)}, \quad g \in \Gamma, \tag{6.1}$$

where vg is the given Γ-action. Denote by $\widetilde{E}'$ the vector bundle $\widetilde{E}$ equipped with this new action, and denote by $g[\xi]$ the action of $g \in \Gamma$ on $\xi \in C_c^{\infty,0}(\widetilde{E}')$. Then

$$g[\xi] = \lambda_g g(\xi). \tag{6.2}$$

With respect to the new action (6.1), the Hermitian metric of $\widetilde{E}$ is no more Γ-invariant. However, we have the relation

$$(v \cdot g, w \cdot g) = \lambda_g(x)^{-2}(v, w), \qquad v, w \in \widetilde{E}_{(m,x)}.$$

This enables us to obtain continuous fields of tangential Sobolev spaces.

Let $P \in \Psi_\Gamma^r(\widetilde{E})$. Then

$$\begin{aligned} g[P(\xi)] &= \lambda_g g(P(\xi)) = g[\xi] = \lambda_g P(g(\xi)) \\ &= P(\lambda_g g(\xi)) \\ &= P(g[\xi]), \end{aligned}$$

here we used the fact that P commutes with the multiplication operator λ_g. Thus $P \in \Psi_\Gamma^r(\widetilde{E}')$.

Conversely, if $Q \in \Psi_\Gamma^r(\widetilde{E}')$, then

$$\begin{aligned} \lambda_g g(Q(\xi)) &= g[Q(\xi)] = Q(g[\xi]) \\ &= Q(\lambda_g g(\xi)) \\ &= \lambda_g Q(g(\xi)). \end{aligned}$$

Since $\lambda_g > 0$, we have $g(Q(\xi)) = Q(g(\xi))$, i.e. $Q \in \Psi_\Gamma^r(\widetilde{E})$.

Denote by $\partial_2\varphi$ the partial derivative of φ in the direction of S^1. Regard the pointwise multiplication by $\partial_2\varphi$ as an operator $C_c^{\infty,0}(\widetilde{E}) \to C_c^{\infty,0}(\widetilde{E}')$, and consider the commutator of operators

$$[\partial_2\varphi, P] = (\partial_2\varphi)P - P(\partial_2\varphi) \quad \text{for} \quad P \in \Psi_\Gamma^r(\widetilde{E}).$$

Proposition 6.3. *We have* $[\partial_2\varphi, P] \in \Psi_\Gamma^{r-1}(\widetilde{E}, \widetilde{E}')$.

Proof. The proof is similar to that of Proposition 4.9. We only have to show that $[\partial_2\varphi, P]$ is Γ-equivariant. We then have

$$\begin{aligned} g[(\partial_2\varphi)P - P(\partial_2\varphi)] &= \lambda_g g((\partial_2\varphi)P - P(\partial_2\varphi)) \\ &= \lambda_g(g(\partial_2\varphi)gP - Pg(\partial_2\varphi)g) \\ &= \lambda_g g(\partial_2\varphi)Pg - \lambda_g Pg(\partial_2\varphi)g \\ &= (\partial_2\varphi + \partial_2(\log\lambda_g))Pg - P(\partial_2\varphi + \partial_2(\log\lambda_g))g \\ &= ((\partial_2\varphi)P - P(\partial_2\varphi))g + \partial_2(\log\lambda_g)Pg - P\partial_2(\log\lambda_g)g. \end{aligned}$$

Since $\log\lambda_g$ is constant along $\widetilde{M}_x$, $x \in S^1$, so is $\partial_2(\log\lambda_g)$. Thus $\partial_2(\log\lambda_g)$ commutes with P. Hence

$$g[\partial_2\varphi, P] = [\partial_2\varphi, P]g,$$

i.e. $[\partial_2\varphi, P]$ is Γ-equivariant. □

Let N be as in Section 3, and let $N \geq r \geq 0$. As in Section 3, we can define a norm $|||\cdot|||$ on $\Psi_\Gamma^{-r}(\widetilde{E}, \widetilde{E}')$ by

$$|||P||| = \max\{||P||_{-N+r,-N}, ||P||_{N,N-r}\}.$$

Denote by $OP_\Gamma^{-r}(\widetilde{E}, \widetilde{E}')$ the completion of $\Psi_\Gamma^{-r}(\widetilde{E}, \widetilde{E}')$ with respect to $|||\cdot|||$. It is easy to see that if $P \in \Psi_\Gamma^p(\widetilde{E})$, $Q \in \Psi_\Gamma^q(\widetilde{E}, \widetilde{E}')$, then PQ, $QP \in \Psi_\Gamma^{p+q}(\widetilde{E}, \widetilde{E}')$.

Proposition 6.4. *The space* $OP_\Gamma^{-2}(\widetilde{E}, \widetilde{E}')$ *is a Banach* $\mathfrak{A}$*-module.*

Proof. Straightforward. □

Notice that the correspondence $P \to [\partial_2\varphi, P]$ is an unbounded derivation from $\mathfrak{A}$ into $OP_\Gamma^{-2}(\widetilde{E}, \widetilde{E}')$ with domain $\Psi_\Gamma^{-1}(\widetilde{E})$. Closability of the multiplication operator $\partial_2\varphi$ implies that the derivation $P \to [\partial_2\varphi, P]$ is closable. Denote by δ_1 its closure with domain $\mathrm{Dom}(\delta_1)$.

Consider the multiplication operator Δ^{it} on both $\widetilde{E}$ and $\widetilde{E}'$.

Proposition 6.5. *If* $Q \in \Psi_\Gamma^r(\widetilde{E}, \widetilde{E}')$, *then* $\Delta^{it}Q\Delta^{-it} \in \Psi_\Gamma^r(\widetilde{E}, \widetilde{E}')$.

Proof. It is sufficient to show that $\Delta^{it}Q\Delta^{-it}$ is Γ-equivariant. Let $\xi \in C_c^{\infty,0}(\widetilde{E})$. Then

$$\begin{aligned} g[(\Delta^{it}Q\Delta^{-it})\xi] &= \lambda_g g(\Delta^{it}Q\Delta^{-it}\xi) = \lambda_g g(\Delta^{it})g(Q\Delta^{-it}\xi) \\ &= g(\Delta^{it})\lambda_g g(Q\Delta^{-it}\xi) = g(\Delta^{it})g[Q\Delta^{-it}\xi] \\ &= g(\Delta^{it})Q(g(\Delta^{-it}))g(\xi). \end{aligned} \tag{6.6}$$

By (4.10), $g(\Delta^{it}) = \Delta^{it}$. Hence the equality (6.6) is equal to

$$\lambda_g^{-it}\Delta^{it}Q(\lambda_g^{it}\Delta^{-it}g(\xi)) = (\Delta^{it}Q\Delta^{-it})(\xi).$$

This is the end of the proof. $\square$

For $Q \in \Psi_\Gamma^{-2}(\widetilde{E}, \widetilde{E}')$, set $\widehat{\sigma}_t'(Q) = \Delta^{it}Q\Delta^{-it}$, $t \in \mathbb{R}$.

Lemma 6.7. *The linear operator $\widehat{\sigma}_t'$ extends to an automorphism of $OP_\Gamma^{-2}(\widetilde{E}, \widetilde{E}')$.*

Proof. By Lemma 4.8, the operator Δ^{it} is bounded on Sobolev spaces. Therefore

$$||\Delta^{it}Q\Delta^{-it}||_{s,s-2} \leq C_s||Q||_{s,s-2}.$$

In particular, there exists $C > 0$ such that

$$||\Delta^{it}Q\Delta^{-it}||_{-N+2,-N} \leq C||Q||_{-N+2,-N},$$
$$||\Delta^{it}Q\Delta^{-it}||_{N,N-2} \leq C||Q||_{N,N-2}.$$

It follows that $|||\widehat{\sigma}_t'(Q)||| \leq C|||Q|||$. $\square$

It is clear that $(\widehat{\sigma}_t')$ is a one-parameter group of automorphisms. Denote by δ_2' the generator of $(\widehat{\sigma}_t')$, and by $\mathrm{Dom}(\delta_2')$ its domain.

Proposition 6.8. *If $Q \in \Psi_\Gamma^{-2}(\widetilde{E}, \widetilde{E}')$, then $Q \in \mathrm{Dom}(\delta_2')$, and $\delta_2'(Q) = [\varphi, Q]$.*

Proof. Same as that for the derivation δ_2. $\square$

Proposition 6.9. *If $P \in \Psi_\Gamma^{-1}(\widetilde{E})$, then $\delta_1(\delta_2(P)) = \delta_2'(\delta_1(P))$.*

Proof. From Proposition 6.8 and the definition of δ_1, the conclusion follows. $\square$

Recall that the underlying Hermitian vector bundle structures of $\widetilde{E}$ and $\widetilde{E}'$ are the same. Therefore $L^2(\widetilde{E}) = L^2(\widetilde{E}')$. Then, if $Q \in \Psi_\Gamma^{-r}(\widetilde{E}, \widetilde{E}')$, $r \geq 0$, the operator P_x can be regarded as a bounded operator on $L^2(\widetilde{E}_x)$. Let σ be a compactly supported smooth function on $\widetilde{M} \times S^1$, and let σ_x be the restriction of σ to $\widetilde{M}_x$, $x \in S^1$.

Proposition 6.10. *Let $s > \dim M$. Then $\sigma_x\Lambda_x^{-s/2}$ and $\Lambda_x^{-s/2}\sigma_x$ are Hilbert-Schmidt class operators.*

Proof. Recall that $\Lambda = (I + \Delta)^{1/2}$. For the Laplacian Δ' on M, we have that

$$(I + \Delta')^{-1/2} \in \mathcal{L}^p \quad \text{for any} \quad p > \dim M.$$

From this, $((I+\Delta')^{1/2})^{-s/2}$ is a Hilbert-Schmidt class operator. If Q is a ψDO of order $-1/2$ on M, then Q is a Hilbert-Schmidt class operator, because

$$Q = Q\left((I+\Delta')^{1/2}\right)^{s/2}\left((I+\Delta')^{1/2}\right)^{-s/2}.$$

In particular, the Schwartz kernel of Q is measurable and square-integrable.

Let $P \in \Psi_\Gamma^{-s/2}(\widetilde{E})$. Then the Schwartz kernel of P is measurable [**6**]. The observation above, combined with Γ-compactness of the support of the Schwartz kernel, implies that $\sigma_x P_x$ and $P_x \sigma_x$ are Hilbert-Schmidt class operators.

Let $P \in \Psi_\Gamma^{-s/2}(\widetilde{E})$ be a parametrix of $\Lambda^{s/2}$, so that $T = P\Lambda^{s/2} - I$ is a compactly smoothing operator. We have

$$\Lambda_x^{-s/2} = P_x - T_x\Lambda_x^{-s/2},$$

as operators on $L^2(\widetilde{E}_x)$. From this, $\sigma_x\Lambda_x^{-s/2} = \sigma_x P_x - \sigma_x T_x \Lambda_x^{-s/2}$. Since both $\sigma_x P_x$ and $T_x\sigma_x$ are Hilbert-Schmidt class operators, so is $\sigma_x\Lambda_x^{-s/2}$. As $\Lambda_x^{-s/2}$ is self-adjoint, we see that $\Lambda_x^{-s/2}\sigma_x$ is also a Hilbert-Schmidt class operator. □

Corollary 6.11. *Let σ, σ' be compactly supported smooth functions on $\widetilde{M} \times S^1$. Then for every $P \in \Psi_\Gamma^{-s}(\widetilde{E}, \widetilde{E}')$ with $s > \dim M$, the operator $\sigma_x P_x \sigma'_x$ is a trace class operator on $L^2(\widetilde{E}_x)$, for any $x \in S^1$. Moreover, there exists a constant $C > 0$ such that*

$$|Tr(\sigma_x P_x \sigma'_x)| \leq C||P||_{s/2,-s/2}.$$

Proof. We have

$$\sigma_x P_x \sigma'_x = (\sigma_x\Lambda_x^{-s/2})(\Lambda_x^{s/2}P_x\Lambda_x^{s/2})(\Lambda_x^{-s/2}\sigma'_x).$$

Consequently, $\sigma_x P_x \sigma'_x$ is of trace class, and

$$\begin{aligned} |Tr(\sigma_x P_x \sigma'_x)| &\leq ||(\sigma_x\Lambda_x^{-s/2})(\Lambda_x^{s/2}P_x\Lambda_x^{s/2})(\Lambda_x^{-s/2}\sigma'_x)||_1 \\ &\leq ||\sigma_x\Lambda_x^{-s/2}||_2||\Lambda_x^{s/2}P_x\Lambda_x^{s/2}||_{0,0}||\Lambda_x^{-s/2}\sigma'_x||_2 \\ &\leq ||\sigma_x\Lambda_x^{-s/2}||_2||\Lambda_x^{-s/2}\sigma'_x||_2||\Lambda_x^{s/2}P_x\Lambda_x^{s/2}||_{s/2,-s/2}, \end{aligned}$$

where $||\cdot||_1$ (resp. $||\cdot||_2$) is the trace class norm (resp. Hilbert-Schmidt norm).

Continuity of the family $(\Lambda_x^{-s/2})_x$ implies the existence of $C > 0$ such that

$$||\sigma_x\Lambda_x^{-s/2}||_2,\ ||\Lambda_x^{-s/2}\sigma'_x||_2 < C.$$

Thus

$$|Tr(\sigma_x P_x \sigma'_x)| \leq C||P||_{s/2,-s/2}.$$

□

Let σ be a compactly supported smooth function on $\widetilde{M} \times S^1$ such that

$$\sum_{g\in\Gamma} g(\sigma)^2 = 1;$$

i.e. $\{g(\sigma)^2\}_{g\in\Gamma}$ is a Γ-invariant partition of unity on $\widetilde{M} \times S^1$.

