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ISOMETRIC IMMERSIONS OF H_1^n INTO H_1^{n+1}

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The complete codimension 1 totally geodesic laminations of the hyperquadrics of constant curvature 1 or −1 are completely determined. Also the set of all codimension 1 isometric immersions of a hyperquadric into another of the same constant curvature are determined and characterized in terms of the naturally associated totally geodesic laminations and the curvature of the laminations. Note that the hyperquadrics of constant curvature 1 and −1 are often called the de Sitter space-time and the anti de Sitter space-tmie, respectively.

1. Introduction.

A classical problem in differential geometry is to characterize and determine all the possible submanifolds in space forms. To this end, there have been various geometrically sensible conditions imposed upon in an effort to make the problem more realistically viable. One of such is to restrict the submanifolds to being of codimension 1 and of the same constant curvature as the ambient space form.

In this restricted situation, the original problem has received much attention and, indeed, has seen much progress. In particular, the problem has fundamentally been settled in the positive definite case. In the indefinite case, Graves [**5**] gave the answer when the metric is Lorentzian and the constant curvature equals 0. For non-zero constant curvature cases, there have been a number of attempts made with limited success, see [**6**] for instance. One of the main causes hampering a further advance in the indefinite case was the lack of a completeness theorem on the leaves of the associated totally geodesic foliations. Now that the completeness theorem [**1**] is available, the time has come to address the problem in the indefinite case from a more comprehensive point of view.

In this paper, we restrict ourselves to the Lorentzian case with non-zero constant curvatures, namely, 1 or −1. The rationale for this restriction will become self-evident later in the paper.

Let S_p^n and H_p^n be the n-dimensional hyperquadrics of signature p and of constant curvatures 1 and −1, respectively. Among others, we will show the following:

a) *Only hyperquadrics which can support a complete totally geodesic foliation of codimension* 1 *in an open subset are either* $H^n_0 = H^n$, H^n_1, $n \geq 1$ *or* S^n_{n-1}, $2 \geq n \geq 1$.

b) *For* $n \geq 3$, *only isometric immersions* $f : S^n_1 \to S^{n+1}_1$ *are totally geodesic ones; therefore, they are congruent to the standard imbedding of* S^n_1 *into* S^{n+1}_1.

c) *We determine all the complete degenerate totally geodesic foliations in any open subset of* H^n_1, $n \geq 1$.

d) *We give a characterization of the space of isometric immersions from* H^n_1 *into* H^{n+1}_1 *in terms of totally geodesic laminations and curvature functions.*

In particular, (d) may be regarded as a form of answer to the problem posed at the beginning for the Lorentzian hyperbolic (or so-called anti-De Sitter) spaces.

2. Hyperquadrics.

Let R^{n+1}_p be R^{n+1} equipped with the quadratic form

$$q(X) = -\sum_{j=1}^{p} {x_j}^2 + \sum_{j=p+1}^{n+1} {x_j}^2, \quad \forall X = (x_1, \cdots, x^{n+1}) \in R^{n+1}_p. \tag{1}$$

The quadratic form q then naturally induces an indefinite but non-degenerate symmetric bilinear functional g by

$$g(X,Y) = -\sum_{j=1}^{p} x_j y_j + \sum_{j=p+1}^{n+1} x_j y_j, \quad \forall X = (x_1, \cdots, x_{n+1}), Y = (y_1, \cdots, y_{n+1}) \in R^{n+1}.$$

Set

$$S^n_p = \left\{ \forall X \in R^{n+1}_p : q(X) = 1 \right\}$$

and

$$H^n_p = \left\{ \forall X \in R^{n+1}_{p+1} : q(X) = -1 \right\}.$$

It is quite elementary to see that the map $\sigma : R^{n+1}_p \to R^{n+1}_{n-p+1}$ given by

$$\sigma(x_1, \cdots, x_{n+1}) = (x_{p+1}, \cdots, x_{n+1}, x_1, \cdots, x_p), \quad \forall X = (x_1, \cdots, x_{n+1}) \in R^{n+1}_p$$

is an anti-isometry from S_p^n onto H_{n-p}^n. It is also equally elementary to see that S_p^n is diffeomorphic to $R^p \times S^{n-p}$ and H_p^n is diffeomorphic to $S^p \times R^{n-p}$, respectively. The first diffeomorphism ϕ , for instance, is given by

$$\phi(x,y) = \left(x, \sqrt{1+|\,x\,|^2}y\right) \in R_p^p \times R^{n+1-p} \approx R_p^{n+1}.$$

We will call S_p^n and H_p^n the (unit) hyperquadrics of dimension n and the signature p. It is well-known that the quadratic form q or the symmetric bilinear form g induces an indefinite Riemannian structure, denoted by the same letter g , on the hyperquadrics, see [**2**], [**7**]. With the induced indefinite Riemannian structure, S_p^n and H_p^n have constant curvatures 1 and -1, respectively. Note that $S_0^n = S^n$, the ordinary unit n-sphere and $H_0^n = H^n$, the hyperbolic space of dimension n. These are the only hyperquadrics which are Riemannian.

Among the hyperquadrics introduced above, only ones that are not simply connected are $H_1^n \approx S_{n-1}^n$.

