

ON THE FAILURE CYCLES FOR THE QUADRATIC NORMALITY OF A PROJECTIVE VARIETY

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Let X be a smooth projective surface and L a very ample line bundle on X which is not quadratically normal; set $r + 1 = h^0(X, L)$. Here we give numerical conditions on X and L which imply the existence of a finite subscheme T of X with $\text{length}(T) \geq 2s + 2$ and contained in a dimension $s \leq r - 2$ linear subspace of $P(H^0(X, L))$ and such that $L|_T$ is not quadratically normal.

Introduction.

It is very classical the following problem (with several variations). Suppose that a curve $C \subset \mathbf{P}^r$ has some bad property, e.g. it is not projectively normal. Show the existence of a finite subscheme S of C contained in a smaller linear subspace such that S explains the failure of C to be projectively normal. In modern times there is the important paper [4]. Here we consider the corresponding problem when the scheme C has $\dim(C) > 1$. We were also motivated from the notion of k -ampleness and k -very ampleness introduced in [2]. By definition these conditions fail for a scheme C if and only if there is a zero dimensional subscheme S of C with a bad property. We were interested (see e.g. [1]) in showing that under suitable conditions there are many such subschemes. A natural question was if there is some bad positive dimensional proper subscheme Y containing all of them for a natural reason (for example if it were the union of them) or if there was some bad "free" zero dimensional subscheme. Here we consider the condition of quadratic normality and give a positive answer if $\dim(C) = 2$ under suitable numerical conditions. These numerical conditions are strange, far from optimal and just come from the proof. We will state them below as Theorem 0.2. But first and most important: the proofs are essentially technical variations on an alternative proof ([5, §2.5]) of a theorem in [4]; hence the idea originates ultimately with Robert Lazarsfeld. After the present results were proven, we checked the references and found that exactly that subsection was deleted in the printed version [6] of [5]. After a while we decided to rewrite a little bit the paper, but to write it anyway.

We fix an integral variety X and a very ample line bundle L on X ; set $r + 1 := h^0(X, L)$ and $\mathbf{O} := \mathbf{O}_X$; let $\phi_L : X \rightarrow \mathbf{P}^r$ be the embedding associated to $H^0(X, L)$ into a projective space. Recall that a subvariety U of $\mathbf{P}(V)$ is called quadratically normal if the restriction map $V \otimes V \rightarrow H_0(U, \mathbf{O}_U(2))$ is surjective. The pair (X, L) (or just L) is called quadratically normal if $\phi_L(X)$ is quadratically normal.

Definition 0.1. If L is not quadratically normal, we will call *amount of failure of quadratic normality* the integer $\dim(\text{coker}(H^0(L) \otimes H^0(L) \rightarrow H^0(L^2)))$.

Let $G = G(r + 1 - \dim(X), r + 1)$ be the Grassmannian of codimension $\dim(X)$ linear subspaces of \mathbf{P}^r ; set

$$B := \{U \in G : X \cap U \text{ is not zero dimensional}\}.$$

Here is the main result proven in this paper.

Theorem 0.2. *Assume $\dim(X) = 2$ and that L is not quadratically normal. Let $\mathbf{f} > 0$ be the amount of failure of the quadratic normality of X . If $h^1(\mathbf{O}_X) < \mathbf{f} + \text{codim}(B) - 1$, then there is a codimension 2 linear subspace $[U] \in G \setminus B$ such that the scheme $X \cap U$ is 0-dimensional and is not quadratically normal with respect to $L|_{(X \cap U)}$. Furthermore, there is an integer $s \leq r - 2$, a linear subspace V of U with $\dim(V) = s$ and a subscheme T of $U \cap X$ contained in V with $\text{length}(T) = 2s + 2$ such that T is not quadratically normal with respect to $L|_T$.*

In particular Theorem 0.2 applies to all linearly normal but not quadratically normal embedded surfaces with $h^1(\mathbf{O}_X) = 0$.

For other related results proven within the same framework, see 2.2 and 2.3. In §1 (after fixing the notations) we will give the framework and the main ingredients for the proofs of all the results of this paper. In §2 we will prove Theorem 0.2.

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1. Preliminaries and general set up.

We work over an algebraically closed base field. We fix an integral variety X and a very ample line bundle L on X ; set $r + 1 := h^0(X, L)$; let $\phi_L : X \rightarrow \mathbf{P}^r$ the embedding associated to $H^0(X, L)$. If A is a sheaf on X , we will often

write $H^i(A)$ or $h^i(A)$ for $H^i(X, A)$ or $h^i(A)$. Set $Y := \phi_L(X)$. Let Ω be the cotangent sheaf of \mathbf{P}^r . Set $M_L := \phi_L^*(\Omega(1))$. By the dual of the Euler sequence of $T\mathbf{P}^r$ and the completeness of the embedding of X we obtain the following exact sequence on X :

$$(1) \quad 0 \rightarrow M_L \rightarrow H^0(X, L) \otimes \mathbf{O}_X \rightarrow L \rightarrow 0$$

which contains a lot of informations on the cohomology of I_Y .

