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MIXED AUTOMORPHIC VECTOR BUNDLES ON SHIMURA VARIETIES

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Let $S^0(G,X)$, $S^0(G',X')$ be connected Shimura varieties associated to semisimple algebraic groups G, G' defined over $\mathbb{Q}$ and Hermitian symmetric domains X, X'. Let $\rho : G \to G'$ be a homomorphism of algebraic groups over $\mathbb{Q}$ that induces a holomorphic map $\omega : X \to X'$ mapping special points of X to special points of X'. Given equivariant vector bundles $\mathcal{J}$, $\mathcal{J}'$ on the compact duals $\check{X}$, $\check{X}'$ of the symmetric domains X, X', we can construct a mixed automorphic vector bundle $\mathcal{M}(\mathcal{J},\mathcal{J}',\rho)$, on $S^0(G,X)$ whose sections can be interpreted as mixed automorphic forms. We prove that the space of sections of a certain mixed automorphic vector bundles is isomorphic to the space of holomorpic forms of the highest degree on the fiber product of a finite number of Kuga fiber varieties. We also prove that for each automorphism τ of $\mathbb{C}$ the conjugate $\tau\mathcal{M}(\mathcal{J},\mathcal{J}',\rho)$ of a mixed automorphic vector bundle $\mathcal{M}(\mathcal{J},\mathcal{J}',\rho)$ on a connected Shimura variety $S^0(G,X)$ can be canonically realized as a mixed automorphic vector bundle $\mathcal{M}(\mathcal{J}_1,\mathcal{J}_1',\rho_1)$ on another connected Shimura variety $S^0(G_1,X_1)$ associated to a semisimple algebraic group G_1 and a Hermitian symmetric domain X_1.

1. Introduction.

Mixed automorphic forms generalize automorphic forms, and certain types of mixed automorphic forms occur naturally as holomorphic differential forms of the highest degree on certain fiber varieties over arithmetic varieties whose fibers are abelian varieties (see e.g. [**8**], [**14**], [**16**] [**17**], [**18**], [**19**]). Holomorphic automorphic forms can be interpreted as the sections of automorphic vector bundles on a Shimura variety (see [**6**], [**7**], [**21**], [**22**]) just as automorphic functions can be considered as sections of the sheaf of germs of functions on a Shimura variety. In this paper, we introduce mixed automorphic vector bundles on connected Shimura varieties whose sections can be interpreted as mixed automorphic forms.

Let E be an elliptic surface and let $\pi : E \to X$ be an elliptic fibration in the sense of Kodaira (cf. [**11**]). Thus E is a compact smooth surface over $\mathbb{C}$, X

is a compact Riemann surface, and the generic fiber of π is an elliptic curve. We assume that π has a global section and that there are no exceptional curves of the first kind in the fibers of π. Let E_0 be the union of the regular fibers of π and let $X_0 = \pi(E_0)$. We identify the universal covering space of X_0 with the Poincaré upper half plane $\mathcal{H}$, and the fundamental group $\pi_1(X_0)$ with a subgroup Γ of $PSL(2,\mathbb{R})$. Thus we have $X_0 = \Gamma\backslash\mathcal{H}$, where Γ acts on $\mathcal{H}_0$ by linear fractional transformations. Given a point $z \in X_0$, we choose a holomorphic 1-form on $E_z = \pi^{-1}(z)$ and a basis $\{\alpha_z, \beta_z\}$ of $H_1(E_z,\mathbb{Z})$ that depends on $z \in X_0$ in a continuous manner. Then the many-valued function

$$\omega(z) = \frac{\int_{\alpha_z}\Phi}{\int_{\beta_z}\Phi}$$

on X_0 can be lifted to a holomorphic function $\omega : \mathcal{H} \to \mathcal{H}$ satisfying $\omega(\gamma z) = \chi(\gamma)\omega(z)$ for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$, where $\chi : \Gamma \to SL(2,\mathbb{R})$ is the monodromy representation of $\Gamma = \pi_1(X_0)$ for the elliptic fibration $\pi : E \to X$. Hunt and Meyer [**8**] defined mixed cusp forms using the automorphy factor

$$j(\gamma, z) = (cz+d)^2(c_\chi\omega(z) + d_\chi),$$

where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \quad \text{and} \quad \chi(\gamma) = \begin{pmatrix} a_\chi & b_\chi \\ c_\chi & d_\chi \end{pmatrix} \in SL(2,\mathbb{R}).$$

They proved that the space $S_{2,1}(\Gamma,\omega,\chi)$ of mixed cusp forms of type $(2,1)$ associated to Γ, ω and χ is canonically isomorphic to the space $H^0(E,\Omega^2)$ of holomorphic 2-forms on E. In [**14**] mixed automorphic forms of type $(2,n)$ for $n > 1$ were defined using the automorphy factor

$$j(\gamma, z) = (cz+d)^2(c_\chi\omega(z) + d_\chi)^n,$$

and it was proved that the space $S_{2,n}(\Gamma,\omega,\chi)$ of mixed cusp forms of type $(2,m)$ associated to Γ, ω and χ is canonically isomorphic to the space $H^0(E^n,\Omega^{n+1})$ of holomorphic $(n+1)$-forms on the elliptic variety E^n, where E^n is obtained by resolving the singularities of the compactification of the n-fold fiber product of E_0 over X_0. Assuming that $\Gamma \subset SL(2,\mathbb{R})$ with $-1 \notin \Gamma$ and that χ is an inclusion $\Gamma \hookrightarrow SL(2,\mathbb{R})$, the above result of Hunt and Meyer was proved by Shioda [**31**] and the higher weight case was proved by Sŏkurov [**32**] (see also [**33**], [**34**]).

More general definition of mixed automorphic forms were given in [**18**], and mixed Siegel modular forms were treated in [**19**]. In this paper, we generalize the notion of mixed automorphic forms further by considering mixed automorphic vector bundles on connected Shimura varieties. Then the mixed automorphic forms are sections of mixed automorphic vector bundles of special type. A Kuga fiber variety is a fiber variety over a Shimura variety whose fibers are isomorphic to a polarized abelian variety (see [**12**], [**30**, Chapter IV]). One of the goals of this paper is to prove Theorem 4.2 which states that the space of sections of a certain type of mixed automorphic vector bundle is canonically isomorphic to the space of holomorphic forms of the highest degree on the fiber product of a finite number of Kuga fiber varieties.

Another goal of this paper is to study the conjugates of mixed automorphic vector bundles. One of the main theorems for automorphic vector bundles proved by Milne ([**21**], [**22**]) is about their conjugation by an automorphism τ of $\mathbb{C}$. More precisely, given a Shimura variety $S(G, X)$ associated to a semisimple algebraic group G over $\mathbb{Q}$ and a Hermitian symmetric domain X, the conjugate $\tau\mathcal{V}(\mathcal{J})$ of every automorphic vector bundle $\mathcal{V}(\mathcal{J})$ on $S(G, X)$ determined by a $G(\mathbb{C})$-vector bundle $\mathcal{J}$ on the compact dual $\check{X}$ of X by an automorphism $\tau$of $\mathbb{C}$ is an automorphic vector bundle of the form $\mathcal{V}({}^{\tau}\mathcal{J})$ determined by the conjugate ${}^{\tau}\mathcal{J}$ for some explicitly determined automorphic vector bundle of $\mathcal{J}$. In other words, this means that to each automorphic form f on the symmetric domain X, a special point x of X (see §4 for the definition of a special point) and an automorphism τ of $\mathbb{C}$, we can associate another automorphic form ${}^{\tau,x}f$ on another symmetric domain ${}^{\tau,x}X$ in such a way that the association $f \mapsto {}^{\tau,x}f$ commutes with the Hecke operators and $\tau(f(x))$ is equal to ${}^{\tau,x}f(y)$ for some explicitly defined special point y of ${}^{\tau,x}X$. A similar problem for automorphic functions instead of forms was conjectured by Langlands [**13**] and was later proved by Milne [**20**] and Borovoi [**1**] (see also [**9**], [**10**], [**15**]).

Let $S^0(G, X)$, $S^0(G', X')$ be connected Shimura varieties associated to semisimple algebraic groups G, G' defined over $\mathbb{Q}$ and Hermitian symmetric domains X, X'. Let $\rho : G \to G'$ be a homomorphism of algebraic groups over $\mathbb{Q}$ that induces a holomorphic map $\omega : X \to X'$ mapping special points of X to special points of X'. Given equivariant vector bundles $\mathcal{J}$, $\mathcal{J}'$ on the compact duals $\check{X}$, $\check{X}'$ of the symmetric domains X, X', we can construct a mixed automorphic vector bundle $\mathcal{M}(\mathcal{J}, \mathcal{J}', \rho)$, on $S^0(G, X)$ (see §3). In this paper, we prove Theorem 5.5 which states that the conjugate $\tau\mathcal{M}$ of a mixed automorphic vector bundle $\mathcal{M}$ on a connected Shimura variety $S^0(G, X)$ can be canonically realized as a mixed automorphic vector bundle $\mathcal{M}(\mathcal{J}_1, \mathcal{J}_1', \rho_1)$ on another connected Shimura variety $S^0(G_1, X_1)$ associated to a semisimple algebraic group G_1 and a Hermitian symmetric domain X_1. In classical terms, this theorem implies that to each mixed automorphic form f on X, a special point x of X and an automorphism τ of $\mathbb{C}$, we can associate another mixed automorphic form ${}^{\tau,x}f$ on the domain X_1 in such a way that $\tau(f(x))$

is related to ${}^{\tau,x}f(x_1)$ for some explicitly defined special point x_1 of X_1 and the association $f \mapsto {}^{\tau,x}f$ commutes with Hecke operators.