Define a map $F : R^n \to R_2^{n+1}$ by

$$F(x) = \left(\sqrt{1+\sum_{j=2}^{n}(x_j)^2}\cos x_1, \sqrt{1+\sum_{j=2}^{n}(x_j)^2}\sin x_1, x_2, \cdots, x_n\right), \tag{2}$$

where $x = (x_1, \cdots, x_n) \in R^n$. It is easy to see that F actually induces a map R^n onto H_1^n. Its derivative, for example for $n = 3$, is given by :

$$\frac{\partial F}{\partial x_1} = \left(-\sqrt{1+\sum_{j=2}^{3}(x_j)^2}\sin x_1, \sqrt{1+\sum_{j=2}^{3}(x_j)^2}\cos x_1, 0, 0\right)$$

$$\frac{\partial F}{\partial x_2} = \left(\frac{x_2}{\sqrt{1+\sum_{j=2}^{3}(x_j)^2}}\cos x_1, \frac{x_2}{\sqrt{1+\sum_{j=2}^{3}(x_j)^2}}\sin x_1, 1, 0\right)$$

$$\frac{\partial F}{\partial x_3} = \left(\frac{x_3}{\sqrt{1+\sum_{j=2}^{3}(x_j)^2}}\cos x_1, \frac{x_3}{\sqrt{1+\sum_{j=2}^{3}(x_j)^2}}\sin x_1, 0, 1\right).$$

The metric with respect to the basis $\{\partial F/\partial x_1, \partial F/\partial x_2, \partial F/\partial x_3\}$ is given in the following matrix form:

$$\begin{bmatrix} -\left(1+\sum_{j=2}^{n}(x_j)^2\right) & 0 & 0 \\ 0 & \frac{1+(x_3)^2}{1+\sum_{j=2}^{n}(x_j)^2} & -\frac{x_2x_3}{1+\sum_{j=2}^{n}(x_j)^2} \\ 0 & -\frac{x_2x_3}{1+\sum_{j=2}^{n}(x_j)^2} & \frac{1+(x_2)^2}{1+\sum_{j=2}^{n}(x_j)^2} \end{bmatrix}.$$

The derivative for the general case can be expressed in a similar way. Rank F_* is easily seen to be 3 in the above case and n in the general case. Hence, F is an immersion. Indeed, it is an isometric immersion with the above induced metric. Therefore, F is a covering projection. We denote by $\tilde{H}_1^n$ the space R^n endowed with the above induced metric and call it the n-dimensional hyperbolic space form of constant curvature -1 with signature 1. Note that $\tilde{H}_1^n$ is simply connected. All other hyperquadrics except S^1 are simply connected.

Any geodesic in $\tilde{H}_1^n$ is mapped under F onto a geodesic in H_1^n and the preimage of a geodesic in H_1^n is a disjoint union of geodesics in $\tilde{H}_1^n$.

A totally geodesic submanifold in a hyperquadric is, in the spirit of the Kleinian approach, defined as follows: A complete totally geodesic submanifold of dimension k in a hyperquadric is the intersection of the hyperquadric and a (k+1)-dimensional linear subspace of R_p^{n+1}. Obviously, a totally geodesic submanifold of a hyperquadric is also a hyperquadric in the linear subspace used to define it. From now on in this paper, a totally geodesic submanifold always means a complete one unless otherwise specified.

Given an open subset U in a hyperquadric, a (complete) c^k-totally geodesic foliation of U is a c^k-foliation of U whose leaves are (complete) totally geodesic submanifolds of constant dimension.

Let $\{U_\alpha\}, \alpha \in \mathcal{A}$ be a collection of mutually disjoint connected open subsets of the hyperquadric. Let $\mathcal{F}_\alpha$, $\forall \alpha \in \mathcal{A}$ be a c^k-totally geodesic foliation of dimension q defined in U_α. Then the collection of the pairs $\{U_\alpha, \mathcal{F}_\alpha\}$, $\alpha \in \mathcal{A}$, will be called a c^k-**lamination** of dimension q on the hyperquadric. We will often call them just a differentiable lamination. Note that the original definition of a lamination is due to Thurston .

Now let $V(n,r)$ be the Stiefel manifold of ordered r-frames in R^n. It is well known that $V(n,r) \to V(n,1)$ forms a principal fiber bundle in a natural way. Denote by ρ the largest integer such that the fibration $V(n,\rho) \to V(n,1)$ has a global section. Define by $\nu(n)$ the largest number so that $\rho(n-\nu(n)) \geq \nu(n)+1$. Some of the numerical values for $\nu(n)$ are given as follows: $\nu(1)=0$, $\nu(2)=0$, $\nu(3)=1$, $\nu(4)=0$, $\nu(5)=1$, $\nu(6)=2$, $\nu(7)=3$ etc.

Theorem 1. *Let $\mathcal{F}$ be a (geodesically) complete totally geodesic foliation of an open subset $U \subset S_p^n$ of dimension q, then $q-p \leq \nu(n-p)$. In particular, a codimension 1 complete totally geodesic foliation on U can exist only when $n-p=0$ or 1.*

Proof. Decompose $R_p^{n+1} = R_p^p \bigoplus R_0^{n+1-p}$. Given a (q+1)-dimensional subspace $L \subset R_p^{n+1}$, $L \cap R_0^{n+1-p}$ has dimension at least $q-p+1$. The totally geodesic submanifold of S_p^n defined by the linear subspace R_0^{n+1-p}

is the ordinary (n-p)-sphere. Thus the subspace $L \cap R_0^{n+1-p}$ defines a complete totally geodesic submanifold of dimension $q-p$ in S_0^{n-p}. Hence, $\mathcal{F}$ induces a complete totally geodesic foliation on $U \cap S_0^{n-p}$. The desired inequality then follows from the arguments given in [**1**]. For the second half, if $q = n-1$, the induced foliation will have dimension $n-p-1$. From the above inequality, this is possible only when $n = p$ or $n = p+1$.

In view of the above anti-isometry between S_p^n and H_{n-p}^n, only hyperquadrics that can support a complete totally geodesic foliation of codimension 1 in an open subset are either H_0^n or H_1^n, $n = 1, 2, \cdots$.

The totally geodesic foliations of codimension 1 in H^n were treated in [**3**]. We will, in this paper, concentrate on those in H_1^n.

Let U be a connected open subset of H_1^n. Let $\mathcal{F}$ be a complete differentiable totally geodesic foliation of U. Given a leaf L_x of the foliation passing through x , one can find a regular curve $\mu(s)$, $s \in (-\epsilon, \epsilon)$, $\epsilon > 0$, transversal to the foliation in a neighborhood of x. If the leaf L_x is non-degenerate, take the unit normal vector field to the foliation along the curve $\mu(s)$. Translate parallelly the vector field along the leaves passing through $\mu(s)$. Denote by X the resulting vector field in a neighborhood of the leaf L. If L happens to be a degenerate leaf, there is a light-like vector which is transversal to the leaf L at x. Extend that vector to a vector field along $\mu(s)$ through the parallel displacement in a neighborhood of x. Then as before, translate it parallelly along the leaves. Denote the resulting vector field by X. Also denote by ∇ the Levi-Civita connection in $\tilde{H}_1^n$ (or in H_1^n), induced from the quadratic form q given in (1) .

