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WEYL'S LAW FOR $SL(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3,\mathbb{R})$

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In this paper we derive a Weyl's law, or asymptotic description of the distribution of eigenvalues of the Laplacian, on $SL(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3,\mathbb{R})$. Our main tool in this derivation is the Selberg trace formula for the space. Our Weyl's law, which refines the present theory for the space in question, is also seen to coincide with known results in the case where $SL(3,\mathbb{Z})$ is replaced by a cocompact discrete subgroup.

§1. Introduction; statement of results.

The distribution of eigenvalues of the Laplacian for a general bounded domain B in $\mathbb{R}^n$ was first studied systematically by H. Weyl [**We**]. There he gave a precise asymptotic formula, of the form $N_B(x) \sim \kappa_0 x^{n/2}$ (κ_0 a constant), for the number $N_B(x)$ of eigenvalues less than x of this Laplacian. It is now standard to refer to such a result as a "Weyl's law" for the domain B.

Since then, many authors have considered generalizations of Weyl's law to other kinds of Riemannian manifolds. Particular attention, in large part because of their relevance to the theory of automorphic forms, has been given to the spaces $\Gamma\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R})$, where Γ is a cofinite (but, usually, not cocompact) discrete subgroup of $SL(2,\mathbb{R})$. Results concerning eigenvalues of the Laplacian on such spaces, and on products of such spaces, may be found in [**Ef**, **Hej1**, **Hej2**, **Hu1**], and [**Hu–Te**] (among other works).

In the present article, we wish to develop a Weyl's law in the higher–rank setting of $\Gamma\backslash H^3$, where $\Gamma = SL(3,\mathbb{Z})$ and $H^3 = SL(3,\mathbb{R})/SO(3,\mathbb{R})$. Actually, what we shall count here is the number $N(x)$ of linearly independent *cusp forms* on $\Gamma\backslash H^3$ whose eigenvalue for the Laplacian is less than x. By "cusp form" we mean a non–constant, square–integrable, *joint eigenfunction* (on $\Gamma\backslash H^3$) of the algebra D of *all* $SL(3,\mathbb{R})$–invariant differential operators on H^3. (In higher–rank settings, the definition of "cusp form" also requires the vanishing of certain "cuspidal integrals" of the eigenfunction. However, in the current situation, such a definition of cusp form coincides with the one we have given.) The cusp forms are central to harmonic analysis on $L^2(\Gamma\backslash H^3)$, and to the corresponding theory of automorphic forms on this space. (Technically, our count will also include the constant functions on

$\Gamma\backslash H^3$, which have eigenvalue 0 under the Laplacian. However, note that the result of Theorem 1 below is unaltered by the addition or exclusion of a subspace of dimension one.)

The algebra D is known [**Hel**, p. 432] to have two generators. We shall denote these by Δ_1 and Δ_2 respectively, where Δ_1 is the Laplacian on our space and Δ_2 is a third-order differential operator. (See §2 below for details.) Because D comprises substantially more than the Laplacian alone, it may seem that counting joint eigenfunctions of this algebra is a problem somewhat different than that of obtaining a Weyl's law for $\Gamma\backslash H^3$. However the problems are in fact the same, according to the following observations: Δ_1 and Δ_2 commute, and any eigenspace of Δ_1 (in $L^2(\Gamma\backslash H^3)$) is finite-dimensional. (The first statement is a consequence of [**Hel**, p. 432]; the second is a general fact regarding Laplacian operators that also follows, in the present setting, from Theorem 1 below.) So any Δ_1-eigenspace has a basis consisting of finitely many eigenvectors for Δ_2—such an eigenvector is, by the above, either a cusp form or a constant. Therefore, $N(x)$ in fact equals the dimension of the subspace of $L^2(\Gamma\backslash H^3)$ spanned by functions that are eigenvectors of Δ_1, and have Δ_1-eigenvalue $< x$. (The above argument was pointed out to us by David Farmer.)

To state our Weyl's law, let us denote by $\mathrm{vol}(\Gamma\backslash H^3)$ the "hyperbolic volume" (computed according to the $SL(3,\mathbb{R})$-invariant volume element given in §2) of this space. We then have

Theorem 1. *Let all notation be as above. Then*

$$N(x) \sim \frac{\mathrm{vol}(\Gamma\backslash H^3)}{(4\pi)^{5/2}\,\Gamma(7/2)}\,x^{5/2}$$

as $x \to \infty$.

Remark. The asymptotic result of our theorem is identical to the one appearing in Weyl's law for bounded domains in $\mathbb{R}^5$, and for *compact* Riemannian 5-manifolds (see Weyl [**We**] and Chavel [**Ch**]). Moreover, our theorem sharpens results of Donnelly [**Do**], and of Huntley [**Hu2**] (who consider more general situations). Specifically, Donnelly obtains the value $\mathrm{vol}(\Gamma\backslash H^3)/((4\pi)^{5/2}\,\Gamma(7/2))$ as an upper bound for $\overline{\lim}\, x^{-5/2}N(x)$; according to our theorem, the $\overline{\lim}$ is actually a limit, and is equal to the stated upper bound. Huntley does show that $x^{-5/2}N(x)$ has a finite limit, but does not determine this limit explicitly.

We further observe that Huntley's count includes certain functions that are square-integrable, but *not* cusp forms. Specifically, for the eigenfunctions of Δ_1 contributing to his Weyl's law, the relevant cuspidal integrals need not vanish identically, but only outside a specified compact subset of $\Gamma\backslash H^3$.

(Such functions are not generally eigenfunctions in the true sense, or even in the sense of distributions, but rather in a certain L^2 sense described in [**Hu2**].) As our Weyl's law counts only the cusp forms, it is technically somewhat different than his.

We now turn to the proof of Theorem 1. Our principal tool in this proof will be the Selberg trace formula—as realized by Wallace [**Wa**]—for nice functions g on the space $SO(3,\mathbb{R})\backslash SL(3,\mathbb{R})/SO(3,\mathbb{R})$. The details of this trace formula are given below; the basic idea behind its application is as follows. The left–hand, or "spectral," side of the trace formula may be interpreted, for appropriate choice of g, as the Laplace transform $L_N(T)$ of $N(x)$. By studying the various "orbital integrals" on the right–hand side of the trace formula, one may obtain an asymptotic equation, as $T \to 0^+$, for $L_N(T)$. Then, by applying a Tauberian theorem [**Wi**] for the Laplace transform, information on $L_N(T)$ may be translated into the desired formula for $N(x)$ itself.

The arguments just described will be fleshed out in §3 below, following a brief discussion, in §2, of harmonic analysis on H^3. In §4, an inversion formula for a certain integral transform, known as the "Helgason transform" $\widehat{g}$ (see §2), of functions g will be proved. This formula will be required in what follows, since the trace formula relates expressions involving g to others involving $\widehat{g}$, and moreover the particular g of interest to us will in fact be defined as a function with a given $\widehat{g}$. In §§5–7 we will carry out the computation of $L_N(T)$, by considering the orbital terms on the right–hand side of the trace formula. In the last section, we will combine this computation with the relevant Tauberian theorem to obtain the result embodied by Theorem 1.

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§2. Harmonic analysis on $SL(3,\mathbb{R})/SO(3,\mathbb{R})$.

In this section, we recall some basic ideas concerning coordinate systems and differential operators on $H^3 = SL(3,\mathbb{R})/SO(3,\mathbb{R})$.