This paper is organized as follows. In §2 we review connected Shimura varieties, Serre groups and the conjugation of Shimura varieties. In §3 we describe the Borel embedding of a symmetric domain into its compact dual, and construct mixed automorphic vector bundles on connected Shimura varieties. In §4 we discuss the connection of mixed automorphic vector bundles and mixed automorphic forms, and prove the theorem about the realization of mixed automorphic vector bundles as holomorphic forms on fiber products of Kuga fiber varieties. The theorem concerning conjugates of mixed automorphic vector bundles is stated and proved in §5.

2. Connected Shimura varieties.

In this section we review the definition of connected Shimura varieties and Serre groups, and describe the theorem about conjugates of Shimura varieties (see **[2]**, **[3]**, **[20]**, **[21]**, **[22]**, **[23]**, **[27]** for details). Let G be a semisimple algebraic group defined over $\mathbb{Q}$, let G^{ad} be the associated adjoint group, and let $G^{\mathrm{ad}}(\mathbb{R})^+$ be the identity component of $G^{\mathrm{ad}}(\mathbb{R})$. Let $\mathbb{S}$ be a real algebraic group $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$, where Res is the Weil's restriction map, and let X be a $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class of homomorphisms $h : \mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$ that satisfy the following conditions:

(i) When composed with $G^{\mathrm{ad}}_{\mathbb{R}} \to GL(\mathfrak{g})$, each h in X defines a Hodge structure of type $\{(0,0),(-1,1),(1,-1)\}$ on the Lie algebra $\mathfrak{g}$ of $G^{\mathrm{ad}}_{\mathbb{R}}$.

(ii) For each h in X, $\mathrm{ad}\,(h(i))$ is a Cartan involution of $G_{\mathbb{R}}$.

(iii) G^{ad} possesses no nontrivial factor defined over $\mathbb{Q}$ whose real points form a compact group.

If x is a point of X regarded as a symmetric domain, we shall denote by h_x the corresponding homomorphism from $\mathbb{S}$ to $G^{\mathrm{ad}}_{\mathbb{R}}$; thus we have $h_{g\cdot x} = \mathrm{ad}\,(g) \circ h_x$ for $g \in G^{\mathrm{ad}}(\mathbb{R})^+$ and $x \in X$. Fix a point x_0 in X and let $h_{x_0} : \mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$ be the homomorphism corresponding to x_0. Let $G(\mathbb{R})^+$ be the identity component of $G(\mathbb{R})$ and let K_0 be the subgroup of $G(\mathbb{R})^+$ that fixes x_0. Since K_0 is fixed by $h_{x_0}(i)$, axiom (ii) above implies that it is compact. The Lie algebra $\mathfrak{g}$ of $G(\mathbb{R})^+$ has the Cartan decomposition $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ with $\mathfrak{k} = \mathrm{Lie}\,K_0$, where $\mathfrak{k}$ and $\mathfrak{p}$ are $+1$ and -1 eigenspaces for ad $h(i)$ acting on $\mathfrak{g}$. The action of $G(\mathbb{R})^+$ on X determines a bijection of $G(\mathbb{R})^+/K_0$ with X. If we use this bijection to provide X with a real analytic structure, then $\mathfrak{p}$ can be identified with the tangent space $T_{x_0}(X)$ of X at x_0. There is a unique homogeneous complex structure on X such that the action of i on $T_{x_0}(X)$ corresponds to the action of $h(e^{\pi i/4})$ on $\mathfrak{p}$. Relative to this structure, X becomes a Hermitian symmetric domain.

A congruence subgroup of $G(\mathbb{Q})$ is a subgroup of the form $\Gamma = K \cap G(\mathbb{Q})$ with K a compact open subgroup of $G(\mathbb{A}^f)$. Consider a topology on $G^{\mathrm{ad}}(\mathbb{Q})$ in which the images of the congruence subgroups in $G(\mathbb{Q})$ form a fundamental system of neighborhoods of the identity element, and let $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$ be the completion of $G^{\mathrm{ad}}(\mathbb{Q})^{+}$ relative to this topology. Let $\Sigma(G)$ be the set of torsion-free arithmetic subgroups of $G^{\mathrm{ad}}(\mathbb{Q})^{+}$ that contains the image of a congruence subgroup of $G(\mathbb{Q})$. For each $\Gamma \in \Sigma(G)$, the quotient $\Gamma\backslash X$ is a locally symmetric algebraic variety. The group $G^{\mathrm{ad}}(\mathbb{Q})^{+}$ acts on the projective system $(\Gamma\backslash X)_{\Gamma\in\Sigma(G)}$ as follows: for each $\Gamma \in \Sigma(G)$ and $g \in G^{\mathrm{ad}}(\mathbb{Q})^{+}$, g defines a map

$$\Gamma\backslash X \to g^{-1}\Gamma g\backslash X, \quad [x] \mapsto [g^{-1}x].$$

This map is holomorphic and therefore algebraic. The action of $G^{\mathrm{ad}}(\mathbb{Q})^{+}$ on $(\Gamma\backslash X)_{\Gamma\in\Sigma(G)}$ extends by continuity to $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$. The connected Shimura variety $S^0(G,X)$ is defined to be the projective system $(\Gamma\backslash X)_{\Gamma\in\Sigma(G)}$, or its limit, together with the continuous right action of $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$.

When G is simply connected, $G(\mathbb{R})$ is connected and $G(\mathbb{Q}) \cdot K = G(\mathbb{A}^f)$. For any congruence subgroup $\Gamma = G(\mathbb{Q}) \cap K$ of $G(\mathbb{Q})$, the map

$$\Gamma\backslash X \to G(\mathbb{Q})\backslash X \times G(\mathbb{A}^f)/K, \quad [x] \mapsto [x,1]$$

is an isomorphism. Taking the limit, we have

$$S^0(G,X)(\mathbb{C}) = \varprojlim \Gamma\backslash X = G(\mathbb{Q})\backslash X \times G(\mathbb{A}^f).$$

The semi-direct product $G(\mathbb{A}^f) \rtimes G^{\mathrm{ad}}(\mathbb{Q})^{+}$ acts on this scheme by

$$[x,a](g,q) = [q^{-1}x, \mathrm{ad}\,(q^{-1})(ag)]$$

for $x \in X$, $a, g \in G(\mathbb{A}^f)$ and $q \in G^{\mathrm{ad}}(\mathbb{Q})^{+}$. The homomorphism $q \mapsto (q^{-1}, \mathrm{ad}\; q)$ identifies $G(\mathbb{Q})$ with a normal subgroup of $G(\mathbb{A}^f) \rtimes G^{\mathrm{ad}}(\mathbb{Q})^{+}$, and the quotient group $G(\mathbb{A}^f) *_{G(\mathbb{Q})} G^{\mathrm{ad}}(\mathbb{Q})^{+}$ continues to act on $S^0(G,X)$. In this case, we have

$$G(\mathbb{A}^f) *_{G(\mathbb{Q})} G^{\mathrm{ad}}(\mathbb{Q})^{+} = G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge},$$

and the action just described agrees with that defined in the preceding paragraph.

A real Hodge structure is a real vector space V with a decomposition

$$V \otimes \mathbb{C} = \bigoplus V^{p,q}, \quad \text{with} \quad V^{p,q} = \overline{V^{p,q}}.$$

The category of real Hodge structures has a Tannakian structure, and the affine group scheme attached to the category and the forgetful functor is

$\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_m$ (see [4], [28], [29]). If V has a real Hodge structure with decomposition $V \otimes \mathbb{C} = \bigoplus V^{p,q}$, then an element $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ acts on $V^{p,q}$ as multiplication by $z^{-p}\overline{z}^{-q}$. Thus we can consider a real Hodge structure as a pair (V, h) consisting of a real vector space V and a homomorphism $h : \mathbb{S} \to GL(V)$.

If W is a vector space over $k \subset \mathbb{C}$, a homomorphism $\nu : \mathbb{G}_m \to GL(W)$ defines a decomposition $W = \bigoplus W^i$ with

$$W^i = \{w \in W \mid \nu(z) = z^i v \quad \text{for all} \quad z \in k^\times\}$$

and a decreasing filtration $F^\bullet$ of W with $F^pW = \bigoplus_{i \geq p} W^i$. Given a real Hodge structure, let $\mu_h : \mathbb{G}_m \to GL(V_\mathbb{C})$ be the map defined by $\mu_h(z) = h_\mathbb{C}(z, 1)$, where we identified $\mathbb{S}_\mathbb{C}$ with $\mathbb{C}^\times \times \mathbb{C}^\times$ in such a way that the embedding $\mathbb{S}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{C})$ becomes $z \mapsto (z, \overline{z})$. Then the Hodge filtration on V is simply the decreasing filtration defined by $\mu_h : \mathbb{G}_m \to GL(V_\mathbb{C})$, and the weight grading $w_h : \mathbb{G}_m \to GL(V)$ is defined by $w_h(r) = h(r^{-1})$ for all $r \in \mathbb{R}^\times$. A Hodge structure is a vector space V over $\mathbb{Q}$ together with a real Hodge structure on $V \otimes_\mathbb{Q} \mathbb{R}$ such that the weight grading is defined over $\mathbb{Q}$.