Given a vector field Z tangent to the leaves, define a field of linear operator $c_Z(x)$ from Span $\{X\}$ into itself at x by

$$c_Z(x)(X) = -P(\nabla_X Z), \tag{3}$$

where P is the natural projection of $T_x H_1^n$ onto Span $\{X\}$ with respect to the direct sum decomposition $T_x H_1^n = T_x L \bigoplus \text{Span } \{X\}$.

Let $\gamma = \gamma(t)$, $-\infty < t < \infty$, be the geodesic in L_x such that $\gamma(0) = x$ and $\dot\gamma(0) = Z$. It is then shown in [**1**] that the operator c_Z along L_x satisfies a Ricatti type differential equation

$$\frac{dy}{dt} = y^2 - \langle Z(t), Z(t) \rangle. \tag{4}$$

This differential equation has two types of global solutions

$$c_Z(t) = -\frac{\sinh t - c_Z(0)\cosh t}{\cosh t - c_Z(0)\sinh t}, \text{ if } \langle Z, Z \rangle = -1$$

or

$$c_Z(t) = 0,\ \forall t \in R,\ \text{if}\ \langle Z, Z\rangle = 0.$$

In the former case, $c_Z(0)$ is a constant satisfying $|c_Z(0)| \leq 1$. As a consequence, we have that $|c_Z(t)| \leq 1$, $\forall t \in R$ and $|c_Z(0)| = 1$ if and only if $c_Z(t) \equiv 1$ or -1 $\forall t \in R$.

The laminations on $H^n = H^n_0$ have been taken up in various occasions from the point of recent developments in hyperbolic geometry. From differential geometric point of view, we have made use of the notion to describe the isometric immersions of codimension 1 of a hyperbolic space into another hyperbolic space of the same constant curvature -1 [3].

In what follows, we will describe all the codimension 1 laminations on $H^n_0 = H^n$ and $H^n_1 \approx S^n_{n-1}$. From the Kleinian point of view, there is a striking dual nature between these two spaces.

In order to explain our construction most effectively, we will start out with the lowest dimension, that is, when n = 2. In this case, H^2_1 is anti-isometric to S^2_1. The anti-isometry maps the totally geodesic submanifolds of one to those of the other. The codimension 1 lamination is exactly a geodesic lamination in this case.

Denote by P^2 the real projective space of dimension 2. According to the Klein model of H^2, it is given as the unit disk in P^2 represented by the homogeneous coordinates $\left\{(1, \xi_1, \xi_2)\ :\ {\xi_1}^2 + {\xi_2}^2 < 1\right\}$, where the first coordinate 1 represent the time like coordinate in R^3_1. It is well-known that the boundary of the unit disk is represented by the the null cone. The complement of H^2 in P^2 then represents S^2_1. The geodesics of H^2, the chords in the disk, are given as the segments of lines inside the disk in $R^2(\xi_1, \xi_2)$. Similarly, a geodesic in S^2_1 is represented by the portion of a line outside the disk in $R^2(\xi_1, \xi_2)$. Two geodesics in S^2_1 do not meet in S^2_1 if and only if the lines representing them intersect inside the closed unit disk. Note that the lines are considered to be projective lines; hence they always intersect in P^2.

Now let us denote by S^1 the center circle in S^2_1, which is represented by the homogeneous coordinates $(0, y_1, y_2)$ in P^2. Let $\{U, \mathcal{F}\}$ be a complete totally geodesic foliation in an open subset $U \subset S^2_1$. Since all the leaves of the foliation meet S^1 at a unique point (projectively), the foliation will be uniquely characterized by a function from $U \cap S^1$ into the set of all geodesics in S^2_1.

The set of all line in P^2 consists of the set of all lines in $R^2(\xi_1, \xi_2)$ and the line of infinity. The set of all oriented lines in $R^2(\xi_1, \xi_2)$ is parametrized by two parameters $(a, \theta) \in R \times S^1$. The actual correspondence is elementary and is given by

$$(a, \theta) \mapsto \xi_1 \cos\theta + \xi_2 \sin\theta = a.$$

θ and $\pi + \theta$ give the same line with the opposite orientations. With this parametrization the set of all oriented lines in $R^2(\xi_1, \xi_2)$ becomes a differentiable manifold, i.e., the cylinder built on the the unit circle. For an open subset $U \subset S_1^2$, $U \cap S^1$ is a disjoint union of open intervals or the center circle itself. Since the center circle always intersects any other complete geodesics, it cannot be a leaf of a complete totally geodesic foliation of any open subset of S_1^2. Thus any such a foliation is completely determined by a differentiable function from a connected open subset of the center circle into the cylinder. As any two lines in P^2 always intersect, we can further restrict the range space of the function by requiring that $0 \leq a \leq 1$. Thus, the the range should be in $[0,1] \times S^1$. The function must be an immersion as we shall see later. The following lemma will be useful later.

Lemma 1. *Let $\mathcal{F}$ be a codimension one totally geodesic foliation defined in a connected open subset $U \subset \tilde{H}_1^n$. Assume that the leaves of the foliation are degenerate. Let L_1 and L_2 be two leaves of the foliation. Then they contain a unique light like line and the tangent space to the leaves at the points in the center circle are parallel to each other along the center circle.*

Proof. By definition L_1 and L_2 are given as the intersections of n-planes P_1 and P_2 with H_1^n in R_2^{n+1}, respectively. Then $P_1 \cap P_2$ has dimension at least $n-1$. Let L_1 and L_2 meet the center circle at x_1 and x_2, respectively. If we also denote the tanent planes of L_1 and L_2 at x_1 and x_2 by $T_{x_1}L_1$ and $T_{x_2}L_2$, respectively, there are unique degenerate lines l_1 and l_2 in $T_{x_1}L_1$ and $T_{x_2}L_2$, respectively. For, if there is another degenerate line, say l, in $T_{x_1}L_1$ then light-like vectors in these two lines will have non-vanishing inner product; hence contradicting the degeneracy.

