MOON HYPERSURFACES AND SOME RELATED EXISTENCE RESULTS OF CAPILLARY HYPERSURFACES WITHOUT GRAVITY AND OF ROTATIONAL SYMMETRY

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Let $\Omega_*(R)$ be a domain in $\mathbb{R}^n$ bounded by two spherical caps Σ_1 and Σ_2 of respective radii $\dfrac{n-1}{n}$ and R, with $\dfrac{n-1}{n} < R < 1$. (cf. Figure 1 for $n = 3$). We consider the vertical cylinder Z over $\partial\Omega_*(R)$ and seek a hypersurface $u_R(x_1,\ldots,x_n)$ over $\Omega_*(R)$ of constant mean curvature $H \equiv 1$ which meets Z in the angle π (vertically downward) over $\Sigma_1(R)$ and the angle 0 (vertically upward) over $\Sigma_2(R)$; intuitively and essentially, this amounts to seeking a solution to the problem

$$(0.1) \qquad \begin{cases} \operatorname{div} Tu_R = & n \\ \nu \cdot Tu_R = & \begin{cases} -1 & \text{on } \Sigma_1(R) \\ 1 & \text{on } \Sigma_2(R), \end{cases} \end{cases}$$

ν being outward unit normal.

0. Introduction.

In view of the shape of the base domain $\Omega_*(R)$, we shall, as in [FG] for $n = 2$, refer to $\Omega_*(R)$ as n-*dimensional moon domains* and as in [F2], refer to the solution of (0.1) as *moon (hyper)-surfaces*. Such a moon surface $(n = 2)$ is chosen to majorize the gradient of solution $u(x)$ of

$$(0.2) \qquad\qquad \operatorname{div} Tu = 2$$

in $B_R, R_0^{(2)} < R < 1$, with $R_0^{(2)} = 0.565406\ldots$ being the unique value of R for which $\Sigma_1(R)$ passes through the center of the circle including $\Sigma_2(R)$. This enables us to show the existence of apriori gradient bounds for solution of the equation (0.2) in $B_R, R_0^{(2)} < R < 1$, in [FG].

0.1. We note that, an integration of (0.1) over the section $\Omega_*(R)$ yields

$$(0.3) \qquad\qquad |\Sigma_2(R)| - |\Sigma_1(R)| = n|\Omega_*(R)|.$$

Thus, the condition (0.3) is necessary for existence of the moon hypersurfaces u_R.

In §3 and §5.1 of this paper, the existence of n-*dimensional moon domains* $\Omega_*(R)$, $1 > R > \dfrac{n-1}{n}$, characterized by *the condition* (0.3), will be verified, for $n = 3$ and $n > 3$, respectively. The existence of *moon hypersurfaces*, for $n = 3$ and $n > 3$, will be proved in §1 and §5.2, respectively. These results may help us to extend the above-mentioned apriori gradient estimates to higher dimensions.

0.2. As in [**F2**] and §3 of [**L1**] for $n = 2$, we shall, in §2 and §5.3, for $n = 3$ and $n > 3$, respectively, in a suitable sense indicated there, *construct the moon (hyper)-surface as a limit of solutions u_ϵ to* (1.2) *defined throughout the sphere B_R including $\Sigma_2(R)$.* This result will also be applied in [**L2**] to show that absolute gradient estimates cannot hold for solutions of

$$(0.4) \qquad\qquad\qquad \operatorname{div} Tu = n$$

in B_R, $R < R_0^{(n)}$, $R_0^{(n)}$ being the unique value of R for which $\Sigma_1(R)$ passes through the center of the sphere including $\Sigma_2(R)$. As calculated in the ending of §4, we have

$$R_0^{(3)} = \frac{-2 + 2\sqrt{19}}{9} = 0.746421987\cdots (\text{cf. } (4.11)).$$
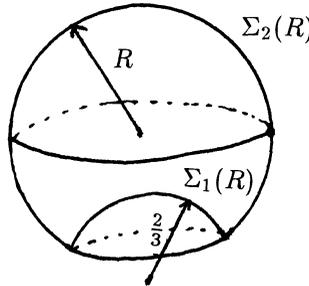
For $n > 3$, $R_0^{(n)}$ is determined as in §5.1.1.

Figure 1. (n=3)

0.3. The proof of the existence of the moon hypersurfaces u_R and the existence of that sequence of solutions converging to it are reduced to the general existence results in Finn [**F1**]. That is, in §1, we shall verify, for $n = 3$,

$$(0.5.1)$$
$$\phi[\Omega^o] \equiv |\partial\Omega^o \cap \Omega_*| + |\partial\Omega^o \cap \Sigma_1| - |\partial\Omega^o \cap \Sigma_2| + n|\Omega^o| > 0$$

(0.5.2)
$$\psi[\Omega^o] \equiv |\partial\Omega^o \cap \Omega_*| - |\partial\Omega^o \cap \Sigma_1| + |\partial\Omega \cap \Sigma_2| - n|\Omega| > 0$$

for every Caccioppoli set $\Omega^0 \subseteq \Omega_*$, $\Omega^0 \neq \phi$, Ω_*; in §2, we shall verify, for $n = 3$, for ϵ sufficiently small

(0.6.1)
$$\phi[\Omega^0] \equiv |\partial\Omega^0 \cap \Omega_*| - (1-\epsilon)|\partial\Omega^0 \cap \Sigma_2| - \widehat{\beta}_\epsilon \left|\partial\Omega^0 \cap \widehat{\Sigma}\right| + n|\Omega^0| > 0,$$

(0.6.2)
$$\psi[\Omega^0] \equiv |\partial\Omega^0 \cap \Omega_*| + (1-\epsilon)|\partial\Omega^0 \cap \Sigma_2| + \widehat{\beta}_\epsilon \left|\partial\Omega^0 \cap \widehat{\Sigma}\right| - n|\Omega^0| > 0,$$

for every Caccioppoli set $\Omega^0 \subseteq B_R$, $1 > R > 2/3$, $\Omega^0 \neq \phi$, B_R where $\widehat{\Sigma} = \partial B_R - \Sigma_2$ and $\widehat{\beta}_\epsilon(R)$ is a constant depending on R, and defined by the equation (2.1); $-1 < \widehat{\beta}_\epsilon < 1$ for $1 > R > 2/3$ and $-1 < \beta_\epsilon < 0$ for $1 > R > R_0^{(3)}$. The verification of (0.5.1), (0.5.2), (0.6.1) and (0.6.2), however, is not a straightforward generalization of that of the two dimensional case, *due to the fact that the hypersurfaces of constant mean curvature are in general not spherical.* A new approach is inexcusably required. We will draw on the technique of the rearrangement of level curves. The rotational symmetry of both the boundary surface ∂B_R and the boundary data will therefore play a crucial role in our investigation. Also, in this connection, we find that, in both cases of §1 and §2, it is more easy and natural to discuss $\psi[\Omega^0]$ than $\phi[\Omega^0]$; thus because of the respective equivalence of (0.5.1), (0.6.1) and (0.5.2), (0.6.2), we will restrict our attention to (0.5.2) and (0.6.2). In either case, a minimizing body for $\psi[\Omega^0]$ exists and, using our new technique, the only possible non-empty minimizing body for $\psi[\Omega^0]$ is shown to have a spherical cap of radius $2/3$ and passing through $\partial\Sigma_1$ as its boundary in the sphere B_R (obtained by completing Σ_2). This only possible non-empty minimizing body includes or is included in a hemisphere in the case of §1 or §2, respectively, and has $\psi > 0$ in either case, thereby proving that the empty set is the one and only minimizing body for $\psi[\Omega^0]$. (0.5.2) and (0.6.2) are immediate consequences of this.

The main tool used in this case of §1 is, what is known as the classical isoperimetric inequality. We, however, find difficulties in applying this technique to the case of §2, mainly due to the boundary data $1 - \epsilon$ being unequal to 1. *Steiner symmetrization* is suitably modified to prove that the minimizing body for $\psi[\Omega^0]$ in (0.6.2) is a surface of revolution, with the extremely useful help of the analyticity of the boundary surface in B_R of a minimizing body for $\psi[\Omega]$ and $n = 3$, (which is provided by Massari [**Ma**]).

0.4. For simplicity of writing and convenience of visualization, we deal exclusively with the case of three dimensional domains in §1, §2, §3 and §4. In the chapter §5, we will extend the results in these chapters to domains of dimension higher than three. We note that, for $n > 7$, Massari's Theorem [**Ma**] does not yield the analyticity of the boundary surface in B_R of a minimizing body for $\psi[\Omega]$. This difficulty of extension, however, as we shall observe in §5.3, is insubstantial. Reviewing the argument used in §2 and §5.3, incidentally, will enable us to formulate in §6 some existence results of capillary hypersurfaces whose domain of definition and boundary data are of rotational symmetry about the same axis.

1. Existence of the Moon Hypersurfaces for $n = 3$.

In this section, we shall prove.

Theorem 1.1. *Let $\Omega_* \subset \mathbb{R}^3$ be a "moon domain", bounded by two spherical caps Σ_1 and Σ_2 with the respective radii $\frac{2}{3}$ and R, $1 > R > \frac{2}{3}$, which satisfies the condition*

$$(1.1) \qquad\qquad |\Sigma_1| - |\Sigma_2| = 3|\Omega_*|.$$

Then the problem

(1.2) $\operatorname{div} Tu = 3$ *in* $\Omega_*,$

$$\int_{\Omega_*} (w_{p_i}\zeta_i + 3\eta)\, dx + \int_{\Sigma_1} \eta\, d\sigma - \int_{\Sigma_2} \eta\, d\sigma = 0 \quad \text{for all } \eta \in H^{1,1}(\Omega_x)$$

where $\quad \zeta_i = \eta_{x_i}, \quad w = \sqrt{1 + |p|^2}, \quad p = (p_1, p_2, p_3), \quad p_i = v_{x_i}$

has a solution $u(x)$, unique up to an additive constant.