The *Mumford-Tate group* $MT(V, h)$ of a Hodge structure (V, h) is the smallest $\mathbb{Q}$-rational algebraic subgroup of $GL(V) \times \mathbb{G}_m$ such that $MT(V, h)_\mathbb{C}$ contains the image of $(\mu_h, 1) : \mathbb{G}_m \to GL(V) \times \mathbb{G}_m$ (see [4], [26]). The Hodge structure (V, h) of weight n is said to be polarizable if there is a morphism of Hodge structures $\psi : V(\mathbb{R}) \otimes V(\mathbb{R}) \to \mathbb{R}(-n)$ such that the real-valued form $(x, y) \mapsto (2\pi i)^n \psi(x, h(i)y)$ is symmetric and positive definite.

A Hodge structure is said to be of CM-type if it is polarizable and its Mumford-Tate group is commutative. The category of Hodge structures of CM-type is a Tannakian category. The *Serre group* $\mathfrak{S}$ is the affine group scheme attached to this Tannakian category and the forgetful fiber functor. The functor sending a Hodge structure (V, h) to the real Hodge structure $(V \otimes \mathbb{R}, h)$ defines a homomorphism $h_{\mathrm{can}} : \mathbb{S} \to \mathfrak{S}_\mathbb{R}$. The Serre group $\mathfrak{S}$ and the homomorphism h_{can} have the following universal property: For any torus T over $\mathbb{Q}$ and homomorphism $h : \mathbb{S} \to T_\mathbb{R}$ whose cocharacter is defined over a CM-field and whose weight is defined over $\mathbb{Q}$ there is a unique $\mathbb{Q}$-rational homomorphism $\rho : \mathfrak{S} \to T$ such that $\rho_\mathbb{R} \circ h_{\mathrm{can}} = h$ (see [25]).

The category of CM-motives over $\mathbb{Q}$ is a $\mathbb{Q}$-linear Tannakian category (see [22], [24] for the definition of CM-motives). The affine group scheme $\mathfrak{T}$ attached to this category and the Betti fiber functor H_B is called the *Taniyama group* (see [5]). The fully faithful tensor functor from the category of Artin motives over $\mathbb{Q}$ to the category of CM-motives determines a surjective homomorphism $\pi : \mathfrak{T} \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The Betti functor H_B is an essentially surjective functor from the category of CM-motives to the category of Hodge structures of CM-type; hence it determines an injective homomorphism $i : \mathfrak{S} \to \mathfrak{T}$. Each $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ determines an automorphism $\mathrm{sp}(\tau)$ of the fiber functor $H_B \otimes \mathbb{Q}_\ell$ whose image in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is τ. The map sp is a homomorphism from $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to $\mathfrak{T}(\mathbb{Q}_\ell)$ that is continuous with respect to the Krull and ℓ-adic topologies, and the product of the homomorphisms sp_ℓ defines a homomorphism $\mathrm{sp} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathfrak{T}(\mathbb{A}^f)$ (see [**22**] for details). Then there is an exact sequence

$$1 \to \mathfrak{S} \xrightarrow{i} \mathfrak{T} \xrightarrow{\phi} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1$$

of affine group schemes (see [**5**]).

Let $S^0(G, X)$ be a connected Shimura variety, and let x be a special point of X. This means that there is a maximal $\mathbb{Q}$-rational torus T in G such that h_x factors through $(T/Z)(\mathbb{R})$, where Z is the center of G. By the universal property of $\mathfrak{S}$ there is a unique $\mathbb{Q}$-rational homomorphism $\rho_x : \mathfrak{S} \to T/Z$ such that $h_x = (\rho_x)_\mathbb{R} \circ h_{\mathrm{can}}$. The map $\rho_x : \mathfrak{S} \to G^{\mathrm{ad}}$ defines an action of $\mathfrak{S}$ on G. For each $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the association $M \mapsto H_\tau(M) = H_B(\tau M)$ is a fiber functor from the category of CM-motives over $\overline{\mathbb{Q}}$ to the category of vector spaces over $\overline{\mathbb{Q}}$; hence $\mathrm{Isom}(H_B, H_\tau)$ is a torsor for $\mathfrak{S}$, and it is represented by ${}^\tau\mathfrak{S} = \pi^{-1}(\tau)$. Using the $\mathfrak{S}$-torsor ${}^\tau\mathfrak{S}$ to twist G, we obtain an algebraic group ${}^{\tau,x}G = {}^\tau\mathfrak{S} \times_{\mathfrak{S},\rho_x} G$ over $\mathbb{Q}$ such that

$${}^{\tau,x}G(\overline{\mathbb{Q}}) = \{s \cdot g \mid s \in {}^\tau\mathfrak{S}(\overline{\mathbb{Q}}),\ g \in G(\overline{\mathbb{Q}})\}/\mathfrak{S}(\overline{\mathbb{Q}})$$

on which $\mathfrak{S}(\overline{\mathbb{Q}})$ acts by

$$(s \cdot g)\sigma = s\sigma \cdot \mathrm{ad}(\rho_x(\sigma^{-1}))g$$

for $\sigma \in \mathfrak{S}(\overline{\mathbb{Q}})$ and $(s \cdot g) \in {}^{\tau,x}G(\overline{\mathbb{Q}})$. Then ${}^{\tau,x}G$ is a semisimple group that contains ${}^\tau\mathfrak{S} \times_{\mathfrak{S},\rho_x} T = T$ as a subtorus. The point $\mathrm{sp}(\tau)$ in ${}^\tau\mathfrak{S}(\mathbb{A}^f)$ defines a canonical isomorphism of $\tilde{G}(\mathbb{A}^f)$ onto ${}^{\tau,x}\tilde{G}(\mathbb{A}^f)$ that maps g to ${}^{\tau,x}g = [\mathrm{sp}(\tau) \cdot g]$.

There is a canonical isomorphism

$$G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge} \xrightarrow{\approx} {}^{\tau,x}G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}, \quad g \mapsto {}^{\tau,x}g$$

that is compatible with the preceding isomorphism. Define ${}^\tau h$ to be the homomorphism $\mathbb{S} \to {}^{\tau,x}G^{\mathrm{ad}}_\mathbb{R}$ associated to the cocharacter ${}^\tau\mu_x$ of $T/Z \subset {}^{\tau,x}G^{\mathrm{ad}}$, and let ${}^{\tau,x}X$ be the ${}^{\tau,x}G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class of maps $\mathbb{S} \to {}^{\tau,x}G^{\mathrm{ad}}_\mathbb{R}$ containing ${}^\tau h$. Then the pair $({}^{\tau,x}G, {}^{\tau,x}X)$ defines a Shimura variety.

Proposition 2.1. *If x and y are special points of X, then there is a canonical isomorphism*

$$\phi^0(\tau; y, x) : S^0({}^{\tau,x}G, {}^{\tau,x}X) \to S^0({}^{\tau,y}G, {}^{\tau,y}X)$$

such that

$$\phi^0(\tau; y, x) \circ ({}^{\tau,x}g) = ({}^{\tau,y}g) \circ \phi^0(\tau; y, x)$$

for all $g \in G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$.

Proof. See [**21**, Proposition 1.3]. □

The following is the main theorem about the conjugates of Shimura varieties which was conjectured by Langlands in [**13**].

Theorem 2.2. *For each* $\tau \in \mathrm{Aut}(\mathbb{C})$, *there is a unique isomorphism*

$$\phi^0_{\tau,x} : \tau S^0(G, X) \to S^0({}^{\tau,x}G, {}^{\tau,x}X)$$

satisfying the following conditions:

(i) *The point* $\tau[x]$ *is mapped to* $[{}^{\tau}x]$, *where* ${}^{\tau}x \in {}^{\tau,x}X$ *is the point corresponding to* ${}^{\tau}h$.

(ii) $\phi^0_{\tau,x} \circ \tau(g) = ({}^{\tau,x}g) \circ \phi^0_{\tau,x}$ *for all* $g \in G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$.

Furthermore, if y *is another special point of* X, *then*

$$\phi^0(\tau; y, x) \circ \phi^0_{\tau,x} = \phi^0_{\tau,y}.$$

Proof. See [**20**, Theorem 1.1] (see also [**1**]). □

3. Mixed automorphic vector bundles.

In this section, we construct automorphic vector bundles on connected Shimura varieties. We first describe the Borel embedding of a Hermitian symmetric space into its compact dual space (see [**21**], [**22**] for details). Let G be a reductive group over a field k, and let $r : G \to GL(V)$ be a representation of G. Then a homomorphism $\mu : \mathbb{G}_m \to G$ defines a decomposition

$$V = \bigoplus V^i, \quad V^i = \{v \in V \mid r \circ \mu(z)v = z^i v \quad \text{for all} \quad z \in k^\times\}$$

and a decreasing filtration $F^\bullet$ of V with $F^pV = \bigoplus_{i \geq p} V^i$. These filtrations are compatible with the formation of tensor products and duals. Conversely, any functor $(r, V) \mapsto (F^\bullet, V)$ from representations of G to filtrations compatible with tensor products and duals arises from a homomorphism $\mu : \mathbb{G}_m \to G$. Such a functor is called a filtration of $\mathbf{Rep}_k(G)$, and the filtration determined by μ is denoted by $\mathrm{Filt}(\mu)$. When k is $\mathbb{C}$ and μ is a cocharacter μ_0 of G, we define the compact dual $\check{X}$ of X to be the set of filtrations of $\mathbf{Rep}_{\mathbb{C}}(G)$ that are $G(\mathbb{C})$-conjugate to $\mathrm{Filt}(\mu_0)$. Then the action of $G(\mathbb{C})$ on $\check{X}$ given by $g \cdot \mu = \mathrm{Filt}(\mathrm{ad}(g) \circ \mu)$ defines a bijection between $G(\mathbb{C})/P_0(\mathbb{C})$ and $\check{X}$, where P_0 is the parabolic subgroup F^0G of G (see [**21**, Proposition 2.2]); hence the bijection induces the structure of a smooth projective variety over $\mathbb{C}$ on $\check{X}$.