First, using the Iwasawa decomposition for $SL(3,\mathbb{R})$, we may identify H^3 with the space AN, where A consists of diagonal matrices

$$a = \operatorname{diag}(a_1, a_2, (a_1a_2)^{-1}) \quad (a_i > 0)$$

and N consists of upper triangular unipotents

$$n = \begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix}$$

($x_i \in \mathbb{R}$). In these coordinates, the $SL(3,\mathbb{R})$–invariant measure on H^3 is given by

$$dn\,da = 54\,dx_1\,dx_2\,dx_3\frac{da_1\,da_2}{a_1\,a_2}. \tag{2.1}$$

The factor of 54 is included to provide a measure equal to the one specified in [**Ch**] and [**Do**].

As mentioned in §1, the algebra D of $SL(3,\mathbb{R})$–invariant differential operators on H^3 has two generators: the Laplacian Δ_1—which, properly normalized, is a *positive* operator (cf. [**Bo**])—and a third–order operator Δ_2. The former may be computed according to a formula in Chavel [**Ch**]. On the other hand, both operators have been determined explicitly by Bump [**Bu**] (using a fundamentally different coordinatization of H^3). In our coordinates, we have

Proposition 2.1.

$$\begin{aligned}
\Delta_1 &= -\frac{1}{3}\left(H_1^2 - H_1 + H_2^2 - H_2 - H_1H_2 + X_1^2 + X_2^2 + Z_0^2\right);\\
\Delta_2 &= -H_1^2H_2 + H_1H_2^2 + H_1^2 - H_2^2 - H_1 + H_2\\
&\quad - X_1^2H_2 + X_2^2H_1 - Z_0^2H_1 + Z_0^2H_2 + Z_0X_2X_1 + Z_0X_1X_2,
\end{aligned}$$

where

$$\begin{aligned}
H_1 &= \frac{a_1}{3}\frac{\partial}{\partial a_1} + \frac{a_2}{3}\frac{\partial}{\partial a_2} - x_2\frac{\partial}{\partial x_2} - x_3\frac{\partial}{\partial x_3};\\
H_2 &= \frac{2a_1}{3}\frac{\partial}{\partial a_1} - \frac{a_2}{3}\frac{\partial}{\partial a_2} - x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2};\\
X_1 &= x_1\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}; \qquad X_2 = \frac{\partial}{\partial x_1}; \qquad Z_0 = \frac{\partial}{\partial x_2}.
\end{aligned}$$

Proof. These are exactly equations (2.31) and (2.36) in Bump [**Bu**], except that we have multiplied by $-1/3$ his definition of Δ_1: this normalization yields the same Laplacian as appears in [**Ch**] and [**Do**]. We must restate Bump's differential operators H_1, H_2, etc. in terms of our coordinates on H^3, but this is straightforward: Bump gives us these operators as matrices in the Lie algebra of $SL(3,\mathbb{R})$, where such a matrix X defines a differential operator in the usual way:

$$Xf(z) = \frac{d}{dt}f(ze^{tX})\bigg|_{t=0}$$

($f \in C^\infty(SL(3,\mathbb{R}))$, $z \in H^3$). In particular, we have

$$H_1 = \operatorname{diag}\left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right); \quad H_2 = \operatorname{diag}\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right);$$

$$X_1 = E_{2,3}; \quad X_2 = E_{1,2}; \quad Z_0 = E_{1,3},$$

where $E_{i,j}$ denotes the matrix with a one in the i, j slot and zeroes elsewhere. We merely choose z to have the AN–coordinates given above, and our proposition follows after a few brief calculations. □

We now wish to consider integral transforms on H^3. (See, for example, [**Te2**] §4.2.) Let $K = SO(3,\mathbb{R})$. Suppose g is a bi–K–invariant function on $SL(3,\mathbb{R})$: that is, we may consider g as a function on H^3 satisfying $g(kz) = g(z)$ for all $k \in K$, $z \in H^3$. Then the *Helgason transform* $\widehat{g}$ of g is defined, for $s, t \in \mathbb{C}$, by

$$\widehat{g}(s,t) = \int_{AN} g(an)\, a_1^{3s+t}\, a_2^{3s-t}\, dn\, da. \tag{2.2}$$

(Our definition of the Helgason transform certainly makes sense for g infinitely differentiable and of compact support; however it is pointed out in [**Te2**] that this transform extends to an isometry on all of $L^2(K\backslash G/K)$.) We remark that the function $H_{s,t}(an) = a_1^{3s+t}\, a_2^{3s-t}$ is an eigenfunction of D, for any s, t; this follows, for example, from Proposition 2.1. We write $\Delta_1 H_{s,t} = \lambda H_{s,t}$ and $\Delta_2 H_{s,t} = \mu H_{s,t}$; then it is readily computed that

$$\lambda = s(1-s) + \frac{1}{3}t(1-t); \qquad \mu = (1-2s)(s+t-1)(s-t). \tag{2.3}$$

(The seemingly unusual normalization of the exponents in the definition of $H_{s,t}$ is justified by the simple form that this provides for the inversion formula of §4. Moreover, it may be seen that $H_{s,t}$ has the same eigenvalues under D as does the Eisenstein series $E(z,s,t)$, as defined in [**Wa**]; therefore we may use results from the latter work without significant modification.)

A related transform of g is the *Harish–Chandra transform* $\widetilde{g}$, defined for a diagonal matrix a as above by

$$\widetilde{g}(a) = \int_N g(an)\, dn. \tag{2.4}$$

Note that Mellin inversion applied to (2.2) gives us

$$\widetilde{g}(a) = \frac{1}{(2\pi i)^2} \int_{\mathrm{Re}(v)=c_2} \int_{\mathrm{Re}(u)=c_1} \widehat{g}\left(\frac{u+v}{6}, \frac{u-v}{2}\right) a_1^{-u}\, a_2^{-v}\, du\, dv, \tag{2.5}$$

for appropriate c_1, c_2.

To state the required version of the trace formula for $SL(3,\mathbb{Z})\backslash H^3$, we will need a convenient way of indexing eigenvalues of cusp forms on this space. We do this in a way consistent with the notation previously introduced for $H_{s,t}$. Specifically, we let $\{\varphi_n\}_{n=1}^{\infty}$ denote a maximal orthonormal set of such cusp forms, arranged in order of increasing eigenvalue for the Laplacian, and let φ_0 denote the constant function 1. Writing $\Delta_1\varphi_n = \lambda_n\varphi_n$ for all n, we then have

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots .$$

We will also write μ_n for the eigenvalue of φ_n under Δ_2. Finally, we introduce "eigenparameters" s_n, t_n in a way analogous to equation (2.3): namely, we write

$$\text{(2.6)}\quad \lambda_n = s_n(1-s_n) + \frac{1}{3}t_n(1-t_n); \quad \mu_n = (1-2s_n)(s_n+t_n-1)(s_n-t_n).$$

It is not hard to show that, for arbitrary λ_n, μ_n, these equations may be solved for s_n, t_n.

We now turn to our discussion of the trace formula for $SL(3,\mathbb{Z})\backslash H^3$, and of its application to Weyl's law.

§3. The trace formula and Weyl's law on $SL(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3,\mathbb{R})$.

It will be instructive to recall the notion of a trace formula in a quite general setting. To this end, we begin with a noncompact real semisimple Lie group G, a maximal compact subgroup K, and a cofinite discrete subgroup Γ. Then G/K is a Riemannian symmetric space on which Γ acts. Let $g(x) : G \to \mathbb{C}$ be a K–bi–invariant function. One constructs the kernel of an integral operator on $L^2(\Gamma\backslash G/K)$ by setting

$$F(z,w) = \sum_{\gamma\in\Gamma} g(z^{-1}\gamma w).$$

The Selberg trace formula, introduced by Selberg in the 1950's (cf., for example, [**Se1**, **Se2**]), equates two expressions for the trace of this operator on the discrete part of the spectrum of D. As above in the setting of $SL(3,\mathbb{R})/SO(3,\mathbb{R})$, D denotes the algebra of G–invariant differential operators on G/K, or on $\Gamma\backslash G/K$.