Let $\gamma = \gamma(s)$, $a \leq s \leq b$ be the portion of the center circle given by $\gamma = U \cap S^1$ such that $\gamma(a) \in L_1$ and $\gamma(b) \in L_2$. From the above argument one sees that each leaf along the curve $\gamma(s)$ contains a unique light-like line which form a line bundle over $\gamma(s)$.

At this point, we digress to discuss a special but useful choice of a basis for R_1^n. Let $(x_1, x_2, \cdots, x_n)$ be the standard coordinates of R_1^n so that the inner product is given by $q(x) = -{x_1}^2 + \sum_{j=2}^n {x_j}^2$. Given a null direction in R_1^n, we may assume that the null direction is given by the vector of the form $(y, y, 0, \cdots, 0) \in R_1^n$ up to an isometry of R_1^n. Now let $e = (1/\sqrt{2})(1, 1, 0, \cdots, 0)$ and $\bar{e} = (1/\sqrt{2})(1, -1, 0, \cdots, 0)$. Clearly these two vectors are null vectors and $\langle e, \bar{e} \rangle = -1$. They span the space

$R_1^2 = \{(x_1, x_2, 0, \cdots, 0)\}$. If we denote by $e_j,\ 1 \leq j \leq n-2,$ the standard orthonormal basis for the orthogonal complement of R_1^2 in R_1^{n+1}, our desired basis consists of $\{e, \bar{e}, e_j,\ 1 \leq j \leq n-2\}$. We call this basis an adapted basis for the degenerate space, which is the span of $\{e, e_j,\ 1 \leq j \leq n-2\}$. Conversely, given a degenerate subspace of codimension one, one can always obtain an adapted basis up to an isometry.

Going back to the line bundle of null vectors along $\gamma = \gamma(s)$, we can choose a frame $\{e(s), \bar{e}(s), e_j(s),\ 1 \leq j \leq n-2\}$ along the curve $\gamma(s),\ a \leq s \leq b,$ such that $e(s)$ spans the null direction of the leaf at $\gamma(s)$ and the frame forms an adapted basis for the leaf.

As before, denote by ∇ the Levi-Civita connection in $\tilde{H}_1^n$ induced from the quadratic form q in (1). Then we have

$$\nabla_{\dot{\gamma}(s)} e(s) = e_L(s) + c_{e(s)}(\dot{\gamma}(s)).$$

Here $c_{e(s)}$ is the conullity operator at $\gamma(s)$ and $e_L(s)$ is the leaf component of $\nabla_{\dot{\gamma}(s)} e(s)$. See [**1**] for more details about the conullity operator and its properties in the indefinite setting. Since $\langle e(t), e(t)\rangle = 0,\ \forall t,$ the Ricatti equation is $dy/dt = y^2$ and the trivial solution $y(t) = 0$ is the only global solution along the null line spanned by $e(s)$. Thus $c_{e(s)} = 0,\ a \leq s \leq b$. This tells us that the parallel displacement of $e(s)$ along the curve $\gamma = \gamma(s)$ always stays in the leaves. But the parallel displacement of a null vector is a null vector; hence, by the above uniqueness, $e(s),\ a \leq s \leq b$ are parallel to each other along the curve $\gamma = \gamma(s)$. Next let $e(s)$ and $\bar{e}(s)$ be the parallel displacement of $e(a)$ and $\bar{e}(a)$ along the curve $\gamma = \gamma(s)$, respectively. Denote by $V(s)$ the orthogonal complement of the span of $e(s)$ and $\bar{e}(s)$ in $T_{\gamma(s)}\tilde{H}_1^n,\ s \in [a, b]$. Then we see that the span of $e(s)$ and $V(s)$ is precisely the tangent space of the leaf at the point $\gamma(s)$. For, suppose that there is a vector in the tangent space of the leaf given in the form of $ue(s) + v\bar{e}(s) + W,\ u, v \in R;\ W \in V(s)$. $\langle e(s),\ ue(s) + v\bar{e}(s) + W\rangle = -v = 0$ implies that the vector is in the span of $e(s)$ and $V(s)$. Since the span of $e(s)$ and $\bar{e}(s)$ is parallel along $\gamma(s)$, $V(s)$ is also parallel; hence, the span of $e(s)$ and $V(s)$ is parallel along the curve $\gamma(s)$.

The parallel displacement in H_1^n along the center curve is given by the rotations in the plane spanned by two coordinates of R_2^{n+1} :

$$(x_1, x_2, x_3, \cdots, x_{n+2}) \mapsto (x_1 \cos s - x_2 \sin s, x_1 \sin s + x_2 \cos s, x_3, \cdots, x_{n+2}).$$

Let L be the tangent space to a leaf at a point in the center circle, say $x = (x_1 \cos s - x_2 \sin s, x_1 \sin s + x_2 \cos s, x_3, \cdots, x_{n+2})$. Then the leaf itself is given as the intersection of H_1^n and the n-plane spanned by L and $x = (x_1 \cos s - x_2 \sin s, x_1 \sin s + x_2 \cos s, x_3, \cdots, x_{n+2})$.

Remark. The arguments in the proof can easily be applied to more general situations. Indeed, we may choose any transversal curve to the foliation . We will still have the same conclusion as in Lemma 1.

We will give some useful examples of complete totally geodesic foliations.

Example 1. Note that any differentiable function f from a connected open subset $\mathcal{O} \subset S^1$ into the set of non-super parallel geodesics in H^2 will define a differentiable totally geodesic foliation of the open subset $U = \cup_{s\in\mathcal{O}} f(s) \subset S_1^2$. Let $\mathcal{P}_w$ be the differentiable family of lines in $R^2(\xi_1, \xi_2)$, where w is in the unit circle and is the common intersection point of all of the lines in the family. The family is otherwise known as the parabolic pencil in H^2, consisting of the parallel geodesics whose end points are w. For a given point $s \in \mathcal{O}$, there is a (projectively) unique line $\ell_s \in \mathcal{P}_w$ which meets s in the line of infinity. We define $f(s) = \ell_s$, $\forall s \in \mathcal{O}$. We see readily that f is an immersion.

