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A NOTE ON A PAPER OF E. BOASSO AND A. LAROTONDA

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E. Boasso and A. Larotonda recently introduced a spectrum for complex solvable Lie algebras of operators, which agrees in the abelian case with the Taylor spectrum for several commuting operators, and proved that their spectrum also satisfies the projection property. This result is correct, but there seems to be a misunderstanding of a theorem of Cartan and Eilenberg in the proof. In this paper we prove the projection property of this generalized Taylor spectrum with the help of the projection property of the approximate point spectrum.

1. Introduction.

Let L be a finite dimensional complex Lie algebra, X a complex Banach space and $\rho : L \to L(X)$ a representation of L in X. As usual we denote the exterior algebra of L by $\Lambda(L)$. For all $p \in \mathbb{Z}$ we define a linear map $d_p(\rho) : X \otimes \Lambda_p(L) \to X \otimes \Lambda_{p-1}(L)$ by

$$d_p(\rho)(x \otimes a_1 \cdots a_p) = \sum_{i=1}^{p}(-1)^i \rho(a_i)x \otimes a_1 \cdots \widehat{a}_i \cdots a_p + \sum_{1 \le i < k \le p} (-1)^{i+k} x \otimes [a_i, a_k] \cdot a_1 \cdots \widehat{a}_i \cdots \widehat{a}_k \cdots a_p$$

for all $x \in X$ and $a_1, \ldots , a_p \in L$ (where $\widehat{\cdot}$ means deletion). Then $d_{p-1}(\rho)d_p(\rho) = 0$ for all $p \in \mathbb{Z}$, i.e.

$$\Lambda(\rho) := (X \otimes \Lambda_p(L), d_p(\rho))_{p \in \mathbb{Z}}$$

is a chain complex, the *Koszul complex of* ρ (see [**Ko50**] or [**CE**]). J.L. Taylor used this chain complex for his functional calculus, first in the abelian case (see [**Ta70a**] and [**Ta70b**]), but later also in the nonabelian case (see [**Ta72**]).

Let $L' := [L, L]$ be the commutator subalgebra of L and $\widehat{L} := \{\alpha \in L^* : \alpha|_{L'} = 0\}$ the set of all characters of L. For each $\alpha \in \widehat{L}$ one gets a new representation $\rho - \alpha := \rho - \alpha \cdot id$ of L in X. For each $p \in \mathbb{Z}$ define

$$\sigma^p(\rho) := \left\{\alpha \in \widehat{L} : \operatorname{ran} d_{p+1}(\rho - \alpha) \neq \ker d_p(\rho - \alpha)\right\}.$$

Then the set $\sigma(\rho) := \bigcup_{p\in\mathbb{Z}} \sigma^p(\rho)$ is called the *Taylor spectrum* of ρ. If L is a Lie subalgebra of $L(X)$, the inclusion map $i : L \to L(X)$ is a representation of L in X and the above defined set $\sigma(i)$ agrees with the spectrum $\mathrm{Sp}(L, X)$ defined in Definition 1 of [**BL93**].

Boasso and Larotonda stated the projection property of this Taylor spectrum in the following form ([**BL93**, Theorem 3]): For each solvable Lie subalgebra L of $L(X)$ and each ideal I of L it holds:

$$\pi\,\mathrm{Sp}(L, X) = \mathrm{Sp}(I, X),$$

where $\pi : L^* \to I^*$ is the restriction map. This result is correct (see the proof in Section 3), but there is at least a gap in the proof of [**BL93**]. Boasso and Larotonda claimed that the coadjoint representation Ad^* of the simply connected Lie group $G(L)$ of L leaves $\mathrm{Sp}(I, X)$ invariant, i.e. that $h := \mathrm{Ad}^*(g)f = f \circ \mathrm{Ad}(g^{-1}) \in \mathrm{Sp}(I, X)$ for all $f \in \mathrm{Sp}(I, X)$ and $g \in G(L)$. To prove this claim they use Theorem VIII.3.1 from [**CE**] for the automorphism $\varphi := \mathrm{Ad}(g)$ of the universal enveloping algebra $U(I)$ of I. But in this situation the theorem gives only the obvious isomorphism