Let (G, X) be a pair defining a connected Shimura variety and let μ_0 be the cocharacter corresponding to a point $o \in X$. We apply the above construction of $\check{X}$ for G^{ad}. Thus $\check{X}$ is the set of filtrations of $\mathbf{Rep}_{\mathbb{C}}(G^{\mathrm{ad}})$ that are conjugate to $\mathrm{Filt}(\mu_0)$ under $G^{\mathrm{ad}}(\mathbb{C})$. Such $\check{X}$ is in fact the compact dual Hermitian symmetric space of X in the usual sense. The Borel embedding $\beta : X \to \check{X}$ is the map that sends a point $x \in X$ to the filtration $\mathrm{Filt}(\mu_x)$ of $\mathbf{Rep}_{\mathbb{C}}(G^{\mathrm{ad}}_{\mathbb{C}})$ determined by μ_x. It is indeed an embedding of X onto an open complex submanifold of $\check{X}$.

Proposition 3.1. *The map $\beta : X \to \check{X}$ sending a point $x \in X$ to the filtration of $\mathbf{Rep}_{\mathbb{C}}(G^{\mathrm{ad}}_{\mathbb{C}})$ defined by μ_x embeds X onto an open complex submanifold of $\check{X}$. For $o \in X$, let K_0 be the isotropy group at o in $G(\mathbb{R})^+$, and let P_0 be the isotropy group at $o \in \check{X}$ in $G(\mathbb{C})$; then the inclusion of K_0 into P_0 identifies $(K_0)_{\mathbb{C}}$ with a Levi subgroup of P_0; there is an equivariant commutative diagram*

$$\begin{array}{ccc} G^{\mathrm{ad}}(\mathbb{R})^+/K_0 & \longrightarrow & G^{\mathrm{ad}}(\mathbb{C})/P_0(\mathbb{C}) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \check{X}, \end{array}$$

where the vertical maps are isomorphisms and the horizontal maps are embeddings.

Proof. See [**21**, Proposition 2.6]. □

Since $\check{X}$ is an algebraic variety over $\mathbb{C}$, for each $\tau \in \mathrm{Aut}(\mathbb{C})$ we can consider the conjugate variety $\tau\check{X}$.

Proposition 3.2. *Let ${}^{\tau,x}\check{X}$ be the dual Hermitian symmetric space associated to the pair $({}^{\tau,x}G, {}^{\tau,x}X)$. For each special point $x \in X$, there is a unique isomorphism $\phi^{\vee}_{\tau,x} : \tau\check{X} \to {}^{\tau,x}\check{X}$ such that*

(i) *the point τx is mapped to ${}^{\tau}\check{x}$, and*

(ii) *$\phi^{\vee}_{\tau,x} \circ \tau(g) = ({}^{\tau,x}g) \circ \phi^{\vee}_{\tau,x}$ for all $g \in G^{\mathrm{ad}}(\mathbb{C})$.*

Proof. See [**21**, Proposition 2.7]. □

Let $S^0(G, X)$ be a connected Shimura variety associated to an algebraic $\mathbb{Q}$-group G and a symmetric domain X. If $\beta : X \hookrightarrow \check{X}$ is the Borel embedding of X, the action of $G(\mathbb{R})$ on X extends to a transitive action of $G(\mathbb{C})$ on

$\check{X}$. Let $(\mathcal{J}, p)$ be a $G_{\mathbb{C}}$-vector bundle on $\check{X}$ with a $G_{\mathbb{C}}$-action satisfying the following conditions:

(i) $p(g \cdot w) = g \cdot p(w)$ for all $g \in G(\mathbb{C})$ and $w \in \mathcal{J}$.

(ii) The maps $g : \mathcal{J}_x \to \mathcal{J}_{gx}$ are linear for all $g \in G(\mathbb{C})$ and $x \in \check{X}$.

Let $G(\mathbb{R})_+$ be the inverse image of $G^{\text{ad}}(\mathbb{R})^+$ in $G(\mathbb{R})$ and let $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$. A $G_{\mathbb{C}}$-vector bundle $(\mathcal{J}, p)$ satisfying (i), (ii) restricts to a $G(\mathbb{R})_+$-vector bundle $\tilde{\mathcal{V}} = \beta^* \mathcal{J}$ on X, and, for each congruence subgroup Γ of $G(\mathbb{Q})$, $\mathcal{V}_\Gamma = \Gamma \backslash \tilde{\mathcal{V}}$ is a vector bundle on $\Gamma \backslash X = S^0_\Gamma(G, X)(\mathbb{C})$. We set

$$\mathcal{V} = \{\mathcal{V}_\Gamma \mid \Gamma \text{ is a congruence subgroup of } G(\mathbb{Q})\}.$$

Then $\mathcal{V}$ is a projective system, and there is a natural action of $G(\mathbb{Q})_+$ on $\mathcal{V}$ that sends an element $v \in \tilde{\mathcal{V}}$ modulo Γ to an element $gv \in \tilde{\mathcal{V}}$ modulo $g\Gamma g^{-1}$. This action extends by continuity to the closure $G(\mathbb{Q})_+^-$ of $G(\mathbb{Q})_+$ in $G(\mathbb{A}^f)$. A $G(\mathbb{Q})_+^-$-vector bundle $\mathcal{V}$ on $S^0(G, X)$ arising in this way from a $G(\mathbb{C})$-vector bundle $\mathcal{J}$ on $\check{X}$ is called an *automorphic vector bundle* (see [**6**], [**7**], [**21**], [**22**]; see also [**35**]). When G is simply connected, $\mathcal{V}$ is the $G(\mathbb{A}^f)$-vector bundle

$$\mathcal{V} = G(\mathbb{Q}) \backslash \tilde{V} \times G(\mathbb{A}^f)$$

on $S^0(G, X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}^f)$, and the action of $g \in G(\mathbb{A}^f)$ on $\mathcal{V}$ is given by

$$g \cdot [v, a] = [v, ag^{-1}]$$

for all $v \in \mathcal{V}$ and $a \in G(\mathbb{A}^f)$. On the other hand, if the action of $G(\mathbb{C})$ on $\mathcal{J}$ factors through $G^{\text{ad}}(\mathbb{C})$, then we can consider $\mathcal{V}$ as the projective system $\{\Gamma \backslash \tilde{\mathcal{V}} \mid \Gamma \in \Sigma(G)\}$, where $\Sigma(G)$ is the set of torsion-free arithmetic subgroups of $G^{\text{ad}}(\mathbb{Q})^+$ that contains the image of a congruence subgroup of $G(\mathbb{Q})$ as in §2, and the action of $G^{\text{ad}}(\mathbb{Q})^{+\wedge}$ on $S^0(G, X)$ lifts to $\mathcal{J}$.

Now we consider another Shimura variety $S^0(G', X')$ determined by an algebraic $\mathbb{Q}$-group G' and a symmetric domain X'. Let $\rho : G \to G'$ be a homomorphism of algebraic groups over $\mathbb{Q}$ carrying the conjugacy class X into X'. The map $\omega : X \to X'$ defined by $\omega(h) = \text{ad}(\rho) \circ h$ for $h \in X$ sends special points of X to special points of X'. As in the case of $S^0(G, X)$ we construct a projective system

$$\mathcal{V}' = \{\mathcal{V}'_{\Gamma'} \mid \Gamma' \text{ is a congruence subgroup of } G'(\mathbb{Q})\}$$

of vector bundles $\mathcal{V}'_{\Gamma'} = \Gamma' \backslash \tilde{\mathcal{V}}'$ on $\Gamma' \backslash X' = S^0_{\Gamma'}(G', X')$ determined by a $G'(\mathbb{C})$-vector bundle associated to the Borel embedding $\beta' : X' \hookrightarrow \check{X}'$.