The two sides of the trace formula take the following forms. On the one hand (the left–hand side of the trace formula), one may use Mercer's Theorem for Hilbert–Schmidt operators to write this trace as a sum over eigenvalues of D. The quantity being summed is the "Helgason" or "Selberg" transform $\widehat{g}$ of g: $\widehat{g}$ is an integral of g against an eigenfunction φ of D on G/K. (The arguments of $\widehat{g}$ will be the eigenvalues of φ under generators for D, or certain parameters indexing these eigenvalues, as we have seen above in the specific case of interest to us.) On the other hand (the right-hand side of the trace formula), one can break up the sum defining $F(z,w)$ according to the conjugacy class of γ, and use this decomposition to break up the integral defining the trace of F. Each of the resulting integrals is called an "orbital integral" for g. (Care must be exercised with the contribution from the "parabolic" orbital integrals; it is only because the "continuous" part of the kernel F has been subtracted that this contribution is finite.) The various orbital integrals may themselves be expressed as integrals and sums of the Helgason transform, and the related Harish–Chandra transform, of g.

Let us now restrict our attention to the situation considered in §1: namely, to the case $G = SL(3,\mathbb{R})$; $K = SO(3,\mathbb{R})$; $\Gamma = SL(3,\mathbb{Z})$. In this setting, the above generalities regarding a trace formula have been made explicit by Wallace [**Wa**]. These results may be summarized by

Proposition 3.1. *Let $g \in C_c^\infty(SO(3,\mathbb{R})\backslash SL(3,\mathbb{R})/SO(3,\mathbb{R}))$ (however, see the remarks below); let all notation be as in §§ 1 and 2 above. Then*

$$\sum_{n=0}^{\infty} \widehat{g}(s_n,t_n) = \mathrm{vol}(\Gamma\backslash H^3)\, g(I)$$
$$+ \sum_{(\varepsilon_1,\varepsilon_2,\varepsilon_3)} \frac{\mathrm{Reg}(\mathbb{Z}[\varepsilon_1])\ \mathrm{Cl}(\mathbb{Z}[\varepsilon_1])\, |\varepsilon_1^2\varepsilon_2|}{|(\varepsilon_1-\varepsilon_2)(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)|}\, \widetilde{g}(\mathrm{diag}(|\varepsilon_1|,|\varepsilon_2|,|\varepsilon_3|))$$
$$+ \sum_{(r,\theta)} \frac{|\ln r_0|\ \mathrm{Cl}(\mathbb{Z}[r])}{|1-2r^{-3}\cos\theta + r^{-6}|} \int_{Re(s)=1/2}\int_{Re(t)=1/2} \widehat{g}(s,t)\, \frac{r^{1-s}e^{-2\theta\, Im(t)}}{1+e^{-2\pi Im(t)}}\, ds\, dt$$
$$+ K_1\,\widehat{g}\Big(\frac{1}{2},\frac{1}{2}\Big) + K_2\,\widehat{g}\Big(\frac{1}{3},1\Big) + K_3\,\widehat{g}\Big(\frac{2}{3},1\Big)$$

where:

a. *I denotes the identity coset (the first term on the right-hand side of Proposition 3.1 is called the "identity" or "central" term).*

b. *The sum in $(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ runs over all* distinct *real triples with $|\varepsilon_1| > |\varepsilon_2| > |\varepsilon_3|$ (no $\varepsilon_i = 1$), such that there is some (by definition, hyperbolic) matrix in $SL(3,\mathbb{Z})$ with eigenvalues $\varepsilon_1,\varepsilon_2,\varepsilon_3$. Also,* Reg *denotes the regulator of the order, and* Cl *the narrow class number. (The second term on the right-hand side above is called the "hyperbolic" term.)*

c. *The sum in (r,θ) runs over all* distinct *pairs with $r > 0$ ($r \neq 1$) and $0 < \theta \leq \pi$, such that there is some (by definition, loxodromic) matrix g in $SL(3,\mathbb{Z})$ with eigenvalues $r^{-2}, re^{i\theta}, re^{-i\theta}$. Again,* Cl *denotes the narrow class* *number; also, r_0 is such that the centralizer of g (in $SL(3,\mathbb{Z})$), which is infinite cyclic, has generator conjugate (in $SL(3,\mathbb{C})$) to* $\operatorname{diag}(r_0^{-2}, r_0 e^{i\theta_0}, r_0 e^{-i\theta_0})$. *(The third term on the right-hand side above is called the "loxodromic" term.)*

d. *K_1, K_2, K_3 are universal constants. (The sum of the last three terms on the right-hand side above is called the "parabolic" term.)*

Remarks on Proposition 3.1. We note that the version of the Selberg trace formula that we have stated above is slightly different than that given in [**Wa**, §8]. Namely, in the latter work both the hyperbolic and loxodromic terms are broken down into sums over powers of "primitive" conjugacy classes. The forms given above for the hyperbolic and loxodromic terms will be more convenient for our purposes. (See §§ 6 and 7, below.)

We observe further that the hyperbolic term above is, as in [**Wa**], actually a sum over all distinct *sets* of eigenvalues of hyperbolic matrices. The stipulation $|\varepsilon_1| > |\varepsilon_2| > |\varepsilon_3|$ precisely assures that each such set is counted only once. Similarly, the loxodromic term is a sum over all sets of eigenvalues of loxodromic matrices; in this case the requirements $r > 0$ and $0 < \theta \leq \pi$ assure that no set of eigenvalues occurs twice.

Finally, we make the crucial observation that the trace formula may be extended to include a broader class of "test function" g than that stipulated in the above proposition. In particular, it is straightforward to show, using techniques analogous to those employed in [**He1**, Chapter 1] and [**He2**, Chapter 8], that Proposition 3.1 remains valid for smooth functions $g \in L^2(K\backslash G/K)$, such that $\widehat{g}(s,t)$ decays exponentially as $\operatorname{Im}(s)$ or $\operatorname{Im}(t)$ approaches $\pm\infty$ (and the corresponding real part remains bounded). We will apply Proposition 3.1 to such a function g, as described immediately below (the class of allowable g may be made broader still, with a small amount of additional work, but we will not require such a generalization).

The application of the above trace formula to Weyl's law will proceed as follows. First, for $T > 0$, we will let g_T be a left K-invariant function on H^3 such that

$$\widehat{g_T}(s,t) = e^{-\lambda T} = e^{-T[s(1-s)+t(1-t)/3]}. \tag{3.1}$$

Such a function g_T must exist for the following reason: we have seen that $s(1-s) + t(1-t)/3$ is the eigenvalue of Δ_1 belonging to the eigenfunction $H_{s,t}(z) = a_1^{3s+t} a_2^{3s-t}$. Such an eigenvalue is invariant under the action of the Weyl group on (s,t), as described in [**Te2**], Chapter 4 or [**Bu**], Chapter 2 (in the respective coordinate systems). But it follows from [**Te2**], Theorem 1, p. 88 (after restriction to the determinant-one surface $\mathcal{S}P_3$), that the Helgason transform is an isometry from square-integrable, K-invariant functions on H^3 *onto* the space of functions of (s,t) that are square–integrable with respect to the spectral measure (cf. §4) and are invariant under this action.

We will then apply Proposition 3.1 to our function $\widehat{g_T}$: on the left–hand side we get

$$L_N(T) \equiv \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda_n T}.$$

In particular $L_N(T)$ is a Laplace, or Laplace–Stieltjes, transform of $N(x)$. By estimating each term occuring on the right–hand side of the trace formula, we may obtain asymptotic information (as $T \to 0^+$) regarding $L_N(T)$. Theorem 1 will then follow, by the Tauberian theorem mentioned in §1.