Definition 6.12. For $P \in \Psi_\Gamma^{-s}(\widetilde{E}, \widetilde{E}')$ with $s > \dim M$, set

$$\text{trace}_\Gamma(P) = \int_{S^1} Tr(\sigma_x P_x \sigma_x)\, dx. \tag{6.13}$$

Notice that the integrand in (6.13) is continuous. A modification of the proof of Lemma 4.9 of [**1**] shows that $\text{trace}_\Gamma(P)$ is independent of the choice of σ.

Let $P \in \Psi_\Gamma^{-s}(\widetilde{E}, \widetilde{E}')$, $Q \in \Psi_\Gamma^{-r}(\widetilde{E}) = \Psi_\Gamma^{-r}(\widetilde{E}')$.

Then PQ, $QP \in \Psi_\Gamma^{-s+r}(\widetilde{E}, \widetilde{E}')$.

Proposition 6.14. *Let $r + s > \dim M + 2$. Assume that either $0 \leq r \leq 2$, or $0 \leq s \leq 2$. Then*

$$\text{trace}_\Gamma(PQ) = \text{trace}_\Gamma(QP).$$

Proof. Since P and Q have Γ-compact Schwartz kernels, there exists a finite subset S of Γ satifying:

(i) $S = S^{-1}$,

(ii) $\operatorname{supp} g(\sigma)_x \cap \operatorname{supp} \sigma \neq \varnothing \Rightarrow g \in S$,

(iii) $\sigma_x P_x \Sigma g(\sigma)_x = \sigma_x P_x \Sigma' g(\sigma)_x$, and
$\sigma_x Q_x \Sigma g(\sigma)_x = \sigma_x Q_x \Sigma' g(\sigma)_x$,

where the summation Σ (resp. Σ') is taken over all $g \in \Gamma$ (resp. $g \in S$). Then

$$\begin{aligned} Tr(\sigma_x(P_x Q_x)\sigma_x) &= Tr(\sigma_x(P_x \Sigma g(\sigma)_x^2 Q_x)\sigma_x) \\ &= Tr(\sigma_x(P_x \Sigma' g(\sigma)_x^2 Q_x)\sigma_x) \\ &= \Sigma' Tr(\sigma_x P_x g(\sigma)_x g(\sigma)_x Q_x \sigma_x). \end{aligned}$$

The last expression is equal to

$$\Sigma' Tr(g(\sigma)_x Q_x \sigma_x \sigma_x P_x g(\sigma)_x),$$

because either $\sigma_x P_x g(\sigma)_x$ or $g(\sigma)_x Q_x \sigma_x$ is a trace class operator, by Corollary 6.11 and our assumption on s, r.

Let $U(g)$ be the canonical unitary mapping $L^2(\widetilde{E}_{xg}) \to L^2(\widetilde{E}_x)$. It is easy to check that, as multiplication operator,

$$g(\sigma)_x = U(g)\sigma_{xg}U(g)^{-1}.$$

Then we have

$$\begin{aligned} &g(\sigma)_x Q_x \sigma_x \sigma_x P_x g(\sigma)_x \\ &\quad = U(g)\sigma_{xg}U(g)^{-1}Q_x U(g)U(g)^{-1}(\sigma_x)^2 U(g)U(g)^{-1}P_x U(g)\sigma_{xg}U(g)^{-1}. \end{aligned}$$

Since Q is Γ-equivariant, $U(g)^{-1}Q_x U(g) = Q_{xg}$. As for P, we have

$$U(g)^{-1}P_x U(g) = \lambda_{g^{-1}}(x)^{-1}P_{xg} = \lambda_g(x)P_{xg}.$$

Hence

$$\begin{aligned} \mathrm{trace}_\Gamma(PQ) &= \int_{S^1} \Sigma' Tr(g(\sigma)_x Q_x \sigma_x \sigma_x P_x g(\sigma)_x)\, dx \\ &= \int_{S^1} \Sigma' Tr(\sigma_{xg} Q_{xg} g^{-1}(\sigma)_{xg} g^{-1}(\sigma)_{xg} \lambda_g(x) P_{xg} \sigma_{xg})\, dx \\ &= \int_{S^1} \Sigma' Tr(\sigma_{xg} Q_{xg} g^{-1}(\sigma)_{xg} g^{-1}(\sigma)_{xg} P_{xg} \sigma_{xg})\, d(xg) \\ &= \int_{S^1} \Sigma' Tr(\sigma_x Q_x g^{-1}(\sigma)_x g^{-1}(\sigma)_x P_x \sigma_x)\, dx \\ &= \int_{S^1} \Sigma' Tr(\sigma_x Q_x g(\sigma)_x g(\sigma)_x P_x \sigma_x)\, dx \\ &= \int_{S^1} Tr(\sigma_x Q_x \Sigma' (g^{-1}(\sigma)_x)^2 P_x \sigma_x)\, dx \\ &= \int_{S^1} Tr(\sigma_x Q_x \Sigma (g^{-1}(\sigma)_x)^2 P_x \sigma_x)\, dx \\ &= \mathrm{trace}_\Gamma(QP). \end{aligned}$$

□

By Corollary 6.11, trace_Γ is continuous with respect to $\|\cdot\|_{s/2,-s/2}$, provided that $s > \dim M$. This implies that trace_Γ extends to a continuous linear functional on $OP_\Gamma^{-s}(\widetilde{E}, \widetilde{E}')$ with $s > \dim M$. (Caution: our trace_Γ is not the same as trace_Γ of [**1**]. Our trace_Γ is not an actual trace on any algebra, it is just a linear functional, while Atiyah's trace_Γ is an actual trace on an algebra.)

Lemma 6.15. (1) $\mathrm{trace}_\Gamma([\partial_2\varphi, P]) = 0$ *for all* $P \in \Psi_\Gamma^{-s}(\widetilde{E})$ *with* $s > \dim M$.

(2) $\operatorname{trace}_\Gamma(\delta_2'(Q)) = 0$ *for all* $Q \in \Psi_\Gamma^{-s}(\widetilde{E}, \widetilde{E}')$ *with* $s > \dim M$.

Proof. (1) Notice that $P_x\sigma_x$ and $P_x(\partial_2\varphi)_x\sigma_x$ are trace class operators. Then

$$\begin{aligned} Tr(\sigma_x(\partial_2\varphi)_x P_x\sigma_x) &= Tr(P_x\sigma_x\sigma_x(\partial_2\varphi)_x) \\ &= Tr(P_x(\partial_2\varphi)_x\sigma_x\sigma_x) \\ &= Tr(\sigma_x P_x(\partial_2\varphi)_x\sigma_x). \end{aligned}$$

Thus $Tr(\sigma_x[\partial_2\varphi, P]_x\sigma_x) = 0$. Hence $\operatorname{trace}_\Gamma([\partial_2\varphi, P]) = 0$.

(2) The proof is the same as that of (1). □

Furnish $\mathfrak{E} = \operatorname{Dom}(\delta_1) \cap \operatorname{Dom}(\delta_2)$ with the locally convex topology given by the graph norms associated with δ_1 and δ_2.

We will construct a densely defined cyclic cocycle on $\mathfrak{A}$. Let us first consider the case where $\dim M = 2$. Set

$$\begin{aligned} (6.16)\quad \tau_2(P^0, P^1, P^2) &= \operatorname{trace}_\Gamma(P^0\delta_1(P^1)\delta_2(P^2)) \\ &\quad - \operatorname{trace}_\Gamma(P^0\delta_2(P^1)\delta_1(P^2)) \quad \text{for} \quad P^0, P^1, P^2 \in \mathfrak{E} \subseteq \mathfrak{A}. \end{aligned}$$

Proposition 6.17. *The trilinear functional* τ_2 *is a cyclic* 2*-cocycle.*

Proof. If $P^0, P^1, P^2 \in \mathfrak{E}$, then the products

$$P^0\delta_1(P^0)\delta_2(P^0) \quad \text{and} \quad P^0\delta_2(P^0)\delta_1(P^0)$$

belong to $OP_\Gamma^{-5}(\widetilde{E}, \widetilde{E}')$. Since δ_1 and δ_2 are derivations, τ_2 is a Hochschild cocycle. By Proposition 6.14 and Lemma 6.15, τ_2 is a cyclic cocycle on $\Psi_\Gamma^{-1}(\widetilde{E}) \subset \mathfrak{E}$. Then by continuity and the fact that $\Psi_\Gamma^{-1}(\widetilde{E})$ is dense in $\mathfrak{E}$, we can see that τ_2 is a cyclic cocycle on $\mathfrak{E}$. □

Proposition 6.18. *The densely defined cyclic cocycle* τ_2 *is a* 2*-trace on* $\mathfrak{A}$ *in the sense of* [**8**].

Proof. We have that

$$\widehat{\tau}_2(a^0dx^1a^1dx^2) = \operatorname{trace}_\Gamma(a^0\delta_1(x^1)a^1\delta_2(x^2)) - \operatorname{trace}_\Gamma(a^0\delta_2(x^1)a^1\delta_1(x^2)),$$

and

$$\begin{aligned} &|\operatorname{trace}_\Gamma(a^0\delta_1(x^1)a^1\delta_2(x^2))| \le C\|a^0\delta_1(x^1)a^1\delta_2(x^2)\|_{3/2,-3/2} \\ &\le C\|a^0\|_{3/2,1/2}\|\delta_1(x^1)\|_{1/2,1/2}\|a^1\|_{1/2,-1/2}\|\delta_2(x^2)\|_{-1/2,-3/2} \\ &\le C_{1,2}|||a^0|||\ |||a^1|||, \end{aligned}$$

for some constant $C_{1,2}$ depending only on x^1 and x^2.

Similarly

$$|\operatorname{trace}_\Gamma(a^0\delta_2(x^1)a^1\delta_1(x^2))| \le C'_{1,2}|||a^0|||\,|||a^1|||.$$

This completes the proof. □

Let us now consider higher dimensional cases. Let $\dim M = 2n$. The formula (6.16) defines a cyclic cocycle on $\Psi_\Gamma^{-\infty}(\widetilde{E})$, but not on $\mathfrak{C}$ when $n > 1$. Consider the cyclic $2n$-cocycle $S^{n-1}\tau_2$, instead. For $P^0, \ldots, P^{2n}$, we have

$$\begin{aligned}
&((n-1)!)^{-1}(2\pi i)^{1-n}S^{n-1}\tau_2(P^0,\ldots,P^{2n}) \\
&= \sum_{1\le i\le j\le n} \{\operatorname{trace}_\Gamma(P^0P^1\cdots P^{2i-2}\delta_1(P^{2i-1})P^{2i} \\
&\qquad \cdots P^{2j-1}\delta_2(P^{2j})P^{2j+1}\cdots P^{2n}) \\
&\qquad - \operatorname{trace}_\Gamma(P^0P^1\cdots P^{2i-2}\delta_2(P^{2i-1})P^{2i} \\
&\qquad \cdots P^{2j-1}\delta_1(P^{2j})P^{2j+1}\cdots P^{2n})\}.
\end{aligned} \tag{6.19}$$

Denote by $\tau_{2n}(P^0,\ldots,P^{2n})$ the right-hand side of (6.19). Notice that $\tau_{2n}(P^0,\ldots,P^{2n})$ makes sense when $P^0,\ldots,P^{2n} \in \mathfrak{C}$.

The proof of Proposition 6.18 can be generalized to show that τ_{2n} is a $2n$-trace on $\mathfrak{A}$.

Definition 6.20. When $\dim M = 2n$, the Godbillon-Vey cyclic cocycle gv is the $2n$-trace

$$gv = (n-1)!\tau_{2n}.$$

By [**8**, Lemma 2.3; Corollary 2.4], gv extends to a cyclic $2n$-cocycle on a holomorphically closed dense subalgebra of $\mathfrak{A}$, consequently it induces an additive map from $K_0[\mathfrak{A}]$ into the scalars. By Proposition 3.6, the canonincal inclusion $\mathfrak{A} \subseteq C^*(X,\mathcal{F},E)$ induces an isomorphism of K_0-groups. Hence gv induces a map $K_0[C^*(X,\mathcal{F},E)] \to \mathbb{C}$. In Section 8 we will compute the value of this map on a specific class in $K_0[C^*(X,\mathcal{F},E)]$.

7. Dirac Operators and Graph Projections.

In this section we will show that the graph projection of a longitudinal Dirac operator belongs to the domain of the $2n$-trace gv on $\mathfrak{A}$.