Let $\{\Gamma_i\}$, $\{\Gamma'_i\}$ be projective systems of congruence subgroups of $G(\mathbb{Q})$ and $G'(\mathbb{Q})$ respectively such that $\rho(\Gamma_i) \subset \Gamma'_i$ for each i. Then $\mathcal{V}$ can be considered as the projective system of vector bundles $\mathcal{V}_{\Gamma_i} = \Gamma_i \backslash \beta^*(\mathcal{J})$ on $S^0_{\Gamma_i}(G, X)$, and similarly $\mathcal{V}'$ is the projective system of vector bundles $\mathcal{V}'_{\Gamma'_i} = \Gamma'_i\backslash\beta'^*(\mathcal{J}')$ on $S^0_{\Gamma'_i}(G', X')$. If $\omega_i : S^0_{\Gamma_i}(G, X) \to S^0_{\Gamma'_i}(G', X')$ denotes the map induced by $\omega : X \to X'$, we define $\mathcal{V} \otimes \omega^*\mathcal{V}'$ to be the projective system of vector bundles $\mathcal{V}_{\Gamma_i} \otimes \omega_i^*\mathcal{V}'_{\Gamma'_i}$ on $S^0_{\Gamma_i}(G, X)$. Then the natural action of $G(\mathbb{Q})_+$ on $\mathcal{V} \otimes \omega^*\mathcal{V}'$ induces an action of $G(\mathbb{Q})^-_+$.

Definition 3.3. The $G(\mathbb{Q})^-_+$-vector bundle $\mathcal{V} \otimes \omega^*\mathcal{V}'$ described above is called a *mixed automorphic vector bundle of type* $(\mathcal{J}, \mathcal{J}', \rho)$, and the space of mixed automorphic vector bundles of type $(\mathcal{J}, \mathcal{J}', \rho)$ will be denoted by $\mathcal{M}(\mathcal{J}, \mathcal{J}', \rho)$.

4. Mixed automorphic forms.

In this section we define mixed automorphic forms, and discuss their relation with mixed automorphic vector bundles. We also review the construction of Kuga fiber varieties, and describe the relation between the mixed automorphic vector bundles and holomorphic forms on fiber products of Kuga fiber varieties. Let Γ be a discrete subgroup of $\mathrm{Aut}(X)$, and let $J : \Gamma \times X \to GL(V)$ be an automorphy factor for (Γ, X) with values in a complex vector space V such that

(i) the map $z \mapsto J(\gamma, z)$ is holomorphic on X for each $\gamma \in \Gamma$;

(ii) $J(\gamma_1\gamma_2, z) = J(\gamma_1, \gamma_2 z) \cdot J(\gamma_2, z)$ for all $\gamma_1, \gamma_2 \in \Gamma$ and $z \in X$.

Let $\rho : \mathrm{Aut}(X) \to \mathrm{Aut}(X')$ be a homomorphism and let $\omega : X \to X'$ be a holomorphic map such that $\omega(\rho z) = \rho(g)\omega(z)$ for all $g \in \mathrm{Aut}(X)$ and $z \in X$; let $J' : \Gamma' \times X' \to GL(V')$ be an automorphy factor for (Γ', X') with values in V', where Γ' is a discrete subgroup of $\mathrm{Aut}(X')$ containing $\rho(\Gamma)$. Then a *mixed automorphic form of type* (J, J', ρ, ω) is defined to be a holomorphic function $f : X \to V \otimes V'$ such that

(i) $f(\gamma z) = \big(J(\gamma, z) \otimes J'(\rho(\gamma), \omega(z))\big) f(z)$ for all $\gamma \in \Gamma$ and $z \in X$;

(ii) f is holomorphic at infinity.

Various types of mixed automorphic forms have been investigated (see [**8**], [**14**], [**16**], [**17**], [**18**], [**19**]), and certain types of mixed automorphic forms occur naturally as holomorphic differential forms on certain fiber varieties over arithmetic varieties whose fibers are abelian varieties ([**8**], [**14**], [**19**]).

Example 4.1. Let X be the Poincaré upper half plain $\mathcal{H}$ and let Γ be a discrete subgroup of $PSL(2, \mathbb{R})$. Let $\chi : \Gamma \to SL(2, \mathbb{R})$ be a homomorphism and let $\omega : \mathcal{H} \to \mathcal{H}$ be a holomorphic map such that $\omega(\gamma z) = \chi(\gamma)\omega(z)$ for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$. We set $J(\gamma, z) = (j(\gamma, z))^k$ and $J'(\delta, w) = (j(\delta, w))^l$, where k, l are nonnegative integers with k even and $j(\gamma, z) = (cz + d)$ if

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \quad \text{or} \quad PSL(2, \mathbb{R}).$$

Then the mixed automorphic form of type (J, J', ρ, ω) is the mixed automorphic form of type (k, l) for Γ associated to χ, ω considered in [**8**], [**14**], [**16**], [**17**], and [**18**], and certain types of such mixed automorphic forms arise as holomorphic differential forms of the highest order on elliptic varieties (see [**14**]). A similar problem for Siegel modular forms was treated in [**19**].

Let $\rho : G \to G'$ and $\omega : X \to X'$ be as in §3, and let $p : \mathcal{J} \to \check{X}$ be a $G(\mathbb{C})$-vector bundle on $\check{X}$. Fix a point $x_0 \in X$ and let V be the fiber $\mathcal{J}_{\beta(x_0)}$ of $\mathcal{J}$ over $\beta(x_0) \in \check{X}$. The isomorphism of V with $\beta^{-1}(\mathcal{J})_{x_0}$ extends to an isomorphism of $X \times V$ with $\beta^{-1}(\mathcal{J})$ and we can transfer the action of $G(\mathbb{R})^+$ on $\beta^{-1}(\mathcal{J})$ to the one on $X \times V$. We write

$$\gamma(x, v) = (\gamma x, J(\gamma, x)v)$$

for $\gamma \in G(\mathbb{R})^+$, $x \in X$ and $v \in V$. Similarly, given a $G'_{\mathbb{C}}$-vector bundle $p' : \mathcal{J}' \to \check{X}'$ on $\check{X}'$, we define $J' : G'(\mathbb{R})^+ \times X \to GL(V')$ by

$$\gamma'(x', v') = (\gamma' x', J'(\gamma', x')v')$$

for $\gamma' \in G'(\mathbb{R})^+$, $x' \in X'$ and $v' \in V'$. Then the maps J and J' are automorphy factors, and a section of $(\mathcal{V} \otimes \omega^* \mathcal{V})_K$ on the connected Shimura variety $S^0_{\Gamma_K}(G, X)$ associated to $\mathcal{J}$, $\mathcal{J}'$ and a compact open subgroup K of $G(\mathbb{A}^f)$ can be identified with a mixed automorphic form for $\Gamma_K = K \cap G(\mathbb{Q})$ of type (J, J', ρ, ω).

In the rest of this section, we describe a relation between mixed automorphic vector bundles and fiber products of Kuga fiber varieties. We first review the construction of Kuga fiber varieties over connected Shimura varieties (see [**12**], [**30**, Chapter IV]). Let W be a vector space over $\mathbb{Q}$ of dimension $2m$, and let β be an alternating bilinear form on W. We set

$$Sp(\beta) = \{g \in GL(W) \mid \beta(gx, gy) = \beta(x, y) \quad \text{for all} \quad x, y \in W\}$$

and let $\mathcal{H}(\beta, \mathbb{R})$ be the set of all complex structures J on $W(\mathbb{R})$ such that the bilinear form $\beta(x, Jy)$ on $W(\mathbb{R})$ is symmetric and positive definite. Then the group $Sp(\beta, \mathbb{R})$ of real points of $Sp(\beta)$ acts on $\mathcal{H}(\beta, \mathbb{R})$ transitively by

$$(g, J) \longmapsto gJg^{-1} \quad \text{for} \quad g \in Sp(\beta, \mathbb{R}), \quad J \in \mathcal{H}(\beta, \mathbb{R}).$$

Let $\{e_1, \dots, e_m, f_1, \dots, f_m\}$ be a basis for $W(\mathbb{R})$ such that

$$\beta(e_i, e_j) = \beta(f_i, f_j) = 0, \qquad \beta(e_i, f_j) = -\delta_{ij}$$

for $1 \leq i, j \leq m$, where δ_{ij} is the Kronecker delta. Then we can identify $Sp(\beta, \mathbb{R})$ with

$$Sp(m, \mathbb{R}) = \{g \in GL(2m, \mathbb{R}) \mid {}^t gEg = E\}, \quad E = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right),$$

and $\mathcal{H}(\beta, \mathbb{R})$ with the Siegel upper half space

$$\mathcal{H}^m = \{ z \in M_m(\mathbb{C}) \mid z = {}^t z, \quad \text{Im } z >> 0 \}$$

(see [**30**, §II.8]).

Let G be a semisimple algebraic $\mathbb{Q}$-group as before, and let K_0 be a maximal compact subgroup of $G(\mathbb{R})^+$. We assume that the symmetric space $X = G(\mathbb{R})^+/K_0$ has a $G(\mathbb{R})^+$-invariant complex structure. Let $\rho : G(\mathbb{R})^+ \to Sp(\beta, \mathbb{R})$ be a homomorphism of Lie groups and let $\omega : X \to \mathcal{H}(\beta, \mathbb{R}) = \mathcal{H}_m$ be a holomorphic map such that

$$\omega(gz) = \rho(g)\omega(z)$$

for all $g \in G(\mathbb{R})^+$ and $z \in X$. Then ρ determines the semidirect product $G(\mathbb{R})^+ \ltimes_\rho W(\mathbb{R})$ in which the multiplication is given by

$$(g_1, v_1) \cdot (g_2, v_2) = (g_1 g_2, \rho(g_1)v_2 + v_1)$$

for all $g_1, g_2 \in G(\mathbb{R})^+$ and $v_1, v_2 \in W(\mathbb{R})$. The group $G(\mathbb{R})^+ \ltimes_\rho W(\mathbb{R})$ acts on $X \times W(\mathbb{R})$ by

$$(g, v) \cdot (x, w) = (gx, \rho(g)w + v)$$

for $(g, v) \in G(\mathbb{R})^+ \ltimes_\rho W(\mathbb{R})$ and $(x, w) \in X \times W(\mathbb{R})$.