We now proceed with our derivation of the inversion formula for the Helgason transform.

§4. Inversion of the Helgason transform on $H^3 = SL(3,\mathbb{R})/SO(3,\mathbb{R})$.

In this section, we will prove

Proposition 4.1. *Let g be a smooth, square-integrable function on H^3 that is $K = SO(3,\mathbb{R})$-invariant, i.e. $g(kz) = g(z)$ for all $k \in K$. Then*

$$g(I) = \frac{1}{2^4 3^3 \pi^3} \int_{r_1, r_2 \in \mathbb{R}} \widehat{g}\Big(\frac{1}{2} + ir_1, \frac{1}{2} + ir_2\Big) \cdot r_2\,(r_2^2 - 9r_1^2)\, \tanh \pi r_2 \tanh \pi \frac{r_2 + 3r_1}{2} \tanh \pi \frac{r_2 - 3r_1}{2}\, dr_1\, dr_2.$$

Proof. Our starting point for the proof of Proposition 4.1 will be the "Helgason inversion formula" that appears in [**Te2**] (Theorem 1, p. 88). This formula pertains to functions on the space

$$\mathcal{P}_n = \{\text{positive definite, symmetric real } n \times n \text{ matrices}\},$$

which is a $GL(n,\mathbb{R})$-space via the action $(Y,g) \to Y[g] = {}^t g Y g$ ($Y \in \mathcal{P}_n$, $g \in GL(n,\mathbb{R})$). We will be concerned with the case $n = 3$. By first identifying integrals on AN with integrals on the determinant–one surface $\mathcal{S}P_3$ of $\mathcal{P}_3$, and then restricting the inversion formula in [**Te2**] to $\mathcal{S}P_3$, we will deduce from Helgason inversion the required inversion formula for H^3. □

We begin with

Lemma 4.1. *Let f be integrable on $\mathcal{S}P_3$. Then, for a suitably normalized $SL(3,\mathbb{R})$-invariant measure dW,*

$$\int_{W\in\mathcal{S}P_3} f(W)\,dW = \frac{2}{27}\int_{AN} f(I[(an)^{-1}])\,dn\,da.$$

Proof of Lemma. We begin with formula (1.39), Chapter 4 in [**Te2**] for integrable functions on $\mathcal{P}^3$:

$$\int_{Y\in\mathcal{P}_3} F(Y)\,dY \;=\; 8\int_{A'N} F\left(I\left[\left(\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3\end{pmatrix}\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1\end{pmatrix}\right)^{-1}\right]\right) dn'\,da'$$

where dY is $GL(3,\mathbb{R})$–invariant measure on $\mathcal{P}_3$, $A' = \{\mathrm{diag}(a_1,a_2,a_3)\,|\,a_i > 0\}$, $dn' = dx_1dx_2dx_3$, and $da' = da_1da_2da_3/a_1a_2a_3$. We choose the $SL(3,\mathbb{R})$–invariant measure dW on $\mathcal{S}P_3$ so that $dY = v^{-1}dv\,dW$, where $v = \det Y$ and $Y = v^{1/3}W$. (That such a choice of measures is possible is pointed out in [**Te2**], §4.1.) Also let k be a function on $\mathbb{R}^+$ whose integral with respect to the Haar measure dv/v exists and is nonzero; put $F(Y) = k(v)f(W)$. Then

$$\int_{v>0} k(v)\,\frac{dv}{v}\int_{W\in\mathcal{S}P_3} f(W)\,dW = 8\int_{A'N} k((a_1a_2a_3)^{-2})$$
$$\cdot f\left(I\left[\left(\begin{pmatrix} (a_1^2/a_2a_3)^{1/3} & & \\ & (a_2^2/a_1a_3)^{1/3} & \\ & & (a_3^2/a_1a_2)^{1/3}\end{pmatrix}\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1\end{pmatrix}\right)^{-1}\right]\right) dn'\,da'.$$

We substitute $v = (a_1a_2a_3)^{-2}$; $u_1 = (a_1^2/a_2a_3)^{1/3}$; $u_2 = (a_2^2/a_1a_3)^{1/3}$ into the integral on the right. It is seen that, under this substitution,

$$da' \to \frac{1}{2}\frac{dv\,du_1\,du_2}{v\,u_1\,u_2}.$$

So we get

$$\int_{v>0} k(v)\,\frac{dv}{v}\int_{W\in\mathcal{S}P_3} f(W)\,dW = 4\int_{v>0} k(v)\frac{dv}{v}$$
$$\cdot\int_{u_1,u_2>0}\int_N f\left(I\left[\left(\begin{pmatrix} u_1 & & \\ & u_2 & \\ & & (u_1u_2)^{-1}\end{pmatrix}\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1\end{pmatrix}\right)^{-1}\right]\right) dn'\,\frac{du_1\,du_2}{u_1u_2}$$

whence, upon replacing u_i by a_i and recalling the factor of 54 present in (2.1), the lemma. □

Next, using Lemma 4.1, we wish to write the Helgason transform $\widehat{g}$ in terms of an analogous transform for functions on $\mathcal{S}P_3$. We begin by writing, for any $Y \in \mathcal{P}_3$ and $s = (s_1, s_2, s_3) \in \mathbb{C}^3$, $p_s(Y) = \prod_{i=1}^3 |Y_i|^{s_i}$ where Y_i denotes the top left–hand $i \times i$ corner of Y. Note that $p_s(W)$ is independent of s_3 for $W \in \mathcal{S}P_3$; we will write $p_{(s_1,s_2)}(W)$ for $p_s(W)$. We have:

Lemma 4.2. *Let g be a function on H^3 whose Helgason transform* (2.2) *converges absolutely; let f be the function on $\mathcal{S}P_3$ defined by $f(I[(an)^{-1}]) = g(an)$. Then*

$$\frac{2}{27}\,\widehat{g}(s,t) = f^*\left(-\bar{t}, \overline{\frac{t-3s}{2}}\right)$$

where, by definition,

$$f^*(s_1, s_2) = \int_{W \in \mathcal{S}P_3} f(W)\,\overline{p_{(s_1,s_2)}(W)}\,dW.$$

Proof of Lemma. This follows immediately from Lemma 4.1, upon noting that

$$H_{s,t}(z) \equiv a_1^{3s+t}\,a_2^{3s-t} = p_{(-t,(t-3s)/2)}(I[(an)^{-1}]).$$

$\square$

In our next lemma, we obtain an inversion formula on $\mathcal{S}P_3$.

Lemma 4.3. *If f is square-integrable and K-invariant on $\mathcal{S}P_3$, i.e. $f(W[k]) = f(W)$ for all $k \in K, W \in \mathcal{S}P_3$, then*

$$f(I) = 2\pi i \omega_3 \int_{Re(s_i)=-1/2} f^*(s_1, s_2)\,|c_3(s)|^{-2} ds_1\,ds_2$$

where the "spectral measure" $\omega_3\,|c_3(s)|^{-2}$ is given in [**Te2**] *Theorem* 1, *p.* 88 (*or equations* (4.1), *below*).