1.1. Background information. As in §2 of [**L1**], we reduce the proof of Theorem 1.1 to the general existence results in Finn [**F1**], which, although have been formulated for two dimensional domains, can be easily extended to higher dimensions by the same argument.

  As in [**F1**], the capillary problem in the absence of gravity can be reduced to the variational problem for a functional

$$\xi[u] = \int_\Omega \sqrt{1 + |Du|^2} + nH \int_\Omega u\, dx - \int_{\partial\Omega} \beta(s)u\, d\sigma,$$

with $\beta(s)$, $-1 \le \beta(s) \le 1$, being piecewise Lipschitz continuous on the boundary $\partial\Omega$ of a bounded domain $\Omega \subseteq \mathbb{R}^n$, and H being a constant. As in §2 of [**L1**], for future reference we formulate

Proposition 1. *If, for a piecewise Lipschitz domain Ω, both the conditions*

$$(1.3) \qquad \phi\left[\Omega^0\right] = \left|\partial\Omega^0 \cap \Omega\right| - \int_{\partial\Omega^0 \cap \partial\Omega} \beta\,ds + nH\left|\Omega^0\right| > 0$$

and

$$(1.4) \qquad \psi\left[\Omega^0\right] = \left|\partial\Omega^0 \cap \Omega\right| + \int_{\partial\Omega^0 \cap \partial\Omega} \beta\,ds - nH\left|\Omega^0\right| > 0$$

hold for every Caccioppoli set $\Omega^0 \neq \phi$, $\Omega\,(\Omega^0 \subseteq \Omega)$. Then there is a minimizing function $u(x) \in BV_{\mathrm{loc}}(\Omega)$ for $\xi[u]$. Furthermore, the minimizing function is unique up to an additive constant, is regular and locally bounded in Ω, satisfies in Ω the Eq. (0.1) and the variational condition

$$(1.5) \qquad \int_\Omega \left(w_{p_i}\zeta_i + nH\eta\right)dx - \int_{\partial\Omega}\beta\eta = 0$$

for any $\eta \in H^{1,1}(\Omega)$, $\zeta_i = \eta_{x_i}$; here

$$W = \sqrt{1 + |p|^2}, \quad p = (p_1, p_2, \ldots, p_n), \quad p_i = u_{x_i}.$$

1.2. The Proof of Theorem 1.1.

In view of Proposition 1, it suffices to show (0.5.1) and (0.5.2) for every Caccioppoli set $\Omega^0 \subseteq \Omega_*$, $\Omega^0 \neq \phi, \Omega_*$.

To show this, we first observe that if $\overline{\Omega}^0 \subset \Omega_*$, then

$$(1.6) \qquad \begin{aligned} \phi\left[\Omega^0\right] &\equiv \left|\partial\Omega^0 \cap \Omega\right| + 3\left[\Omega^0\right] > 0 \\ \psi\left[\Omega^0\right] &\equiv \left|\partial\Omega^0 \cap \Omega\right| - 3\left[\Omega^0\right] > 0 \end{aligned}$$

where the last inequality is an immediate consequence of the following Proposition. (Henceforth, we denote the characteristic function of a Caccioppoli set E as $\varphi_{E'}$, and the integral $\int_{B_R}|D\varphi_E|$, denoted as the perimeter of E in B_R, is defined by $\int_{B_R}|D\varphi_E| = \sup\int_{B_R}\varphi_E\operatorname{div}g$ among all vector functions $g \in C_0^1(B_R)$, $|g| \leq 1$. This integral equals the surface area of ∂E in B_R whenever this boundary is smooth.)

Proposition 2. *If A is a Caccioppoli set with $A \subseteq B_R$, $0 < R < 1$, then*

$$\int_{B_R}|D\varphi_A| - 3\int_{B_R}\varphi_A\,dx > 0.$$

Proof. Let $v(x)$, defined on B_1, describe the lower unit hemisphere, then

$$(1.7) \qquad \operatorname{div}Tv = 3 \quad \text{in } B_1.$$

If $A \subseteq B_R$, $0 < R < 1$, we can integrate the Eq. (1.7) in A, obtaining

$$3 \int_{B_R} \varphi_A \, dx = \int_A \operatorname{div} Tv \, dx = -\langle D\varphi_A, Tv \rangle,$$

and hence, as $|Tv| < 1$ in A, $3 \int_{B_R} \varphi_A \, dx < \int_{B_R} |D\varphi_A|$. $\qquad\qquad \square$

(We note that this result is alternatively obtained in Giusti [**G1**], pages 114 and 115.)

Thus it now suffices to consider all those sets intersecting $\partial\Omega_*$ with a set of positive area. We shall show that (0.5.2) *holds for all those* $\Omega^0 \subsetneqq \Omega_*$ *which have either or both of* $|\partial\Omega^0 \cap \Sigma_1|$ *and* $|\partial\Omega^0 \cap \Sigma_2| > 0$. Once we show this, since, for all the Caccioppoli sets $\Omega^0 \subseteq \Omega_*$,

$$
\begin{aligned}
(1.8) \quad \phi\left[\Omega^0\right] &\equiv \left|\partial\Omega^0 \cap \Omega\right| + \left|\partial\Omega^0 \cap \Sigma_1\right| - \left|\partial\Omega^0 \cap \Sigma_2\right| + 3\left|\Omega^0\right| \\
&= \left|\partial\Omega^0 \cap \Omega\right| - \left|\Sigma_1 \cap \partial\left(\Omega_* - \Omega^0\right)\right| \\
&\quad + \left|\Sigma_2 \cap \partial\left(\Omega_* - \Omega^0\right)\right| - 3\left|\Omega - {}_*\Omega^0\right|, \qquad \text{(by (1.1))} \\
&= \psi\left[\Omega_* - \Omega^0\right],
\end{aligned}
$$

(0.5.2) implies that there also holds (0.5.1) for all the Caccioppoli sets $\Omega^0 \subseteq \Omega_*$, $\Omega^0 \neq \phi$, Ω_*. The proof of Theorem 1.1 can thus be completed.

To show this, we first observe that if Ω^0 has $|\partial\Omega^0 \cap \Sigma_1| = 0$, then $\psi[\Omega^0] = |\partial\Omega^0 \cap \Omega_*| + |\partial\Omega^0 \cap \Sigma_2| - 3|\Omega_0| > 0$, again due to *Proposition 2*.
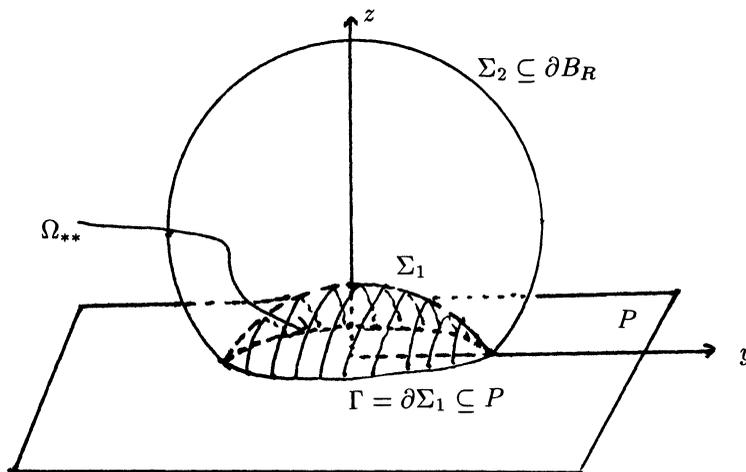

Figure 2.

Thus, it suffices to consider all those Caccioppoli sets Ω^0 *with* $|\partial\Omega^0 \cap \Sigma_1| > 0$ *and* $|\partial\Omega^0 \cap (\Omega_* \cup \Sigma_2)|$ *being connected. We observe also that, for all such sets we can always assume that* $\partial\Omega^0 \cap \Sigma_1 = \Sigma_1$, *for otherwise we could add*

to Ω^0 and ϵ-neighbourhood of Σ_1 and then pass to limit as $\epsilon \to 0^+$. (Here we note that the *boundary data* β_ϵ *being* $\equiv -1$ *on* Σ_1 enables us to do so.) *We call the collection of these sets as* S. For sets in S, we have

$$(1.9) \qquad \psi\left[\Omega^0\right] \equiv \left|\partial\Omega^0 \cap \Omega_*\right| - \left|\Sigma_1\right| + \left|\partial\Omega^0 \cap \Sigma_2\right| - 3\left|\Omega^0\right|;$$

to minimize this expression (1.9) among all these sets in S, however, is equivalent to minimizing

$$(1.10) \qquad \psi^*\left[\Omega^0\right] = \left|\partial\Omega^0 \cap \Omega_*\right| + \left|\partial\Omega^0 \cap \Sigma_2\right| - 3\left|\Omega^0 \cup \Omega_{**}\right|$$

in the same collection of sets, where Ω_{**} is that part of $B_R - \Omega_*$ lying above the unique plane P passing through the circle $\Gamma = \partial\Sigma_1$ (see Figure 2). Here and in the following, we assume B_R to be the sphere that is obtained by completing Σ_2, P to be the x, y plane and that side of P containing the center of B_R to be "above" P.

As in §2 of [**L1**], we consider a minimizing sequence $\left\{\Omega_j^0\right\}$ for the functional $\psi^*[\Omega^0]$ in (1.10), and use the same argument to conclude from Theorem 1.19 in Giusti [**G2**] that there is a subsequence of $\{\varphi_{\Omega_j^0}\}$ that converges in $L^1(\Omega)$ to $\varphi_{\widetilde{\Omega}'}$ and that setting $\widetilde{\Sigma} = \partial\widetilde{\Omega} \cap \Omega_*$

$$\left|\widetilde{\Sigma}\right| = \int_{\Omega_*} \left|D\varphi_{\widetilde{\Omega}}\right| \le \inf \int_{\Omega_*} \left|D\varphi_{\Omega_j^0}\right|.$$

Further, we have

$$\psi^*\left[\widetilde{\Omega}\right] \le \inf \psi^*\left[\Omega_j^0\right],$$

by a reasoning similar to that used for the proof of Lemma 6.3 in Finn [**F1**].