Now we generalize the Remark in [6] given at page 510 (between the statement of [6], Prop. 1.3.3, and its proof).

Lemma 1.1. *With the notations X, L, ϕ_L, M_L , and so on introduced at the beginning, we have:*

- (i) *Fix an integer $k > 0$ and assume $H^1(L^k) = 0$; the multiplication map $H^0(L) \otimes H^0(L^k) \rightarrow H^0(L^{k+1})$ is surjective if and only if $H^1(M_L \otimes L^k) = 0$. In particular if $h^1(L^s) = 0$ for every $s > 0$, then L is normally generated if and only if $H^1(M_L \otimes L^t) = 0$ for every $t > 0$.*
- (ii) *The amount of failure for the quadratic normality of L is*

$$\dim(\ker(H^1(M_L \otimes L) \rightarrow H^0(L) \otimes H^1(L))).$$

- (iii) *If $H^1(L^2) = 0$ the amount of failure of quadratic normality is*

$$h^1(M_L \otimes L) - h^0(L) \cdot h^1(L).$$

Proof. Just use a twist of the exact sequence (1).

Let $G := G(r - x + 1, r + 1)$ be the Grassmannian of codimension x linear subspaces of $\mathbf{P}(H^0(X, L))$ and $F := \{(y, U) \in X \times G : y \in U\} \subset X \times G$ be the incidence variety. On G we have the exact sequence

$$(2) \quad 0 \rightarrow S \rightarrow H^0(X, L) \otimes \mathbf{O}_G \rightarrow Q \rightarrow 0$$

with Q tautological quotient bundle and S tautological rank x subbundle. Let $f : X \times G \rightarrow G$ and $p : X \times G \rightarrow X$ be the projections. The incidence variety F is defined by the vanishing of the induced morphism $s : f^*S \rightarrow p^*L$ i.e., its ideal sheaf \mathbf{I} in $X \times G$ is the image of the associated map $f^*S \otimes p^*L^* \rightarrow \mathbf{O}_{X \times G}$. Note that this ideal sheaf \mathbf{I} has a resolution:

$$(3) \quad \begin{aligned} \dots f^* \Lambda^{t+1} S \otimes p^* L^{*t} \rightarrow f^* \Lambda^t S \otimes p^* L^{*(t-1)} \rightarrow f^* \Lambda^{(t-1)} S \otimes p^* L^{*(t-2)} \dots \\ \dots \rightarrow f^* \Lambda^2 S \otimes p^* L^* \rightarrow f^* S \rightarrow \mathbf{I} \otimes p^* L \rightarrow 0. \end{aligned}$$

On $X \times G$ there is an important commutative diagram. First, we will write it as formula (4) in the particular case $x = \text{rank}(S) = 2$ needed in the proof

of 0.2.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & f^*\Lambda^2 S \otimes p^*L^* & \longrightarrow & p^*M_L & & \\
 & & \downarrow & & \downarrow & & \\
 (4) \quad 0 & \longrightarrow & f^*S & \longrightarrow & H^0(L) \otimes \mathbf{O}_{X \times G} & \longrightarrow & f^*Q \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & p^*L & \longrightarrow & p^*L \otimes \mathbf{O}_F \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

□

In the general case this commutative diagram has 3 columns. The first column of this diagram is the resolution (3) of \mathbf{I} (without \mathbf{I}). The second column of the diagram is the pull-back p^* of the exact sequence (1) and the third column is just the tautological surjection $f^*Q \rightarrow p^*L \otimes \mathbf{O}_F$. These columns are connected so that the only long row in the diagram is the pull-back by f^* of the exact sequence (1); just above this exact sequence there is a map $f^*\Lambda^2 S \otimes p^*L^* \rightarrow p^*M_L$ and just below the exact sequence there is the surjection $p^*L \rightarrow p^*L \otimes \mathbf{O}_F$ coming from the surjection $\mathbf{O}_{X \times G} \rightarrow \mathbf{O}_F$. Follow the first column of the diagram till the term $f^*\Lambda^2 S \otimes p^*L^*$; then go on the right one step and find f^*Q ; then go down one step and find $f^*\Lambda^2 S \otimes p^*L^*$. In this way from this diagram we obtain an exact sequence obtained from the exact sequence (3) substituting the last part $f^*S \rightarrow \mathbf{I} \otimes p^*L \rightarrow 0$ with

$$p^*M_L \rightarrow f^*Q \rightarrow p^*L \otimes \mathbf{O}_F \rightarrow 0.$$

Call (§§)(k) the exact sequence obtained twisting by p^*L_k the sequence just described. If $x = 2$ the complex (§§)(1) is the following exact sequence:

$$(5) \quad 0 \rightarrow f^*\Lambda^2 S \rightarrow p^*(M_L \otimes L) \rightarrow f^*Q \otimes p^*L \rightarrow p^*L^2 \otimes \mathbf{O}_F \rightarrow 0.$$