$$\mathrm{Tor}_*^{U(I)}(X^\varphi, \mathbb{C}(f)) \cong \mathrm{Tor}_*^{U(I)}(X, \mathbb{C}(h)),$$

where $\mathbb{C}(f)$ (resp. $\mathbb{C}(h)$) is the one dimensional left $U(I)$-module induced by the character $f \in \widehat{I}$ (resp. $h \in \widehat{I}$), X carries the natural right $U(I)$-module structure (defined by $xa := -a(x)$ for all $x \in X$ and $a \in I$) and for each $U(I)$-module A, the module A^φ is obtained by carrying back the structure of A by φ (so that for example $\mathbb{C}(h)^\varphi = \mathbb{C}(f)$). Boasso and Larotonda used the following isomorphism instead:

$$\mathrm{Tor}_*^{U(I)}(X, \mathbb{C}(f)) \cong \mathrm{Tor}_*^{U(I)}(X, \mathbb{C}(h)),$$

which is not established by Theorem VIII.3.1 from [**CE**] (but true if you prove the projection property because in this case $h = f$).

2. The Approximate Point Spectrum.

Let L be a finite dimensional complex Lie algebra, X a non-zero complex Banach space and $\rho : L \to L(X)$ a representation of L in X.

Definition 2.1. Let $\alpha \in L^*$. Define

$$E_\alpha(\rho) := \{x \in X : \rho(a)x = \alpha(a)x \text{ for all } a \in L\}.$$

α is said to be in the *point spectrum* $\sigma_p(\rho)$ of ρ if $E_\alpha(\rho) \neq \{0\}$. α is said to be in the *approximate point spectrum* $\sigma_{ap}(\rho)$ of ρ, if there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in X such that $||x_k|| = 1$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} (\rho(a) - \alpha(a))x_k = 0 \text{ for all } a \in L.$$

Remark 2.2.

$$\sigma_p(\rho) \subseteq \sigma_{ap}(\rho) \subseteq \widehat{L}.$$

The following lemma is a Banach space version of a well known lemma of Lie theory. The proof is motivated by the proof of Lemma 1 from [**Po92**].

Lemma 2.3. *Let I be an ideal of L and $\alpha \in \sigma_p(\rho|_I)$. Then ρ leaves the eigenspace $E_\alpha(\rho|_I)$ invariant.*

Proof. It is enough to show that $\alpha|_{[I,L]} = 0$. Let $a \in I$ and $b \in L$. The commutator subalgebra $I' = [I, I]$ of I is a characteristic subalgebra of I and therefore also an ideal of L. So one can easily prove that ρ leaves the null eigenspace $W := E_0(\rho|_{I'})$ invariant. Since $[a, [a, b]] \in I'$ it follows that

$$[\rho(a)|_W, [\rho(a)|_W, \rho(b)|_W]] = \rho([a, [a, b]])|_W = 0.$$

Therefore the operator $[\rho(a)|_W, \rho(b)|_W] = \rho([a, b])|_W$ is quasi-nilpotent by the Kleinecke-Širokov Theorem (see [**Kl57**]). Since $\rho|_I$ leaves the subspace $E := E_\alpha(\rho|_I)$ of W invariant the operator $\rho([a, b])|_E$ is also quasi-nilpotent. But $\rho([a, b])|_E = \alpha([a, b]) \cdot \mathrm{id}_E$ by the definition of E. So $\alpha([a, b]) = 0$. □

The ultrapower technique (see for example [**He80**]) helps us to reduce the proof of the projection property of the approximate point spectrum to the above lemma about the point spectrum. We only need the following two elementary lemmas about ultrapowers. For each nontrivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$ let $X_\mathcal{U}$ be the ultrapower of X with respect to $\mathcal{U}$, and for each Banach space Y and every linear operator $T \in L(X, Y)$ let $T_\mathcal{U} \in L(X_\mathcal{U}, Y_\mathcal{U})$ be the linear operator induced by T with respect to $\mathcal{U}$.

