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SINGULAR MODULI SPACES OF STABLE VECTOR BUNDLES ON $\mathbf{P}^3$

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The goal of this paper is to give an example of singular moduli space of rank 3 stable vector bundles on $\mathbf{P}^3$.

Introduction.

In 1977/78, M. Maruyama proved the existence of a moduli scheme $M_{\mathbf{P}^n}(r; c_1, ..., c_{\min(r,n)})$ parametrizing isomorphic classes of rank r stable vector bundles on $\mathbf{P}^n$ with given Chern classes c_1,..., $c_{\min(n,r)}$ (cf. [**M1, M2**]). The goal of this note is to give, to the best of my knowledge, the first example of singular moduli space of stable vector bundles on $\mathbf{P}^3$. It has been motivated by a recent work of Ancona and Ottaviani where they show that the moduli space $MI_{\mathbf{P}^5}(k)$ of stable instanton bundles on $\mathbf{P}^5$ with quantum number k=3 or 4 is singular. Moreover they claim that $MI_{\mathbf{P}^5}(3)$ and $MI_{\mathbf{P}^5}(4)$ are the first examples of singular moduli spaces of stable vector bundles on projective spaces (cf. [**AO**]). Ancona-Ottaviani's result together with the well known fact that $M_{\mathbf{P}^2}(r; c_1, c_2)$ is a smooth quasi-projective variety of dimension $2rc_2 - (r-1)c_1^2 + 1 - r^2$ gives rise the following question:

Is there any example of singular moduli space of stable vector bundles on $\mathbf{P}^3$?

As I pointed out before my aim is to give an affirmative answer to this question (cf. Theorem 2.10).

1. Preliminaries.

In this section we recall some well known results needed later on.

1.1. Let H(18, 39) be the open subscheme of $Hilb\mathbf{P}^3_k$ parametrizing smooth connected curves $C \subset \mathbf{P}^3$ of degree 18 and genus 39. (See [**EF**] for a precise description of H(18, 39).) Let $H_1 \subset H(18,39)$ be the 72-dimensional irreducible, generically smooth component whose general point parametrizes an arithmetically Cohen-Macaulay curve $X \subset \mathbf{P}^3$ having a locally free resolution of the following type:

$$0 \to \mathcal{O}(-7)^4 \to \mathcal{O}(-6)^4 \oplus \mathcal{O}(-4) \to I_X \to 0. \tag{1}$$

Let $H_2 \subset H(18,39)$ be the 72-dimensional irreducible, generically smooth component whose general point parametrizes an arithmetically Cohen-Macaulay curve $Y \subset \mathbf{P}^3$ having a locally free resolution of the following type:

$$0 \to \mathcal{O}(-6)^2 \oplus \mathcal{O}(-8) \to \mathcal{O}(-5)^4 \to I_Y \to 0. \tag{2}$$

It is well known that there exits an irreducible subset $H = H_1 \cap H_2 \subset H(18,39)$ of dimension 71 whose general point parametrizes an arithmetically Buchsbaum curve $C \subset \mathbf{P}^3$ having a locally free resolution of the following type:

$$0 \to \mathcal{O}(-8) \to \mathcal{O}(-7)^4 \oplus \mathcal{O}(-8) \to \mathcal{O}(-6)^4 \oplus \mathcal{O}(-4) \to I_C \to 0. \tag{3}$$

1.2. Remark. For all curve $Z \in H_1 \cup H_2$, $\omega_Z(2)$ is globally generated. From now on, for all curve $Z \in H_1 \cup H_2$, we set $\alpha := \dim H^0(\omega_Z(2))$ (=74; by Riemann-Roch's Theorem).

1.3. Fact. Let $0 \to E \to F \to G \to 0$ be an exact sequence of vector bundles. Then, we have the following exact sequence involving alternating and symmetric powers:

$$0 \to S^qE \to S^{q-1}E \otimes F \to .... \to E \otimes \Lambda^{q-1}F \to \Lambda^qF \to \Lambda^qG \to 0.$$

1.4. Hoppe's criterion for the stability of a vector bundle. Let X be a projective manifold with $Pic(X) \cong \mathbf{Z}$ and let E be a vector bundle on X. If $H^0(X, (\Lambda^q E)_{\text{norm}}) = 0$ for $1 \leq q \leq rk(E) - 1$, then E is stable. As usual, given a vector bundle E on X, we denote by E_{norm} the twist of E whose first Chern class c_1 verifies $-rk(E) + 1 \leq c_1 \leq 0$.

2. Main results.

2.1. Let us call $\mathcal{L}_1$ the irreducible family of sheaves E on $\mathbf{P}^3$ constructed as an extension:

$$\sigma = (\sigma_1, ..., \sigma_\alpha) : \qquad 0 \to \mathcal{O}^\alpha \to E(1) \to I_X(2) \to 0$$

where $X \in H_1$ and $\sigma_1, ..., \sigma_\alpha \in H^0(\omega_X(2)) \cong Ext^1(I_X(2), \mathcal{O})$ are general global sections which generate the sheaf $\omega_Z(2)$ everywhere.