Let $u(x) = (u_1(x), \dots, u_k(x))$ be a global complex analytic coordinate system of the bounded symmetric domain X. Define the map $z : X \times W(\mathbb{R}) \to \mathbb{C}^m$ by

$$z(x, w) = (\omega(x), 1)E\varphi(w),$$

where $E = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right) \in M_{2m}(\mathbb{R})$. This induces the map $\mu : X \times W(\mathbb{R}) \to \mathbb{C}^{k+m}$ given by

$$\mu(x, w) = (u(x), z(x, w)).$$

Thus μ is a diffeomorphism of $X \times W(\mathbb{R})$ onto $u(X) \times \mathbb{C}^m$. If J is the natural complex structure on $u(X) \times \mathbb{C}^m$, then $\mathcal{J} = \mu^{-1}(J)$ defines a complex structure on $X \times W(\mathbb{R})$ with global coordinates

$$u_1, \dots, u_k, z_1 \dots, z_m.$$

Let L be a lattice in $W(\mathbb{R})$ and let Γ be a torsion-free co-compact discrete subgroup of $G(\mathbb{R})^+$ such that $\rho(\Gamma)L \subset L$. Then the semidirect product $\Gamma \ltimes_\rho L$ operates on $X \times W(\mathbb{R})$ properly discontinuously, and the complex structure $\mathcal{J}$ on $X \times W(\mathbb{R})$ determined by the holomorphic map $\omega : X \to \mathcal{H}(\beta, \mathbb{R})$ induces a complex structure on the manifold $\Gamma \ltimes_\rho L \backslash X \times W(\mathbb{R})$. We denote by A_ρ the complex manifold $\Gamma \ltimes_\rho L \backslash X \times W(\mathbb{R})$ obtained this way. Then the projection map $X \times W(\mathbb{R}) \to X$ induces a fiber bundle $\pi : A_\rho \to S^0(G, X)(\mathbb{C})$ known as a Kuga fiber variety over the complex manifold $S^0(G,X)(\mathbb{C}) = \Gamma\backslash X$ whose fibers are complex tori of dimension m (see [**12**] and [**30**, Chapter IV] for details; see also [**15**]). We denote by A^n_ρ the n-fold fiber product

$$A_\rho \times_\pi A_\rho \times_\pi \cdots \times_\pi A_\rho$$

of A_ρ over $S^0(G,X)(\mathbb{C})$.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $G(\mathbb{R})^+$ as in §2, and let J_0 be the complex structure on X such that

$$[J_0\xi, \eta] = [\xi, J_0\eta] \quad \text{for all} \quad \xi \in \mathfrak{k}, \quad \eta \in \mathfrak{p},$$
$$[J_0\xi, J_0\eta] = [\xi, \eta] \quad \text{for all} \quad \xi, \eta \in \mathfrak{p}.$$

Let $\mathfrak{p}_+, \mathfrak{p}_- \subset \mathfrak{p}_\mathbb{C}$ be the eigenspaces of J_0 belonging to the eigenvalues $i, -i$ respectively. Then we have

$$\mathfrak{g}_\mathbb{C} = \mathfrak{p}_+ + \mathfrak{k}_\mathbb{C} + \mathfrak{p}_-, \quad \overline{\mathfrak{p}}_+ = \mathfrak{p}_-$$

and $G(\mathbb{C}) = P_+K_0(\mathbb{C})P_-$, where P_+, $K_0(\mathbb{C})$, P_- are the Lie groups whose Lie algebras are $\mathfrak{p}_+$, $\mathfrak{k}_\mathbb{C}$, $\mathfrak{p}_-$, respectively. The map $J_c : G(\mathbb{C}) \times \mathfrak{p}_+ \to K_0(\mathbb{C})$ defined by

$$J_c(g, z) = (g \cdot \exp z)_0$$

for $(g, z) \in G(\mathbb{C}) \times \mathfrak{p}_+$, where $(\ )_0$ denotes the $K_0(\mathbb{C})$-part in the decomposition $G(\mathbb{C}) = P_+K_0(\mathbb{C})P_-$, is called the *canonical automorphy factor* of $G(\mathbb{C})$ (see [**30**, §II.5]). Now we define the function $j_H : G(\mathbb{R})^+ \times X \to \mathbb{C}$ by

$$j_H(g, z) = \det[\mathrm{ad}_{\mathfrak{p}_+}(J_H(g, z))]$$

for $(g, z) \in G(\mathbb{R})^+ \times X$, where J_H is the restriction of J_c to $G(\mathbb{R})^+ \times X$. Then the map $z \mapsto j_H(g, z)$ is the Jacobian map for the transformation $z \mapsto gz$ of X.

We also consider another automorphy factor $j_V : Sp(m, \mathbb{R}) \times \mathcal{H}^m \to \mathbb{C}$ defined by

$$j_V(\sigma, z) = \det(cz + d)$$

for $z \in \mathcal{H}^m$ and $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Sp(m, \mathbb{R})$.

Given a positive integer l, let $\mathcal{J}$ (resp. $\mathcal{J}'_l$) be $G(\mathbb{C})$-vector (resp. $Sp(m, \mathbb{C})$-vector) bundle on the compact dual $\check{X}$ of X (resp. $\check{\mathcal{H}}^m$ of $\mathcal{H}^m$) that induces a $G(\mathbb{R})^+$-vector bundle $\hat{\mathcal{J}}$ (resp. $\hat{\mathcal{J}}'_l$) on $\mathcal{H}^m$ that has a trivialization $\eta : X \times \mathbb{C} \to \hat{\mathcal{J}}$ (resp. $\eta' : \mathcal{H}^m \times \mathbb{C} \to \hat{\mathcal{J}}'_l$) given by

$$g(\eta(z, v)) = \eta(z, j_H(g, z)^{-1}v) \quad (\text{resp. } g'(\eta'(z', v')) = \eta'(z', j_V(g', z')^l v'))$$

for $g \in G$, $z \in X$ and $v \in \mathbb{C}$ (resp. $g' \in Sp(m,\mathbb{R})$, $z' \in \mathcal{H}^m$ and $v' \in \mathbb{C}$). Then, for each positive integer l, the mixed automorphic vector bundle $\mathcal{M}(\mathcal{J}, \mathcal{J}'_l, \rho)$ is the projective system

$$\left\{\Gamma\backslash\hat{\mathcal{J}} \otimes \omega^* \hat{\mathcal{J}}'_l\right\},$$

where each discrete subgroup Γ of $G(\mathbb{R})$ is of the form $\Gamma_K = G(\mathbb{Q}) \cap K$ for some compact open subgroup K of $G(\mathbb{A}_f)$ as before. We denote by $\mathcal{M}_K(\mathcal{J}, \mathcal{J}'_l, \rho)$ the vector bundle $\Gamma_K\backslash\hat{\mathcal{J}} \otimes \omega^* \hat{\mathcal{J}}'_l$ on $S^0_{\Gamma_K}(G,X)(\mathbb{C}) = \Gamma_K\backslash X$, and by $\widetilde{\mathcal{M}}_K(\mathcal{J}, \mathcal{J}'_l, \rho)$ the sheaf of sections of $\mathcal{M}_K(\mathcal{J}, \mathcal{J}'_l, \rho)$.

Theorem 4.2. *Let $\pi : A^n_\rho \to S^0_{\Gamma_K}(G,X)(\mathbb{C})$ be the fiber product of a Kuga fiber variety determined by (ρ, ω) described above. Then the cohomology space*

$$H^0\left(S^0_{\Gamma_K}(G,X)(\mathbb{C}), \widetilde{\mathcal{M}}_K(\mathcal{J}, \mathcal{J}'_n, \rho)\right)$$

is canonically isomorphic to the cohomology space $H^0(A^n_\rho, \Omega^{k+mn})$, where Ω^{k+mn} is the sheaf of holomorphic $(k+mn)$-forms on A^n_ρ.