Now we push-forward the complex (§§)(1) to the Grassmannian; since (§§)(1) is exact, its higher pushforwards vanish and we obtain a spectral sequence (call it (#)) converging to zero.

$$(\#) \quad E_1^{qt} = R^t f_*(C^q) \Rightarrow 0$$

where

$$0 \rightarrow C^{2-x} \rightarrow \dots \rightarrow C^0 \rightarrow C^1 \rightarrow C^2 \rightarrow C^3 \rightarrow 0$$

is the complex

$$0 \rightarrow f^* \Lambda^x S \otimes p^* L^{*(x-2)} \rightarrow \dots \rightarrow f^* \Lambda^2 S \rightarrow p^*(M_L \otimes L) \rightarrow f^* Q \otimes p^* L \rightarrow p^* L^2 \otimes \mathbf{O}_F \rightarrow 0.$$

In Section 2 we will write the E_1 -part of (#) as formulas (7), (8) and (9) in the case $\dim(X) = x = \text{rank}(S) = 2$ we need for the proof of 0.2. Use the projection formula $R^i f_*(f^* A' \otimes p^* A) = H^i(X, A) \otimes A'$ for all locally free sheaves A on X . We normalize the indices of the complex (§§)(1) in such a way that the term E_1^{00} of the spectral sequence (#) is $H^0(X, \mathbf{O}_X) \otimes \Lambda^2 S$. With this normalization the term E_1^{qt} of (#) is 0 if either $t < 0$ or $q < 2 - x$, it is $H^t(X, \mathbf{O}_X) \otimes \Lambda^{-q+2} S$ for $2 - x \leq q \leq 0$, $H^t(X, M_L \otimes L) \otimes \mathbf{O}_G$ for $q = 1$, $H^t(X, L) \otimes Q$ for $q = 2$ and $R^t f_*(p^* L^2 \otimes \mathbf{O}_F)$ for $q = 3$.

Remark 1.2. Note that over $G \setminus B$ we have $R^i f_*(p^* L^j \otimes \mathbf{O}_F) = 0$ for every j and every $i > x - 2$ because the fibers of f over $G \setminus B$ have dimension $\leq x - 2$. Fix a point $[U] \in G \setminus B$ corresponding to a codimension x linear subspace U of \mathbf{P}^r . Then for every integer k the fiber of the sheaf $f_*(p^* L^{k+1} \otimes \mathbf{O}_F)$ at $[U]$ is canonically isomorphic to the vector space $H^0(U, \mathbf{O}_{U \cap X}(k + 1))$ and fiber over $[U]$ of the homomorphism $u := d_1^{20} : H^0(X, L) \otimes Q \rightarrow f_*(p^* L^2 \otimes \mathbf{O}_F)$ in the E_1 -part of the spectral sequence (#) is identified at $[U]$ with the natural multiplication map

$$(6) \quad H^0(X, L) \otimes H^0(U, \mathbf{O}_U(1)) \rightarrow H^0(U \mathbf{O}_{X \cap U}(2)).$$

2. Proof of Theorem 0.2.

Now we specialize the situation of §1 to the situation of Theorem 0.2, whose proof will be given now.

Proof of Theorem 0.2. First, note that the ‘‘Furthermore part’’ of the statement of 0.2 follows from the first part and [6, Lemma 2.4.4].

Now we will prove the first part of 0.2. We write as formulas (7), (8) and (9) the 3 non trivial lines of the E_1 -term of the spectral sequence (#) under the assumptions of 0.2; in particular we have $x = 2, \text{rank}(S) = 2, \Lambda^2 S \cong \mathbf{O}_G(-1), \dim(X) = 2$.

$$(7) \quad H^2(\mathbf{O}_X) \otimes \mathbf{O}_G(-1) \rightarrow H^2(M_L \otimes L) \otimes \mathbf{O}_G \rightarrow H^2(L) \otimes Q$$

$$(8) \quad H^1(\mathbf{O}_X) \otimes \mathbf{O}_G(-1) \xrightarrow{\alpha} H^1(M_L \otimes L) \otimes \mathbf{O}_G \xrightarrow{\beta} H^1(L) \otimes Q \rightarrow R^1 f_*(p^* L^2 \otimes \mathbf{O}_F)$$

$$(9) \quad H^0(\mathbf{O}_X) \otimes \mathbf{O}_G(-1) \rightarrow H^0(M_L \otimes L) \otimes \mathbf{O}_G \rightarrow H^0(L) \otimes Q \xrightarrow{\gamma} R^0 f_*(p^* L^2 \otimes \mathbf{O}_F).$$