Similarly, we may consider a family of lines in $R^2(\xi_1, \xi_2)$, where w is in H^2. This family corresponds to the elliptic pencil. As in the above case, one can uniquely define a differentiable immersion from $\mathcal{O}$ into the family, defining a complete totally geodesic foliation.

Among the examples, let us choose a very special one. Set $w = (0,0) \in R^2(\xi_1, \xi_2)$. Then the elliptic pencil $\mathcal{P}_{(0,0)}$ gives the complete totally geodesic foliation of S_1^2 consisting of the vertical time-like geodesics perpendicular to the center circle. This is the foliation studied in [**6**].

Example 2. Define $\mathcal{L}$ to be the family of lines which are tangent to the unit circle in $R^2(\xi_1, \xi_2)$. Each line in this family projectively represents two opposite generators in the well-known two rulings of S_1^2 by two families of null lines. Every pair in the family $\mathcal{L}$ intersect in the complement of the closed unit disk in P^2, but this point of intersection projectively represents the intersection points of lines in the pair which belong to the opposite rulings. Thus $\mathcal{L}$ projectively represents the two rulings which are obviously complete totally geodesic foliations. In fact it is easy to see that all the leaves are null lines of S_1^2.

Variations of this construction can be readily obtained. For example, consider the interval $(-1,1) \subset R$. Then $(a, \theta) \in (-1,1) \times S^1$ represents a line in P^2 which intersects the unit disk. Rotate the line about the origin in $R^2(\xi_1, \xi_2)$. The resulting one parameter family of lines gives rise a complete totally geodesic foliation of S_1^2 by time-like lines.

It is easy to see that there are families of lines which are not always given by a line under a one parameter family of rotations. Let $\mathcal{L} = \ell_s$, $s \in \mathcal{O} \subset S^1$ be a given differentiable one parameter family of lines in $R^2(\xi_1, \xi_2)$. We assume that the one parameter family is orientable, that is, all the line in the family can be oriented at least continuously. We also assume for an obvious reason that every line in the family intersects the closed unit disk in $R^2(\xi_1, \xi_2)$. Furthermore, we require that any pair of lines in the family meet either in the closed unit disk or the point of intersection outside of the disk occurs in the incoming portion of one line toward the disk and in the outgoing portion of the other from the disk. There will be, indeed, infinitely many choices of such one parameter families. Any one of such families gives rise to a complete totally geodesic foliation of the open subset of S_1^2, which is basically given as the union of the geodesics in the family.

Example 3. In H_1^n similar constructions as in the previous examples give higher dimensional cases of totally geodesic foliations. Obviously, there are many more totally geodesic foliations exist. We will give a totally geodesic foliation of H_1^n, which will be of particular interest later on. Let the the quadratic form be given by

$$q(x) = -\sum_{j=1}^{2}(x_j)^2 + \sum_{j=3}^{n+1}(x_j)^2.$$

Then P^n is divided into three disjoint components S_2^n, ∂H_1^n and $H_1^n = S_{n-1}^n$. In particular, if $(\xi_1, \cdots, \xi_n)$ is the inhomogeneous coordinate system for $x_1 = 1$, we have in the n-plane $x_1 = 1$ that

$$S_2^n = \left\{(\xi_1, \cdots, \xi_n) : -{\xi_1}^2 + \sum_{j=2}^{n}\xi_j > 1\right\},$$
$$\partial H_1^n = \left\{(\xi_1, \cdots, \xi_n) : -{\xi_1}^2 + \sum_{j=2}^{n}\xi_j = 1\right\},$$
$$H_1^n = \left\{(\xi_1, \cdots, \xi_n) : -{\xi_1}^2 + \sum_{j=2}^{n}\xi_j < 1\right\}.$$

Now set

$$e_1(t) = (1, \sinh t, \pm\cosh t, 0, \cdots, 0),$$
$$e_2(t) = (0, \cosh t, \pm\sinh t, 0, \cdots, 0),$$
$$e_3(t) = (0, 0, 0, 1, 0, \cdots, 0),$$
$$\vdots$$
$$e_n(t) = (0, 0, 0, 0, \cdots, 0, 1).$$

Then $\{e_j(t),\ t \in R,\ 1 \le j \le n\}$ is a frame field along a branch of the hyperbola in $H_1^2(x_1, x_2, x_3) \subset H_1^n$.

$$e_i(t) \perp e_j(t) \qquad (1 \le i, j \le n)$$
$$\langle e_1(t), e_1(t)\rangle = 0, \quad \langle e_2(t), e_2(t)\rangle = -1, \quad \langle e_j(t), e_j(t)\rangle = 1,\ (3 \le j \le n).$$

Set $V(t)$ to be the span of $e_j(t)$, $1 \leq j \leq n$. Then $V(t)$ is an n-dimensional subspace in R_2^{n+1}. In terms of the original rectangular coordinates, an element $x = \sum_{j=1}^n a_j e_j(t) \in V(t)$ is expressed as

$$x = (a_1, a_1 \sinh t + a_2 \cosh t, a_1 \cosh t + a_2 \sinh t, a_3, \cdots, a_n).$$

Then we have

$$\begin{aligned} x \in H_1^n &\iff \langle x, x\rangle = -1 \\ &\iff -(a_2)^2 + \sum_{j=3}^n (a_j)^2 = -1. \end{aligned} \tag{5}$$

$V(t) \cap H_1^n$ is a hyperboloid of two sheets in H_1^n. Suppose that $V(t) \cap V(t') \cap H_1^n \neq \emptyset$. Then for some pairs of n-tuples $(a_1, \cdots, a_n)$ and $(a_1{}', \cdots, a_n{}')$,

$$\begin{aligned} &(a_1, a_1 \sinh t + a_2 \cosh t, a_1 \cosh t + a_2 \sinh t, a_3, \cdots, a_n) \\ &\quad = (a_1{}', a_1{}' \sinh t' + a_2{}' \cosh t', a_1{}' \cosh t' + a_2{}' \sinh t', a_3{}', \cdots, a_n{}'). \end{aligned}$$