It is easy to see that E is a vector bundle on $\mathbf{P}^3$ of rank $\alpha + 1$.

2.2. Let us call $\mathcal{L}_2$ the irreducible family of sheaves F on $\mathbf{P}^3$ constructed as an extension:

$$\lambda = (\lambda_1, ..., \lambda_\alpha) : \qquad 0 \to \mathcal{O}^\alpha \to F(1) \to I_Y(2) \to 0$$

where $Y \in H_2$ and $\lambda_1, ..., \lambda_\alpha \in H^0(\omega_Y(2)) \cong Ext^1(I_Y(2), \mathcal{O})$ are general global sections which generate the sheaf $\omega_Z(2)$ everywhere.

Again it is easy to see that F is a vector bundle on $\mathbf{P}^3$ of rank $\alpha+1$.

2.3. And let $\mathcal{L} \subset \mathcal{L}_1 \cap \mathcal{L}_2$ be the irreducible family of sheaves G on $\mathbf{P}^3$ constructed as an extension:

$$\mu = (\mu_1, ..., \mu_\alpha): \qquad 0 \to \mathcal{O}^\alpha \to G(1) \to I_C(2) \to 0$$

where $C \in H \subset H_1 \cap H_2$ and $\mu_1, ..., \mu_\alpha \in H^0(\omega_C(2)) \cong Ext^1(I_C(2), \mathcal{O})$ are general global sections which generate the sheaf $\omega_Z(2)$ everywhere.

Again it is easy to see that G is a vector bundle on $\mathbf{P}^3$ of rank $\alpha+1$.

Proposition 2.4.

(1) *A general vector bundle $E \in \mathcal{L}_1$ has a locally free resolution of the following type:*

$$0 \to \mathcal{O}(-5)^4 \to \mathcal{O}(-4)^4 \oplus \mathcal{O}(-2) \oplus \mathcal{O}^\alpha \to E(1) \to 0.$$

(2) *A general vector bundle $F \in \mathcal{L}_2$ has a locally free resolution of the following type:*

$$0 \to \mathcal{O}(-6) \oplus \mathcal{O}(-4)^2 \to \mathcal{O}(-3)^4 \oplus \mathcal{O}^\alpha \to F(1) \to 0.$$

(3) *A general vector bundle $G \in \mathcal{L}$ has a locally free resolution of the following type:*

$$0 \to \mathcal{O}(-6) \to \mathcal{O}(-6) \oplus \mathcal{O}(-5)^4 \to \mathcal{O}(-4)^4 \oplus \mathcal{O}(-2) \oplus \mathcal{O}^\alpha \to G(1) \to 0.$$

Proof. (1) From the exact sequence:

$$0 \to \mathcal{O}^\alpha \to E(1) \to I_X(2) \to 0$$

and the locally free resolution of $I_X(2)$ (See 1.1):

$$0 \to \mathcal{O}(-5)^4 \to \mathcal{O}(-4)^4 \oplus \mathcal{O}(-2) \to I_X(2) \to 0$$

we get the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}(-5)^4 & = & \mathcal{O}(-5)^4 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}^\alpha & \longrightarrow & \mathcal{O}^\alpha \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-4)^4 & \longrightarrow & \mathcal{O}(-2) \oplus \mathcal{O}(-4)^4 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}^\alpha & \longrightarrow & E(1) & \longrightarrow & I_X(2) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

which gives what we want.

(2) and (3) Analogous. □

Corollary 2.5. *Given a vector bundle $E \in \mathcal{L}_1 \cup \mathcal{L}_2$, $E(t)$ is globally generated for all $t \geq 5$.*

2.6. Let $\mathcal{F}_1$ be the irreducible family of rank 3 vector bundles P on $\mathbf{P}^3$ defined as the cokernel:

$$0 \to \mathcal{O}(-5)^{\alpha-2} \xrightarrow[s_1,\dots,s_2]{} E \to P \to 0$$

where $E \in \mathcal{L}_1$ and $s_i \in H^0(E(5))$ are general global sections of $E(5)$.

2.7. Let $\mathcal{F}_2$ be the irreducible family of rank 3 vector bundles Q on $\mathbf{P}^3$ defined as the cokernel:

$$0 \to \mathcal{O}(-5)^{\alpha-2} \xrightarrow[f_1,\dots,f_2]{} F \to Q \to 0$$

where $F \in \mathcal{L}_2$ and $f_i \in H^0(F(5))$ are general global sections of $F(5)$.