Let $\alpha := d_1^{01}, \beta := d_1^{11}, u := d_1^{20}$ be the maps indicated above. By Remark 1.2 to prove 0.2 it is sufficient to prove that the map u is not surjective on $G \setminus B$. We use that the spectral sequence $(\#)$ converges to 0 because the complex $(\S\S)(1)$ is exact. We have $\text{coker}(u) = E_2^{30}$. We divide the proof into two parts.

(A) Here we assume $h^1(L) = 0$, hence $\beta = 0$ and $\text{coker}(a) = E_2^{11}$. Since the spectral sequence $(\#)$ abuts to 0, we have

$$0 = E_\infty^{11} = E_3^{11} = \ker(d_2^{11} : E_2^{11} \rightarrow E_2^{30}).$$

Hence $\text{coker}(\alpha)$ injects onto $\text{coker}(u)$. Hence it is sufficient to prove that the codimension of the support of $\text{coker}(\alpha)$ is at most $h^1(\mathbf{O}_X) - \mathbf{f} + 1$. Since $\alpha : H^1(\mathbf{O}_X) \otimes \mathbf{O}_G(-1) \rightarrow \mathbf{O}_G^{\mathbf{f}}$ and $\mathbf{O}_G(1)$ is ample, this follows from [3, Th. 1.1(a)].

(B) Now we make no assumption on $H^1(L)$. As in the corresponding case of [5], the exact sequence (1) gives a homomorphism

$$c : H^1(M_L \otimes L) \rightarrow H^1(L) \otimes H^0(L)$$

and $\dim(\ker(c))$ is the amount of failure \mathbf{f} of quadratic normality of L by Lemma 1.1 (ii). On G there is an inclusion of sheaves $(\ker(c)) \otimes \mathbf{O}_G \rightarrow \ker(\beta)$. Since $\ker(\beta)$ is a subsheaf of a trivial sheaf, this inclusion is an isomorphism of $(\ker(c)) \otimes \mathbf{O}_G$ onto a direct summand of $\ker(\beta)$. Hence projecting $\ker(\beta)$ onto this summand we obtain a surjection from $E_2^{11} = \ker(\beta) / \text{im}(\alpha)$ onto $\text{coker}(H^1(\mathbf{O}_X) \otimes \mathbf{O}_G(-1) \rightarrow \mathbf{O}_G^{\mathbf{f}})$. We conclude as in part (A).

The proof of 0.2 is over. \square

Remark 2.1. The proof of 0.2 depends only on $\dim(B)$. If we want to exclude a bigger subset of G , then we obtain a corresponding result in a suitable range. Viceversa, if we may control a dense part of B the corresponding result is true in a larger range. The proof of Theorem 0.2 gives with no change the following result.

Proposition 2.2. *Fix an integer $k \geq 1$. Assume $\dim(X) = 2$. Assume the surjectivity of the restriction map $H^0(\mathbf{P}^r, \mathbf{O}_{\mathbf{P}}(k)) \rightarrow H^0(\phi_L(X), \mathbf{O}(k)) \cong H^0(L^k)$. Let $\mathbf{f}(k)$ be the dimension of the cokernel of the multiplication map $H^0(L) \otimes H^0(L^k) \rightarrow H^0(L^{k+1})$. Assume $\mathbf{f}(k) > 0$ and*

$$h^1(L^{k-1}) < \mathbf{f}(k) + \text{codim}(B) - 1.$$

Then there is a codimension 2 linear subspace $[U] \in G \setminus B$ such that the scheme $X \cap U$ is 0-dimensional and the multiplication map

$$H^0(L) \otimes H^0(X \cap U, (L|_{(X \cap U)})^k) \rightarrow H^0(X \cap U, (L|_{(X \cap U)})^{k+1})$$

is not surjective.

Remark 2.3. Note that for a complete but not projectively normal embedding the machine can start (and give informations on (X, L)) using the proposition just given exactly at the first step, say the $(k + 1)^{\text{th}}$ step, at which the embedding is not $(k + 1)$ -normal. However, it can also be used at an intermediate step with large $h^0(L^k)$, obtaining a result of Castelnuovo - Mumford type.

If we look at the proof of Theorem 0.2 when X is a smooth curve with $H^1(L) \leq 1$, we find exactly the proof of [5, §2.5]. In the statement we have the small precision about the amount of failure of quadratic normality of L .

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