Lemma 2.4. *Let $\mathcal{U}$ be a nontrivial ultrafilter on $\mathbb{N}$. For each $a \in L$ define*

$$\rho_\mathcal{U}(a) := \rho(a)_\mathcal{U} \in L(X_\mathcal{U}).$$

Then $\rho_\mathcal{U}$ is a representation of L in $X_\mathcal{U}$ and

$$\sigma_{ap}(\rho) = \sigma_{ap}(\rho_\mathcal{U}) = \sigma_p(\rho_\mathcal{U}).$$

Lemma 2.5. *Let $\mathcal{U}$ be a nontrivial ultrafilter on $\mathbb{N}$, Y a Banach space and $T \in L(X)$, $S \in L(X, Y)$ linear operators such that the operator $T_\mathcal{U} \in L(X_\mathcal{U})$ leaves $\ker S_\mathcal{U}$ invariant. Then*

$$\sigma_{ap}(T_\mathcal{U}|_{\ker S_\mathcal{U}}) = \sigma_p(T_\mathcal{U}|_{\ker S_\mathcal{U}}).$$

Now we are in a position to prove the projection property of the approximate point spectrum. Our proof is a nonabelian version of Lyubichs proof in the abelian case (see [**Ly**], "basic lemma" on page 115 or [**Ly71**, Lemma 2]).

Theorem 2.6. *Let I be an one codimensional ideal of L. Then*

$$\sigma_{ap}(\rho|_I) = \sigma_{ap}(\rho)|_I.$$

Proof. The inclusion "$\supseteq$" is obvious. Let $\alpha \in \sigma_{ap}(\rho|_I)$, $(a_0, a_1, \dots, a_n)$ a basis of L such that $(a_1, \dots, a_n)$ is a basis of I and let $\mathcal{U}$ be a nontrivial ultrafilter on $\mathbb{N}$. Then $\alpha \in \sigma_p(\rho_{\mathcal{U}}|_I)$ by Lemma 2.4. Therefore $\rho_{\mathcal{U}}$ leaves the (nontrivial) eigenspace $E_\alpha(\rho_{\mathcal{U}}|_I)$ invariant by Lemma 2.3. Now consider the operators $T := \rho(a_0) \in L(X)$ and $S \in L(X, X^n)$, defined by

$$Sx := ((\rho(a_1) - \alpha(a_1))x, \dots, (\rho(a_n) - \alpha(a_n))x)$$

for all $\in X$. Then $T_{\mathcal{U}} = \rho_{\mathcal{U}}(a_0)$ leaves $\ker S_{\mathcal{U}} = E_\alpha(\rho_{\mathcal{U}}|_I)$ invariant. Therefore

$$\sigma_p(T_{\mathcal{U}}|_{\ker S_{\mathcal{U}}}) = \sigma_{ap}(T_{\mathcal{U}}|_{\ker S_{\mathcal{U}}}) \neq \emptyset$$

by Lemma 2.5. Let $\lambda \in \sigma_p(T_{\mathcal{U}}|_{\ker S_{\mathcal{U}}})$ and define $\beta \in L^*$ by $\beta|_I := \alpha$ and $\beta(a_0) := \lambda$. Then $\beta \in \sigma_p(\rho_{\mathcal{U}}) = \sigma_{ap}(\rho)$, and so $\alpha \in \sigma_{ap}(\rho)|_I$. $\square$

Corollary 2.7. *Let I be an ideal of L such that L/I is solvable. Then*

$$\sigma_{ap}(\rho|_I) = \sigma_{ap}(\rho)|_I.$$

In particular if L is solvable then the above equation holds for every ideal I of L and $\sigma_{ap}(\rho) \neq \emptyset$.

The following Banach space version of the well known theorem of Lie will be the key step in our proof of the projection property of the Taylor spectrum in Section 3. In [**BL93**] it is proven as a corollary of the projection property of the Taylor spectrum.

Corollary 2.8. *Let L be solvable. Then $\rho(a)$ is quasi-nilpotent for all $a \in L'$.*

Proof. Let $a \in L' \setminus \{0\}$ and $\lambda \in \sigma_{ap}(\rho(a))$. Because L is solvable, L' is nilpotent. In this situation it is well known that there exists a subideal chain

$$\{0\} = L_0 \trianglelefteq L_1 \trianglelefteq \cdots \trianglelefteq L_k = L' \trianglelefteq \cdots \trianglelefteq L_n = L$$

with $a \in L_1$ and $\dim L_i = i$ for all $i \in \{0, 1, \dots, n\}$. By Theorem 2.6 there exists a character $\alpha \in \sigma_{ap}(\rho)$ with $\alpha(a) = \lambda$. Because of $\sigma_{ap}(\rho) \subseteq \widehat{L}$ and $a \in L'$ it follows that $\lambda = \alpha(a) = 0$. Therefore $\rho(a)$ is quasi-nilpotent. $\square$

3. The Taylor Spectrum.

Let L, X and ρ be as in Chapter 2. The following remark, which gives an inductive description of the Koszul complex, is an obvious generalization of Lemma 1.3 from [**Ta70a**] (see also [**Sl77**, Definition 1.1]).