Proof. Since we are assuming that $S^0_{\Gamma_K}(G,X)(\mathbb{C}) = \Gamma_K\backslash X$ is compact, we do not need to consider the holomorphy condition at infinity. Let $u = (u_1, \dots, u_k)$ be a global coordinates for the symmetric domain X, and let $z^{(j)} = (z^{(j)}_1, \dots, z^{(j)}_m)$ be the canonical coordinates for $\mathbb{C}^m$ for $1 \le j \le n$. If ψ is a holomorphic $(k+mn)$-form on A^n_ρ, then ψ can be considered as a holomorphic $(k+mn)$-form on $X \times (\mathbb{C}^m)^n$ that is invariant under the action of $\Gamma_K \ltimes_\rho L^n$, where L is a lattice in $\mathbb{C}^m$. Then there is a holomorphic function $f_\psi(u,z)$ on $X \times (\mathbb{C}^m)^n$ such that

$$\psi = f_\psi(u,z)\, du \wedge dz^{(1)} \wedge \cdots \wedge dz^{(n)},$$

where $u = (u_1, \dots, u_k) \in X$, $z = (z^{(1)}, \dots, z^{(n)}) \in (\mathbb{C}^m)^n$, and $z^{(j)} = (z^{(j)}_1, \dots, z^{(j)}_n) \in \mathbb{C}^n$ for $1 \le j \le n$. Given $x \in X$, ψ descends to a holomorphic mn-form on the fiber $A^n_{\rho,x}$ over x. The fiber $A^n_{\rho,x}$ is the n-fold product of a complex torus of dimension m, and hence the dimension of the space of holomorphic mn-forms on $A^n_{\rho,x}$ is one. Since any holomorphic function on a compact complex manifold is constant, the restriction of $f_\psi(u,z)$ to the compact complex manifold $A^n_{\rho,x}$ is constant. Thus $f_\psi(u,z)$ depends only on u; and hence ψ can be written in the form

$$\psi = f_\psi(u)\, du \wedge dz^{(1)} \wedge \cdots \wedge dz^{(n)},$$

where f_ψ is a holomorphic function on X. To consider the invariance of ψ under the group $\Gamma_K \ltimes_\rho L^n$, we first notice that the action of $\Gamma_K \ltimes_\rho L^n$ on $du = du_1 \wedge \cdots \wedge du_k$ is given by

$$(\gamma, v) \cdot du = j_H(\gamma, v) du$$

for all $(\gamma, v) \in \Gamma_K \ltimes_\rho L^n$, because $j_H(\gamma, *)$ is the Jacobian map of the transformation $u \mapsto \gamma u$ of X. On the other hand, from Equation (17) in [**12**, §II.6], the action of $\Gamma_K \ltimes_\rho L^n$ on $dz^{(j)} = dz_1^{(j)} \wedge \cdots \wedge dz_m^{(j)}$ is given by

$$\begin{aligned}(\gamma, v) \cdot dz^{(j)} &= d\left[{}^t(c_\rho \omega(v) + d_\rho)^{-1} z^{(j)} + (\omega(\gamma v), 1)Ev\right] \\ &= \det(c_\rho \omega(v) + d_\rho)^{-1} dz^{(j)} \\ &= j_V(\rho(\gamma), \omega(v))^{-1} dz^{(j)}\end{aligned}$$

for $1 \leq j \leq n$, where

$$E = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \in M_{2m}(\mathbb{R}) \quad \text{and} \quad \rho(\gamma) = \begin{pmatrix} a_\rho & b_\rho \\ c_\rho & d_\rho \end{pmatrix} \in Sp(m, \mathbb{R});$$

hence we obtain

$$(\gamma, v) \cdot \psi = f_\psi(\gamma v)\, j_H(\gamma, v) j_V(\rho(\gamma), \omega(v))^{-n} du \wedge dz^{(1)} \wedge \cdots \wedge dz^{(n)}.$$

Thus we have

$$f_\psi(\gamma u) = j_H(\gamma, u)^{-1} j_V(\rho(\gamma), \omega(u))^n f_\psi(u)$$

for all $\gamma \in \Gamma_K$ and $u \in X$. On the other hand, each element

$$h \in H^0(S^0_{\Gamma_K}(G, X)(\mathbb{C}), \widetilde{\mathcal{M}}_K(\mathcal{J}, \mathcal{J}'_n, \rho))$$

is a Γ_K-invariant section of the vector bundle $\hat{\mathcal{J}} \otimes \omega^* \hat{\mathcal{J}}'$ on $S^0_{\Gamma_K}(G, X)(\mathbb{C})$; hence it is a function satisfying

$$h(\gamma z) = j_H(\gamma, u)^{-1} j_V(\rho(\gamma), \omega(u))^n h(u)$$

for $u \in X$ and $\gamma \in \Gamma_K$. Therefore the assignment $\psi \mapsto f_\psi(u)$ determines an isomorphism between the space $H^0(A^n_\rho, \Omega^{k+mn})$ of holomorphic $(k + mn)$-forms on A^n_ρ and the space

$$H^0(S^0_{\Gamma_K}(G, X)(\mathbb{C}), \widetilde{\mathcal{M}}_K(\mathcal{J}, \mathcal{J}'_n, \rho))$$

of sections of the automorphic vector bundle $\mathcal{M}_K(\mathcal{J}, \mathcal{J}'_n, \rho)$. □

Example 4.3. If G is SL_2, then X is the Poincaré upper half plane $\mathcal{H}$, and $j_H(\gamma, z) = (cz + d)^{-2}$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$ and $z \in \mathcal{H}$. If, furthermore, Γ_K is a cocompact arithmetic subgroup of $SL(2, \mathbb{Q})$, then A^n_ρ becomes the elliptic variety E^n considered in [**14**], and Theorem 4.2 above reduces to Theorem 3.2 in [**14**] which states that the space $H^0(E^n, \Omega^{n+1})$ of holomorphic $(n + 1)$-forms on E^n is canonically isomorphic to the space $S_{2,n}(\Gamma_K, \omega, \rho)$ of mixed cusp forms of type $(2, n)$; it reduces to Theorem 1.2 in [**8**] for n=1.

5. Conjugates of mixed automorphic vector bundles.

In this section we state and prove the theorem about the conjugates of mixed automorphic vector bundles. Let (G, X) be a pair defining a connected Shimura variety $S^0(G, X)$ as in §1. Each $\Gamma \in \tilde{\Sigma}(G)$ defines a principal $G(\mathbb{C})$-bundle $P^0_\Gamma(G, X) = \Gamma \backslash X \times G(\mathbb{C})$ over $S^0_\Gamma(G, X)$. Such bundles form a projective system $P^0(G, X)$ which can be considered as a principal $G(\mathbb{C})$-bundle over $S^0(G, X)$. When G is simply connected, we have

$$P^0(G, X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{C}) \times G(\mathbb{A}^f)$$

with $g \in G(\mathbb{Q})$ acting on $(x, c, a) \in X \times G(\mathbb{C}) \times G(\mathbb{A}^f)$ by the rule $g(x, c, a) = (gx, gc, ga)$.

Proposition 5.1. *The principal $G(\mathbb{C})$-bundle $P^0(G, X)$ is algebraic, and there is a canonical $G(\mathbb{C})$-equivariant map $\gamma = \gamma(G, X) : P^0(G, X) \to \check{X}$.*

Proof. See Propositions 3.2 and 3.5 in [**21**]. □

Theorem 5.2. *Let (G, X) be a pair defining a connected Shimura variety, and let x be a special point of X. From each automorphism τ of $\mathbb{C}$, there is a unique isomorphism $\varphi^P_{\tau,x} : \tau P^0(G, X) \to P^0({}^{\tau,x}G, {}^{\tau,x}X)$ that lies over $\varphi^0_{\tau,x} : \tau S^0(G, X) \to S^0({}^{\tau,x}G, {}^{\tau,x}X)$ and satisfies the following conditions:*
(i) *The point τw is mapped to ${}^\tau w$.*
(ii) $\varphi^P_{\tau,x} \circ \tau(g) = ({}^{\tau,x}g) \circ \varphi^P_{\tau,x}$ *for all* $g \in (G(\mathbb{C}) \times G(\mathbb{Q})^-_+) *_{G(\mathbb{Q})_+} G(\mathbb{Q})^+$.

Proof. See [**21**, Theorem 3.10]. □

Proposition 5.3. *If $\varphi^P_{\tau,x}$ is as in Theorem 5.2, then there is a commutative diagram*

$$\begin{array}{ccc}
\tau\check{X} & \xrightarrow{\varphi^\vee_{\tau,x}} & {}^{\tau,x}\check{X} \\
{\scriptstyle \tau\gamma}\uparrow & & \uparrow{\scriptstyle \gamma} \\
\tau P^0(G, X) & \xrightarrow{\varphi^P_{\tau,x}} & P^0({}^{\tau,x}G, {}^{\tau,x}X) \\
\downarrow & & \downarrow \\
\tau S^0(G, X) & \xrightarrow{\varphi^0_{\tau,x}} & S^0({}^{\tau,x}G, {}^{\tau,x}X).
\end{array}$$

Furthermore, the two maps $\varphi^\vee_{\tau,x}$ and $\varphi^P_{\tau,x}$ in the upper square are compatible with the map $g \mapsto {}^{\tau,x}g$ form $G(\mathbb{C})$ to ${}^{\tau,x}G(\mathbb{C})$.