This implies that

$$\begin{gathered} a_1 \sinh t + a_2 \cosh t = a_1{}' \sinh t + a_2{}' \cosh t, \\ a_1 \cosh t + a_2 \sinh t = a_1{}' \cosh t + a_2{}' \sinh t, \\ a_j{}' = a_j,\ 1 \leq j \leq n, /; j \neq 2. \end{gathered} \tag{6}$$

Combining (5) and (6), we get

$$(a'_2)^2 = (a_2)^2 \iff a'_2 = \pm a_2.$$

But $a'_2 = a_2$ implies that $t = t'$ and $V(t) \equiv V(t')$; hence, the subspaces $V(t)$ and $V(t')$ meet in the $(n-1)$-dimensional plane determined by $a'_2 = -a_2$. This equality is the precisely the condition that the two sheets of the hyperboloids represented by the n-planes $V(t)$ and $V(t')$ meet each other in the opposite sheets. Thus, for instance, by choosing the sheets of the hyperboloid corresponding to the half plane determined by $a_2 > 0$, we obtain a foliation of codimension 1 in $\tilde{H}_1^n$ consisting of degenerate leaves, as t runs from $-\infty$ to $+\infty$ along the original hyperbola. This foliation is the higher dimensional version of the double rulings in S_1^2. Note that the above construction can be carried out starting with any geodesic in ∂H_1^n instead of the particular hyperbolas of the above kind.

Any complete totally geodesic submanifold of codimension 1 intersects the center circle S^1 of $H_1^n = S_{n-1}^n$ once and only once in the projective sense. Thus again any completely totally geodesic foliation of codimension 1 of a connected open subset U in $H_1^n = S_{n-1}^n$ is parametrized by the open interval $U \cap H_1^n \cap S^1 = U \cap S_{n-1}^n \cap S^1$ in the center circle S^1. Consequently, any differential lamination on $H_1^n = S_{n-1}^n$ will be parametrized by a family of mutually disjoint connected open subsets in the center circle. As before the set of all oriented (n-1)-planes corresponds to the differentiable manifold $R \times S^{n-1}$, where S^{n-1} is the unit sphere in R^n. A parametrization by a connected open subset of the center circle means an differentable immersion of the open subset into the parameter space $R \times S^{n-1}$, which satisfies appropriate conditions to generate a complete totally geodesic foliation.

Unlike in H^n the space of differentiable laminations on H_1^n is rather simple. Since any complete totally geodesic foliation of an open subset is uniquely parametrized by one parameter in the intersection of the open subset and the center circle, any mutually disjoint family of connected open subsets of the center circle parametrizes a lamination on H_1^n and vice versa. As seen in the above examples, a connected open subset of the center circle may parametrize, in general, infinitely many complete totally geodesics foliations. Thus, a given family of mutually disjoint connected open subsets in the center circle can parametrize infinitely many mutually distinct differentiable laminations.

3. Isometric immersions.

Let M_p^n and $\bar{M}_q^{n+1}$ be two (indefinite) Riemannian manifolds of dimensions n and $n+1$ and signatures p and q, respectively (p = 0 and q = 0 inclusive). We will often denote them and their differential geometric quantities without superscripts or subscripts when there is no fear of confusion. Thus, TM_x and $T\bar{M}_y$ denote the tangent spaces of M_p^n and $\bar{M}_q^{n+1}$ at the points $x \in M_p^n$ and $y \in \bar{M}_q^{n+1}$, respectively. Denoted by $\langle\, ,\, \rangle$ the (indefinite) Riemannian metrics on M and $\bar{M}$. As usual, ∇ and $\bar{\nabla}$ denote the corresponding Levi-Civita connections on M and $\bar{M}$.

Let $f : M \to \bar{M}$ be an isometric immersion of M into $\bar{M}$. Given X, Y, and $Z \in TM_x$, and the unit normal e_x at $x \in M$, there are two fundamental equations of the immersion :

$$R(X,Y)Z = \bar{R}(X,Y)Z + (A_{e_x}X \wedge A_{e_x}Y)Z; \tag{7}$$

$$(\nabla_X A_{e_x})Y = (\nabla_Y A_{e_x})X. \tag{8}$$

Here, R and $\bar{R}$ are the curvature tensors of M and $\bar{M}$ and A_{e_x} is the shape operator of the immersion f in the direction of e_x at $x \in M$. The first equation (7) is called the Gauss equation and the second (8) the Codazzi equation.

The dimension of the null space of A_{e_x} is called the conullity of the immersion and is denoted by ν_x, $x \in M$. The open subset of M where the conullity assumes the minimum ν is then foliated by (geodesically) complete totally geodesic submanifolds of dimension ν if M is geodesically complete. See [**1**] for more details of these accounts.

In particular, if both M and $\bar{M}$ are space forms of the same constant curvature, then $\nu \geq n-1$. Unless $f : M \to \bar{M}$ is a totally geodesic immersion, $\nu = n-1$, and the immersion naturally defines a differentiable lamination on M. In particular, non-totally geodesic isometric immersions from a space form into another space form of the same curvature can exist only if $M = H^n$ or H_1^n by Theorem 1. When $M = H^n$, we have obtained in [**3**] that any differentiable lamination can be realized as a conullity lamination of an isometric immersion $f : H^n \to H^{n+1}$. Furthermore, any differentiable lamination on H^n can be the conullity foliation of infinitely many mutually non-rigid isometric immersions.

Note that H_1^n may possibly isometrically and non-totally geodesically immersed only into either H_1^{n+1} or H_2^{n+1}.

In [**6**] Graves and Nomizu proved that there was no umbilic free isometric imbedding of S_1^n into S_1^{n+1} if $n \geq 4$. The following is an improvement of their result.