Remark 3.1. *Let I be an one codimensional ideal of L and $a_0 \in L \setminus I$. For each $p \in \mathbb{Z}$ define a linear map $f_p : X \otimes \Lambda_p(I) \to X \otimes \Lambda_p(I)$ by*

$$f_p(x \otimes a_1 \cdots a_p) = \rho(a_0)x \otimes a_1 \cdots a_p + \sum_{i=1}^{p} (-1)^i x \otimes [a_i, a_0] \cdot a_1 \cdots \widehat{a}_i \cdots a_p$$

for all $a_1, \ldots, a_p \in I$ and $x \in X$. Then $f := (f_p)_{p \in \mathbb{Z}}$ is a chain map of the Koszul complex $\Lambda(\rho|_I)$, and the Koszul complex $\Lambda(\rho)$ is isomorphic to the mapping cone $M(f)$ of f in the following sense: For each $p \in \mathbb{Z}$ consider the decomposition

$$\begin{aligned} X \otimes \Lambda_p(L) &= X \otimes \Lambda_p(I) \oplus X \otimes \Lambda_{p-1}(I) \cdot a_0 \\ &\cong X \otimes \Lambda_p(I) \oplus X \otimes \Lambda_{p-1}(I). \end{aligned}$$

Then the operator $d_p(\rho) : X \otimes \Lambda_p(L) \to X \otimes \Lambda_{p-1}(L)$ has the following matrix form with respect to this decomposition:

$$d_p(\rho) = \begin{pmatrix} d_p(\rho|_I) & (-1)^p f_{p-1} \\ 0 & d_{p-1}(\rho|_I) \end{pmatrix}.$$

In particular one gets a short exact sequence

$$0 \longrightarrow \Lambda(\rho|_I) \xrightarrow{i} \Lambda(\rho) \xrightarrow{\pi} \Lambda(\rho|_I)^- \longrightarrow 0$$

of chain complexes, where $i_p : X \otimes \Lambda_p(I) \to X \otimes \Lambda_p(L)$, $\omega \mapsto (\omega, 0)$ is the inclusion map, $\pi_p : X \otimes \Lambda_p(L) \to X \otimes \Lambda_{p-1}(I)$, $(\omega, \nu) \mapsto \nu$ is the projection map onto the second component with respect to the above decomposition for all $p \in \mathbb{Z}$ and

$$\Lambda(\rho|_I)^- := (X \otimes \Lambda_{p-1}, d_{p-1}(\rho|_I))_{p \in \mathbb{Z}}.$$

The homology lemma now gives an exact sequence

$$\cdots \longrightarrow H_p(\rho|_I) \xrightarrow{i_p^*} H_p(\rho) \xrightarrow{\pi_p^*} H_{p-1}(\rho|_I) \xrightarrow{f_{p-1}^*} H_{p-1}(\rho|_I) \xrightarrow{i_{p-1}^*} H_{p-1}(\rho) \longrightarrow \cdots,$$

where $H_p(\rho) := \ker d_p(\rho) / \operatorname{ran} d_{p+1}(\rho)$.

This remark immediately implies the easy half of the projection property of the Taylor spectrum, which we state in the version of Slodkowski (see [**Sl77**], Theorem 1.7 in the abelian case).

Proposition 3.2. *Let I be an one codimensional ideal of L. Then*

$$\sigma^p(\rho)|_I \subseteq \sigma^p(\rho|_I) \cup \sigma^{p-1}(\rho|_I)$$

for each $p \in \mathbb{Z}$. In particular $\sigma(\rho)|_I \subseteq \sigma(\rho|_I)$.