We proceed to characterize the geometry of $\widetilde{\Sigma}$.

Proposition 3. *If* $\widetilde{\Sigma} \ne \phi$, *then* $\widetilde{\Sigma}$ *must be a spherical cap passing through* $\partial\Sigma_1$.
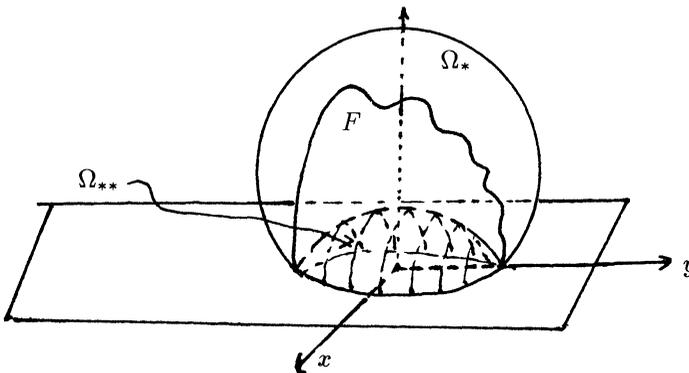

Figure 3.

Proof of Proposition 3. We consider an arbitrary body F in $\Omega_* \cup \Omega_{**} \cup \Sigma_2$ (cf.
Figure 3), passing through $\Gamma = \partial\Sigma_1$, and bounded below by the disk $P \cap B_R$.
From the discussion below Figure 2 and above (1.9), we may, without loss of
generality, assume that $F \setminus \Omega_{**}$ is in the collection S. Now that $\psi^*[F \cap \Omega_*] =$
$|\partial F \cap (\Omega_* \cup \Sigma_2)| - 3|F|$, we shall prove Proposition 3 by constructing a body
$\widehat{F}$ such that $\widehat{F} \setminus \Omega_{**}$ is in the collection S, and that

$$\left|\widehat{F}\right| = |F|,$$

$$|\partial F \cap (\Omega_* \cup \Sigma_2)| \geq \left|\partial\widehat{F} \cap (\Omega_* \cup \Sigma_2)\right|,$$

where the last equality holds only when $\partial F \cap (\Omega_* \cup \Sigma_2)$ is a spherical cap
passing through Γ.

We observe first that, for each value V with

$$|\Omega_{**}| < V < \left|\overline{\Omega}_* \cup \Omega_{**}\right|,$$

a spherical cap passing through Γ and situating above P exists, the volume
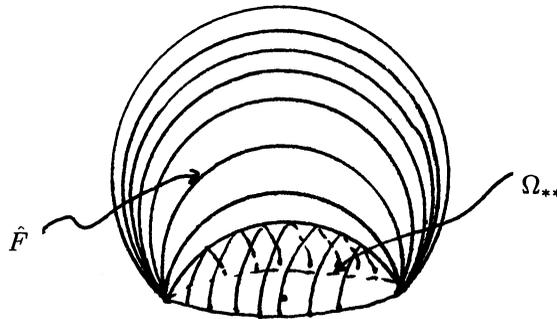enclosed by which and disk $P \cap B_R$ is equal to V. (Cf. Figure 4).

Figure 4.

Now that

$$|\Omega_{**}| < |F| < \left|\overline{\Omega}_* \cup \Omega_{**}\right|,$$

a body with

$$\left|\widehat{F}\right| = |F|$$

exists which has a spherical cap $\widehat{\Sigma}$ as its boundary in Ω_*. Obviously, $\widehat{F} \setminus \Omega_{**}$
is in the collection S. Furthermore, we may extend the spherical cap $\widehat{\Sigma}$ to a
full sphere Σ which is the boundary of a ball B. Then

$$|F| + |B - F| = \left|\widehat{F}\right| + \left|B - \widehat{F}\right| = |B|,$$

and the isoperimetric inequality for three dimensions (Cf. [**MM**], p. 92) asserts that

$$|\partial F \cap (\Omega_* \cup \Sigma_2)| + \left|\Sigma \setminus \widehat{\Sigma}\right| \geq |\Sigma|,$$

that is,

$$|\partial F \cap (\Omega_* \cup \Sigma_2)| \geq \left|\widehat{\Sigma}\right| = \left|\partial\widehat{F} \cap (\Omega_* \cup \Sigma_2)\right|,$$

and equality holds only when $F = \widehat{F}$. □

Also, by the analyticity of $\overline{\Sigma}$ (see [**Ma**]), we may use an argument similar to that one used to prove Lemma 6.4 in page 148 of [**F1**] to conclude.

Lemma 1. *If $\overline{\Sigma} \neq \phi$ then $\overline{\Sigma}$ must consist of surfaces of constant mean curvature $3/2$ and $\overline{\Omega}$ lies on the side of $\overline{\Sigma}$ into which the curvature vector points.*

Putting Proposition 3 and Lemma 1 together, we see that a non-empty $\widetilde{\Sigma}$ must be a spherical cap of radius $2/3$, which can possibly occur only when Σ_1 is a subset of a hemisphere of radius $2/3$ and $\widetilde{\Sigma}$ strictly includes a hemisphere of radius $2/3$. In case that Σ_1 is included in a hemisphere, denting Σ_0 as the spherical cap of radius $2/3$, included in Ω_* and Ω_0 as the body enclosed by Σ_0 and Σ_1, we shall show

$$\psi^* \left[\overline{\Omega}_0 \cup \Omega_{**}\right] - \psi^* \left[\Omega_{**}\right] > 0, \quad \text{where } \psi^* \left[\Omega_{**}\right] \equiv |\Sigma_1| - 3\left|\Omega_{**}\right|$$

and hence

$$\begin{aligned}
\psi\left[\Omega_0\right] &= \psi^* \left[\Omega_0 \cup \Omega_{**}\right] - |\Sigma_1| + 3\left|\Omega_{**}\right| \\
&> \psi^* \left[\Omega_{**}\right] - |\Sigma_1| + 3\left|\Omega_{**}\right| \\
&= 0,
\end{aligned}$$

thereby proving (0.5.2), as minimizing ψ and ψ^* are one and the same matter.

In fact, adopting spherical coordinates with origin at the center 0 of $B_{2/3}$ including Σ_0, we choose $\theta_1 < \pi/2$ so that the equation $r = \dfrac{2\sin(\pi - \theta_1)}{3}$ is that for the circle Γ ($\equiv \partial\Sigma_1$). Thus, (cf. Figure 5) as calculated in (3.7) and (3.8) for $R = \frac{2}{3}$ (cf. (3.1), (3.3))

$$\begin{aligned}
\psi^* \left[\overline{\Omega}_0 \cup \Omega_{**}\right] - \psi^* \left[\Omega_{**}\right] &= \left(\frac{8}{27}\pi + \frac{8}{27}\pi\cos^3\theta_1\right) - \left(\frac{8}{27}\pi - \frac{8}{27}\pi\cos^3\theta_1\right) \\
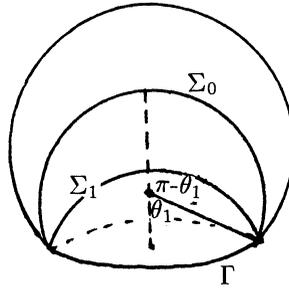&= \frac{16}{27}\pi\cos^3\theta_1 \\
&> 0,
\end{aligned}$$

as desired.

Figure 5.

2. Moon Hypersurfaces constructed as a generalized solution over B_R in the sense of Miranda for $n = 3$.

As in Sec. 7.11 of Finn [**F1**], II of Finn [**F2**], or §3 of [**L1**], let us extend the spherical cap Σ_2 to a full sphere ∂B_R, and write $\widehat{\Sigma} = \partial B_R - \Sigma_2$ (cf. Figure 6). Then if ϵ is small enough, it will be verified in §5 that there is unique $\widehat{\beta}_\epsilon(R)$, $-1 < \widehat{\beta}_\epsilon < 1$ for $1 > R \geq 2/3$ and $-1 < \widehat{\beta}_\epsilon < 0$ for $1 > R \geq R_0^{(3)}$, such that data

$$\beta_\epsilon = \begin{cases} 1 - \epsilon & \text{on } \Sigma_2 \\ \widehat{\beta}_\epsilon & \text{on } \widehat{\Sigma}, \end{cases}$$

satisfies the necessary condition

(2.1) $$(1 - \epsilon) |\Sigma_2| + \widehat{\beta}_\epsilon \left|\widehat{\Sigma}\right| = 3 |B_R|$$

for the existence of a minimizing function $u_\epsilon(x) \in BV_{\text{loc}}(B_R)$, which minimizing the functional

$$\xi^\epsilon[u] = \int_{B_R} \sqrt{1 + |\nabla u|^2} + 3 \int_{B_R} u \, dx - \int_{\partial B_R} \beta_\epsilon(s) u \, ds,$$

and thus (cf. Proposition 1) satisfies

$$\text{div}\, Tu_\epsilon = 3$$

in B_R; here (2.1) is necessary because substituting $\eta(x) = 1$ (in $\overline{B}_R$) into the variational condition (1.5) for this particular function $\xi^\epsilon[u]$ yields (2.1).