Proof. See [**21**, Corollary 3.11]. □

Let $\rho : G \to G'$ be a homomorphism of algebraic groups over $\mathbb{Q}$ that determines a mixed automorphic vector bundle $\mathcal{M}(\mathcal{J}, \mathcal{J}', \rho)$ as described in §3, where $\mathcal{J}$ (resp. $\mathcal{J}'$) is a $G_{\mathbb{C}}$-vector (resp. $G'_{\mathbb{C}}$-vector) bundle on $\check{X}$ (resp $\check{X}'$). Thus ρ carries the conjugacy class X into X', and the map $\omega : X \to X'$ defined by $\omega(h) = \mathrm{ad}(\rho) \circ h$ for $h \in X$ sends special points of X to special points of X'. We fix a special point x in X, and denote the special point $\omega(x)$ in X' by x'. The homomorphism ρ induces a homomorphism ${}^{\tau,x}\rho : {}^{\tau,x}G \to {}^{\tau,x}G'$ and a map ${}^{\tau,x}\omega : {}^{\tau,x}X \to {}^{\tau,x}X'$. The $G_{\mathbb{C}}$-vector (resp. $G'_{\mathbb{C}}$-vector) bundle $\tau\mathcal{J}$ (resp. $\tau\mathcal{J}'$) on $\tau\check{X}$ (resp. $\tau\check{X}'$) corresponds under the isomorphism $\check{\varphi}_{\tau,x} : \tau\check{X} \to {}^{\tau,x}\check{X}$ (resp. $\check{\varphi}_{\tau,x'} : \tau\check{X}' \to {}^{\tau,x'}\check{X}'$) to a ${}^{\tau,x}G(\mathbb{C})$-vector (resp. ${}^{\tau,x'}G'(\mathbb{C})$-vector) bundle ${}^{\tau,x}\mathcal{J}$ (resp. ${}^{\tau,x'}\mathcal{J}'$) on ${}^{\tau,x}\check{X}$ (resp. ${}^{\tau,x'}\check{X}'$). The vector bundles ${}^{\tau,x}\mathcal{J}$, ${}^{\tau,x'}\mathcal{J}'$ define automorphic vector bundles ${}^{\tau,x}\mathcal{V}$, ${}^{\tau,x'}\mathcal{V}'$ on the connected Shimura varieties $S^0({}^{\tau,x}G, {}^{\tau,x}X)$, $S^0({}^{\tau,x'}G', {}^{\tau,x'}X')$ respectively, and they also determine the mixed automorphic vector bundle $\mathcal{M}({}^{\tau,x}\mathcal{J}, {}^{\tau,x'}\mathcal{J}', {}^{\tau,x}\rho)$ on the connected Shimura variety $S^0(G, X)$.

Proposition 5.4. *If y is another special point of X, then there is a canonical isomorphism $\varphi^{\mathcal{M}}(\tau; y, x) : \mathcal{M}({}^{\tau,x}\mathcal{J}, {}^{\tau,x'}\mathcal{J}', {}^{\tau,x}\rho) \to \mathcal{M}({}^{\tau,y}\mathcal{J}, {}^{\tau,y'}\mathcal{J}', {}^{\tau,y}\rho)$ lying over $\varphi^0(\tau; y, x)$ and such that*

$$\varphi^{\mathcal{M}}(\tau; y, x) \circ ({}^{\tau,x}g) = ({}^{\tau,y}g) \circ \varphi^{\mathcal{M}}(\tau; y, x)$$

for all ${}^{\tau,x}g \in {}^{\tau,x}\tilde{G}(\mathbb{A}^f)$.

Proof. By [**21**, Lemma 5.1] there is a canonical isomorphism $\varphi^{\mathcal{V}}(\tau; y, x) : {}^{\tau,x}\mathcal{V} \to {}^{\tau,y}\mathcal{V}'$ lying over $\varphi^0(\tau; y, x)$ and such that

$$\varphi^{\mathcal{V}}(\tau; y, x) \circ ({}^{\tau,x}g) = ({}^{\tau,y}g) \circ \varphi^{\mathcal{V}}(\tau; y, x)$$

for all ${}^{\tau,x}g \in {}^{\tau,x}\tilde{G}(\mathbb{A}^f)$. Thus the proposition follows easily from this and the $G(\mathbb{R})_+$-equivariance of the map $\omega : X \to X'$ used in the construction of $\mathcal{M}(\mathcal{J}, \mathcal{J}', \rho)$. ∎

Theorem 5.5. *Let $\mathcal{M}(\mathcal{J}, \mathcal{J}', \rho)$ be the mixed automorphic vector bundle on a connected Shimura variety $S^0(G, X)$ associated to a homomorphism $\rho : G \to G'$, a $G_{\mathbb{C}}$-vector bundle $\mathcal{J}$ on $\check{X}$, and $G'_{\mathbb{C}}$-vector bundle $\mathcal{J}'$ on $\check{X}'$. Then there is a canonical isomorphism*

$$\varphi^{\mathcal{M}}_{\tau,x} : \tau\mathcal{M}(\mathcal{J}, \mathcal{J}', \rho) \to \mathcal{M}({}^{\tau,x}\mathcal{J}, {}^{\tau,x'}\mathcal{J}', {}^{\tau,x}\rho)$$

such that the diagram

$$\begin{array}{ccc} \tau\mathcal{M}(\mathcal{J},\mathcal{J}',\rho) & \xrightarrow{\varphi^{\mathcal{M}}_{\tau,x}} & \mathcal{M}({}^{\tau,x}\mathcal{J},{}^{\tau,x'}\mathcal{J}',{}^{\tau,x}\rho) \\ \downarrow & & \downarrow \\ \tau S^0(G,X) & \xrightarrow{\varphi^{0}_{\tau,x}} & S^0({}^{\tau,x}G,{}^{\tau,x}X) \end{array}$$

is commutative and $\varphi^{\mathcal{M}}_{\tau,x}\circ\tau(g) = ({}^{\tau,x}g)\circ\varphi^{\mathcal{M}}_{\tau,x}$ *for all* $g \in G(\mathbb{Q})^-_+$. *Furthermore, if* y *is a second special point of* X, *then*

$$\varphi^{\mathcal{M}}(\tau;y,x)\circ\varphi^{\mathcal{M}}_{\tau,x} = \varphi^{\mathcal{M}}_{\tau,y}.$$

Proof. From the construction of ${}^{\tau,x}\mathcal{J}$ and ${}^{\tau,x'}\mathcal{J}'$ we obtain the commutative diagrams

$$\begin{array}{ccc} \tau\mathcal{J} & \longrightarrow & {}^{\tau,x}\mathcal{J} \\ \downarrow & & \downarrow \\ \tau\check{X} & \xrightarrow{\check{\varphi}_{\tau,x}} & {}^{\tau,x}\check{X} \end{array}$$

and

$$\begin{array}{ccc} \tau\mathcal{J}' & \longrightarrow & {}^{\tau,x'}\mathcal{J}' \\ \downarrow & & \downarrow \\ \tau\check{X}' & \xrightarrow{\check{\varphi}_{\tau,x'}} & {}^{\tau,x'}\check{X}'. \end{array}$$

Let $\check{\omega} : \check{X} \to \check{X}'$ be the $G(\mathbb{C})$-equivariant extension of $\omega : X \to X'$, and let ${}^{\tau,x}\check{\omega} : {}^{\tau,x}\check{X} \to {}^{\tau,x}\check{X}'$ be the map induced by $\tau\omega : \tau X \to \tau X'$ and the isomorphisms $\check{\varphi}_{\tau,x} : \tau\check{X} \to {}^{\tau,x}\check{X}$ and $\check{\varphi}_{\tau,x'} : \tau\check{X}' \to {}^{\tau,x'}\check{X}'$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \tau(\mathcal{J}\otimes\check{\omega}^*\mathcal{J}') & \longrightarrow & {}^{\tau,x}\mathcal{J}\otimes({}^{\tau,x}\check{\omega})^*\,{}^{\tau,x'}\mathcal{J}' \\ \downarrow & & \downarrow \\ \tau\check{X} & \xrightarrow{\check{\varphi}_{\tau,x}} & {}^{\tau,x}\check{X}. \end{array}$$

Pulling back these vector bundles via the upward vertical maps $\tau\gamma$ and γ in Proposition 5.3, and using the lower square in the commutative diagram in

Proposition 5.3, we obtain

$$\begin{array}{ccc}
\tau(\gamma)^*(\mathcal{J} \otimes \check{\omega}^*\mathcal{J}') & \longrightarrow & \gamma^*({}^{\tau,x}\mathcal{J} \otimes ({}^{\tau,x}\check{\omega})^*\, {}^{\tau,x'}\mathcal{J}') \\
\downarrow & & \downarrow \\
\tau P^0(G,X) & \xrightarrow{\varphi^P_{\tau,x}} & P^0({}^{\tau,x}G, {}^{\tau,x}X) \\
\downarrow & & \downarrow \\
\tau\check{X} & \xrightarrow{\check{\varphi}_{\tau,x}} & {}^{\tau,x}\check{X}.
\end{array}$$

From the construction of mixed automorphic vector bundles it follows that $\mathcal{M}(\mathcal{J},\mathcal{J}',\rho)$ and $\mathcal{M}({}^{\tau,x}\mathcal{J}, {}^{\tau,x'}\mathcal{J}', {}^{\tau,x}\rho)$ are obtained by descent from the vector bundles $\gamma^*(\mathcal{J} \otimes \check{\omega}^*\mathcal{J}')$ and $\gamma^*({}^{\tau,x}\mathcal{J} \otimes ({}^{\tau,x}\check{\omega})^*\, {}^{\tau,x'}\mathcal{J}')$, respectively. Thus the commutativity of the diagram in the theorem follows from the above commutative diagram. By Theorem 5.2(ii) the map $\varphi^P_{\tau,x}$ commutes with the Hecke operators $\tau(g)$ and ${}^{\tau,x}g$, and the map $\varphi^{\mathcal{M}}(\tau;y,x)$ commutes with operators ${}^{\tau,x}g$ and ${}^{\tau,y}g$; hence, using the compatibility of $\varphi^{\vee}_{\tau,x}$ and $\varphi^P_{\tau,x}$, we have

$$\varphi^{\mathcal{M}}(\tau;y,x) \circ \varphi^{\mathcal{M}}_{\tau,x} = \varphi^{\mathcal{M}}_{\tau,y},$$

where y is a second special point of X. □

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