Proof. Let $p \in \mathbb{Z}$ and $\alpha \in \widehat{L}$ with $\alpha|_I \notin \sigma^p(\rho|_I) \cup \sigma^{p-1}(\rho|_I)$. Consider the representation $\mu := \rho - \alpha$ instead of ρ. Then the above remark gives a long exact sequence

$$\cdots \longrightarrow H_p(\mu|_I) = \{0\} \longrightarrow H_p(\mu) \longrightarrow H_{p-1}(\mu|_I) = \{0\} \longrightarrow \cdots .$$

It follows $H_p(\mu) = \{0\}$, i.e. $\alpha \notin \sigma^p(\rho)$. □

Corollary 3.3. *Let L be nilpotent and let $\alpha \in \sigma(\rho)$. Then $\alpha(a) \in \sigma(\rho(a))$ for each $a \in L$.*

Proof. Let $a \in L \setminus \{0\}$. Since L is nilpotent there exists a subideal chain

$$\{0\} = L_0 \trianglelefteq L_1 \trianglelefteq \cdots \trianglelefteq L_n = L$$

with $a \in L_1$ and $\dim L_i = i$ for all $i \in \{o, \dots, n\}$. By Proposition 3.2 it follows that $\alpha|_{L_1} \in \sigma(\rho|_{L_1})$, i.e. $\alpha(a) \in \sigma(\rho(a))$. □

Now we can prove the other half of the projection property of the Taylor spectrum (in the version of Slodkowski, see [**Sl77**, Theorem 1.7]).

Theorem 3.4. *Let L be solvable and let I be an ideal of L. Then*

$$\sigma^p(\rho|_I) \subseteq \sigma^p(\rho)|_I$$

for all $p \in \mathbb{Z}$. In particular $\sigma(\rho|_I) \subseteq \sigma(\rho)|_I$, and by Proposition 3.2 we have

$$\sigma(\rho|_I) = \sigma(\rho)|_I.$$

Proof. Since L is solvable we can assume I to be one codimensional. Let $\alpha \in \sigma^p(\rho|_I)$. Since I is one codimensional, I contains L'. So $\alpha|_{L'} \in \sigma(\rho|_{L'})$ by Proposition 3.2. Since L' is nilpotent, Corollary 3.3 implies that $\alpha(a) \in \sigma(\rho(a))$ for all $a \in L'$. But $\sigma(\rho(a)) = \{0\}$ for each $a \in L'$ by Corollary 2.8. Therefore $\alpha|_{L'} = 0$, i.e. $\beta \in \widehat{L}$ for all $\beta \in L^*$ with $\beta|_I = \alpha$. The remainder of the proof is as in the abelian case (see [**Sl77**, Theorem 1.7]). □

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References

[CE] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, 1956.

[BL93] E. Boasso and A. Larotonda, *A spectral theory for solvable Lie algebras of operators*, Pacific J. Math., **158** (1993), 15-22.

[He80] S. Heinrich, *Ultraproducts in Banach space theory*, J. für reine und ang. Math., **313** (1980), 72-104.

[Kl57] D.C. Kleinecke, *On operator commutators*, Proc. Amer. Math. Soc., **8** (1957), 535-536.

[Ko50] J.L. Koszul, *Homologie et Cohomologie de Algebres de Lie*, Bull. Soc. Math. France, **78** (1950), 65-127.

[Ly] Y.I. Lyubich, *Introduction to the theory of Banach representations of groups*, Birkhäuser, 1988.

[Ly71] ———, *On the spectrum of a representation of an Abelian topological group*, Soviet Math. Dokl., **12** (1971), 1482-1486.

[Po92] D. Poguntke, *The theorem of Lie and hyperplane subalgebras of Lie algebras*, Geom. Ded., **43** (1992), 83-91.

[Sl77] Z. Slodkowski, *An infinite family of joint spectra*, Stud. Math., **61** (1977), 239-255.

[Ta70a] J.L. Taylor, *A joint spectrum for several commuting operators*, J. Funct. Anal., **6** (1970), 172-191.

[Ta70b] ———, *The analytic functional calculus for several commuting operators*, Acta Math., **125** (1970), 1-38.

[Ta72] ———, *A general framework for a multi-operator functional calculus*, Adv. in Math., **9** (1972), 183-252.

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