Proof of Lemma. Let F be square–integrable and $O(3,\mathbb{R})$–invariant on $\mathcal{P}_3$. The Helgason transform $\widehat{F}(s)$ of F, for $s \in \mathbb{C}^3$ as above, may be defined by

$$\widehat{F}(s) = \int_{v>0} v^{\bar{r}} \int_{W \in \mathcal{S}P_3} F(v^{1/3}W)\,\overline{p_s(W)}\,dW\,\frac{dv}{v}$$

where $r = (1/3)\sum_{j=1}^3 js_j$ (cf. [**Te2**], p. 94). We now take $F(Y) = k(v)f(W)$, where k is a suitable function on $\mathbb{R}^+$ that does not vanish at $v = 1$. (Note that f K–invariant $\Rightarrow$ F $O(3,\mathbb{R})$–invariant.) For simplicity, let us in fact take $k(v) = e^{-v}$. Then

$$\widehat{F}(s) = \int_{v>0} e^{-v}\,v^{\bar{r}}\,\frac{dv}{v} \int_{W \in \mathcal{S}P_3} f(W)\,\overline{p_s(W)}\,dW = \Gamma(\bar{r}) f^*(s_1, s_2)$$

(cf. Lemma 4.2 above). We recall the Helgason inversion formula for $\mathcal{P}_3$ (cf. [**Te2**], Theorem 1, p. 88):

$$F(Y) = \omega_3 \int_{\substack{\mathrm{Re}(s_1)=\mathrm{Re}(s_2)=-1/2 \\ \mathrm{Re}(s_3)=1/2}} \widehat{F}(s)\, h_s(Y)\, |c_3(s)|^{-2}\, ds$$

where h_s is a "spherical function" obtained by averaging the power function p_s over the orthogonal group. For our purposes, it suffices to know that $h_s(I) = 1$. Also, one notes from the definition that $c_3(s)$ is independent of s_3. So the above inversion formula gives us

$$\begin{aligned} F(I) = k(1) f(I) = &\omega_3 \int_{\mathrm{Re}(s_1)=\mathrm{Re}(s_2)=-1/2} f^*(s_1, s_2)\, |c_3(s)|^{-2} \\ &\cdot \left[\int_{\mathrm{Re}(s_3)=1/2} \Gamma\left(\overline{s_3} + \frac{2}{3}\overline{s_2} + \frac{1}{3}\overline{s_1}\right) ds_3 \right] ds_1 ds_2. \end{aligned}$$

But $k(1) = 1/e$, which is $(2\pi i)^{-1}$ times the value of the integral in s_3 (we are merely invoking Mellin inversion to evaluate this integral). So

$$f(I) = 2\pi i \omega_3 \int_{\mathrm{Re}(s_1)=\mathrm{Re}(s_2)=-1/2} f^*(s_1, s_2)\, |c_3(s)|^{-2}\, ds_1 ds_2,$$

as was to be proved. □

We now put the above lemmas together to deduce our inversion formula on H^3. Let g be square–integrable and K–invariant on H^3; let f be the function on $\mathcal{S}P_3$ with $g(an) = f(I[(an)^{-1}])$. Noting that f is K–invariant on $\mathcal{S}P_3$, we have by Lemma 4.3

$$g(I) = f(I) = 2\pi i \omega_3 \int_{\mathrm{Re}(s_i)=-1/2} f^*(s_1, s_2)\, |c_3(s)|^{-2} ds_1\, ds_2.$$

We substitute $s_i \to -\overline{s_i}$, to get

$$g(I) = 2\pi i \omega_3 \int_{\mathrm{Re}(s_i)=1/2} f^*(-\overline{s_1}, -\overline{s_2})\, |c_3(-\overline{s_1}, -\overline{s_2})|^{-2} ds_1\, ds_2;$$

following this with the substitution

$$s = \frac{s_1 + 2s_2}{3}; \qquad t = s_1,$$

we get

$$\begin{aligned} g(I) &= 3\pi i \omega_3 \int_{\mathrm{Re}(s)=\mathrm{Re}(t)=1/2} f^*\left(-\overline{t}, \overline{\frac{t-3s}{2}}\right) \left| c_3\left(-\overline{t}, \overline{\frac{t-3s}{2}}\right) \right|^{-2} ds\, dt \\ &= \frac{2}{9} \pi i \omega_3 \int_{\mathrm{Re}(s)=\mathrm{Re}(t)=1/2} \widehat{g}(s,t) \left| c_3\left(-\overline{t}, \overline{\frac{t-3s}{2}}\right) \right|^{-2} ds\, dt \end{aligned}$$

by Lemma 4.2. Let us finally substitute $s = \frac{1}{2} + ir_1$, $t = \frac{1}{2} + ir_2$. We find from [**Te2**], p. 88 (and equation (3.23), p. 144 in [**Te1**]), that

(4.1a)
$$\left| c_3 \left(-\bar{t}, \overline{\frac{t-3s}{2}} \right) \right|^{-2} = \pi r_2 \left(r_2^2 - 9r_1^2\right) \tanh \pi r_2 \, \tanh \pi \frac{r_2 + 3r_1}{2} \, \tanh \pi \frac{r_2 - 3r_1}{2}.$$

Since our substitution takes $ds\,dt$ to $-dr_1\,dr_2$, and as

$$\omega_3 = \prod_{j=1}^{3} \frac{\Gamma(j/2)}{j(2\pi i)\pi^{j/2}} = \frac{\Gamma(1/2)\Gamma(1)\Gamma(3/2)}{6(2\pi i)^3\pi^3} = \frac{i}{96\pi^5} \tag{4.1b}$$

(cf. [**Te2**], p. 88, again), we get

$$g(I) = \frac{1}{2^4 3^3 \pi^3} \int_{r_1, r_2 \in \mathbb{R}} \widehat{g}\left(\frac{1}{2} + ir_1, \frac{1}{2} + ir_2\right) \cdot r_2 \left(r_2^2 - 9r_1^2\right) \tanh \pi r_2 \tanh \pi \frac{r_2 + 3r_1}{2} \tanh \pi \frac{r_2 - 3r_1}{2} \, dr_1\, dr_2.$$

This proves Proposition 4.1.

§5. The identity term.

The first term that contributes to the right–hand side of the trace formula, for a given test function g, is the identity term

$$\mathrm{vol}(\Gamma \backslash H^3)\, g(I)$$

(cf. Proposition 3.1). For $g = g_T$ having the Helgason transform given in (3.1), we will prove

Proposition 5.1.
$$g_T(I) = \frac{T^{-5/2}}{(4\pi)^{5/2}} + O(T^{-3/2})$$

as $T \to 0^+$.

Proof. Recalling Proposition 4.1, we have

$$\begin{aligned} g_T(I) &= \frac{1}{2^4 3^3 \pi^3} \int_{r_1, r_2 \in \mathbb{R}} \widehat{g_T}\left(\frac{1}{2} + ir_1, \frac{1}{2} + ir_2\right) r_2 \left(r_2^2 - 9r_1^2\right) \\ &\qquad \cdot \tanh \pi r_2 \tanh \pi \frac{r_2 + 3r_1}{2} \tanh \pi \frac{r_2 - 3r_1}{2} \, dr_1\, dr_2 \\ &= \frac{1}{2^4 3^3 \pi^3} \int_{r_1, r_2 \in \mathbb{R}} \mathrm{e}^{-T(1+3r_1^2+r_2^2)/3}\, r_2 \left(r_2^2 - 9r_1^2\right) \\ &\qquad \tanh \pi r_2 \tanh \pi \frac{r_2 + 3r_1}{2} \tanh \pi \frac{r_2 - 3r_1}{2} \, dr_1\, dr_2. \end{aligned}$$

To simplify this expression, we use the trig identity

$$\tanh(a+b)\tanh(a)\tanh(b) = \tanh(a) + \tanh(b) - \tanh(a+b)$$

to obtain

$$g_T(I) = \frac{1}{2^4 3^3 \pi^3} \int_{r_1, r_2 \in \mathbb{R}} \mathrm{e}^{-T(1+3r_1^2+r_2^2)/3}\, r_2 (r_2^2 - 9r_1^2) \cdot \left(\tanh \pi \frac{r_2 + 3r_1}{2} + \tanh \pi \frac{r_2 - 3r_1}{2} - \tanh \pi r_2 \right) dr_1\, dr_2.$$