2.8. Let $\mathcal{F} \subset \mathcal{L}_1 \cap \mathcal{L}_2$ be the irreducible family of rank 3 vector bundles R on $\mathbf{P}^3$ defined as the cokernel:

$$0 \to \mathcal{O}(-5)^{\alpha-2} \xrightarrow[g_1,\dots,g_2]{} G \to R \to 0$$

where $G \in \mathcal{L}$ and $g_i \in H^0(G(5))$ are general global sections of $G(5)$.

Proposition 2.9.

(1) *A general vector bundle P of $\mathcal{F}_1$ is a rank 3 stable vector bundle on $\mathbf{P}^3$ with Chern classes $(287, 42065, 4195775)$.*

(2) *A general vector bundle Q of $\mathcal{F}_2$ is a rank 3stable vector bundle on $\mathbf{P}^3$ with Chern classes $(287, 42065, 4195775)$.*

(3) *A general vector bundle R of $\mathcal{F}$ is a rank 3 stable vector bundle on $\mathbf{P}^3$ with Chern classes $(287, 42065, 4195775)$.*

Proof. (1) Using the exact sequence:

$$(*) \qquad 0 \to \mathcal{O}(-5)^{\alpha-2} \to E \to P \to 0$$

and the locally free resolution of E given in Proposition 2.4(1) we get:

$$c_t(P) = (1-3t)(1-t)^{74} / \left((1-6t)^4(1-5t)^{68}\right).$$

Hence $c_1(P) = 287$, $c_2(P) = 42065$ and $c_3(P) = 4195775$.

Let us see that P is stable. Using Hoppe's criterion we need to prove that $H^0(P)_{\mathrm{norm}} = H^0(\Lambda^2 P)_{\mathrm{norm}} = 0$. Since $c_1(P) > 0$ and $c_1(\Lambda^2 P) > 0$, we have $(P)_{\mathrm{norm}} = P(\lambda)$ and $(\Lambda^2 P)_{\mathrm{norm}} = (\Lambda^2 P)(\rho)$ for some $\rho, \lambda \leq -1$. So it siffices to prove that $H^0(P)(-1) = H^0(\Lambda^2 P)(-1) = 0$.

Using the exact sequence (*) and the locally free resolution of E given in Proposition 2.4(1) we easily get that $H^0 E(-1) = H^0 P(-1) = 0$. Again using the exact sequence (*) and taking wedge powers we get the exact sequence

$$0 \to S^2\mathcal{O}(-5)^{\alpha-2} \to \mathcal{O}(-5)^{\alpha-2} \otimes E \to \Lambda^2 E \to \Lambda^2 P \to 0$$

cutting in short exact sequences we get $H^0(\Lambda^2 P)(-1) = H^0(\Lambda^2 E)(-1) = 0$ where the last equality follows from the locally free resolution of E given in Proposition 2.4(1) taking wedge powers and cutting in short exact sequences.

(2) and (3) are analogous. □

Theorem 2.10. *The moduli space* $M_{\mathbf{P}^3}(3; -1, 14609, 1917791)$ *is singular.*

Proof. We have constructed two irreducible families $\mathcal{F}_1$ and $\mathcal{F}_2$ of rank 3 stable vector bundles on $\mathbf{P}^3$ with Chern classes (287,42065,4195775) which meets along an irreducible family $\mathcal{F}$. Hence in order to see that $M := M_{\mathbf{P}^3}(-1, 14609, 1917791) \cong M_{\mathbf{P}^3}(-287, 42065, 4195775)$ is singular it is enough to prove that $\mathcal{F}_1$ and $\mathcal{F}_2$ belongs to two different components of M. Using proposition 2.9 and 2.4 we get:

(1) If P is a general vector bundle of $\mathcal{F}_1$ then:

$$H^1_* P = H^3 P(3) = 0$$
$$h^0 P(3) = 1 + 10\alpha,\ h^2 P(3) = 0.$$

(2) If Q is a general vector bundle of $\mathcal{F}_2$ then:

$$H^1_* Q = H^3 Q(3) = 0$$
$$h^0 Q(3) = 10\alpha,\ h^2 Q(3) = 1.$$

Therefore, by semicontinuity $\mathcal{F}_1$ and $\mathcal{F}_2$ are contained in different components of M which gives what we want. □

References

[AO] V. Ancona and G. Ottaviani, *On moduli of instanton bundles on* $\mathbf{P}^{2n+1}$, Pacific J. Math., to appear.

[EF] Ph. Ellia and M. Fiorentini, *Défaut de postulation et singularités du schéme de Hilbert*, Ann. Univ. Ferrara, **30** (1984), 185-198.

[H] H. Hoppe, *Generischer spaltungstypum un zweite Chernklasse stabiler Vektorraumbündel vom rang* 4 *auf* $\mathbf{P}^4$, Math. Z., **187** (1984), 345-360.

[M1] M. Maruyama, *Moduli of stable sheaves*, I, J. Math. Kyoto Univ., **17** (1977), 91-126.

[M2] ———, *Moduli of stable sheaves*, II, J. Math. Kyoto Univ., **18** (1978), 557-614.

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