Let $\widetilde{M}$ be as in the preceding sections. Assume further that $\widetilde{M}$ is even-dimensional and is furnished with a Γ-invariant spin structure. Denote by $\widetilde{D}$ the associated Dirac operator on $\widetilde{M}$ acting on the bundle $\widetilde{S}$ of (complex) spinors. Since $\widetilde{M}$ is even, the bundle $\widetilde{S}$ has a $\mathbb{Z}_2$-grading ε. Thus

$$\widetilde{S} = \widetilde{S}^+ \oplus \widetilde{S}^-, \tag{7.1}$$

where $\widetilde{S}^{\pm}$ are ± 1 eigenspaces of ε, respectively. With respect to the decomposition (7.1), the operator $\widetilde{D}$ has the form

$$\widetilde{D} = \begin{pmatrix} 0 & \widetilde{D}^{-} \\ \widetilde{D}^{+} & 0 \end{pmatrix},$$

where $\widetilde{D}^{\pm}$ are first-order, elliptic differential operators. Since the Γ-action on $\widetilde{S}$ preserves $\widetilde{S}^{\pm}$ respectively, $\widetilde{D}^{\pm}$ are Γ-equivariant operators. Moreover, $\widetilde{D}$ is essentially selfadjoint and has a closed extension. The closure $\widetilde{D}^{**}$ of $\widetilde{D}$ has the form

$$\widetilde{D}^{**} = \begin{pmatrix} 0 & T^{*} \\ T & 0 \end{pmatrix},$$

where T is the closure of $\widetilde{D}^{*}$, and $\widetilde{D}^{**}$ is selfadjoint.

The graph $G(T)$ of T is, by the definition of T, a closed subspace of $L^2(\widetilde{S}^{+}) \oplus L^2(\widetilde{S}^{-}) = L^2(\widetilde{S})$. Denote the corresponding orthogonal projection by e, and set

$$X = \widetilde{D}^{**}\varepsilon = \begin{pmatrix} 0 & T^{*} \\ T & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -T^{*} \\ T & 0 \end{pmatrix}.$$

Lemma 7.2. *We have*

$$e = \begin{pmatrix} (I+T^{*}T)^{-1} & (I+T^{*}T)^{-1}T^{*} \\ T(I+T^{*}T)^{-1} & T(I+T^{*}T)^{-1}T^{*} \end{pmatrix} = (I+X) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (I+X)^{-1}.$$

Proof. Define $\imath : L^2(\widetilde{S}^{+}) \to L^2(\widetilde{S}^{+}) \oplus L^2(\widetilde{S}^{-})$ by

$$\imath = \begin{pmatrix} I \\ T \end{pmatrix} (I+T^{*}T)^{-1/2} = \begin{pmatrix} (I+T^{*}T)^{-1/2} \\ T(I+T^{*}T)^{-1/2} \end{pmatrix}.$$

It is easy to see that $\imath^{*}\imath = 1$. Since $(I+T^{*}T)^{-1/2}$ is an isomorphism from $L^2(\widetilde{S}^{+})$ onto the domain $\mathrm{Dom}(T)$ of T, the image of $\imath$ is precisely the graph $G(T)$. Thus the projection e is given by

$$e = \imath\imath^{*} = \begin{pmatrix} (I+T^{*}T)^{-1} & (I+T^{*}T)^{-1}T^{*} \\ T(I+T^{*}T)^{-1} & T(I+T^{*}T)^{-1}T^{*} \end{pmatrix}.$$

As for the second equality, from the equality

$$(I+X)^{-1} = (I-X^2)^{-1}(I-X) = \begin{pmatrix} (I+T^{*}T)^{-1} & 0 \\ 0 & (I+TT^{*})^{-1} \end{pmatrix} \begin{pmatrix} 0 & T^{*} \\ -T & 0 \end{pmatrix},$$

it follows that

$$e = (I + X) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (I + X)^{-1}.$$

□

Set

$$p_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad p_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Set $u = (I + X)\varepsilon$. Then

$$u^2 = (I + X)\varepsilon(I + X)\varepsilon = (I + X)(I - X) = I - X^2,$$

because $X\varepsilon = -\varepsilon X$.

Let $\widehat{e} = e - p_-$. Then using the equality

$$\varepsilon = (I + X)p_+ - p_-(I + X),$$

we can see that

$$\begin{aligned} \widehat{e} = e - p_- &= (I + X)p_+(I + X)^{-1} - p_- \\ &= ((I + X)p_+ - p_-(I + X))(I + X)^{-1} \\ &= \varepsilon(I + X)^{-1} \\ &= u^{-1}. \end{aligned} \tag{7.3}$$

From this,

$$\widehat{e}^2 = u^{-2} = (I - X^2)^{-1}. \tag{7.4}$$

A straightforward computation shows that

$$\widehat{e} = \begin{pmatrix} (I + T^*T)^{-1} & (I + T^*T)^{-1}T^* \\ T(I + T^*T)^{-1} & -(I + TT^*)^{-1} \end{pmatrix}. \tag{7.5}$$

As in the preceding sections, suppose that Γ acts on S^1 by orientation preserving diffeomorphisms. For each $x \in S^1$, identify $\widetilde{M}_x = \widetilde{M} \times \{x\}$ with $\widetilde{M}$ in a natural way. Via this identification, we obtain a vector bundle $\widetilde{S}_x$ and a differential operator $\widetilde{D}_x$. By abuse of language, denote the family $(\widetilde{D}_x)$ by $\widetilde{D}$. It is clear that $\widetilde{D}$ is a Γ-equivariant family of elliptic operators, acting on a Γ-equivariant vector bundle $\widetilde{S} = (\widetilde{S}_x)$, i.e.

$$\widetilde{D} \in \Psi^1_\Gamma(\widetilde{S}).$$

The Γ-equivariant differential operator $\widetilde{D}$ on $\widetilde{M} \times S^1$ descends to a longitudinal elliptic operator D on $X = \widetilde{M} \times_\Gamma S^1$, which we call a *longitudinal Dirac operator.*

The operator $\widetilde{D}$ is of the form

$$\widetilde{D} = \begin{pmatrix} 0 & \widetilde{D}^- \\ \widetilde{D}^+ & 0 \end{pmatrix},$$

and $\widetilde{D}^+ = (\widetilde{D}^+_x) \in \Psi^1_\Gamma(\widetilde{S}^+, \widetilde{S}^-)$. Consequently, we can consider a continuous field $e = (e_x)$ of projections: each e_x is the orthogonal projection of $L^2(\widetilde{S}^+_x) \oplus L^2(\widetilde{S}^-_x)$ onto the graph of the closure of $\widetilde{D}^+_x$. The matrix p_- can be regarded as the orthogonal projection of $\left(L^2(\widetilde{S}_x)\right)_{x\in S^1}$ onto $\left(L^2(\widetilde{S}^-_x)\right)_{x\in S^1}$. Then, obviously $p_- \in \Psi^0_\Gamma(\widetilde{S})$.

We devote the rest of the section to show that $\widehat{e}$ belongs to the domain of the cyclic cocycle gv. For this purpose we employ the method of bounded propagation [**20**], [**21**], [**23**]. Since the Dirac operator $\widetilde{D}$ is the lifting of the Dirac operator on a closed manifold M, it has bounded propagation speed.

Recall that the space $S^0(\mathbb{R})$ of symbols of order zero is the collection of all C^∞-functions f on $\mathbb{R}$ such that for each $j = 0, 1, 2, \dots$, it holds that

$$\sup\{(1+|x|)^j |f^{(j)}(x)| : x \in \mathbb{R}\} < \infty.$$

We need the following:

Proposition 7.6. ([**15**, Thm. 7.25], [**20**, Thm. 21]). *Let $P \in \Psi^1_\Gamma(\widetilde{E})$ be a longitudinal, tangentially essentially selfadjoint, first-order elliptic differential operator of bounded propagation speed. If the Fourier transform $\widehat{f}$ of $f \in S^0(\mathbb{R})$ is compactly supported, then*

$$f(P) \in \Psi^0_\Gamma(\widetilde{E}).$$

If the Fourier transform $\widehat{g}$ of a Schwartz function g is compactly supported, $g(P)$ is compactly smoothing.

Let $\rho_+ : \mathbb{R} \to [0,1]$ be a C^∞-function such that

$$\rho_+ \equiv 1 \qquad \text{on} \quad t \le 1 - \delta,$$

and

$$\rho_+ \equiv 0 \qquad \text{on} \quad t \ge 1 + \delta$$

for some sufficiently small $0 < \delta < 1$. Set $\rho_-(t) = \rho_+(-t)$. For $\lambda \ge 2$, set

$$\rho_\lambda(t) = \rho_+(t - (\lambda - 1))\rho_-(t + \lambda - 1)$$

to obtain a C^∞-function $\rho_\lambda(t) : \mathbb{R} \to [0,1]$ such that

$$\rho_\lambda(t) \equiv 1 \qquad \text{on} \quad |t| \le \lambda - 1 - \delta,$$

and

$$\rho_\lambda(t) \equiv 0 \qquad \text{on} \quad |t| \ge \lambda - 1 + \delta.$$

Lemma 7.7. *For any positive integer i, there exists a positive constant C_i such that*

$$|\rho_\lambda^{(i)}(t)| \le C_i \qquad \textit{for all} \quad \lambda, t.$$

Proof. By the construction of ρ_λ, it is straightforward. ☐

Set

$$\varphi_\lambda(x) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{ixt} \rho_\lambda(t) e^{-|t|}\, dt. \tag{7.8}$$

Lemma 7.9. (1) *The function φ_λ belongs to $S^0(\mathbb{R})$, and its Fourier transform is $\rho_\lambda(t)e^{-|t|}$.*

(2) *The function $\psi_\lambda(x) = (2\pi)^{-1/2}(1+x^2)\varphi_\lambda(x) - 1$ is a Schwartz function with compactly supported Fourier transform.*

(3) *As $\lambda \to \infty$, ψ_λ converges to zero in $C_0(\mathbb{R})$.*

Proof. (1) Using integration by parts twice, we get that

$$\begin{aligned} (2\pi)^{-1/2}\varphi_\lambda(x) = \frac{1}{1+x^2} &+ \int_0^\infty \rho_\lambda''(t) e^{(ix-1)t}(ix-1)^{-2}\, dt \\ &+ \int_{-\infty}^0 \rho_\lambda''(t) e^{(ix+1)}(ix+1)^{-2}\, dt. \end{aligned} \tag{7.10}$$

From this, it follows that $\sup\{(1+x^2)|\varphi_\lambda(x)|;\ x \in \mathbb{R}\} < \infty$. This, in turn, means that $\varphi_\lambda \in L^1(\mathbb{R})$, because φ_λ is continuous. Then by the Fourier inversion formula,

$$\widehat{\varphi}_\lambda(t) = \rho_\lambda(t) e^{-|t|}.$$

For a given nonnegative integer j, consider

$$h_\lambda(t) = (it)^j \rho_\lambda(t).$$

Notice that $h_\lambda^{(k)}(0) = 0$ for $k = 0, 1, \dots, j$. Then

$$\begin{aligned}(2\pi)^{-1/2}\varphi_\lambda^{(j)}(t) &= \int_{\mathbb{R}} (it)^{-j} e^{ixt} \rho_\lambda(t) e^{-|t|}\, dt \\ &= \int_0^\infty h_\lambda(t) e^{(ix-1)t}\, dt + \int_{-\infty}^0 h_\lambda(t) e^{(ix+1)t}\, dt \\ &= (-1)^j \int_0^\infty h_\lambda^{(j)}(t) e^{(ix-1)t} (ix-1)^{-j}\, dt \\ &\quad + \int_{-\infty}^0 h_\lambda^{(j)}(t) e^{(ix+1)t}(ix+1)^{-j}\, dt.\end{aligned}$$

It is easy to see that there exists a constant $C > 0$ such that

$$|\varphi_\lambda^{(j)}(x)| \leq C(|ix-1|^{-j} + |ix+1|^{-j}) \quad \text{for all} \quad x.$$

Thus

$$\sup\{(1+|x|)^j |\varphi_\lambda^{(j)}(x)|; \quad x \in \mathbb{R}\} < \infty.$$

(2) The equality (7.10) implies that $\psi_\lambda \in C_0(\mathbb{R})$. We need the following Sublemma, which we will prove later.

Sublemma. *As distributions, we have the identity*

$$\left(1 - \frac{d^2}{dt^2}\right) e^{-|t|} = 2\delta_0,$$

where δ_0 is the delta function at $t = 0$.

We now have that

$$\begin{aligned}(7.11) \qquad \widehat{\psi}_\lambda &= \left(1 - \frac{d^2}{dt^2}\right) \rho_\lambda e^{-|t|} - \delta_0 \\ &= -\rho_\lambda'' e^{-|t|} + 2\rho_\lambda' e^{-|t|} sgn(t) \qquad \text{(as distributions).}\end{aligned}$$

Since both sides of (7.11) are compactly supported C^∞-functions, they are actually equal as C^∞-functions. It is now clear that ψ_λ is a Schwartz function.