Theorem 2. *Only isometric immersions of S_1^n into S_1^{n+1} for $n \geq 3$ are totally geodesic ones; hence they are all congruent to the standard imbedding induced from the imbedding $(x_1, \cdots, x_{n+1}) \mapsto (x_1, \cdots, x_{n+1}, 0)$ of R_1^n into R_1^{n+1}.*

Proof. Since S_1^n is anti-isometric to H_{n-1}^n, S_1^n can have a complete totally geodesic lamination of codimension 1 only if either $n-1=0$ or 1. If n = 1, we have an isometric immersion of a curve; hence any regular and non-totally geodesic immersion of R with arclength parametrization will do. When n = 2, there are many isometric immersions of S_1^2 into S_1^3, which are not totally geodesic. Some examples are given in [**6**].

Lemma 2. *Let $\mathcal{F}$ be a complete totally geodesic foliation of a connected open subset $U \subset \tilde{H}_1^n$. Then U admits a global Frobenius coordinate system with respect to $\mathcal{F}$.*

Proof. Let x_1 and x_2 be the first and second coordinate functions of R_2^{n+1}. Let L a leaf of $\mathcal{F}$. Then L is the intersection of an n-plane P and H_1^n in R_2^{n+1}. Consider $P \cap R_2^2(x_1, x_2) \subset R_2^{n+1}$. The intersection is of either dimension 1 or dimension 2. In the latter case, $P \cap R_2^2(x_1, x_2) = R_2^2(x_1, x_2)$. Thus L contains the entire $R_1^1 = \tilde{H}_1^1 \subset \tilde{H}_1^n$. Hence L meets every totally geodesic submanifold in $\tilde{H}_1^n$, presenting a contradiction. In the former, the line of intersection represents two distinct points in $R_1^1(\xi_1) = \tilde{H}_1^1 \subset \tilde{H}_1^n$. Suppose that these points lie in L. Then they can be joined by a geodesic segment in L. Hence, there is a 2-plane P_2 in P which represents the geodesic segment. Given any leaf L' of $\mathcal{F}$ nearby the leaf L, L' and the geodesic represented by P has non-empty intersection. Thus $L \cap L'$ is non-empty, contradicting that they are leaves of the same foliation. What we have seen so far is that every leaf intersects $\tilde{H}_1^1 \subset \tilde{H}_1^n$ once and only once. Denote by $\mathcal{O}_{\mathcal{F}}$ the intersection $U \cap \tilde{H}_1^1$. $\mathcal{O}_{\mathcal{F}}$ is an open interval, say (a,b), $-\infty \leq a < b \leq \infty$, in $\tilde{H}_1^1 \approx R$. Now let $\{e_j,\ j = 1 \leq j \leq n-1\}$, be a differentiable frame field along the open interval so that the frame at each point in the open interval is a basis for the leaf passing through the point. Define a differentiable map $F : R^{n-1} \times (a,b) \to U$ by

$$(9)\ F(t_1, \cdots, t_{n-1}) = \exp_s \left(\sum_{j=1}^{n-1} t_j e_j \right), \ (t_1, \cdots, t_{n-1}) \in R^{n-1},\ s \in (a,b).$$

Here $\exp_s$ means the exponential map at s. F is clearly a diffeomorphism because of the curvature condition on $\tilde{H}_1^n$. This completes the proof.

A null curve $x = x(s)$, $s \in (a,b)$, $-\infty \leq a < b \leq \infty$, in $\tilde{H}_1^n$ is a differentiable curve such that $\langle dx/ds, dx/ds \rangle = \langle \dot{x}, \dot{x} \rangle = 0, \forall s \in (a,b)$. As before, we locally identify $\tilde{H}_1^n$ with $H_1^n \subset R_2^{n+1}$ under the covering map in (2) .

We will give here a brief account on Cartan frames on $\tilde{H}_1^3$. The idea can be applied to $\tilde{H}_1^n$ with minor modifications. The reader should find more detailed discussions about the Cartan frames in [**4**, **5**] and [**6**].

A null frame in $\tilde{H}_1^3$ is an ordered quadruple

$$F = [A, B, C, x] = \begin{bmatrix} a_1 & b_1 & c_1 & x_1 \\ a_2 & b_2 & c_2 & x_2 \\ a_3 & b_3 & c_3 & x_3 \\ a_4 & b_4 & c_4 & x_4 \end{bmatrix},$$

where A, B, C and x are vectors in R_2^4 with the following additional conditions:

$$\begin{aligned} &\langle A, B \rangle = -1, \langle A, A \rangle = \langle B, B \rangle = 0, \\ &\langle C, C \rangle = 1, \langle A, C \rangle = \langle B, C \rangle = 0, \\ &\langle x, x \rangle = -1, \langle x, A \rangle = \langle x, B \rangle = \langle x, C \rangle = 0, \\ &\det F = \pm 1. \end{aligned}$$

To any null frame there is an associated orthonormal frame defined by

$$L(F) = \left[\frac{1}{\sqrt{2}}(A+B), \frac{1}{\sqrt{2}}(A-B), C, x \right].$$

Let

$$N = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is easy to see that $L(F) = F \cdot N^{-1}$, $\forall F$. Given a null curve $x = x(s)$, $s \in (a,b)$, $-\infty \leq a < b \leq \infty$, in $\tilde{H}_1^n$, we call the following system of differential equations the adapted Frenet equations on the null curve.

$$\begin{aligned} dA/ds &= k_1(s)A(s) + k_2(s)C(s), \\ dB/ds &= -k_1(s)B(s) + k_3(s)C(s) + k_4(s)x(s), \\ dC/ds &= k_3(s)A(s) + k_2(s)B(s), \\ dx/ds &= k_4(s)A(s). \end{aligned}$$

In a matrix form the adapted Frenet equations are stated as

$$F^{-1}\frac{dF}{ds} = \begin{bmatrix} k_1 & 0 & k_3 & k_4 \\ 0 & -k_1 & k_2 & 0 \\ k_2 & k_3 & 0 & 0 \\ 0 & k_4 & 0 & 0 \end{bmatrix}.$$