We shall show that (a) *this minimizing function $u_\epsilon(x)$ indeed exists if ϵ is small enough, and* (b) *as $\epsilon \to 0$, $|\nabla u_\epsilon|$ cannot be bounded in ϵ for any subset of $|\Sigma_1|$ of positive area.*
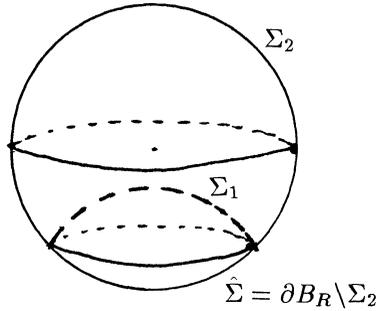
$\widehat{\Sigma} = \partial B_R \backslash \Sigma_2$

Figure 6.

2.1. *To prove* (a), in view of Proposition 1, it suffices to show that, for sufficiently small ϵ, (0.6.1) and (0.6.2) hold for every Caccioppoli set $\Omega^0 \subseteq B_R$, $\Omega^0 \neq \phi$, B_R. To show this, as in §1, we first observe that, if $\overline{\Omega^0} \subseteq B_R$, then

$$\phi\left[\Omega^0\right] \equiv \left|\partial\Omega^0\right| + 3\left|\Omega^0\right| > 0,$$
$$\psi\left[\Omega^0\right] \equiv \left|\partial\Omega^0\right| - 3\left|\Omega^0\right| > 0,$$

where the last inequality readily follows from *Proposition* 2 in §1. Thus, it suffices to consider all those sets whose intersection with ∂B_R is a set of positive area. We shall show that (0.6.2) for $n = 3$ holds for all those $\Omega^0 \subsetneq \Omega_*$ which have $\left|\partial\Omega^0 \cap \Sigma_2\right| > 0$ or $\left|\partial\Omega^0 \cap \widehat{\Sigma}\right| > 0$. As in §1, we note that *proof of* (a) *will be completed once we verify the truth of* (0.6.2), *because there holds by virtue of* (2.1),

$$\phi\left[\Omega^0\right] = \psi\left[B_R - \Omega^0\right],$$

for each Caccioppoli set $\Omega^0 \subseteq B_R$.
   To show (0.6.2) for $n = 3$, we first observe that *if ϵ is small enough,*

$$\psi\left[\Omega^0\right] \equiv \left[\partial\Omega^0 \cap B_R\right] + (1 - \epsilon)\left|\partial\Omega^0 \cap \Sigma_2\right| - 3\left|\Omega^0\right| > 0$$

for all the Caccioppoli sets Ω_0 *with* $\left|\partial\Omega^0 \cap \widehat{\Sigma}\right| = 0$. *This follows from Proposition* 2 *and* Giusti [**G1**], *Lemma* 1.
   Thus it suffices to consider all those Caccioppoli sets Ω^0 *with* $\left|\partial\Omega^0 \cap \widehat{\Sigma}\right| > 0$ *and* $\partial\Omega^0 \cap (B_R \cup \Sigma_2)$ *being connected.*

As in §1, we may try to minimize $\psi[\Omega^0]$ among all the Caccioppoli sets $\overline{\Omega}$ and consider a minimizing sequence $\{\Omega_j^0\}$ for $\psi[\Omega^0]$; the same reasoning concludes that there exists a subsequence of the $\{\varphi_{\Omega_j^0}\}$ converging in $L^1(\Omega)$ to $\varphi_{\tilde{\Omega}}$ such that

$$\psi[\tilde{\Omega}] \leq \inf \psi[\Omega_j^0].$$

Set $\tilde{\Sigma} = \tilde{\Omega} \cap B_R$. If $\tilde{\Sigma} \neq \phi$, we have shown that $\left|\partial \tilde{\Omega} \cap \hat{\Sigma}\right| > 0$.

Due to the very fact that $\beta_\epsilon \neq 1$, it seems infeasible to proceed further as in §1. We may, however, take a different approach and arrive at the same conclusion. The main idea of the following discussion is provided by Steiner's solution to the two dimensional isoperimetric problem.

Our main aim is to show

Proposition 4. *The only non-empty candidate for $\tilde{\Sigma}$ is the spherical cap Σ_1. In other words, the only non-empty candidate for $\tilde{\Omega}$ is $B_R - \Omega_*$.*

We again let P to be the unique plane passing through the circle $\Gamma \equiv \partial \Sigma_1 \cap \partial B_R$ and designate P as the x, y plane so that the center of B_R has the z-coordinate $z > 0$.

To prove Proposition 4, we shall proceed to verify

Proposition 3*. *If $\tilde{\Sigma} \neq \phi$, then $\tilde{\Sigma}$ is made up of surfaces of revolution about the Z-axis.*

We will reduce the proof of Proposition 3* to that of the following

Proposition 3**. *If $\tilde{\Sigma} \neq \phi$, then at each point of $\tilde{\Sigma}$, the tangent of the horizontal cross-section of $\tilde{\Sigma}$ through this point is the normal of the unique vertical plane $ax + by = 0, a, b :$ constants, passing through this point (and the origin).*

The equivalence of Proposition 3* and Proposition 3** is obvious; in fact, at each height z_0, Proposition 3** yields that

$$x\dot{x} + y\dot{y} = 0$$

for each connected subarc $(x(t), y(t), z_0)$ of the horizontal cross-section of $\tilde{\Sigma}$, which holds if and only if

$$x^2 + y^2 = \text{constant},$$

i.e., $(x(t), y(t), z_0)$ describes a circle with the center on the z-axis. This amounts to Proposition 3*.

We thus proceed to give a

Proof of Proposition 3**. Consider a vertical plane $\widehat{P}$: $ax + by = 0$, a, b : constants, which divides $\overline{\Omega}$ into two non-empty parts Ω_1 and Ω_2 (and of course passes through a great circle of ∂B_R). We can assume $\psi[\Omega_1] \leq \psi[\Omega_2]$. Reflecting the body Ω_1 in the plane $\widehat{P}$, we obtain a body Ω_1' on the opposite side of the plane $\widehat{P}$ such that

$$\Omega_1 \cup \Omega_1' \subseteq \overline{B}_R.$$

Then

$$\psi[\Omega_1 \cup \Omega_1'] = \psi[\Omega_1] + \psi[\Omega_1'] - 2\left|\partial\Omega_2 \cap \widehat{P}\right|$$

$$\leq \psi[\Omega_1] + \psi[\Omega_2] - \left|\partial\Omega_1 \cap \widehat{P}\right| - \left|\partial\Omega_2 \cap \widehat{P}\right|$$

$$= \psi\left[\tilde{\Omega}\right]$$

since $\psi[\Omega_1'] = \psi[\Omega_1] \leq \psi[\Omega_2]$ (cf. Remark 1 below) and $\left|\partial\Omega_1 \cap \widehat{P}\right| = \left|\partial\Omega_2 \cap \widehat{P}\right|$, by construction. The minimizing property of $\overline{\Omega}$ yields $\psi[\Omega_1 \cup \Omega_1'] = \psi\left[\overline{\Omega}\right]$ (and hence $\psi[\Omega_1] = \psi[\Omega_2]$). The body $\Omega_1 \cup \Omega_1'$ is therefore another minimizing body for $\psi[\Omega]$ and the theorem of Massari [**Ma**] thus yields the *analyticity* of the boundary surface of $\Omega_1 \cup \Omega_1'$ in B_R. In other words, $\widehat{\Sigma}_1 \cup \widehat{\Sigma}_1'$ is an analytic surface in B_R, where $\widehat{\Sigma}_1 = \partial\Omega_1 \cap B_R$ and $\widehat{\Sigma}_1'$ is the reflection of $\widehat{\Sigma}_1$ in the plane $\widehat{P}$. In particular, each horizontal cross-section of $\widehat{\Sigma}_1 \cup \widehat{\Sigma}_1'$ must consist of smooth arcs, which is possible only if Proposition 3** holds, (for otherwise a cusp would have appeared at a certain horizontal cross-section of $\widehat{\Sigma}_1 \cup \widehat{\Sigma}_1'$). □

Remark 1. We note that $\psi[\Omega_1'] = \psi[\Omega_1]$ because of the rotational symmetry of both the boundary surface ∂B_R and the boundary data β_ϵ.

In Proposition 3*, *we know that* $\partial\tilde{\Sigma} \cap \partial B_R \subseteq \widehat{\Sigma} \cup \partial\Sigma_1$ by the fact that ϵ can be arbitrarily small and the reasoning used in the proof of Proposition 3 in §1. Thus, (0.6.2) yields that $\partial\tilde{\Sigma} \cap \partial B_R$ must be a connected subset of $\widehat{\Sigma}$, for otherwise replacing a part of $\tilde{\Sigma}$ below $\partial\Sigma_1$ by that part of $\widehat{\Sigma}$ surrounding it yields a smaller value for ψ. Thus, the reasoning used in the proof of Proposition 3 yields that $\tilde{\Sigma}$ must be spherical. Also, the reasoning following the proof of Proposition 3 excludes *that spherical cap situated above* Σ_1 *and passing through* $\partial\Sigma_1$.

Furthermore, in Proposition 3*, were $\tilde{\Sigma}$ situating below Σ_1, then a rigid motion of it would result in a body meeting $\widehat{\Sigma}$ with the same surface area and therefore yielding the same value for ψ (cf. Figure 7), which, however, would

by no means be symmetric with respect to the z-axis, violating Proposition 3*. We thus precluded the occurence of $\tilde{\Sigma}$ being a spherical cap other than Σ_1. *Proposition* 4 *is finally proved.*
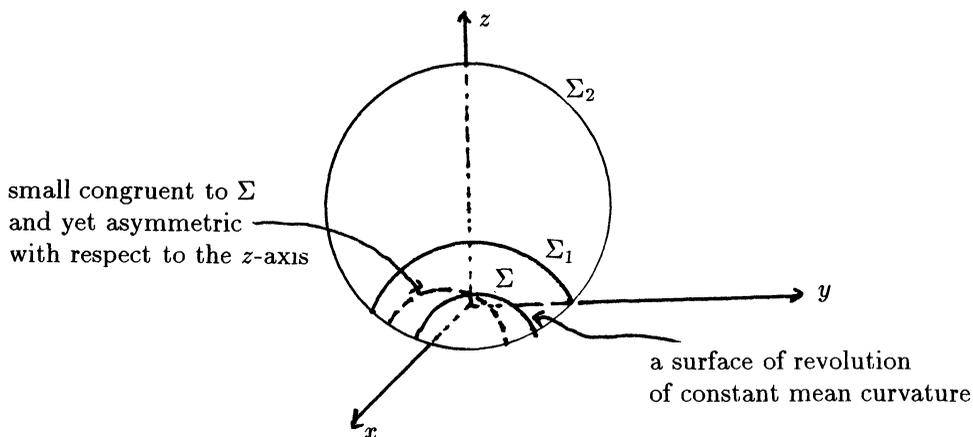

Figure 7.