(3) The Fourier transform induces an isomorphism from $C_0(\mathbb{R})$ onto $C^*(\mathbb{R})$. So

$$||\psi_\lambda||_{C_0(\mathbb{R})} = ||\widehat{\psi}_\lambda||_{C^*(\mathbb{R})} \leq ||\widehat{\psi}_\lambda||_{L^1(\mathbb{R})}.$$

By our construction, $\rho_\lambda'', \rho_\lambda'$ are bounded uniformly in λ. Therefore the equality (7.11) implies that

$$||\widehat{\psi}_\lambda||_{L^1(\mathbb{R})} \to 0 \qquad \text{as} \quad \lambda \to \infty.$$

This concludes the proof of Lemma 7.9. □

Proof of Sublemma. Let $f(t) = e^{-|t|}$. For $g \in C_c^\infty(\mathbb{R})$, applying integration by parts twice, we obtain that

$$\begin{aligned}\int_{\mathbb{R}} f(t)g''(t)\,dt &= \int_0^\infty f(t)g''(t)\,dt + \int_{-\infty}^0 f(t)g''(t)\,dt \\ &= -2g(0) + \int_{\mathbb{R}} f(t)g(t)\,dt.\end{aligned}$$

Therefore

$$\begin{aligned}\left\langle \frac{1}{2}\left(1 - \frac{d^2}{dt^2}\right) f, g \right\rangle &= \left\langle f, \frac{1}{2}\left(1 - \frac{d^2}{dt^2}\right) g \right\rangle \\ &= \frac{1}{2}\int_{\mathbb{R}} f(t)g(t)\,dt - \frac{1}{2}\int_{\mathbb{R}} f(t)g''(t)\,dt \\ &= g(0) \\ &= \langle \delta_0, g\rangle.\end{aligned}$$

□

For $P \in \Psi_\Gamma^r(\widetilde{E})$, by a straightforward computation we get that

$$||P||_{k,k+r} = ||(I+\Delta)^{k/2} P (I+\Delta)^{-(k+r)/2}||_{0,0}. \tag{7.12}$$

In the definition of tangential Sobolev spaces for the bundle $\widetilde{S}$, we can use $\widetilde{D}^2$ in place of the Laplacian, thanks to the standard elliptic estimate. Thus we may assume that the Sobolev s-norm is given by

$$||\xi||_s = \left\| \left(I + \widetilde{D}^2\right)^{s/2} \xi \right\|_0 \qquad \text{for} \quad \xi \in C_c^\infty.$$

Consider an (unbounded) intertwining operator $T = (T_x)$ of $W_\tau^0(\widetilde{S}) = (L^2(\widetilde{S}_x))_x$, where T_x is the closure of $\widetilde{D}_x^+$. As before, set

$$X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}.$$

Then

$$\widehat{e} = (I+X)\varepsilon \left(I + \widetilde{D}^2\right)^{-1}. \tag{7.13}$$

By Proposition 7.5 and Lemma 7.9,

$$\varphi_\lambda(\widetilde{D}) \in \Psi_\Gamma^0(\widetilde{S}),$$

and

$$\psi_\lambda(\widetilde{D}) = \sqrt{2\pi}\left(I + \widetilde{D}^2\right)\varphi_\lambda(\widetilde{D}) - I \in \Psi_\Gamma^{-\infty}(\widetilde{S}).$$

These imply that $\varphi_\lambda(\widetilde{D}) \in \Psi_\Gamma^{-2}(\widetilde{S})$. Hence

$$(I + X)\varepsilon\varphi_\lambda(\widetilde{D}) \in \Psi_\Gamma^{-1}(\widetilde{S}).$$

The equality (7.13) means, in particular that $\widehat{e}$ is an operator of order -1. Therefore we can consider the norm $\left\|\widehat{e} - \sqrt{2\pi}(I + X)\varepsilon\varphi_\lambda(\widetilde{D})\right\|_{k,k-1}$.

By (7.12)

$$\begin{aligned}
&\left\|\widehat{e} - \sqrt{2\pi}(I + X)\varepsilon\varphi_\lambda(\widetilde{D})\right\|_{k,k-1} \\
&= \left\|(I + X)\varepsilon\left(I + \widetilde{D}^2\right)^{-1} - \sqrt{2\pi}(I + X)\varepsilon\varphi_\lambda(\widetilde{D})\right\|_{k,k-1} \\
&= \left\|\left(I + \widetilde{D}^2\right)^{k/2}(I + X)\varepsilon\left(\left(I + \widetilde{D}^2\right)^{-1} - \sqrt{2\pi}\varphi_\lambda(\widetilde{D})\right)\left(I + \widetilde{D}^2\right)^{(1-k)/2}\right\|_{0,0} \\
&= \left\|(I + X)\varepsilon\left(\left(I + \widetilde{D}^2\right)^{-1} - \sqrt{2\pi}\varphi_\lambda(\widetilde{D})\right)\left(I + \widetilde{D}^2\right)^{1/2}\right\|_{0,0} \\
&= \left\|(I + X)\varepsilon\left(I + \widetilde{D}^2\right)^{-1}\left(I - \sqrt{2\pi}\left(I + \widetilde{D}^2\right)\varphi_\lambda(\widetilde{D})\right)\right\|_{0,0} \\
&\leq \left\|(I + X)\varepsilon\left(I + \widetilde{D}^2\right)^{-1}\right\|_{0,0}\left\|I - \sqrt{2\pi}\left(I + \widetilde{D}^2\right)\varphi_\lambda(\widetilde{D})\right\|_{0,0}.
\end{aligned}$$

In this computation we have used the fact that $(I + \widetilde{D}^2)^{1/2}$ commutes with $(I + X)\varepsilon(I + \widetilde{D}^2)^{-1} - \sqrt{2\pi}\varphi_\lambda(\widetilde{D})$. Now by Lemma 7.9, (3),

$$\left\|\widehat{e} - \sqrt{2\pi}(I + X)\varepsilon\varphi_\lambda(\widetilde{D})\right\|_{k,k-1} \to 0 \quad \text{as} \quad \lambda \to \infty.$$

Thus $\widehat{e}$ is in the closure of $\Psi_\Gamma^{-1}(\widetilde{S})$ with respect to the norm $|||\cdot|||$. Therefore $\widehat{e} \in \mathfrak{A}$.

We show that $\widehat{e}$ belongs to the domain of δ_2. Recall that $\widehat{e} = u^{-1} = ((I+X)\varepsilon)^{-1} = (\widetilde{D}+\varepsilon)^{-1}$. If φ is bounded, then the commutator $[\varphi, (\widetilde{D}+\varepsilon)^{-1}]$ is a bounded operator, and

$$\lim_{\lambda\to\infty}\left[\varphi, \sqrt{2\pi}(I + X)\varepsilon\varphi_\lambda(\widetilde{D})\right] = \left[\varphi, (\widetilde{D} + \varepsilon)^{-1}\right].$$

Unfortunately, φ is unbounded in general (see (4.11)). Thus $\left[\varphi, (\widetilde{D} + \varepsilon)^{-1}\right]$ is defined only on a subspace which may not be dense. So, even if $\left[\varphi, (\widetilde{D} + \varepsilon)^{-1}\right]$

extends to a bounded operator, the extension may not be unique. However, "formally" we have the equality

$$\left[\varphi,(\widetilde{D}+\varepsilon)^{-1}\right]=\varphi(\widetilde{D}+\varepsilon)^{-1}-(\widetilde{D}+\varepsilon)^{-1}\varphi$$
$$=(\widetilde{D}+\varepsilon)^{-1}[\widetilde{D}+\varepsilon,\varphi](\widetilde{D}+\varepsilon)^{-1},$$

and $(\widetilde{D}+\varepsilon)^{-1}[\widetilde{D}+\varepsilon,\varphi](\widetilde{D}+\varepsilon)^{-1}$ is a bounded operator, because $[\widetilde{D}+\varepsilon,\varphi]=[\widetilde{D},\varphi]\in\Psi^0_\Gamma(\widetilde{S})$. Thus it is natural to expect that

$$\delta_2(\widehat{e})=(\widetilde{D}+\varepsilon)^{-1}[\widetilde{D}+\varepsilon,\varphi](\widetilde{D}+\varepsilon)^{-1}.$$

Notice that $[\varphi,(\widetilde{D}+\varepsilon)\sqrt{2\pi}\varphi_\lambda(\widetilde{D})]\in\Psi^{-2}_\Gamma(\widetilde{S})$, and that $(\widetilde{D}+\varepsilon)^{-1}[\widetilde{D}+\varepsilon,\varphi](\widetilde{D}+\varepsilon)^{-1}$ is an operator of order -2 (not a ψDO). We will show (Proposition 7.17) that

$$\left\|\left[\varphi,(\widetilde{D}+\varepsilon)\sqrt{2\pi}\varphi_\lambda(\widetilde{D})\right]-(\widetilde{D}+\varepsilon)^{-1}\left[\widetilde{D}+\varepsilon,\varphi\right](\widetilde{D}+\varepsilon)^{-1}\right\|_{s,s-2}\to 0$$

as $\lambda\to\infty$ for any s. It is enough to show that

$$\left\|(\widetilde{D}+\varepsilon)\left[\varphi,(\widetilde{D}+\varepsilon)\sqrt{2\pi}\varphi_\lambda(\widetilde{D})\right](\widetilde{D}+\varepsilon)-[\widetilde{D}+\varepsilon,\varphi]\right\|_{s,s}\to 0$$

as $\lambda\to\infty$. Recall that $\psi_\lambda(x)=1-(1+x^2)\varphi_\lambda(x)$.

Lemma 7.14. *We have*

$$\left\|\left[\varphi,\psi_\lambda(\widetilde{D})\left(I+\widetilde{D}^2\right)\right]\right\|_{0,0}\to 0\quad as\quad \lambda\to\infty.$$

Proof. For simplicity, set $\alpha_\lambda(x)=(1+x^2)\psi_\lambda(x)$. Then

$$\psi_\lambda(\widetilde{D})\left(I+\widetilde{D}^2\right)=\alpha_\lambda(\widetilde{D})=\int\widehat{\alpha}_\lambda(s)e^{is\widetilde{D}}\,ds.$$

Since $[\varphi,\widetilde{D}]$ extends to a bounded operator, by Duhamel's formula,

$$\left[\varphi,\psi_\lambda(\widetilde{D})\left(I+\widetilde{D}^2\right)\right]=\int_{\mathbb{R}}\int_0^1\widehat{\alpha}_\lambda(s)e^{is\widetilde{D}t}[\varphi,\mathrm{is}\,\widetilde{D}]e^{is\widetilde{D}(1-t)}\,dt\,ds.$$

From this

$$\left\|\left[\varphi,\psi_\lambda(\widetilde{D})\left(I+\widetilde{D}^2\right)\right]\right\|_{0,0}\le\|[\varphi,\widetilde{D}]\|_{0,0}\int_{\mathbb{R}}|\widehat{\alpha}_\lambda(s)|\,|s|\,ds.$$

By the definition of ψ_λ, when $\lambda\to\infty$, the integral $\int_{\mathbb{R}}|\widehat{\alpha}_\lambda(s)|\,|s|\,ds$ behaves like $\lambda e^{-\lambda}$; i.e. there exists a constant $C>0$ such that

$$\int_{\mathbb{R}}|\widehat{\alpha}_\lambda(s)|\,|s|\,ds\le C\lambda e^{-\lambda}.$$

Thus

$$\left\| \left[\varphi, \psi_\lambda(\widetilde{D}) \left(I + \widetilde{D}^2\right)\right] \right\|_{0,0} \leq C\lambda e^{-\lambda}.$$

□

Lemma 7.15. *We have*

$$\left\| \left[\varphi, \psi_\lambda(\widetilde{D}) \left(I + \widetilde{D}^2\right)^{1/2}\right] \right\|_{0,0} \to 0 \quad as \quad \lambda \to \infty.$$

Proof. We have

$$\begin{aligned}
&\left[\varphi, \psi_\lambda(\widetilde{D}) \left(I + \widetilde{D}^2\right)^{1/2}\right] \\
&= [\varphi, \psi_\lambda(\widetilde{D})] \left(I + \widetilde{D}^2\right)^{1/2} + \psi_\lambda(\widetilde{D}) \left[\left(I + \widetilde{D}^2\right)^{1/2}, \varphi\right] \\
&= \left[\varphi, \psi_\lambda(\widetilde{D}) \left(I + \widetilde{D}^2\right)\right] \left(I + \widetilde{D}^2\right)^{-1/2} \\
&\quad + \psi_\lambda(\widetilde{D}) \left(I + \widetilde{D}^2\right)^{1/2} \left[\left(I + \widetilde{D}^2\right)^{1/2}, \varphi\right] \left(I + \widetilde{D}^2\right)^{-1/2} \\
&\quad + \psi_\lambda(\widetilde{D}) \left[\left(I + \widetilde{D}^2\right)^{1/2}, \varphi\right].
\end{aligned}$$

Then

$$\begin{aligned}
&\left\| \left[\varphi, \psi_\lambda(\widetilde{D}) \left(I + \widetilde{D}^2\right)^{1/2}\right] \right\|_{0,0} \\
&\leq \left\| \left[\varphi, \psi_\lambda(\widetilde{D}) \left(I + \widetilde{D}^2\right)\right] \right\|_{0,0} \left\| \left(I + \widetilde{D}^2\right)^{-1/2} \right\|_{0,0} \\
&\quad + ||\psi_\lambda(\widetilde{D})||_{0,0} \left\| \left(I + \widetilde{D}^2\right)^{1/2} \left[\left(I + \widetilde{D}^2\right)^{1/2}, \varphi\right] \left(I + \widetilde{D}^2\right)^{-1/2} \right\|_{0,0} \\
&\quad + ||\psi_\lambda(\widetilde{D})||_{0,0} \left\| \left[\left(I + \widetilde{D}^2\right)^{1/2}, \varphi\right] \right\|_{0,0}
\end{aligned}$$

(notice that $[(I + \widetilde{D}^2)^{1/2}, \varphi]$ is an operator of order 0). By Lemma 7.14, we get the conclusion. □

Lemma 7.16. *We have*

$$\left\| \left[\varphi, \psi_\lambda(\widetilde{D})\right] \right\|_{s,s-1} \to 0 \quad as \quad \lambda \to \infty.$$

Proof. By (7.12),

$$\left\| \left[\varphi, \psi_\lambda(\widetilde{D})\right] \right\|_{s,s-1} = \left\| \left(I + \widetilde{D}^2\right)^{s/2} [\varphi, \psi_\lambda(\widetilde{D})] \left(I + \widetilde{D}^2\right)^{(1-s)/2} \right\|_{0,0}.$$

Case (i). $s \geq 0$. In this case

$$\begin{aligned}
&\left(I+\widetilde{D}^2\right)^{s/2}\left[\varphi,\psi_\lambda(\widetilde{D})\right]\left(I+\widetilde{D}^2\right)^{(1-s)/2} \\
&=\left[\left(I+\widetilde{D}^2\right)^{s/2},\varphi\right]\psi_\lambda(\widetilde{D})\left(I+\widetilde{D}^2\right)^{(1-s)/2}+\left[\varphi,\psi_\lambda(\widetilde{D})\right]\left(I+\widetilde{D}^2\right)^{(1-s)/2} \\
&\quad-\psi_\lambda(\widetilde{D})\left[\left(I+\widetilde{D}^2\right)^{s/2},\varphi\right]\left(I+\widetilde{D}^2\right)^{(1-s)/2};
\end{aligned}$$

here we have used the fact that $\left[\left(I+\widetilde{D}^2\right)^{s/2},\varphi\right] \in \Psi_\Gamma^{s-1}(\widetilde{S})$ provided that $s \geq 0$. We have

$$\begin{aligned}
&\left\|\left[\left(I+\widetilde{D}^2\right)^{s/2},\varphi\right]\psi_\lambda(\widetilde{D})\left(I+\widetilde{D}^2\right)^{(1-s)/2}\right\|_{0,0} \\
&\leq\left\|\left[\left(I+\widetilde{D}^2\right)^{s/2},\varphi\right]\right\|_{0,s-1}\|\psi_\lambda(\widetilde{D})\|_{s-1,s-1}\left\|\left(I+\widetilde{D}^2\right)^{(1-s)/2}\right\|_{s-1,0},
\end{aligned}$$

which converges to zero as $\lambda \to \infty$.