Into the first of the resulting three integrals we substitute $p = (-r_1 + r_2)/2, q = (r_2 + 3r_1)/2$; into the second we substitute $p = (r_1 + r_2)/2, q = (r_2 - 3r_1)/2$; and into the third, $p = r_1, q = r_2$. Each of these substitutions has Jacobian equal to one, and moreover the first and second merely negate the quantity

$$\mathrm{e}^{-T(1+3r_1^2+r_2^2)/3}\, r_2\, (r_2^2 - 9r_1^2)$$

(and give the new names p, q to r_1, r_2 respectively). This latter phenomenon is a consequence of a general invariance of Helgason transforms, cf. the discussion following equation (3.1). At any rate, our substitutions yield

(5.1)
$$\begin{aligned} g_T(I) &= -\frac{1}{2^4 3^2 \pi^3} \int_{p, q \in \mathbb{R}} \mathrm{e}^{-T(1+3p^2+q^2)/3}\, q\, (q^2 - 9p^2)\, \tanh \pi q\, dp\, dq \\ &= \frac{1}{2^2 3^2 \pi^3} \int_0^\infty \int_0^\infty \mathrm{e}^{-T(1+3p^2+q^2)/3}\, q\, (9p^2 - q^2)\, \tanh \pi q\, dp\, dq. \end{aligned}$$

Now

$$\tanh \pi q = \frac{\mathrm{e}^{\pi q} - \mathrm{e}^{-\pi q}}{\mathrm{e}^{\pi q} + \mathrm{e}^{-\pi q}} = 1 - 2\frac{\mathrm{e}^{-2\pi q}}{1 + \mathrm{e}^{-2\pi q}} = 1 + O(\mathrm{e}^{-2\pi q}) \tag{5.2}$$

for $q \geq 0$. Since

$$\begin{aligned} &\int_0^\infty \int_0^\infty \mathrm{e}^{-T(1+3p^2+q^2)/3}\, q\, (9p^2 - q^2)\, \mathrm{e}^{-2\pi q}\, dp\, dq \\ &\quad = O\left(\int_0^\infty \int_0^\infty \mathrm{e}^{-Tp^2}\, q (9p^2 + q^2) \mathrm{e}^{-2\pi q}\, dp\, dq \right) \\ &\quad = O\left(\int_0^\infty \mathrm{e}^{-Tp^2} p^2\, dp + \int_0^\infty \mathrm{e}^{-Tp^2}\, dp \right) \\ &\quad = O(T^{-3/2} + T^{-1/2}) = O(T^{-3/2}) \end{aligned}$$

as $T \to 0^+$, we have from (5.1) and (5.2):

(5.3)
$$\begin{aligned} g_T(I) &= \frac{1}{2^2 3^2 \pi^3} \int_0^\infty \int_0^\infty \mathrm{e}^{-T(1+3p^2+q^2)/3}\, q\,(9p^2 - q^2)\, dp\, dq + O(T^{-3/2}) \\ &= \frac{1}{2^4 3^2 \pi^3} \int_0^\infty \int_0^\infty \mathrm{e}^{-T(1+3p+q)/3}\, p^{1/2}\, q\,(9p - q)\, \frac{dp\, dq}{pq} + O(T^{-3/2}) \end{aligned}$$

as $T \to 0^+$.

Making the substitution

$$p = \frac{xy}{y+1}; \qquad q = \frac{3x}{y+1},$$

we get

$$g_T(I) = \frac{\mathrm{e}^{-T/3}}{2^4 \pi^3} \int_0^\infty \int_0^\infty \mathrm{e}^{-Tx}\, x^{5/2}\, y^{1/2}\, \frac{(3y-1)}{(y+1)^{5/2}} \frac{dx\, dy}{xy} + O(T^{-3/2}).$$

The integral in y may be evaluated via the standard beta function formula

$$\int_0^\infty y^a (y+1)^{-(a+b)}\, \frac{dy}{y} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$(\mathrm{Re}(a), \mathrm{Re}(b) > 0)$, whereby

(5.4)
$$\begin{aligned} g_T(I) &= \frac{\mathrm{e}^{-T/3}}{2^4 \pi^3} \left(3\frac{\Gamma(3/2)\Gamma(1)}{\Gamma(5/2)} - \frac{\Gamma(1/2)\Gamma(2)}{\Gamma(5/2)} \right) \\ &\quad \cdot \int_0^\infty \mathrm{e}^{-Tx}\, x^{5/2}\, \frac{dx}{x} + O(T^{-3/2}) \\ &= \frac{\mathrm{e}^{-T/3}}{2^4 \pi^3} \left(3\frac{\sqrt{\pi}/2}{\Gamma(5/2)} - \frac{\sqrt{\pi}}{\Gamma(5/2)} \right) \frac{\Gamma(5/2)}{T^{5/2}} + O(T^{-3/2}) \\ &= \frac{T^{-5/2} \mathrm{e}^{-T/3}}{(4\pi)^{5/2}} + O(T^{-3/2}). \end{aligned}$$

Since $T^{-5/2}\mathrm{e}^{-T/3} = T^{-5/2} + O(T^{-3/2})$ as $T \to 0^+$, Proposition 5.1 clearly follows.

□

§6. The hyperbolic term.

We now wish to consider the contribution from hyperbolic conjugacy classes to the orbital side of the trace formula. In particular we wish to prove, for $g = g_T$ as above (and all notation as in §3):

Proposition 6.1.

$$\sum_{(\varepsilon_1,\varepsilon_2,\varepsilon_3)} \frac{\operatorname{Reg}(\mathbb{Z}[\varepsilon_1])\ \operatorname{Cl}(\mathbb{Z}[\varepsilon_1])\,|\varepsilon_1^2\varepsilon_2|}{|(\varepsilon_1-\varepsilon_2)(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)|}\,\widetilde{g_T}(\operatorname{diag}(|\varepsilon_1|,|\varepsilon_2|,|\varepsilon_3|)) = O(T^{-1})$$

as $T \to 0^+$.

Proof. Let us write, for brevity, $E_\varepsilon = \operatorname{diag}(|\varepsilon_1|,|\varepsilon_2|,|\varepsilon_3|)$. By equation (2.5),

$$\begin{aligned}\widetilde{g_T}(E_\varepsilon) &= \frac{1}{(2\pi i)^2}\int_{\operatorname{Re}(v)=c_2}\int_{\operatorname{Re}(u)=c_1} \widehat{g_T}\left(\frac{u+v}{6},\frac{u-v}{2}\right)|\varepsilon_1|^{-u}\,|\varepsilon_2|^{-v}\,du\,dv\\ &= \frac{6}{(2\pi i)^2}\int_{\operatorname{Re}(t)=(c_1-c_2)/2}\int_{\operatorname{Re}(s)=(c_1+c_2)/6} \widehat{g_T}(s,t)\,|\varepsilon_1|^{-(3s+t)}\,|\varepsilon_2|^{-(3s-t)}\,ds\,dt\end{aligned}$$

for suitable c_1, c_2. Now equation (3.1) for $\widehat{g_T}$ clearly allows us to take $c_1 = 2, c_2 = 1$. Then

$$\begin{aligned}\widetilde{g_T}(E_\varepsilon) &= \frac{6}{(2\pi)^2}\int_{\mathbb{R}^2}\widehat{g_T}\left(\frac{1}{2}+ir_1,\frac{1}{2}+ir_2\right)\\ &\qquad\cdot|\varepsilon_1|^{-(2+i(3r_1+r_2))}\,|\varepsilon_2|^{-(1+i(3r_1-r_2))}\,dr_1\,dr_2\\ &= \frac{6\,|\varepsilon_1^{-2}\varepsilon_2^{-1}|\,e^{-T/3}}{(2\pi)^2}\int_{\mathbb{R}} e^{-Tr_1^2}|\,\varepsilon_1^3\varepsilon_2^3|^{-ir_1}\,dr_1\int_{\mathbb{R}} e^{-Tr_2^2/3}\,|\varepsilon_1/\varepsilon_2|^{-ir_2}\,dr_2.\end{aligned}$$