Now that Proposition 4 has been proved, our proof of (a) is complete by observing that

$$
\begin{aligned}
\psi[\Omega_*] &= \phi[B_R - \Omega_*] \\
&= (3|\Omega_*| + |\Sigma_1| - |\Sigma_2|) + \epsilon|\Sigma_2| \\
&= \epsilon|\Sigma_2|, \qquad \text{by (0.3)} \\
&> 0.
\end{aligned}
$$

2.2. *Next, to prove* (b), *we note that a proof for* (b) *given in* §3.2 *of* [**L1**] *for the two dimensional domains extends in an obvious may to arbitrary dimensional domains* and we do not repeat it here.

We, however, recall that, in the course of our proof, we have incidentally proved

Proposition 5.

$$
\int_{\Sigma_1} \nu \cdot T u_\epsilon \, ds \longrightarrow -|\Sigma_1|, \qquad \text{as } \epsilon \to 0
$$

and $T u_\epsilon(x) \to \nu(x_0)$, *as* $\epsilon \to 0$, *uniformly for* $x_0 \in \Sigma_2$.

We therefore gain the rough impression that *the solution of* (0.6) *in* Ω_* *has been constructed as a limit of solutions* u_ϵ *defined throughout* B_R, as stated in §0.0.2. We may proceed to gain a rigid and precise understanding on this.

As in §3.3.3 of [**L1**], according to a theorem of Miranda [**M**], we know that a subsequence of $\{u_\epsilon\}$ can be found which converges in B_R to a generalized solution $u(x)$ of the equation (0.4), $n = 3$, in $L^1_{\text{loc}}(B_R)$. Set P and N and normalize the solutions u_ϵ in essentially the same way as we have done in §3.3.3 of [**L1**]. We again have

Proposition 6. *Both the sets N and $B_R - P$ minimize the functional*

$$(2.3) \qquad \psi[\Omega^0] \equiv \left|\partial\Omega^0 \cap B_R\right| + \widehat{\beta}_0 \left|\partial\Omega^0 \cap \widehat{\Sigma}\right| + \left|\partial\Omega^0 \cap \Sigma_2\right| - 3|\Omega^0|,$$

among all the Caccioppoli sets $\Omega^0 \subseteq B_R$, $\Omega^0 \neq \phi$ or B_R, where

$$0 < \widehat{\beta}_0 = \frac{3|B_R| - |\Sigma_2|}{\left|\widehat{\Sigma}\right|} < 1.$$

Repeating our reasoning for proving *Proposition 4*, we again know that the minimizing body for (2.3) must be either empty or else $B_R - \Omega_*$. In consideration of our normalization, the results in (b) and the reasoning used in §3.3.4 of [**L1**] therefore again yield that $P = \phi$ and $N = B_R - \Omega_*$. We thus prove that the regularity domain of u coincides with Ω_*. Also, the reasoning used in the ending of §3.3.3 of [**L1**] or Theorem 7.8 in [**F3**] again yields the identity of the function u and the solution to (1.2) (or (0.1)) in Ω_*. We therefore arrive at an accurate interpretation of what we asserted.

3. The Existence of Three Dimensional Moon Domains $\Omega_*(R)$ for $1 > R > 2/3$.

Consider the function

$$(3.1) \qquad\qquad f(r;\theta) = \sigma_\theta(r) - 3v_\theta(r)$$

where $\sigma_\theta(x)$ is the area of the spherical cap $D_{\rho(\theta,r)}$ whose boundary $\partial D_{\rho(\theta,r)}$ is a circle of radius $\rho = r\sin\theta$ on ∂B_r and $V_\theta(r)$ is the volume enclosed by the spherical cap $D_{\rho(\theta,r)}$ and the plane passing through the circle $\partial D_{\rho(\theta,r)}$ (cf. Figure 8). We readily see that, if $\Omega_*(R)$, $1 > R > 2/3$, exists, the equation of the circle $\Gamma \equiv \partial\Sigma_1(R)$ is $\rho = \frac{2}{3}\sin\theta_1(R)$ where $\theta_1(R)$ is the root of the equation

$$(3.2) \qquad f(R;\pi - \psi(\theta)) - f(2/3;\theta) = 0 \qquad (\text{cf. } (1.1) \text{ or } (0.3)),$$

with

$$(3.3) \qquad\qquad \psi(\theta) = \sin^{-1}\frac{2\sin\theta}{3R}.$$
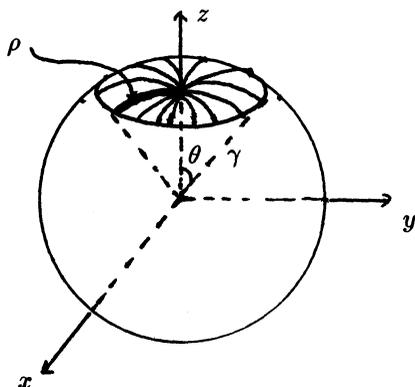
Figure 8.

We shall justify, for $1 > R > 2/3$, the existence of $\Omega_*(R)$ by showing the existence of a root $\theta = \theta_1(R)$ for the equation

$$g(R; \theta) = 0,$$

with

$$(3.4) \qquad g(R; \theta) = f(R; \pi - \psi(\theta)) - f(2/3; \theta).$$

We have

$$(3.5) \qquad \sigma_\theta(x) = 2\pi r^2 \int_0^\theta \sin\theta \, d\theta = 2\pi r^2 (1 - \cos\theta),$$

and

$$(3.6) \qquad V_\theta(r) = \pi \int_{r\cos\theta}^r (r^2 - z^2) \, dz = \pi r^3 (2/3 - \cos\theta + 1/3 \cos^3\theta).$$

Hence, by (3.1),

$$(3.7) \qquad f\left(\frac{2}{3}; \theta\right) = \frac{8\pi}{9}(1 - \cos\theta) - \frac{8}{27}\pi(2 - 3\cos\theta + \cos^3\theta)$$

$$= \frac{8}{27}\pi - \frac{8}{27}\pi \cos^3\theta,$$

and

$$(3.8) \quad f(R, \pi - \psi(\theta))$$

$$= 2\pi R^2 (1 + \cos\psi(\theta)) - \pi R^3 (2 + 3\cos\psi(\theta) - \cos^3\psi(\theta)),$$

with

$$(3.9) \qquad \cos \psi(\theta) = \sqrt{1 - \left[\frac{2\sin\theta}{3R}\right]^2} \qquad \text{(cf. (3.3))}.$$

By (3.4), (3.7), (3.8) and (3.9), we have

$$(3.10) \qquad g(R,0) = 4\pi R^2 - 4\pi R^3 \begin{cases} > 0, & \text{if } R < 1, \\ = 0, & \text{if } R = 1, \end{cases}$$

and

$$(3.11) \quad g(R,\pi) = 4\pi R^2 - 4\pi R^3 - \frac{16}{27}\pi \begin{cases} = 0, & \text{if } R = 2/3, \\ < 0, & \text{if } 1 \geq R > 2/3. \end{cases}$$

The existence of a root $\theta = \theta_1(R)$, $0 \leq \theta_1(R) < \pi$, for the equation $g(R;\theta) = 0$ readily follows from (3.10) and (3.11).

4. In (4.1), if ϵ is sufficiently small, $-1 < \widehat{\beta}_\epsilon(R) < 1$ for $1 > R > 2/3$ and $-1 < \widehat{\beta}_\epsilon(R) < 0$ for $1 > R > R_0^{(3)}$.

In (2.1), we have to set

$$\widehat{\beta}_\epsilon = \frac{3|B_R| - (1-\epsilon)|\Sigma_2|}{\left|\widehat{\Sigma}\right|}.$$

It follows at once that $\widehat{\beta}_\epsilon(R) < 1$ for ϵ sufficiently small, since $3|B_R| = 4\pi R^3 < 4\pi R^2 = |\Sigma_2| + \left|\widehat{\Sigma}\right|$, for $1 > R \geq 2/3$. On the other hand, using (0.3),

$$(4.1) \qquad \widehat{\beta}_\epsilon(R) = \frac{3|B_R - \Omega_*| - |\Sigma_1| + \epsilon|\Sigma_2|}{\left|\widehat{\Sigma}\right|}.$$

To show that $-1 < \widehat{\beta}_\epsilon(R)$ for sufficiently small ϵ we only need verify

$$(4.2) \qquad |\Sigma_1| - 3|B_R - \Omega_*| < \left|\widehat{\Sigma}\right|.$$

To do so, we, as in §1, denote P as the plane passing through the circle $\partial\Sigma_1$ and denote Ω_{**} as the body enclosed by $P \cap B_R$ and Σ_1 (cf. Figure 2). Then, we have

$$(4.3) \qquad \left|\widehat{\Sigma}\right| > |\text{ the planar disk } P \cap B_R|,$$

and

(4.4) $|\Sigma_1| - 3\,|B_R - \Omega_*| < |\Sigma_1| - 3|\Omega_{**}|.$

However, the inequality

(4.5) $|P \cap B_R| > |\Sigma_1| - 3|\Omega_{**}|$

follows immediately from the fact proved in §1 that Ω_{**} strictly minimizes $\psi^*[\Omega^0]$ (cf. (1.10) and (1.12)) among all the Caccioppoli sets passing through the circle $\partial\Sigma_1$ and situating entirely at one side of the plane P (including P). The inequality (4.2) is thus proved.