Similarly,

$$\left\|\psi_\lambda(\widetilde{D})\left[\left(I+\widetilde{D}^2\right)^{s/2},\varphi\right]\left(I+\widetilde{D}^2\right)^{(1-s)/2}\right\|_{0,0} \to 0 \quad \text{as} \quad \lambda \to \infty.$$

Then, by Lemma 7.15, we obtain the conclusion.

Case (ii). $s < 0$. In this case $-s/2+1/2 > 0$ and $[\varphi,(I+\widetilde{D}^2)^{(1-s)/2}]$ is a ψDO. Making use of $[\varphi,(I+\widetilde{D}^2)^{(1-s)/2}]$ in the place of $[(I+\widetilde{D}^2)^{s/2},\varphi]$ in Case (i), we can deduce the conclusion. □

Proposition 7.17. *The element $\widehat{e}$ is in the domain of δ_2, and*

$$\begin{aligned}
\delta_2(\widehat{e}) &= (\widetilde{D}+\varepsilon)^{-1}[\widetilde{D}+\varepsilon,\varphi](\widetilde{D}+\varepsilon)^{-1} \\
&= (\widetilde{D}+\varepsilon)^{-1}[\widetilde{D},\varphi](\widetilde{D}+\varepsilon)^{-1}.
\end{aligned}$$

Proof. As mentioned above, it is sufficient to show that

$$(\widetilde{D}+\varepsilon)\left[\varphi,(\widetilde{D}+\varepsilon)\sqrt{2\pi}\varphi_\lambda(\widetilde{D})\right](\widetilde{D}+\varepsilon)$$

converges to $[\widetilde{D}+\varepsilon,\varphi]$ as $\lambda \to \infty$, as operator of order zero. By a straightforward computation,

$$\begin{aligned}
&\left[\widetilde{D}+\varepsilon,\varphi\right]-(\widetilde{D}+\varepsilon)\left[\varphi,(\widetilde{D}+\varepsilon)\sqrt{2\pi}\varphi_\lambda(\widetilde{D})\right](\widetilde{D}+\varepsilon) \\
&= [\widetilde{D}+\varepsilon,\varphi]\psi_\lambda(\widetilde{D})+[\varphi,\psi_\lambda(\widetilde{D})](\widetilde{D}+\varepsilon).
\end{aligned}$$

We have that

$$||[\varphi, \psi_\lambda(\widetilde{D})](\widetilde{D}+\varepsilon)||_{s,s} \leq ||[\varphi, \psi_\lambda(\widetilde{D})]||_{s,s-1}||\widetilde{D}+\varepsilon||_{s-1,s}.$$

Then by Lemma 7.16,

$$||[\varphi, \psi_\lambda(\widetilde{D})](\widetilde{D}+\varepsilon)||_{s,s} \to 0 \quad \text{as} \quad \lambda \to \infty.$$

By construction, $\psi_\lambda(\widetilde{D})$ commutes with $\left(I+\widetilde{D}^2\right)$. Hence

$$\begin{aligned} ||\psi_\lambda(\widetilde{D})||_{s,s} &= \left\| \left(I+\widetilde{D}^2\right)^{s/2} \psi_\lambda(\widetilde{D}) \left(I+\widetilde{D}^2\right)^{-s/2} \right\|_{0,0} \\ &= ||\psi_\lambda(\widetilde{D})||_{0,0} \to 0 \quad \text{as} \quad \lambda \to \infty. \end{aligned}$$

From these it follows that $||[\widetilde{D}+\varepsilon, \varphi]\psi_\lambda(\widetilde{D})||_{s,s} \to 0$ as $\lambda \to \infty$. Consequently,

$$\left[\varphi, (\widetilde{D}+\varepsilon)\sqrt{2\pi}\varphi_\lambda(\widetilde{D})\right] \to (\widetilde{D}+\varepsilon)^{-1}[\widetilde{D}+\varepsilon, \varphi](\widetilde{D}+\varepsilon)^{-1}.$$

Recall that $(\widetilde{D}+\varepsilon)\sqrt{2\pi}\varphi_\lambda(\widetilde{D}) \to \widehat{e}$ in $\mathfrak{A}$. Therefore, by closedness of δ_2 we obtain that

$$\delta_2(\widehat{e}) = (\widetilde{D}+\varepsilon)^{-1}[\widetilde{D}+\varepsilon, \varphi](\widetilde{D}+\varepsilon)^{-1}.$$

□

By the same argument, we can verify that $\widehat{e}$ is also in the domain of δ_1, and that

$$\delta_1(\widehat{e}) = (\widetilde{D}+\varepsilon)^{-1}[\widetilde{D}+\varepsilon, \partial_2\varphi](\widetilde{D}+\varepsilon)^{-1}. \tag{7.18}$$

8. Main Theorem.

In this section we will compute the pairing between the $2n$-trace gv and the class of the graph projection of the longitudinal Dirac operator. Throughout this section $\dim M = 2n$.

Let D be the longitudinal Dirac operator for the foliated S^1-bundle $(X, \mathcal{F})$. Denote by $C^*(X, \mathcal{F}, S)^\sim$ the C^*-algebra generated by $C^*(X, \mathcal{F}, S)$ and the projection p_- in $\wp_0$. We then have a split exact sequence:

$$0 \to C^*(X, \mathcal{F}, S) \to C^*(X, \mathcal{F}, S)^\sim \to \mathbb{C}p_- \to 0.$$

In Section 7, we showed that

$$\widehat{e} = e - p_- \in \mathfrak{A} \subseteq C^*(X, \mathcal{F}, S).$$

Set $\Theta = [e] - [p_-]$. Then $\Theta \in K_0[C^*(X,\mathcal{F},S)]$.

Proposition 8.1. *The class* Θ *is equal to* $\operatorname{ind}(D^+)$.

Proof. Recall [**9**, Lemma 6.1] that

$$\operatorname{ind}(D^+) = \left[\begin{pmatrix} S_0^2 & S_0(I+S_0)\widetilde{Q} \\ S_1\widetilde{D}^+ & I - S_1^2 \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] \in K_0[C^*(X,\mathcal{F},S)],$$

where $\widetilde{Q}$ is a parametrix of $\widetilde{D}^+$, and

$$S_0 = I - \widetilde{Q}\widetilde{D}^+ \in C^*(X,\mathcal{F},S^+),$$
$$S_1 = I - \widetilde{D}^+\widetilde{Q} \in C^*(X,\mathcal{F},S^-).$$

Set

$$u = \begin{pmatrix} S_0(I+\widetilde{D}^-\widetilde{D}^+)^{-1} & S_0(I+\widetilde{D}^-\widetilde{D}^+)^{-1}\widetilde{D}^- \\ \widetilde{D}^+(I+\widetilde{D}^-\widetilde{D}^+)^{-1} & \widetilde{D}^+(I+\widetilde{D}^-\widetilde{D}^+)^{-1}\widetilde{D}^- \end{pmatrix},$$

and

$$v = \begin{pmatrix} S_0 & (I+S_0)\widetilde{Q} \\ \widetilde{D}^+S_0 & \widetilde{D}^+(I+S_0)\widetilde{Q} \end{pmatrix}.$$

Then,

$$u, v \in C^*(X,\mathcal{F},S)^\sim,$$

and

$$uv = \begin{pmatrix} S_0^2 & S_0(I+S_0)\widetilde{Q} \\ S_1\widetilde{D}^+ & I - S_1^2 \end{pmatrix},$$

and $vu = e$. Thus

$$\operatorname{ind}(D^+) = \Theta \text{ in } K_0[C^*(X,\mathcal{F},S)].$$

$\square$

Denote by $C^*(X,\mathcal{F},S)^+$ the C^*-algebra $C^*(X,\mathcal{F},S)$ with unit adjoined. Notice that $C^*(X,\mathcal{F},S)^+$ is identified with the C^*-subalgebra of $\wp_0$ generated by $C^*(X,\mathcal{F},S)$ and $I \in \wp_0$. The $2n$-cocycle gv, constructed in Section 6, extends to $C^*(X,\mathcal{F},S)^+$ by setting

$$gv(a^0, a^1, \ldots, a^{2n}) = 0,$$

if one of $a^0, a^1, \dots, a^{2n}$ is a scalar multiple of I.

In terms of $C^*(X, \mathcal{F}, S)^+$, the class Θ is expressed as a difference

$$\Theta = [p] - [q],$$

where

$$p = \begin{pmatrix} (I + \widetilde{D}^-\widetilde{D}^+)^{-1} & 0\,0 & (I + \widetilde{D}^-\widetilde{D}^+)^{-1}\widetilde{D}^- \\ 0 & 0\,0 & 0 \\ 0 & 0\,1 & 0 \\ \widetilde{D}^+(I + \widetilde{D}^-\widetilde{D}^+)^{-1} & 0\,0 & \widetilde{D}^+(I + \widetilde{D}^-\widetilde{D}^+)^{-1}\widetilde{D}^- \end{pmatrix},$$

and

$$q = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}.$$

Notice that $p, q \in M_2(C^*(X, \mathcal{F}, S)^+)$. Then it is easy to see that

$$\langle gv, [p] - [q] \rangle = (2\pi i)^{2n} n! gv(\widehat{e}, \dots, \widehat{e}).$$

The main focus of the section is to explicitly compute $gv(\widehat{e}, \dots, \widehat{e})$.

We have

$$\begin{aligned} gv(\widehat{e}, \dots, \widehat{e}) = (n-1)! \sum \{ & \mathrm{trace}_\Gamma(\widehat{e}^{2i+1}\delta_1(\widehat{e})\widehat{e}^{2j}\delta_2(\widehat{e})\widehat{e}^{2n-2i-2j-2}) \\ & - \mathrm{trace}_\Gamma(\widehat{e}^{2i+1}\delta_2(\widehat{e})\widehat{e}^{2j}\delta_1(\widehat{e})\widehat{e}^{2n-2i-2j-2}) \}, \end{aligned}$$

where the summation is taken over all i and j such that $0 \le i, j$ and $i + j \le n - 1$.