Using the fact that

$$\int_{\mathbb{R}} e^{-\alpha r^2}\varepsilon^{-ir}\,dr = \sqrt{\frac{\pi}{\alpha}}\,e^{-(\log^2\varepsilon)/4\alpha}$$

for $\varepsilon, \alpha > 0$ (this identity amounts to the statement that $f(x) = e^{-\pi x^2}$ is invariant under the Fourier transform), we find that

$$\begin{aligned}\widetilde{g_T}(E_\varepsilon) &= \frac{6\,|\varepsilon_1^{-2}\varepsilon_2^{-1}|\,e^{-T/3}}{(2\pi)^2}\left(\sqrt{\frac{\pi}{T}}\,e^{-(\log^2|\varepsilon_1^3\varepsilon_2^3|)/4T}\right)\left(\sqrt{\frac{3\pi}{T}}\,e^{-3(\log^2|\varepsilon_1/\varepsilon_2|)/4T}\right)\\ &= \frac{3\sqrt{3}\,|\varepsilon_1^{-2}\varepsilon_2^{-1}|\,e^{-T/3}}{2\pi T}e^{-(9\log^2|\varepsilon_1|+3\log^2|\varepsilon_1\varepsilon_2^2|)/4T}\\ &= O(|\varepsilon_1^{-2}\varepsilon_2^{-1}|\,T^{-1}e^{-(9\log^2|\varepsilon_1|)/4T}).\end{aligned}$$

The implied constant is absolute and holds for all $T > 0$.

Thus we find that

$$\begin{aligned}&\sum_{(\varepsilon_1,\varepsilon_2,\varepsilon_3)} \frac{\operatorname{Reg}(\mathbb{Z}[\varepsilon_1])\ \operatorname{Cl}(\mathbb{Z}[\varepsilon_1])\,|\varepsilon_1^2\varepsilon_2|}{|(\varepsilon_1-\varepsilon_2)(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)|}\,\widetilde{g_T}(\operatorname{diag}(|\varepsilon_1|,|\varepsilon_2|,|\varepsilon_3|))\\ &\qquad = O\left(T^{-1}\sum_{(\varepsilon_1,\varepsilon_2,\varepsilon_3)}\frac{\operatorname{Reg}(\mathbb{Z}[\varepsilon_1])\ \operatorname{Cl}(\mathbb{Z}[\varepsilon_1])\,e^{-(9\log^2|\varepsilon_1|)/4T}}{|(\varepsilon_1-\varepsilon_2)(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)|}\right).\end{aligned}\tag{6.1}$$

To estimate the sum on the right–hand side of (6.1), we need the following

Lemma 6.1. *For some positive constant b,*

$$\frac{Reg(\mathbb{Z}[\varepsilon_1])\ Cl(\mathbb{Z}[\varepsilon_1])}{|(\varepsilon_1-\varepsilon_2)(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)|}=O(\varepsilon_1^b).$$

Proof of Lemma. Let $d=[(\varepsilon_1-\varepsilon_2)(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)]^2$ be the absolute value of the discriminant of $\mathbb{Z}[\varepsilon_1]$. It follows from [**Bo–Sh**], Lemma 3, p. 121, as well as Lemmas 2 and 3, p. 127, that

$$\mathrm{Reg}(\mathbb{Z}[\varepsilon_1])\ \mathrm{Cl}(\mathbb{Z}[\varepsilon_1])=O(d^{b_1})$$

for some positive number b_1. Therefore

$$\begin{aligned}\frac{\mathrm{Reg}(\mathbb{Z}[\varepsilon_1])\,\mathrm{Cl}(\mathbb{Z}[\varepsilon_1])}{|(\varepsilon_1-\varepsilon_2)(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)|}&\\ &=O\left(|(\varepsilon_1-\varepsilon_2)(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)|^{2b_1-1}\right)=O(\varepsilon_1^b),\end{aligned}$$

for $b=6b_1-3$ (we have used the fact that $|\varepsilon_1|>|\varepsilon_2|>|\varepsilon_3|$). Thus our lemma is proved. ☐

Now let

$$\pi(x)=\left\{\begin{matrix}\text{distinct sets } (\varepsilon_1,\varepsilon_2,\varepsilon_3) \text{ of eigenvalues}\\ \text{of hyperbolic matrices }:\ |\varepsilon_3|<|\varepsilon_2|<|\varepsilon_1|\le x\end{matrix}\right\}.$$

Then, since $\pi(x)=0$ for $x\le 1$, equation (6.1) becomes

$$\begin{aligned}\sum_{(\varepsilon_1,\varepsilon_2,\varepsilon_3)}&\frac{\mathrm{Reg}(\mathbb{Z}[\varepsilon_1])\ \mathrm{Cl}(\mathbb{Z}[\varepsilon_1])\,|\varepsilon_1^2\varepsilon_2|}{|(\varepsilon_1-\varepsilon_2)(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)|}\,\widetilde{g_T}(\mathrm{diag}(|\varepsilon_1|,|\varepsilon_2|,|\varepsilon_3|))\\ &=O\left(T^{-1}\sum_{(\varepsilon_1,\varepsilon_2,\varepsilon_3)}\varepsilon_1^b\,\mathrm{e}^{-(9\log^2|\varepsilon_1|)/4T}\right)\\ &=O\left(T^{-1}\int_1^\infty x^b\,\mathrm{e}^{-(9\log^2 x)/4T}\,d\pi(x)\right).\end{aligned}$$

But $\mathrm{e}^{-(9\log^2 x)/4T}=0(x^{-b-4})$ for $x>1$ and for T bounded above, so

$$\begin{aligned}(6.2)\qquad\sum_{(\varepsilon_1,\varepsilon_2,\varepsilon_3)}&\frac{\mathrm{Reg}(\mathbb{Z}[\varepsilon_1])\ \mathrm{Cl}(\mathbb{Z}[\varepsilon_1])\,|\varepsilon_1^2\varepsilon_2|}{|(\varepsilon_1-\varepsilon_2)(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)|}\,\widetilde{g_T}(\mathrm{diag}(|\varepsilon_1|,|\varepsilon_2|,|\varepsilon_3|))\\ &=O\left(T^{-1}\int_1^\infty x^{-4}\,d\pi(x)\right),\end{aligned}$$

for T bounded above.