We note that, alternatively, (4.5) can be proved by a direct calculation. Namely, using the notations in §3,

$$|P \cap B_R| = \frac{4}{9}\pi \sin^2 \theta_1,$$

and

$$|\Sigma_1| - 3|\Omega_{**}| = f\left(\frac{2}{3}; \theta_1\right) = \frac{8}{27}\pi - \frac{8}{27}\pi \cos^3 \theta_1 \qquad (\text{cf. } (3.7)).$$

Hence

$$
\begin{aligned}
&|P \cap B_R| - (|\Sigma_1| - 3\,|B_R - \Omega_*|) \\
&= \frac{4}{27}\pi(3\sin^2\theta - 2 + 2\cos^3\theta_1) \\
&= \frac{4}{27}\pi[1 - \cos^2\theta(3 - 2\cos\theta)] \\
&\geq 0,
\end{aligned}
$$

for all θ.

We now proceed to prove $\widehat{\beta}_\epsilon(R) < 0$ for $1 > R > R_0^{(3)}$. We have, as $|B_R - \Omega_*| < 2|\Omega_{**}|$, that

$$
\begin{aligned}
&|\Sigma_1| - 3\,|B_R - \Omega_*| \\
&> |\Sigma_1| - 6|\Omega_{**}| \\
&= \frac{8}{27}\pi(-1 + 3\cos\theta_1 - 2\cos^3\theta_1) \\
&= \frac{8}{27}\pi(1 - \cos\theta_1)(2\cos^2\theta_1 + 2\cos\theta_1 - 1) \\
&= \frac{16}{27}\pi(1 - \cos\theta_1)\left[\cos\theta_1 + \frac{\sqrt{3}+1}{2}\right]\left[\cos\theta_1 - \frac{\sqrt{3}-1}{2}\right].
\end{aligned}
$$

In view of (2.1), we therefore only need verify

$$(4.6) \qquad \theta_1(R) < \cos^{-1}\left[\frac{\sqrt{3}-1}{2}\right], \qquad \text{for } 1 > R \geq R_0^{(3)}.$$

To do so, we may observe that, there holds the following

Proposition 7. $\theta_1(R_1) \gneq \theta_1(R_2)$, if $R_1 \lneq R_2$.

Proof. This is an immediate consequence of (0.5.2). *In fact*, if $R_1 \lneq R_2$ and $\theta_1(R_1) \leq B_1(R_2)$, then after a rigid motion, $\Sigma_1(R_1) \subseteq \Sigma_1(R_2)$ and $\Omega_*(R_1) \subseteq \Omega_*(R_2)$ with

$$\psi[\Omega_*(R_1)] = |\Sigma_2(R_1)| - |\Sigma_1(R_1)| - 3|\Omega_*(R_1)| > 0,$$

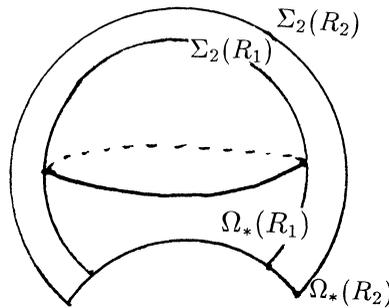in accordance with (0.5.2) and yet contradicting our original definition (0.3)of $\Omega_*(R_1)$. $\qquad\square$

Figure 9.

Thus, to verify (4.6), it suffices to show that

$$(4.7) \qquad \theta_1\left(R_0^{(1)}\right) < \cos^{-1}\left[\frac{\sqrt{3}-1}{2}\right].$$

To do so, we may observe that, as $\Sigma_1\left(R_0^{(3)}\right)$ passes though the center of $B_{R_0^{(3)}}$, we have, using the notation as in §3,

$$(4.8) \qquad \psi\left(\theta_1\left(R_0^{(3)}\right)\right) = \frac{\pi}{2} - \frac{\theta_1\left(R_0^{(3)}\right)}{2}, \qquad \text{(cf. Figure 10)},$$

that is,

$$R_0^{(3)} \cos \frac{\theta_1\left(R_0^{(3)}\right)}{2} = \frac{2}{3} \sin \theta_1\left(R_0^{(3)}\right)$$

$$= \frac{4}{3} \sin \frac{\theta_1 \left(R_0^{(3)} \right)}{2} \cos \frac{\theta_1 \left(R_0^{(3)} \right)}{2}$$

and hence

(4.9) $$\sin \frac{\theta_1 \left(R_0^{(3)} \right)}{2} = \frac{3}{4} R_0^{(3)},$$

which yields

(4.10) $$\cos \theta_1 \left(R_0^{(3)} \right) = 1 - 2 \sin^2 \frac{\theta_1 \left(R_0^{(3)} \right)}{2} = 1 - \frac{9}{8} \left(R_0^{(3)} \right)^2.$$

Substituting (4.8) into (3.2) or (3.4), we shall obtain $R_0^{(3)}$ as the root of the equation

$$f \left(R_0^{(3)}; \frac{\pi}{2} + \frac{\theta_1 \left(R_0^{(3)} \right)}{2} \right) - f \left(\frac{2}{3}; \theta_1 \left(R_0^{(3)} \right) \right) = 0,$$

or, using (3.1), (3.7) and (3.8),

$$2 \left(R_0^{(3)} \right)^2 \left(1 + \sin \frac{\theta_1 \left(R_0^{(3)} \right)}{2} \right)$$

$$- \left(R_0^{(3)} \right)^3 \left(2 + 3 \sin \frac{\theta_1 \left(R_0^{(3)} \right)}{2} - \sin^3 \frac{\theta_1 \left(R_0^{(3)} \right)}{2} \right)$$

$$= \frac{8}{9} \left(1 - \cos \theta_1 \left(R_0^{(3)} \right) \right) - \frac{8}{27} \left(2 - 3 \cos \theta_1 \left(R_0^{(3)} \right) + \cos^3 \theta_1 \left(R_0^{(3)} \right) \right),$$

or, using (4.9) and (4.10)

$$2 \left(R_0^{(3)} \right)^2 \left(1 + \frac{3}{4} R_0^{(3)} \right) - \left(R_0^{(3)} \right)^2 \left(2 + \frac{9}{4} R_0^{(3)} - \frac{27}{64} R_0^{(3)} \right)^3$$

$$= \left(R_0^{(3)} \right)^2 - \frac{9}{8} \left(R_0^{(3)} \right)^4 + \frac{27}{64} \left(R_0^{(3)} \right)^6,$$

that is,

$$\frac{1}{8} \left[9 \left(R_0^{(3)} \right)^4 + 4 \left(R_0^{(3)} \right)^3 - 8 \left(R_0^{(3)} \right)^2 \right] = 0,$$

or,

$$9 \left(R_0^{(3)} \right)^2 + 4 \left(R_0^{(3)} \right)^2 - 8 = 0.$$

Hence,

$$(4.11) \qquad R_0^{(3)} = \frac{-2 + \sqrt{76}}{9} = \frac{-2 + 2\sqrt{19}}{9} = 0.746421987\ldots$$
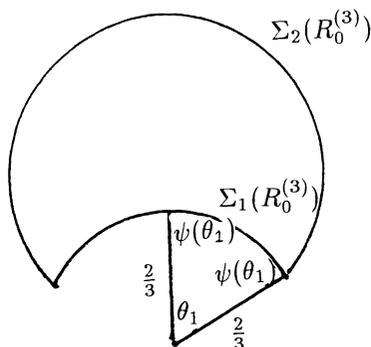

Figure 10.

Thus, using (4.10) and (4.11)

$$\theta_1 \left(R_0^{(3)} \right) = \cos^{-1} \left[1 - \frac{9}{8} \left(R_0^{(3)} \right)^2 \right] = \cos^{-1} \left[\frac{-1 + \sqrt{19}}{9} \right].$$

As $\left[\frac{-1+\sqrt{19}}{9} \right] > \frac{\sqrt{3}-1}{2}$, (4.7) (and hence (4.6)) follows.

5. On Still Higher Dimensional Cases.

5.1. We first verify the *existence of the n-dimensional moon domain $\Omega_*(R)$*, $1 > R > \dfrac{n-1}{n}$, characterized by the equation

$$(5.1) \qquad\qquad |\Sigma_2(R)| - |\Sigma_1(R)| = n|\Omega_*(R)|$$

where $\partial\Omega_* = \Sigma_2 \cup \Sigma_1$, Σ_2 and Σ_1 being spherical caps in $\mathbb{R}^n$ of the respective radii $R_1 = \dfrac{n-1}{n}$ and R. As in §2, we set

$$(5.2) \qquad\qquad f(r;\theta) = \sigma_\theta(r) - nv_\theta(r),$$

where the definition for $\sigma_\theta(r)$ and $v_\theta(r)$ in the beginning of §3 extends to the present setting in an obvious way. If $\Omega_*(R)$ exists, the equation of the $(n-1)$-dimensional sphere $\Gamma \equiv \partial\Sigma_1(R)$ is $\rho = \dfrac{n-1}{n}\sin\theta_1$ where θ_1 is the root of the equation

$$f(R;\pi - \psi(\theta)) - f\left(\frac{n-1}{n};\theta \right) = 0,$$

with

$$\psi(\theta) = \sin^{-1} \frac{(n-1)\sin\theta}{nR}.$$

Denoting ω_N as the volume of the N-dimensional sphere and setting, again,

(5.3) $$g(R;\theta) = f(R; \pi - \psi(\theta)) - f\left(\frac{n-1}{n}; \theta\right),$$

we have

(5.4) $$\begin{aligned} g(R;0) &= f(R;\pi) \\ &= 2(n-1)\omega_{n-1}R^{n-1} - 2(n-1)\omega_{n-1}R^n \\ &> 0, \qquad \text{if } R < 1, \end{aligned}$$

and

(5.5)

$$g(R;\pi) = f(R;\pi) - f\left(\frac{n-1}{n}; \pi\right)$$

$$= 2(n-1)\omega_{n-1}\left\{ (R^{n-1} - R^n) - \left[\left(\frac{n-1}{n}\right)^{n-1} - \left(\frac{n-1}{n}\right)^n\right]\right\}$$

$$\begin{cases} = 0, & \text{if } R = \dfrac{n-1}{n} \\ < 0, & \text{if } 1 > R > \dfrac{n-1}{n}, \end{cases}$$

since $r = \dfrac{n-1}{n}$ is the zero of the derivative of the concave function $h(r) = r^{n-1} - r^n$. From (5.4) and (5.5) follows the existence of a root $\theta = \theta_1(R)$, $0 \leq \theta_1(R) \leq \pi$, for the equation $g(R;\pi) = 0$, of which the existence of $\Omega_*(R)$ is an immediate consequence.