Lemma 8.2. *We have*

$$\begin{aligned} (1)\quad & \widehat{e}^{2i+1}\delta_1(\widehat{e})\widehat{e}^{2j}\delta_2(\widehat{e})\widehat{e}^{2n-2i-2j-2} \\ & = \left(I + \widetilde{D}^2\right)^{-(i+1)} [\widetilde{D}, \partial_2\varphi] \left(I + \widetilde{D}^2\right)^{-(j+1)} [\widetilde{D}, \varphi] \\ & \quad \times \left(I + \widetilde{D}^2\right)^{-(n-i-j-1)} (\widetilde{D} + \varepsilon)^{-1}, \end{aligned}$$

and

$$\begin{aligned} (2)\quad & \widehat{e}^{2i+1}\delta_2(\widehat{e})\widehat{e}^{2j}\delta_1(\widehat{e})\widehat{e}^{2n-2i-2j-2} \\ & = \left(I + \widetilde{D}^2\right)^{-(i+1)} [\widetilde{D}, \varphi] \left(I + \widetilde{D}^2\right)^{-(j+1)} [\widetilde{D}, \partial_2\varphi] \\ & \quad \times \left(I + \widetilde{D}^2\right)^{-(n-i-j-1)} (\widetilde{D} + \varepsilon)^{-1}. \end{aligned}$$

Proof. Recall that $u = (I+X)\varepsilon = \widetilde{D} + \varepsilon$. In Section 7 we showed that

$$\delta_1(\widehat{e}) = (\widetilde{D}+\varepsilon)^{-1}[\widetilde{D}+\varepsilon, \partial_2\varphi](\widetilde{D}+\varepsilon)^{-1} = u^{-1}[\widetilde{D}, \partial_2\varphi]u^{-1},$$

and

$$\delta_2(\widehat{e}) = u^{-1}[\widetilde{D}, \varphi]u^{-1}.$$

Therefore

$$\begin{aligned}
&\widehat{e}^{2i+1}\delta_1(\widehat{e})\widehat{e}^{2j}\delta_2(\widehat{e})\widehat{e}^{2n-2i-2j-2} \\
&= (u^{-1})^{2i+1}u^{-1}[\widetilde{D}, \partial_2\varphi]u^{-1}(u^{-1})^{2j}u^{-1}[\widetilde{D}, \varphi]u^{-1}(u^{-1})^{2n-2i-2j-2} \\
&= \left(I+\widetilde{D}^2\right)^{-(i+1)}[\widetilde{D}, \partial_2\varphi]\left(I+\widetilde{D}^2\right)^{-(j+1)}[\widetilde{D}, \varphi] \\
&\quad \times \left(I+\widetilde{D}^2\right)^{-(n-i-j-1)}(\widetilde{D}+\varepsilon)^{-1}.
\end{aligned}$$

Similarly we obtain the second equality. □

For i, j with $0 \le i, j$, and $i+j \le n-1$, let

$$\begin{aligned}
A^{i,j} &= \widehat{e}^{2i+1}\delta_1(\widehat{e})\widehat{e}^{2j}\delta_2(\widehat{e})\widehat{e}^{2n-2i-2j-2}, \\
B^{i,j} &= \widehat{e}^{2i+1}\delta_2(\widehat{e})\widehat{e}^{2j}\delta_1(\widehat{e})\widehat{e}^{2n-2i-2j-2}.
\end{aligned}$$

Then

$$\begin{aligned}
gv(\widehat{e}, \dots, \widehat{e}) &= (n-1)!\sum\left(\mathrm{trace}_\Gamma\left(A^{i,j}\right) - \mathrm{trace}_\Gamma\left(B^{i,j}\right)\right), \\
\mathrm{trace}_\Gamma\left(A^{i,j}\right) &= \int_{S^1} tr\left(\sigma_x A_x^{i,j}\sigma_x\right)\, dx, \\
\mathrm{trace}_\Gamma\left(B^{i,j}\right) &= \int_{S^1} tr\left(\sigma_x B_x^{i,j}\sigma_x\right)\, dx,
\end{aligned}$$

where $A_x^{i,j}$ (resp. $B_x^{i,j}$) is the restriction of $A^{i,j}$ (resp. $B^{i,j}$) onto $\widetilde{M}_x = \widetilde{M} \times \{x\}$, $x \in S^1$. We must compute $tr(\sigma_x A_x^{i,j}\sigma_x)$ and $tr(\sigma_x B_x^{i,j}\sigma_x)$. In order to do so, we make use of Getzler's symbolic calculus method [**12**]. Fix an arbitrary $x \in S^1$. For a while we do analysis on the manifold $\widetilde{M}_x = \widetilde{M}$. In order to simplify the notation we supress the subindex, as long as it is clear on which manifold we are working on.

Consider a one-parameter family of operators on $\widetilde{M} = \widetilde{M}_x$,

$$\begin{aligned}
A^{i,j}(t) = &\left(I+t^2\widetilde{D}^2\right)^{-(i+1)}[t\widetilde{D}, \partial_2\varphi]\left(I+t^2\widetilde{D}^2\right)^{-(j+1)} \\
&\times [t\widetilde{D}, \varphi]\left(I+t^2\widetilde{D}^2\right)^{-(n-i-j-1)} \\
&\times \begin{pmatrix} (I+t^2\widetilde{D}^-\widetilde{D}^+)^{-1} & (I+t^2\widetilde{D}^-\widetilde{D}^+)^{-1}t\widetilde{D}^- \\ t\widetilde{D}^+(I+t^2\widetilde{D}^-\widetilde{D}^+)^{-1} & -(I+t^2\widetilde{D}^+\widetilde{D}^-)^{-1} \end{pmatrix}, \quad t > 0.
\end{aligned}$$

Similarly, define $B^{i,j}(t)$.

In the symbolic calculus method, a key notion is that of asymptotic order. Assign to the parameter t the order -1, and to a Clifford multiplication the order $+1$. The total order is called the *asymptotic order.* For instance, the following symbols have the asymptotic order 0 [**9**]:

(i) $\sigma\left(\left(\lambda+t^2\widetilde{D}^2\right)^{-1}\right)(m,\xi)$,

(ii) $\sigma([t\widetilde{D},f])(m,\xi)=t\,df_m,\quad f\in C^\infty(\widetilde{M})$.

In (i) the operator $(\lambda+t\widetilde{D}^2)^{-1}$ is a ψDO. However, its distributional kernel does not have Γ-compact support. In (ii) df_m is a Clifford multiplication operator.

Although in [**12**] only compact manifolds are studied, the method developed there works for compactly supported ψDO's. In particular, the following "Fundamental Lemma" is valid for such ψDO's (we use the notation of [**12**] and omit the proof).

Lemma 8.3. ([**9**], [**12**]). (1) *If $A=A(t)$ has asymptotic order 0, then*

$$\sigma_{t^{-1}}(A(t))=\sigma_0(A)+O(t),$$

where $\sigma_{t^{-1}}$ is the rescaled symbol, and $\sigma_0(A)$ is the asymptotic symbol of A.

(2) *If A,B are operators of asymptotic order 0, then*

$$\sigma_0(AB)=\sigma_0(A)*\sigma_0(B),$$

where $*$ is the Getzler multiplication of symbols.

(3) *If $\Pi(t)\in OpS^{-\infty}$, then*

$$Tr_s(\Pi(t))=(2\pi)^{-\dim M}\int_{T^*\widetilde{M}}tr_s(\sigma_{t^{-1}}(\Pi(t)))(m,\xi)\,dm\,d\xi,\qquad t>0,$$

where $dmd\xi$ is the symplectic measure on $T^*\widetilde{M}$.

We return to the computation. It is easy to see that

$$tr\left(\sigma A^{i,j}(t)\sigma\right)=Tr_s\left(\Pi^A_{i,j}(t)\right),$$

and

$$tr\left(\sigma B^{i,j}(t)\sigma\right)=Tr_s\left(\Pi^B_{i,j}(t)\right),$$

where

$$\Pi^A_{i,j}(t)=\sigma\left(I+t^2\widetilde{D}^2\right)^{-(i+1)}[t\widetilde{D},\partial_2\varphi]\left(I+t^2\widetilde{D}^2\right)^{-(j+1)}\times[t\widetilde{D},\varphi]\left(I+t^2\widetilde{D}^2\right)^{-(n-i-j)}\sigma,$$

and

$$\Pi^B_{i,j}(t) = \sigma\left(I + t^2\widetilde{D}^2\right)^{-(i+1)}[t\widetilde{D},\varphi]\left(I + t^2\widetilde{D}^2\right)^{-(j+1)} \times [t\widetilde{D},\partial_2\varphi]\left(I + t^2\widetilde{D}^2\right)^{-(n-i-j)}\sigma.$$

Next, notice that the operators considered in [**9**] and [**12**] are the operator $\sqrt{-1}\widetilde{D}$. For simplicity, let $\not{D} = \sqrt{-1}\widetilde{D}$. Then $\not{D}^* = -\not{D}$ and $\not{D}^2 = -\widetilde{D}^2$. We have $[\widetilde{D},\varphi] = -\sqrt{-1}[\not{D},\varphi]$, and $[\widetilde{D},\partial_2\varphi] = -\sqrt{-1}[\not{D},\partial_2\varphi]$. From this it follows that

$$\Pi^A_{i,j}(t) = -\sigma\left(I - t^2\not{D}^2\right)^{-(i+1)}[t\not{D},\partial_2\varphi]\left(I - t^2\not{D}^2\right)^{-(j+1)} \times [t\not{D},\varphi]\left(I - t^2\not{D}^2\right)^{-(n-i-j)}\sigma.$$

Similarly,

$$\Pi^B_{i,j}(t) = \sigma\left(I - t^2\not{D}^2\right)^{-(i+1)}[t\not{D},\varphi]\left(I - t^2\not{D}^2\right)^{-(j+1)} \times [t\not{D},\partial_2\varphi]\left(I - t^2\not{D}^2\right)^{-(n-i-j)}\sigma.$$

The operators $\Pi^A_{i,j}$ and $\Pi^B_{i,j}$ satisfy the assumption of Lemma 8.3. Therefore

$$\begin{aligned}(8.4)\qquad tr\left(\sigma A^{i,j}(t)\sigma\right) &= Tr_s\left(\Pi^A_{i,j}(t)\right)\\ &= (2\pi)^{-2n}\int_{T^*\widetilde{M}} tr_s\left(\sigma_{t^{-1}}\left(\Pi^A_{i,j}\right)\right)(m,\xi)\,dm\,d\xi\\ &= (2\pi)^{-2n}\int_{T^*\widetilde{M}} tr_s\left(\sigma_0\left(\Pi^A_{i,j}\right)\right)(m,\xi)\,dm\,d\xi + O(t).\end{aligned}$$

Similarly

$$(8.5)\quad tr\left(\sigma B^{i,j}(t)\sigma\right) = (2\pi)^{-2n}\int_{T^*\widetilde{M}} tr_s\left(\sigma_0\left(\Pi^B_{i,j}\right)\right)(m,\xi)\,dm\,d\xi + O(t).$$

We compute the asymptotic symbols $\sigma_0\left(\Pi^A_{i,j}\right)$ and $\sigma_0\left(\Pi^B_{i,j}\right)$. Symbols which are independent of ξ commute with those dependent on ξ, with respect to Getzler multiplication. By [**9**, Example (3.2)], $\sigma_0([t\not{D},\varphi]) = d\varphi$ and $\sigma_0([t\not{D},\partial_2\varphi]) = d(\partial_2\varphi)$. Hence

$$\sigma_0\left(\Pi^A_{i,j}\right) = -\sigma d(\partial_2\varphi)\wedge d\varphi\sigma\sigma_0\left(\left(I - t^2\not{D}^2\right)^{-(n+2)}\right),$$

and

$$\begin{aligned}\sigma_0\left(\Pi^B_{i,j}\right) &= -\sigma d\varphi \wedge d(\partial_2\varphi)\sigma\sigma_0\left(\left(I - t^2\, \not{D}^{\,2}\right)^{-(n+2)}\right)\\ &= \sigma d(\partial_2\varphi) \wedge d\varphi\sigma\sigma_0\left(\left(I - t^2\, \not{D}^{\,2}\right)^{-(n+2)}\right)\\ &= -\sigma_0\left(\Pi^A_{i,j}\right).\end{aligned}$$

Using the formula:

$$\left(I - t^2\, \not{D}^{\,2}\right)^{-k-1} = \frac{1}{k!}\int_0^\infty s^k e^{-s} e^{st^2 \not{D}^2}\, ds,$$

we obtain

$$\sigma_0\left(\Pi^A_{i,j}\right) = -\sigma d(\partial_2\varphi) \wedge d\varphi\sigma \frac{1}{(n+1)!}\int_0^\infty s^{n+1} e^{-s} \sigma_0(e^{-st^2 \not{D}^2})\, ds.$$

By [**9**, p. 362],

$$\int_{T^*\widetilde{M}} \sigma_0(e^{st^2\, \not{D}^{\,2}})\, d\xi = \pi^n s^{-n} \det\left(\frac{sR/2}{\sinh sR/2}\right)^{1/2},$$

where R is the curvature tensor of the Γ-invariant metric on $\widetilde{M}$.

Applying the super trace, which amounts to multiplying $(2/i)^n$ and taking the top degree term, we get that (8.4) is equal to

$$\begin{aligned}&-(2\pi)^{-2n}\int_{T^*\widetilde{M}} \sigma d(\partial_2\varphi) \wedge d\varphi\sigma \frac{1}{(n+1)!}\\ &\qquad \times \int_0^\infty s^{n+1} e^{-s} \pi^n s^{-n} \det\left(\frac{sR/2}{\sinh sR/2}\right)^{1/2} ds\, dm\, d\xi + O(t)\\ &= -\left(\frac{2}{i}\right)^n (2\pi)^{-2n}\pi^n \int_{\widetilde{M}} \sigma^2 d(\partial_2\varphi) \wedge d\varphi\sigma \det\left(\frac{R/2}{\sinh R/2}\right)^{1/2} + O(t).\end{aligned}$$

Therefore

$$\begin{aligned}(8.6)\ &\sum[tr\,(\sigma A^{i,j}(t)\sigma) - tr\,(\sigma B^{i,j}(t)\sigma)]\\ &= -\operatorname{Card}(\{(i,j);\ 0 \le i,j, \text{ and } i+j \le n-1\}) \cdot 2(2\pi)^{-2n}\pi^n\\ &\quad \times \left(\frac{2}{i}\right)^n \times \int_{\widetilde{M}} \sigma^2 d(\partial_2\varphi) \wedge d\varphi\sigma \det\left(\frac{R/2}{\sinh R/2}\right)^{1/2} + O(t)\\ &= -\left(\frac{2}{i}\right)^n \frac{n(n+1)}{2} \cdot 2(2\pi)^{-2n}\pi^n\\ &\quad \times \int_{\widetilde{M}} \sigma^2 d(\partial_2\varphi) \wedge d\varphi\sigma \det\left(\frac{R/2}{\sinh R/2}\right)^{1/2} + O(t).\end{aligned}$$

The piece of degree $(2n-2)$ of $\det\left(\dfrac{R/2}{\sinh R/2}\right)^{1/2}$ is homogeneous of degree $(n-1)$. Hence (8.6) is equal to the following (8.7)

$$-\left(\frac{2}{i}\right)^n (2\pi)^{-2n}\pi^n n(n+1)(-2\pi i)^{n-1}$$
$$\times \int_{\widetilde{M}} \sigma^2 d(\partial_2\varphi)\wedge d\varphi\sigma \det\left(\frac{-(1/2\pi i)(R/2)}{\sinh(-(1/2\pi i)(R/2))}\right)^{1/2} + O(t).$$

Proposition 8.8. *As $t \to 0$, the term $\sum[tr(\sigma_x A^{i,j}(t)_x\sigma_x) - tr(\sigma_x B^{i,j}(t)_x\sigma_x)]$ converges to*

$$-\left(\frac{2}{i}\right)^n (2\pi)^{-2n}\pi^n n(n+1)(-2\pi i)^{n-1}$$
$$\times \int_{\widetilde{M}} \sigma_x^2 d((\partial_2\varphi)_x)\wedge d(\varphi_x) \det\left(\frac{-(1/2\pi i)(R/2)}{\sinh(-(1/2\pi i)(R/2))}\right)^{1/2}.$$

Moreover, convergence is uniform in x.