To complete our proof of Proposition 6.1, we need only show the above integral converges. We do this via

Lemma 6.2. *$\pi(x) = O(x^3)$ for $x \geq 1$.*

Proof. Consider the map

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3) \to \lambda^3 - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\lambda^2 + (\varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_3)\lambda - 1.$$

This map is an injection from the set $\{(\varepsilon_1, \varepsilon_2, \varepsilon_3)\}$ figuring in the definition of $\pi(x)$, into the set of cubic monic polynomials $\lambda^3 - v\lambda^2 + w\lambda - 1 \in \mathbb{Z}[x]$ with

$$|v| \leq |\varepsilon_1| + |\varepsilon_2| + |\varepsilon_3| < 3x; \qquad |w| \leq |\varepsilon_1\varepsilon_2| + |\varepsilon_2\varepsilon_3| + |\varepsilon_1\varepsilon_3| < 3x^2.$$

This set of polynomials clearly has cardinality $O(x^3)$, whence the lemma. $\square$

We now integrate the right–hand side of equation (6.2) by parts. By Lemma 6.2, and since $\pi(1) = 0$, we find

$$\begin{aligned} &\sum_{(\varepsilon_1,\varepsilon_2,\varepsilon_3)} \frac{\operatorname{Reg}(\mathbb{Z}[\varepsilon_1])\ \operatorname{Cl}(\mathbb{Z}[\varepsilon_1])\ |\varepsilon_1^2\varepsilon_2|}{|(\varepsilon_1 - \varepsilon_2)(\varepsilon_1 - \varepsilon_3)(\varepsilon_2 - \varepsilon_3)|}\, \widetilde{g_T}(\operatorname{diag}(|\varepsilon_1|, |\varepsilon_2|, |\varepsilon_3|)) \\ &\qquad = O\left(T^{-1} \int_1^\infty x^{-5}\, \pi(x)\, dx\right) = O\left(T^{-1} \int_1^\infty x^{-2}\, dx\right) = O(T^{-1}) \end{aligned}$$

as $T \to 0^+$. Thus Proposition 6.1 is proved. $\square$

§7. The loxodromic and parabolic terms.

In this section we complete our study of the orbital side of the trace formula. The calculations in this section will be brief, as the arguments regarding the loxodromic term are quite similar to those of the previous section, and the parabolic terms may be evaluated readily.

We begin with the loxodromic term

$$\sum_{(r,\theta)} \frac{|\ln r_0|\ \operatorname{Cl}(\mathbb{Z}[r])}{|1 - 2r^{-3}\cos\theta + r^{-6}|} \int_{\operatorname{Re}(s)=1/2} \int_{\operatorname{Re}(t)=1/2} \widehat{g_T}(s,t)\, \frac{r^{1-s}\mathrm{e}^{-2\theta \operatorname{Im}(t)}}{1 + \mathrm{e}^{-2\pi \operatorname{Im}(t)}}\, ds\, dt$$

(cf. Proposition 3.1). By pairing each summand arising from a conjugacy class $\{g\}$ in the sum with the one arising from $\{g^{-1}\}$ (and noting that r is a unit in $\mathbb{Z}[r]$), we may write this term as

$$\sum_{\substack{(r,\theta)\\ r>1}} \frac{|\ln r_0|\ \mathrm{Cl}(\mathbb{Z}[r])\,(1+r^{-6})}{|1-2r^{-3}\cos\theta+r^{-6}|} \cdot \int_{\mathrm{Re}(s)=1/2}\int_{\mathrm{Re}(t)=1/2} \widehat{g_T}(s,t)\,\frac{(r^{1-s}+r^{s-1})\,\mathrm{e}^{-2\theta \mathrm{Im}(t)}}{1+\mathrm{e}^{-2\pi\mathrm{Im}(t)}}\,ds\,dt.$$

The integral in t above is readily shown to be $O(T^{-1/2})$. The integral in s may be evaluated by techniques very similar to those employed in §6. We may thus show that

$$\int_{\mathrm{Re}(s)=1/2}\int_{\mathrm{Re}(t)=1/2} \widehat{g_T}(s,t)\,\frac{(r^{1-s}+r^{s-1})\,\mathrm{e}^{-2\theta \mathrm{Im}(t)}}{1+\mathrm{e}^{-2\pi\mathrm{Im}(t)}}\,ds\,dt = O(T^{-1}r^{1/2}\mathrm{e}^{-(\log^2 r)/4T}).$$

Next, as in Lemma 6.1, and using the fact that $|\ln r_0| \le |\ln r|$, we may show that $|\ln r_0|\ \mathrm{Cl}(\mathbb{Z}[r]) = O(r^c)$ for some $c>0$. We therefore find that, for T bounded above,

$$\begin{aligned}
&\sum_{(r,\theta)} \frac{|\ln r_0|\ \mathrm{Cl}(\mathbb{Z}[r])}{|1-2r^{-3}\cos\theta+r^{-6}|}\int_{\mathrm{Re}(s)=1/2}\int_{\mathrm{Re}(t)=1/2} \widehat{g_T}(s,t)\,\frac{r^{1-s}\mathrm{e}^{-2\theta \mathrm{Im}(t)}}{1+\mathrm{e}^{-2\pi\mathrm{Im}(t)}}\,ds\,dt\\
&= O\left(T^{-1}\sum_{\substack{(r,\theta)\\ r>1}} r^{c+1/2}\mathrm{e}^{-(\log^2 r)/4T}\right)\\
&= O\left(T^{-1}\sum_{\substack{(r,\theta)\\ r>1}} r^{-4}\right) = O\left(T^{-1}\int_1^\infty x^{-4}\,d\sigma(x)\right)
\end{aligned}$$

where

$$\sigma(x) = \left\{ \begin{array}{c} \text{distinct pairs } (r,\theta) \text{ corresponding} \\ \text{to loxodromic matrices } : 1<r\le x, 0<\theta\le\pi \end{array} \right\}.$$

To complete our study of the loxodromic term, we need to consider the quantity $\sigma(x)$. But minor modifications applied to Lemma 6.2 tell us that $\sigma(x) = O(x^3)$, whence

(7.1)

$$\begin{aligned}
&\sum_{(r,\theta)} \frac{|\ln r_0|\ \mathrm{Cl}(\mathbb{Z}[r])}{|1-2r^{-3}\cos\theta+r^{-6}|} \cdot \int_{\mathrm{Re}(s)=1/2}\int_{\mathrm{Re}(t)=1/2} \widehat{g_T}(s,t)\,\frac{r^{1-s}\mathrm{e}^{-2\theta \mathrm{Im}(t)}}{1+\mathrm{e}^{-2\pi\mathrm{Im}(t)}}\,ds\,dt\\
&= O\left(T^{-1}\int_1^\infty x^{-2}\,dx\right) = O(T^{-1}),
\end{aligned}$$

as $T \to 0^+$.

We now need only evaluate the parabolic terms in Proposition 3.1. It follows immediately from equation (3.1) that

$$(7.2)\quad K_1 \widehat{g}\left(\frac{1}{2},\frac{1}{2}\right) + K_2 \widehat{g}\left(\frac{1}{3},1\right) + K_3 \widehat{g}\left(\frac{2}{3},1\right) = K_1 \mathrm{e}^{-T/3} + (K_2+K_3)\mathrm{e}^{-2T/9},$$

and we are done.

§8. Proof of Theorem 1.

In this section, we put together previous results to obtain an asymptotic expression for $L_N(T)$. A Tauberian theorem is finally used to derive from this the desired formula for $N(x)$.

We begin by combining Proposition 3.1, for $g = g_T$ given by (3.1), with Propositions 5.1, 6.1; equations (7.1) and (7.2). We then have

$$(8.1)\quad \begin{aligned} L_N(T) \equiv \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda_n T} &= \frac{\mathrm{vol}(\Gamma\backslash H^3)}{(4\pi)^{5/2}} T^{-5/2} + O(T^{-3/2} + T^{-1} + 1) \\ &\sim \frac{\mathrm{vol}(\Gamma\backslash H^3)}{(4\pi)^{5/2}} T^{-5/2} \end{aligned}$$

as $T \to 0^+$.

We now note that

$$L_N(T) = \int_0^{\infty} \mathrm{e}^{-xT}\, dN(x),$$

the Laplace transform of $N(x)$. Then a standard Tauberian theorem (see Widder, [**Wi**]) yields

$$N(x) \sim \frac{\mathrm{vol}(\Gamma\backslash H^3)}{(4\pi)^{5/2}\Gamma(7/2)} x^{5/2},$$

as $x \to \infty$. This completes the proof of Theorem 1.

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