5.1.1. Using the above notation, we may here describe a procedure for determining the value $R_0^{(n)}$, $n > 3$. Indeed, since $R_0^{(n)}$ is the unique value of R for which $\Sigma\left(R_0^{(n)}\right)$ passes through the center of $B_{R_0^{(n)}}$, we may, as in §4, obtain $R_0^{(n)}$ as the root of the equation

$$f\left(R; \frac{\pi}{2} + \frac{\theta_1}{2}\right) - f\left(\frac{2}{3}; \theta_1\right) = 0,$$

with

$$\sin\frac{\theta_1}{2} = \frac{n}{2(n-1)}R;$$

and

$$\cos\theta_1 = 1 - 2\sin^2\frac{\theta_1}{2} = 1 - \frac{n^2}{2(n-1)^2}R^2;$$

here $f(r;\theta)$ is defined by (5.2).

The number $R_0^{(n)}$, as mentioned of in the end of 0.2 is of significance once we place it into perspective in the context of the results in [**L2**].

5.2. Having verified the eistence of $\Omega_*(R)$ for $n > 3$, we proceed to prove *the existence of comparison hypersurfaces in* $\Omega_*(R)$, $1 > R > \dfrac{n-1}{n}$, which is the solution to the problem (0.4) and (1.5), setting $H = 1$ and $\beta = -1, +1$ on Σ_1, Σ_2, respectively. We again, using Proposition 2, reduce this to the proof of the ineq.

(5.6) $\psi^*[\Omega^0] \equiv |\partial\Omega^0 \cap \Omega| + |\partial\Omega^0 \cap \Sigma_2| - n|\Omega^0 \cup \Omega_{**}|$

$$> \psi^*[\Omega_{**}] \equiv |\Sigma_1| - n|\Omega_{**}|$$

for all Ω^0 passing through $\Gamma \equiv \partial\Sigma_1$, with the whole Σ_1 as a part of its boundary and situating entirely in one of the two half spaces provided by the hyperplane passing through Γ; here Ω_{**} is the region bounded by Σ_1 and this hyperplane. Repeating the variational procedure indicated in §1, we again justify the existence of a minimizing body for $\psi^*[\Omega^0]$. Set $\tilde{\Sigma} = \partial\tilde{\Omega}\cap\Omega_*$. We readily see that Proposition 3 holds here; that is, nonempty $\overline{\Sigma}$ must be a spherical cap passing through Γ, which as Lemma 1 can also be extended, must be a spherical cap of radius $\dfrac{n-1}{n}$ strictly including a hemisphere and can possibly occur only when Σ_1 is included in a hemisphere. However, if Σ_1 is included in a hemisphere, denoting Ω_0 as the body enclosed by Σ_1 and that spherical cap of radius $\dfrac{n-1}{n}$ included in Ω_*, we have, adopting the notation in §5.1,

$\psi^*[\Omega_0 \cup \Omega_{**}] - \psi^*[\Omega_{**}]$

$= f(2/3; \pi - \theta_1(2/3)) - f(2/3; \theta_1(2/3))$

$= (\sigma_{\pi-\theta_1(2/3)}(2/3) - \sigma_{\theta_1(3/2)}(2/3)) - (v_{\pi-\theta_1(2/3)}(2/3) - v_{\theta_1(2/3)}(2/3))$

$= (n-1)\omega_{n-1}(2/3)^{n-1}\displaystyle\int_{\theta_1}^{\pi-\theta_1}\sin\theta\,d\theta - \omega_{n-1}\int_{2/3\cos(\pi-\theta_1)}^{2/3\cos\theta_1}\left(\frac{4}{9} - Z_1^2\right)^{\frac{n-1}{2}}dz_1$

$= (n-1)\omega_{n-1}\left(\dfrac{2}{3}\right)^{n-1}\displaystyle\int_{\theta_1}^{\pi-\theta_1}\sin\theta\,d\theta - \omega_{n-1}\left(\dfrac{2}{3}\right)^n\int_{\theta_1}^{\pi-\theta_1}\sin^n\theta\,d\theta$

$> 0,$ obviously.

We therefore prove (5.6) and the existence of the comparison hypersurfaces for $n > 3$.

5.3. A careful examination of the work of §5 tells us that; in order to *construct such a sequence of solutions to* (0.4) *in* $B_R \subseteq \mathbb{R}^n$, $n > 3$, *whose limit, in the sense indicated in 2.2 is our moon hypersurface in* $\Omega_*(R) \subseteq \mathbb{R}^n$, we only need to (1) *verify that* $-1 < \widehat{\beta}_\epsilon(R) < 1$, *for* $1 > R > \dfrac{n-1}{n}$ *and* ϵ *sufficiently small, where*

$$\widehat{\beta}_\epsilon(R) = \frac{n|B_R| - (1-\epsilon)|\Sigma_2(R)|}{\left|\widehat{\Sigma}\right|}$$

$\widehat{\Sigma} = \partial B_R - \Sigma_2$, the full sphere ∂B_R being obtained by extending $\Sigma_2(R)$.

(2) Prove the statement of Proposition 3** and Proposition 4 *in spite of the difficulty arised by the possible existence of singular subsets of* $\widetilde{\Sigma} \cap B_R$ *in the case of* $n > 7$. The fact that $\widehat{\beta}_\epsilon < 1$ readily follows from the inequality

$$n|B_R| = n\omega_n R^n < n\omega_n R^{n-1} = |\partial B_R| = |\Sigma_2| + \left|\widehat{\Sigma}\right|.$$

The fact $\widehat{\beta}_\epsilon > -1$, by (5.1), amounts to the fact that

$$\left|\widehat{\Sigma}\right| > |\Sigma_1| - n|B_r - \Omega_*|$$

which is a consequence of the inequality

$$\left|D_{\rho\left(\theta_1, \frac{n-1}{n}\right)}\right| > |\Sigma_1| - n|\Omega_{**}|,$$

(see the beginning of §3 for notation) obtained immediately from the fact that Ω_{**} minimizing $\psi^*[\Omega]$ (cf. (5.6)) among all sets indicated below (5.6).

As of (2), we may, first of all, put Proposition 4, 3* and 3** in a precise form in the higher dimensional setting. In fact, to extend the existence results in §2 to the case where $n > 3$, it suffices to verify that (0.6.2) holds for every Caccioppoli set $\Omega^0 \subseteq B_R$, $\Omega^0 \neq \phi, B_R$, $1 > R > \dfrac{n-1}{n}$. To do so, as in §2, we may observe that it suffices to consider those Caccioppoli sets with $\left|\partial\Omega^0 \cap \widehat{\Sigma}\right| > 0$ and $\partial\Omega^0 \cap (B_R \cup \Sigma_2(R))$ being connected. Thus, as in §2, we may try to minimize $\psi[\Omega^0]$ in (0.6.2) among all the Caccioppoli sets $\Omega \subseteq B_R$ and the same reasoning concludes that a subsequence of minimizing sequence $\{\Omega_j^0\}$ for $\psi\{\Omega^0\}$, $\Omega_j^0 \subseteq B_R$, exists such that $\left\{\varphi_{\Omega_j^0}\right\}$ converges in $L^1(\Omega)$ to $\varphi_{\widetilde{\Omega}}$ such that

$$\psi\left[\widetilde{\Omega}\right] \leq \inf \psi[\Omega_j^0].$$

Set $\widetilde{\Sigma} = \partial\widetilde{\Omega} \cap B_R$. If $\overline{\Sigma} \neq \phi$, we have observed that $\left[\widetilde{\Sigma} \cap \widehat{\Sigma}\right] > 0$, and we may assume $\partial\widetilde{\Omega}^0 \cap (B_R \cup \Sigma_2(R))$ to be connected.

For our present purpose, we only have to show, as in §2.1.

Proposition $\bar{4}$. *The only non-empty candidate for $\widetilde{\Sigma}$ is the spherical cap Σ_1. In other words, the only non-empty candidate for $\widetilde{\Omega}$ is $B_R - \Omega_*$.*

We again let P to be the unique plane passing through the $(n-2)$-dimensional sphere $\Gamma \equiv \partial\Sigma_1 \cap \partial B_R$ and designate P as the $x_1, x_2, \ldots, x_{n-1}$ plane so that the center of B_R has the x_n-coordinate $x_n > 0$.