Proof. Convergence follows from the equality (8.7).

Recall that we are dealing with a family of operators $\widetilde{D} = (\widetilde{D}_x)$ on $\widetilde{M}\times S^1$ such that $\widetilde{D}_x = \widetilde{D}_y$ via the canonical identification of $\widetilde{M}_x$ and $\widetilde{M}_y$, and $\varphi, \partial_2\varphi$ are smooth functions. It follows that, when one applies Lemma 8.3, (1), one obtains an estimate $O(t)$, which is uniform in x. Then the conclusion is immediate. □

Proposition 8.9. *We have that*

$$\sum \int_{S^1} [tr(\sigma_x A_x^{i,j}\sigma_x) - tr(\sigma_x B_x^{i,j}\sigma_x)]\,dx,$$
$$= \sum \int_{S^1} [tr(\sigma_x A^{i,j}(t)_x\sigma_x) - tr(\sigma_x B^{i,j}(t)_x\sigma_x)]\,dx \quad \textit{for all} \quad t > 0.$$

Proof. The right-hand side of the identity above is precisely

$$((n-1)!)^{-1} gv(\widehat{e}_t, \dots, \widehat{e}_t),$$

where $\widehat{e}_t$ is the graph projection of the operator $t\widetilde{D}^+$, and $\widehat{e}_t = e_t - p_-$. Clearly, (e_t) is a continuous path of projections. Therefore

$$[e] - [p_-] = [e_t] - [p_-] \quad \text{in} \quad K_0.$$

Hence

$$gv(\widehat{e}_t, \dots, \widehat{e}_t) = gv(\widehat{e}, \dots, \widehat{e}) \quad \text{for all} \quad t > 0.$$

□

From Propositions 8.8 and 8.9, it follows that

$$
\begin{aligned}
gv(\widehat{e},\dots,\widehat{e}) &= gv(\widehat{e}_t,\dots,\widehat{e}_t)\\
&= \lim_{t\to 0} gv(\widehat{e}_t,\dots,\widehat{e}_t)\\
&= \lim_{t\to 0}(n-1)!\sum\int_{S^1}[tr(\sigma_x A^{i,j}(t)_x\sigma_x) - tr(\sigma_x B^{i,j}(t)_x\sigma_x)]\,dx\\
&= (n-1)!\sum\int_{S^1}\lim_{t\to 0}[tr(\sigma_x A^{i,j}(t)_x\sigma_x) - tr(\sigma_x B^{i,j}(t)_x\sigma_x)]\,dx\\
&= -(n+1)!(2\pi)^{-2n}\pi^n(-2\pi i)^{n-1}\left(\frac{2}{i}\right)^n\\
&\quad\times\int_{S^1}\int_{\widetilde{M}_x}\sigma_x^2 d((\partial_2\varphi)_x)\wedge d(\varphi_x)\det\left(\frac{-(1/2\pi i)(R/2)}{\sinh(-(1/2\pi i)(R/2))}\right)^{1/2}dx\\
&= -(n+1)!(2\pi)^{-2n}\pi^n(-2\pi i)^{n-1}\left(\frac{2}{i}\right)^n\\
&\quad\times\int_{S^1}\int_{\widetilde{M}}\sigma^2 d'd''\varphi\wedge d'\varphi\wedge\widehat{A}(R)\\
&= -(n+1)!(2\pi)^{-2n}\pi^n(-2\pi i)^{n-1}\left(\frac{2}{i}\right)^n\int_X d'd''\varphi\wedge d'\varphi\wedge\widehat{A}(R),
\end{aligned}
$$

where $X = \widetilde{M}\times_\Gamma S^1$, and $\widehat{A}(R)$ is the $\widehat{A}$-polynomial of $\widetilde{M}$ given in terms of the curvature R of the Γ-invariant Riemannian metric on $\widetilde{M}$. Since $d'd''\varphi\wedge d'\varphi$ is Γ-invariant, so is $d'd''\varphi\wedge d'\varphi\wedge\widehat{A}(R)$. Consequently the integration of $d'd''\varphi\wedge d'\varphi\wedge\widehat{A}(R)$ on X is well defined. By Proposition 5.4, the 3-form $-d'd''\varphi\wedge d'\varphi$ represents the Godbillon-Vey class $gv(\mathcal{F})$. On the manifold X, the cohomology class of $\widehat{A}(R)$ is exactly the pullback of $\widehat{A}$-class $\widehat{A}(M)$ of the spin manifold M. Thus

$$
\begin{aligned}
(8.10)\quad gv(\widehat{e},\dots,\widehat{e}) &= -(n+1)!(-1)^{n-1}(2\pi i)^{-1}\int_X d'd''\varphi\wedge d'\varphi\wedge\widehat{A}(R)\\
&= (n+1)!(-1)^{n-1}(2\pi i)^{-1}\left(gv(\mathcal{F})\cup\widehat{A}(M)\right)[X].
\end{aligned}
$$

Summarizing the arguments above, we have the main result:

Theorem 8.11. *Let X be a foliated S^1-bundle over a $2n$-dimensional closed spin manifold M, and let D be the longitudinal Dirac operator. Then*

$$\langle gv, \mathrm{ind}(D^+)\rangle = (-1)^{n-1}(n+1)(2\pi i)^{-n-1}\left(gv(\mathcal{F})\cup\widehat{A}(M)\right)[X].$$

Corollary 8.12. *If* $\left(gv(\mathcal{F}) \cup \widehat{A}(M)\right)[X] \neq 0$, *then the class* $\Theta = \mathrm{ind}(D^+)$ *is nontrivial in* $K_0[C^*(X, \mathcal{F}, S)]$.

Example 8.13. Let $(T_1\Sigma, \mathcal{F}_A)$ be an Anosov foliation associated with the geodesic flow on the unit circle bundle $T_1\Sigma$ over a closed Riemann surface Σ of genus ≥ 2. Since $\dim \Sigma = 2$,

$$\langle gv, \mathrm{ind}(D^+)\rangle = -2(2\pi)^{-2} gv(\mathcal{F}_A)[T_1\Sigma].$$

It is known [**18**] that $gv(\mathcal{F}_A)[T_1\Sigma] \neq 0$. Therefore, $\Theta = \mathrm{ind}(D^+)$ is nontrivial in $K_0[C^*(T_1\Sigma, \mathcal{F}_A, S)]$. In the next section we will show that Θ together with other known elements generates the whole $K_0[C^*(T_1\Sigma, \mathcal{F}_A, S)]$.

Remark 8.14. In (8.10), the righthand side is always purely imaginary. This is due to the fact that the cyclic $2n$-cocycle gv is *purely imaginary*, i.e.

$$gv(a_{2n}^*, a_{2n-1}^*, \dots, a_0^*) = -\overline{gv(a_0, a_1, \dots, a_{2n})}$$

for $a_0, \dots, a_{2n} \in \mathrm{Dom}(gv)$.

9. A relationship between the cocycle gv and Connes's cocycle.

In this section we will study the relationship between the cyclic cocycle gv and Connes's cocycle [**8**].

Let us recall his construction. Denote by τ_1 the transverse fundamental class for $C(S^1) \rtimes \Gamma$. That is

$$\tau_1(f^0, f^1) = \sum_{g_0 g_1 = 1} \int_{S^1} f^0_{g_0}(x g_0)\, df^1_{g_1}(x),$$

where $f^j = \sum f^j_g U_g \in C_c^\infty(S^1 \times \Gamma)$. Its derivative $\dot{\tau}_1$, defined by

$$\dot{\tau}_1(f^0, f^1) = \lim_{t \to 0} \frac{1}{it}\left(\tau_1(\sigma_t(f^0), \sigma_t(f^1)) - \tau_1(f^0, f^1)\right), \tag{9.1}$$

is (σ_t)-invariant. The cocycle which Connes studied is $i_{D\varphi}(\dot{\tau}_1)$. We will see that there exists a homomorphism Π from $C(S^1) \rtimes \Gamma$ into $C^*(X, \mathcal{F}, E)$ such that

$$\Pi^*(gv) = i_{D\varphi}(\dot{\tau}_1)$$

on

$$C_c^\infty(S^1 \times \Gamma) \subset C(S^1) \rtimes \Gamma.$$

There exists a compactly supported C^∞-function σ on $\widetilde{M}$ such that

$$\sum_{g \in \Gamma} g(\sigma) = 1;$$

i.e. $\{g(\sigma)\}_{g\in\Gamma}$ is a Γ-invariant partition of unity for $\widetilde{M}$. We can choose σ so that σ takes the value 1 on some open set U. We may further assume that the fundamental domain $\mathcal{D}$ is contained in $\operatorname{supp}\sigma$. Assume that $\widetilde{E}$ is a Γ-equivariant vector bundle on $\widetilde{M}\times S^1$, which is the pullback of a vector bundle E on M by the composition of two canonical maps

$$\widetilde{M}\times S^1 \to \widetilde{M} \xrightarrow{p} M.$$

The bundle $\widetilde{S}$ of spinors considered in the preceding two sections satisfies this assumption. Choose a section $\xi \in C_c^\infty(p^*E)$ such that $\operatorname{supp}\xi \subset U$, and

$$\int_{\widetilde{M}} \langle \xi, \xi\rangle \, d\mu(m) = 1.$$

In a natural way, ξ can be regarded as a compactly supported section of $\widetilde{E}$. By the choice of ξ, we have that

$$\operatorname{supp}\xi \cap \operatorname{supp} g(\xi) = \varnothing \qquad \text{unless} \quad g = 1. \tag{9.2}$$

Moreover

$$\int_{\widetilde{M}} \langle \xi, \xi\rangle_x \, d\mu_x(m) = 1 \tag{9.3}$$

for all $x \in S^1$. From this follows that

$$\langle \xi, \xi\rangle = 1 \in C(S^1) \rtimes \Gamma,$$

where $\langle\cdot,\cdot\rangle$ is the $C(S^1)\rtimes\Gamma$-valued inner product on ϵ in Section 2.

In general, for a right Hilbert module over a unital C^*-algebra $\mathfrak{A}$, if there exists an $\eta \in \epsilon$ such that $\langle \eta, \eta\rangle_{\mathfrak{A}} = 1$, then the map Π defined by

$$\Pi(a) = \theta_{\eta\cdot a, \eta}, \qquad a \in \mathfrak{A}, \tag{9.4}$$

is a $*$-homomorphism from $\mathfrak{A}$ into $\mathcal{K}(\epsilon)$, which induces an isomorphism of K-groups. Apply this principle to ξ above to obtain a $*$-homomorphism Π from $C(S^1)\rtimes\Gamma$ into $\mathcal{K}(\epsilon) \cong C^*(X, \mathcal{F}, E)$.

Let dx and $\widetilde{\omega}$ be as in Section 5. Let ψ be a real-valued C^∞-function on $\widetilde{M}\times S^1$. It is easy to see that $\omega = \psi\widetilde{\omega}\wedge dx$ is a Γ-invariant volume form on $\widetilde{M}\times S^1$ if and only if ψ is never zero, and $\psi = g(\psi)\lambda_g$ for any $g\in\Gamma$. Set

$$\psi = \sum_{g\in\Gamma} \lambda_g g(\sigma).$$

Since $\{g(\sigma)\}$ is a partition of unity, and $\lambda_g > 0$, the function ψ is always positive. Moreover

$$\begin{aligned}\lambda_g g(\psi) &= \sum_{h\in\Gamma} \lambda_g g(\lambda_h)(gh)(\sigma)\\ &= \sum_{h\in\Gamma} \lambda_{gh}(gh)(\sigma) = \psi.\end{aligned}$$

Thus $\omega = \psi\tilde{\omega} \wedge dx$ is a Γ-invariant volume form. Using the definitions given in Section 5, obtain (Δ^{it}) and $(\widehat{\sigma}_t)$.