To prove Proposition 4, we shall also proceed to verify

Proposition $\widetilde{3}^*$. *If $\widetilde{\Sigma} \neq \phi$, then $\widetilde{\Sigma}$ is of rotational symmetry about the x_n-axis.*

In §2.1, Proposition 3^* is proved with the aid of a theorem of Massari [**Ma**], which, as mentioned above, does not exclude the possibility of existence of singular points of a minimizing body in the case that $n > 7$; however, it gives an estimate for the dimension of singular parts, which has been improved by Federer. Their results yields

Theorem Of Massari And Federer. *If $\overline{\Sigma} \neq \phi$, then the reduced boundary $\partial^*\overline{\Omega}$ of $\overline{\Omega}$ is an analytic manifold of dimension $n-1$ and*

$$H_s\left[\left(\overline{\Sigma} \setminus \partial^*\overline{\Omega}\right) \cap B_R\right] = 0, \quad \forall\, s > n - 7,\; s \in \mathbb{R},$$

where H_s denotes the Hausdorff s-measure.

To prove Proposition $\widetilde{3}^*$, as in §2.1, we consider a vertical plane $\widehat{P}$: $a_1 x^1 + a_2 x^2 + \cdots + a_{n-1} x^{n-1} = 0$, $a_1, \ldots, a_{n-1}$: constants, which divides $\overline{\Omega}$ into two non-empty parts Ω_1 and Ω_2. We may assume, without loss of generality, that $\psi[\Omega_1] \leq \psi[\Omega_2]$. Reflecting the body Ω_1 in the opposite side of the plane $\widehat{P}$, then

$$\Omega_1 \cup \Omega_1' \subseteq B_R,$$

and, as in §2.1, we have $\psi[\Omega_1 \cup \Omega_1'] \leq \psi\left[\widetilde{\Omega}\right]$ and hence $\psi[\Omega_1 \cup \Omega_1'] = \psi\left[\widetilde{\Omega}\right]$, in view of the minimizing property of $\widetilde{\Omega}$. Thus, we have

Proposition $\widetilde{3}^{**}$. *If $\widetilde{\Sigma} \neq \phi$, then at each <u>regular</u> point of $\widetilde{\Sigma}$, the normal of the horizontal cross-section of $\widetilde{\Sigma}$ through this point is orthogonal to the normal of the unique vertical plane $a_1 x^1 + a_2 x^2 + \cdots + a_{n-1} x^{n-1} = 0$, $a_1, a_2, \ldots, a_{n-1}$: constants, passing though this point (and the origin).*

At height x_0^n, if the horizontal cross-section includes regular points of $\widetilde{\Sigma}$, we may choose a regular point $(x_0^1, \ldots, x_0^n)$ of $\widetilde{\Sigma}$, then, for each connected curve $(x^1(t), \ldots, x^{n-1}(t), x_0^n)$ through this point and included in a regular

part of the horizontal cross-section of $\widetilde{\Sigma}$ at the height x_0^n, Proposition 3^{**} yields that

$$\dot{x}^1 x^1 + \dot{x}^2 x^2 + \cdots + \dot{x}^{n-1} x^{n-1} = 0,$$

which holds if and only if

$$(x^1)^2 + (x^2)^2 + \cdots + (x^{n-1})^2 = \text{ constant},$$

i.e. $(x^1(t), \ldots, x^{n-1}(t), x_0^n)$ lies on a sphere with its center at $(0, \ldots, 0, x_0^n)$. Thus, each regular point of this horizontal cross-section of $\widetilde{\Sigma}$ must be included in a region on an $(n-2)$-dimensional sphere with its center at $(0, 0, \ldots, x_0^n)$ and, furthermore, denoting C as the component of this horizontal cross-section including this spherical cap, we note that C must be a whole *closed* sphere; indeed, were C bounding a region in the hyperplane $x^n = x_0^n$ and C includes only a *portion* of and *not* the whole sphere, then C would have to include at least two disjoint spherical regions and *the dimension of singular parts of this cross-section would be $n-2$*, contradicting above-mentioned regularity result of Massari and Federer; however, were C bounding no region then a portion of $\widetilde{\Sigma}$ with positive $(n-1)$-dimensional Hausdorff measure would *not* be a portion of the boundary of any component of $\widetilde{\Omega}$ (with positive n-dimensional Hausdorff measure) and removing this portion of $\widetilde{\Sigma}$ would result in a smaller value of ψ, contradicting the minimality of $\widetilde{\Omega}$ and $\widetilde{\Sigma}$. Thus, the proof of Proposition 3^* is complete. The argument following the proof of Proposition 3^{**} in §2.1 again applies in our present setting and enables us to prove Proposition 1, from which, as indicated above, follows (0.6.2) and the existence of that sequence of solutions to (0.4) in $B_R \subseteq \mathbb{R}^n$, described in §0.0.2 and beginning of this section.

6. Some existence Results of Capillary Hypersurfaces without Gravity and of Rotational Symmetry.

As in Finn [**F1**] and quoted in Proposition §1 of this paper, we may reduce the capillary problem in the absence of grativity to the variational problem

$$(6.1) \qquad \xi[u] = \int_\Omega \sqrt{1 + |\nabla u|^2} + nH \int_\Omega u \, dx - \int_{\partial\Omega} \beta(s) u \, ds,$$

with $\beta(s)$, $-1 \leq \beta(s) \leq 1$, being piecewise Lipschitz on the boundary of a piecewise Lipschitz domain $\Omega \subseteq \mathbb{R}^n$, and H being a constant. As quoted in Proposition 1, a necessary and sufficient condition for the existence of a minimizing function $u(x) \in BV_{\text{loc}}(\Omega)$ for the functional (6.1) is that both the conditions (1.3) and (1.4) hold for every Caccioppoli set $\Omega^0 \neq \phi$, $\Omega(\Omega^0 \subseteq \Omega)$. Furthermore, since H is constant, the conditions (1.3) and (1.4) are equivalent. Thus, in §1, §2, §5.1 and §5.3 of this work, we have restricted

our attention to verifying (1.4); the argument used in §2 and §5.3 yields the existence of a minimizing body $\widetilde{\Omega}$ for $\psi[\Omega^0]$ and setting $\widetilde{\Sigma} = \partial\widetilde{\Omega} \cap \Omega$, the argument used to verify Proposition 3* yields.

Proposition 3***. *Suppose Ω and $\beta(s)$ are rotational symmetry of the same axis. If $\widetilde{\Sigma} \neq \phi$ then $\widetilde{\Sigma}$ is of rotational symmetry about this axis.*

We may, without loss of generality assume that this axis of symmetry is the x_n-axis. Suppose, in addition, that $\beta(s)$ is piecewise constant; i.e., there exist relatively open subsets Σ^i of $\partial\Omega$, such that, if $i < j$, Σ^i is "below" Σ^j (in the sense that, for two arbitrarily chosen points $x^i \in \Sigma^i$ and $x^j \in \Sigma^j$, then x_n component of x^i is less than that of x^j), and,

$$(6.2) \qquad \beta(s)|_{\Sigma^i} = \text{ constant } c_i, \quad \cup\Sigma^i = \partial\Omega.$$

Then, the argument used in §2.1 to exclude those $\widetilde{\Sigma}$ situating below Σ_1 can be applied to yield

Corollary 1. *Suppose, in addition to the hypothesis of Proposition 3***, $\beta(s)$ is piecewise constant, as indicated in (6.2). Then, if $\widetilde{\Omega} \neq \phi$ or Ω, there occurs at least one of the following possibilities:*

Possibility 1. $\partial\widetilde{\Sigma} \cap \Sigma^1 = \phi$ or Σ^1.

Possibility 2. There exists at least one i, $i \geq 1$, such that

$$\partial\Sigma^i \setminus \partial\Sigma^{i+1} \subseteq \partial\widetilde{\Omega}.$$

Possibility 3. In (6.2), $\overset{k}{\underset{i=1}{\cup}} \Sigma^i = \partial\Omega$ for some integer $k < \infty$ and

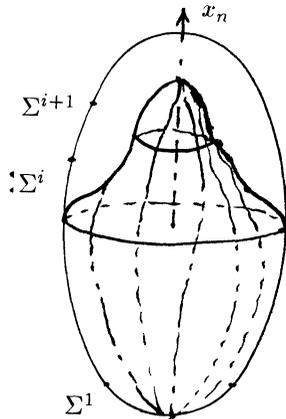$$\partial\widetilde{\Omega} \cap \Sigma^k = \phi \text{ or } \Sigma^k.$$

Figure 11.

Indeed, Σ^i being open, were Corollary 1 false, a rigid motion of $\widetilde{\Omega}$ would result in a body meeting Σ^j, for each j, with the same area as $\widetilde{\Omega}$ and is therefore another minimizing body for the functional ψ, which, however, would *not* be of rotational symmetry of the axis indicated in Proposition 3***.

References

[F1] R. Finn, *Equilibrium Capillary Surfaces*, Springer-verlag, New York-Heidelberg, 1986.

[F2] _____, *Moon surfaces, and boundary behavior of capillary surfaces for perfect wetting and non-wetting*, Proc. London Math. Soc. (3), **57** (1988), 542-576.

[FG] R. Finn and E. Giusti, *On nonparametric surfaces of constant mean curvature*, Ann. Scuola Norm. Sup. Pisa, **4** (1977), 13-31.

[G1] E. Giusti, *On the equation of surfaces of prescribed mean curvature: eistence and uniqueness without boundary conditions*, Invent. Math., **46** (1978), 111-137.

[G2] _____, *Minimal Surfaces and Functions of Bounded Variations*, Birkhauser, Boston, 1984.

[L1] F. Liang, *An absolute gradient bound for nonparametric surfaces of constant mean curvature*, Indiana Univ. Math. J., **41**, no, **3** (1992), 569-604.

[Ma] U. Massari, *Esistenza e regolarita delle ipersuperifici di curvatura media assegnata in $\mathbb{R}^n$*, Arch Rational Mech. Anal., **55** (1974), 357-382.

[MM] U. Massari and M. Miranda, *Minimal Surfaces of codimension One*, North-Holland, Amsterdam-New York-Oxford, 1984.

[M] M. Miranda, *Superfic minime illimitate*, Ann. Scuola Norm. Sup. Pisa (4), **4** (1977), 312-322.

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