Lemma 9.5. *The section ξ (as a section of $\widetilde{E}$ over $\widetilde{M} \times S^1$) has the property that*

$$\Delta^{it}(\xi) = \xi, \qquad t \in \mathbb{R}.$$

Proof. Obvious from the fact that $\psi \equiv 1$ on $\operatorname{supp} \xi$. □

Lemma 9.6. *The $*$-homomorphism Π given by (9.2) is $\mathbb{R}$-equivariant; i.e.*

$$\widehat{\sigma}_t(\Pi(a)) = \Pi(\sigma_t(a)), \quad \textit{for all} \quad a \in C(S^1) \rtimes \Gamma \quad \textit{and} \quad t \in \mathbb{R}.$$

Proof. For each $a \in C(S^1) \rtimes \Gamma$ and $t \in \mathbb{R}$, by Lemma 4.3,

$$\begin{aligned}\widehat{\sigma}_t(\Pi(a)) &= \Delta^{it}\theta_{\xi\cdot a,\xi}\Delta^{-it}\\ &= \theta_{\Delta^{it}(\xi\cdot a),\Delta^{it}(\xi)}\\ &= \theta_{\Delta^{it}(\xi)\cdot\sigma_t(a),\Delta^{it}(\xi)}\\ &= \theta_{\xi\cdot\sigma_t(a),\xi}\\ &= \Pi(\sigma_t(a)).\end{aligned}$$

□

For $a \in C_c^\infty(S^1 \times \Gamma)$, the operator $\Pi(a)$ is a compactly smoothing operator. Therefore $\mathrm{trace}_\Gamma(\Pi(a))$ is well defined.

Proposition 9.7. *For $a^0, a^1 \in C_c^\infty(S^1 \times \Gamma)$, we have*

$$\mathrm{trace}_\Gamma\,(\Pi(a^0)\delta_1(\Pi(a^1))) = \dot{\tau}_1(a^0, a^1).$$

Proof. We have, using (9.2) and (9.3), that

$$\begin{aligned}
&\mathrm{trace}_\Gamma\,(\Pi(a^0)\delta_1(\Pi(a^1)))\\
&= \int_{S^1}\int_{\mathcal{D}}\int_{\widetilde{M}} \sum_{g,h,g',h'} a^0(xg,g^{-1}h)((\partial_2\varphi)(n,x)-(\partial_2\varphi)(m,x))\\
&\qquad \times a^1(xg',g'^{-1}h')(\xi(nh,xh),\xi(ng',xg'))\\
&\qquad \times (\xi(mh',xh'),\xi(mg,xg))\,d\mu_x(n)\,d\mu_x(m)\,dx\\
&= \int_{S^1}\int_{\mathcal{D}}\int_{\widetilde{M}} \sum_{g,h} a^0(xg,g^{-1}h)((\partial_2\varphi)(n,x)-(\partial_2\varphi)(m,x))a^1(xg,g^{-1}h)\\
&\qquad \times ||\xi(nh,xh)||^2\,||\xi(mg,xg)||^2\,d\mu_x(n)\,d\mu_x(m)\,dx\\
&= \int_{S^1}\int_{\mathcal{D}}\int_{\widetilde{M}} \sum_{h} a^0(x,h)a^1(xh,h^{-1})((\partial_2\varphi)(n,x)-(\partial_2\varphi)(m,x))\\
&\qquad \times ||\xi(nh,xh)||^2\,||\xi(mg,xg)||^2\,d\mu_x(n)\,d\mu_x(m)\,dx.
\end{aligned}$$

Since $\psi \equiv 1$ on $\operatorname{supp}\xi$, we have $(\partial_2\varphi)(m,x)=0$ if $m\in\mathcal{D}$. Hence

$$\begin{aligned}
&\mathrm{trace}_\Gamma\,(\Pi(a^0)\delta_1(\Pi(a^1)))\\
&= \int_{S^1}\int_{\mathcal{D}}\int_{\widetilde{M}} \sum_{h} a^0(x,h)a^1(xh,h^{-1})(\partial_2\varphi)(n,x)||\xi(nh,xh)||^2\,||\xi(mg,xg)||^2\\
&\qquad \times d\mu_x(n)\,d\mu_x(m)\,dx\\
&= \int_{S^1}\int_{\widetilde{M}} \sum_{h} a^0(x,h)a^1(xh,h^{-1})(\partial_2\varphi)(n,x)||\xi(nh,xh)||^2\,d\mu_x(n)\,dx.
\end{aligned}$$

If $nh\notin\mathcal{D}$, then $||\xi(nh,xh)||^2=0$. By the choice of ψ, if $||\xi(nh,xh)||^2\neq 0$, then

$$\psi(n,x)=\lambda_{h^{-1}}(x)h(\sigma)(n),$$

and

$$\varphi(n,x)=l(h^{-1})(x)+\log(h(\sigma)(n)).$$

Therefore $(d''\varphi)_{(n,x)}=dl(h^{-1})_x$. Consequently

$$\begin{aligned}
\mathrm{trace}_\Gamma\,(\Pi(a^0)\delta_1(\Pi(a^1))) &= \sum_h \int_{S^1} a^0(x,h)a^1(xh,h^{-1})dl(h^{-1})\\
&= \dot{\tau}_1(a^0,a^1).
\end{aligned}$$

□

Finally we can relate the two cocycles:

Proposition 9.8. *For a^0, a^1, a^2 in $C_c^\infty(S^1 \times \Gamma)$, we have*

$$i_{D_\varphi}(\dot{\tau}_1)(a^0, a^1, a^2) = (\Pi^* gv)(a^0, a^1, a^2).$$

Proof. This is immediate from Lemma 9.6, Proposition 9.7 and [**8**, Lemma 6]. □

Remark 9.9. Suppose that E is the trivial line bundle. Then the formula

$$\int_{S^1}\int_{\mathcal{D}}\int_{\widetilde{M}} k^0(m,n,x)d''k^1(n,m,x)\,dn\,dm\,dx \tag{9.10}$$

defines the transverse fundamental class on $\mathcal{K}_c$. The cocycle $(k^0, k^1) \to \mathrm{trace}_\Gamma(k^0\delta_1(k^1)) = \mathrm{trace}_\Gamma(k^0[d''\varphi, k^1])$ is the derivative, in the sense of (9.1), of the cocycle (9.10) with respect to the modular automorphism group $(\widehat{\sigma}_t)$.

10. The K_0-groups of the C^*-algebras of Foliated S^1-bundles.

In this section we will determine the generators of the group $K_0[C^*(X, \mathcal{F})]$ for an arbitrary foliated S^1-bundle over a closed Riemann surface.

Let Σ be a closed Riemann surface of genus $g \geq 2$, and let $\Gamma = \pi_1(\Sigma)$. To any (right)action of Γ on the circle S^1 by orientation preserving diffeomorphisms, a fibre bundle with fibre S^1 is associated (Section 2). By evaluating the Euler class of this bundle on the fundamental class of Σ, we get an integer χ, which is called the Euler characteristic.

This group Γ is an amalgamated free product $\Gamma = F_2 *_{\mathbb{Z}} F_{2g-2}$. By [**17**] we have an exact sequence, a part of which looks like

$$K_0(A_1) \oplus K_0(A_2) \to K_0(A) \to K_1(A_0) \to K_1(A_1) \oplus K_1(A_2),$$

where $A_0 = C(S^1) \rtimes \mathbb{Z}$, $A_1 = C(S^1) \rtimes F_2$, $A_2 = C(S^1) \rtimes F_{2g-2}$, and $A = C(S^1) \rtimes \Gamma$. The computations done in [**16**] enable us to obtain

$$K_0[A] \cong \mathbb{Z}^{2g} \oplus \mathbb{Z} \oplus \mathbb{Z}/\chi\mathbb{Z}. \tag{10.1}$$

The subgroup $\mathbb{Z}^{2g}$ in (10.1) is generated by Rieffel projections. It is straightforward to see that those $2g$ generators lie in the kernel of the map $K_0(A) \to \mathbb{C}$ induced by the pairing with the cyclic 2-cocycle $i_{D_\varphi}(\dot{\tau}_1)$ described in the preceding section. The torsion subgroup $\mathbb{Z}/\chi\mathbb{Z}$ is generated by the class of the unit. As for the remaining generator, we know only of its existence, by applying an exact sequence to compute the K-groups. We will show that this missing generator is given by the class Θ associated with the Dirac operator.

Recall that the upper half plane $\mathbb{H}_+$ is the universal covering of Σ. The Γ-equivariant Hermitian vector bundles $\widetilde{S} = \widetilde{S}^+ \oplus \widetilde{S}^-$, associated with the Γ-invariant spin structure on $\mathbb{H}_+$, give rise to a Hilbert C^*-module ϵ_1 over $C^*\Gamma$ in the fashion used to create ϵ in Section 2. Let ξ be as in Section 9. Then ξ yields $*$-homomorphisms $\Pi : C(S^1) \rtimes \Gamma \to \mathcal{K}(\epsilon)$ and $\Pi_1 : C^*\Gamma \to \mathcal{K}(\epsilon_1)$, which induce isomorphisms of K-groups.

Proposition 10.2. *There exists a $*$-homomorphism from $\mathcal{K}(\epsilon_1)$ into $\mathcal{K}(\epsilon)$ such that the diagram*

$$\begin{array}{ccc} C^*\Gamma & \xrightarrow{\Pi_1} & \mathcal{K}(\epsilon_1) \\ \Big\downarrow & & \Big\downarrow \\ C(S^1) \rtimes \Gamma & \xrightarrow{\Pi} & \mathcal{K}(\epsilon) \end{array}$$

is commutative, where $C^*\Gamma \to C(S^1) \rtimes \Gamma$ is the canonical inclusion.

Proof. Recall that $\mathcal{K}(\epsilon)$ is generated by operators with Γ-compactly supported, Γ-invariant C^∞-kernels. Let $P \in \mathcal{K}(\epsilon_1)$ have the kernel k. Then the Γ-invariant C^∞-kernel $\widetilde{k}$ defined by

$$\widetilde{k}(m,n,x) = k(m,n), \qquad (m,n,x) \in \mathbb{H}_+ \times \mathbb{H}_+ \times S^1,$$

determines an operator $\widetilde{P} \in \mathcal{K}(\epsilon)$. Using the definition of norm, it is not hard to check that the correspondence $P \to \widetilde{P}$ extends to a $*$-homomorphism $j : \mathcal{K}(\epsilon_1) \to \mathcal{K}(\epsilon)$.

Commutativity of the diagram is also easy. $\square$

The Dirac operator D^+ on Σ lifts to a Γ-equivariant differential operator

$$\widetilde{D}^+ : C_c^\infty(\mathbb{H}_+, \widetilde{S}^+) \to C_c^\infty(\mathbb{H}_+, \widetilde{S}^-).$$

The graph projection $\widetilde{e}^+$ associated with D^+ is a bounded operator on $L^2(\mathbb{H}_+, \widetilde{S}^+ \oplus \widetilde{S}^-)$ and determines a class

$$\Theta_0 = [\widetilde{e}^+] - [p_-] \in K_0[\mathcal{K}(\epsilon_1)].$$

Proposition 10.3. *The class Θ_0 and the class of unit $1 \in C^*\Gamma$ generate*

$$K_0[\mathcal{K}(\epsilon_1)] \cong K_0[C^*\Gamma] \cong \mathbb{Z}^2.$$

Proof. By the fact that the index map from the K-homology of Σ into $K_*[C^*\Gamma]$ is an isomorphism [**2**, Thm. 3], we can see that $K_0[C^*\Gamma]$ is isomorphic to $\mathbb{Z}^2$ and is generated by the class of the unit and the index $\mathrm{ind}_\Gamma(\widetilde{D}^+)$. As in Section 8, it is not hard to see that Θ_0 coincides with $\mathrm{ind}_\Gamma(\widetilde{D}^+)$. $\square$

By the construction of j we can see that

$$j(\widetilde{e}^{+} - p_{-}) = e - p_{-}.$$

From this we see $j_*(\Theta_0) = \Theta$, where $j_* : K_0[\mathcal{K}(\epsilon_1)] \to K_0[\mathcal{K}(\epsilon)]$ is the induced map.

Theorem 10.4. *The class Θ is the missing generator of $K_0[C(S^1) \rtimes \Gamma] \cong K_0[\mathcal{K}(\epsilon_1)]$.*

Proof. We claim that Θ, together with the known generators, spans the K_0-group. Let A_0, A_1, A_2, and A be as above. We have a commutative diagram:

$$\begin{tikzcd} K_0[A_1] \oplus K_0[A_2] \arrow[r] & K_0[A] \arrow[r, "\delta"] & K_1[A_0] \arrow[r] & K_1[A_1] \oplus K_1[A_2] \\ K_0[C^*F_2] \oplus K_0[C^*F_{2g-2}] \arrow[r] \arrow[u] & K_0[C^*\Gamma] \arrow[r, "\delta"] \arrow[u] & K_1[C^*\mathbb{Z}] \arrow[r] \arrow[u] & K_1[C^*F_2] \oplus K_1[C^*F_{2g-2}], \arrow[u] \end{tikzcd}$$

where horizontal rows are exact, and all the vertical arrows are induced from the canonical inclusions of C^*-algebras.

The map $K_1[C^*\mathbb{Z}] \to K_1[C^*F_2] \oplus K_1[C^*F_{2g-2}]$ is a zero map, and the kernel of $K_1[A_0] \to K_1[A_1] \oplus K_1[A_2]$ is an infinite cyclic group generated by the class of the unitary of $C(S^1) \rtimes \mathbb{Z}$ corresponding to the generator of $\mathbb{Z}$.

Since the class of the unit and the class Θ_0 generate $K_0[C^*\Gamma]$, we see that $\delta(\Theta_0)$ must be the generator of $K_1[C^*\mathbb{Z}]$. From this and the observation above, $\delta(\Theta)$ is the generator of the kernel of $K_1[A_0] \to K_1[A_1] \oplus K_1[A_2]$. Therefore the class Θ and the image of the map $K_0[A_1] \oplus K_0[A_2] \to K_0[A]$ generate $K_0[A]$. □

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