It is known that there is a unique framed curve $F = F(s)$ which is a solution to the Frenet equations under given initial conditions $F(0) = F_x$. In particular, a Cartan framed curve is an adapted framed curve $F = F(s)$ whose Frenet equations have the form

$$F^{-1}\frac{dF}{ds} = \begin{bmatrix} 0 & 0 & k_3 & 1 \\ 0 & 0 & k_2 & 0 \\ k_2 & k_3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Given initial conditions, Cartan framed fields uniquely exist. Given a Cartan framed curve F , we define the B-scroll $B_F : R^2 \to H_1^3$ of F by

$$B_F(u,s) = x(s) + uB(s).$$

We see easily that

$$\begin{aligned} B_{F*}(\partial/\partial u) &= B(s), \\ B_{F*}(\partial/\partial s) &= A + uk_3C + ux. \end{aligned}$$

Therefore we have

$$\begin{aligned}\langle B_{F*}(\partial/\partial u), B_{F*}(\partial/\partial u)\rangle &= 0,\\ \langle B_{F*}(\partial/\partial u), B_{F*}(\partial/\partial s)\rangle &= -1,\\ \langle B_{F*}(\partial/\partial s), B_{F*}(\partial/\partial s)\rangle &= u^2(1+{k_3}^2).\end{aligned}$$

R^2 with this induced metric has signature $(-,+)$. The shape operator of the B-scroll relative to the basis $\{\partial/\partial u, \partial/\partial s\}$ will then be calculated to be

$$\begin{bmatrix}-k_3 & -k_2 - udk_3/ds\\ 0 & -k_3\end{bmatrix}.$$

Thus the Lorentz surface $B_F : R^2 \to H_1^3$ has the same constant curvature -1 as H_1^3 if and only if $k_3 = 0$. This surface is often referred to as a generalized null cubic. If, in addition, $k_2 \equiv 1$ everywhere, the B-scroll takes especially simple form with the adapted Cartan framed curve given by

$$F(s) = \frac{1}{\sqrt{2}}\begin{bmatrix}\cosh s & \cosh s & \sinh s & \sinh s\\ \cos s & -\cos s & -\sin s & \sin s\\ \sin s & -\sin s & \cos s & -\cos s\\ \sinh s & \sinh s & \cosh s & \cosh s\end{bmatrix}.$$

The induced metric and the shape operator relative to the basis $\{\partial/\partial u, \partial/\partial s\}$ are respectively given as

$$\begin{bmatrix}0 & -1\\ -1 & u^2\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}.$$

The u coordinate curves are the precisely the geodesics in the relative nullity foliation and are imbedded isometrically onto the the geodesics in the direction of B vector field.

Let $\mathcal{F}$ be a complete totally geodesic foliation of a connected open subset $U \subset \tilde{H}_1^n$. Let $f : U \to H_1^{n+1}$ be an isometric immersion such that its relative nullity foliation is $\mathcal{F}$.

Lemma 3. *The leaves of $\mathcal{F}$ are either all non-degenerate or all degenerate.*

Proof. If a leaf L is non-degenerate, there is a neighborhood of L where all the leaves are non-degenerate. Suppose that there is a maximal open subset $V \subset U$ such that all the leaves in V are non-degenerate. Then the boundary ∂V of V consists of two degenerate complete totally geodesic submanifolds of codimension 1. According to the classification of the shape operator [**8**], there are four possible matrix representations of the shape operator relative to an pseudo orthonormal basis $\{\hat{e}, e, e_1, \cdots, e_{n-2}\}$. In our case the rank of the shape operator is known to be less than or equal to 1. If the shape operator A_{e_x} is diagonalized relative to the above basis, assuming $A_{e_x}\hat{e} \neq 0$, we have that $\langle A_{e_x}\hat{e}, e\rangle = \langle \hat{e}, e\rangle = -1$. On the other hand, $\langle A_{e_x}\hat{e}, e\rangle = \langle \hat{e}, A_{e_x} e\rangle = 0$. This is a contradiction. Therefore, the shape operator has rank n. This also implies that the ranks of the shape operators in the leaf are n everywhere, contradiction. Only other possible case is that it is represented by the matrix:

$$\begin{bmatrix} 0 & 0 & & & \\ 1 & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}.$$

Now suppose that one of the totally geodesic submanifolds in ∂V is the limit set of non-degenerate leaves $\{L_j,\ j = 1, 2, \cdots\}$ in U. The shape operator at a point in a non-degenerate leaf is always diagonalizable. For, if not, the shape operator will have the above matrix representation relative to a pseudo orthonormal basis. It implies that the vectors $\{e, e_1, \cdots, e_{n-2}\}$ span the null space, which will consequently be degenerate, a contradiction.

Let $\{x_j \in L_j,\ j = 1, 2, \cdots\}$ be a sequence of points which converges to a point $x \in L \subset \partial V$. Since the correspondence $x \mapsto A_{e_x}$ is differentiable, the correspondence $x \mapsto (A_{e_x})^2$ is also differentiable, which equals the zero matrix at any point on the leaf in the boundary ∂V. By differentiability of the correspondence $x \mapsto (A_{e_x})^2$, $\left\{A_{e_{x_j}},\ j = 1, 2, \cdots\right\}$ converges to the zero matrix. Consequently, the shape operator at any point in the leaf must also be represented by the zero matrix. This is a contradiction. Hence, the rank of the shape operator at any point in the leaf in ∂V must equal n, if the leaf is a limit leaf of non-degenerate leaves.

This proves that under the assumption in the lemma U is either completely foliated by non-degenerated leaves or else completely foliated by degenerated leaves.

Suppose that a connected open set $U \subset \tilde{H}_1^n$ is foliated by the degenerated leaves of the complete relative nullity foliation $\mathcal{F}$ which is induced by an isometric immersion$f : U \subset \tilde{H}_1^n \to H_1^{n+1}$. Let $\beta = \beta(s),\ s \in (a, b) \subset R,$ the portion of the center circle which is in U. Let $e(s)$ and $\hat{e}(s)$ be the parallel null vector fields along β such as described in the proof of Lemma 1. Recall that $e(s)$ generate the unique null line and $\langle e(s), e(s)\rangle = \langle \hat{e}(s), \hat{e}(s)\rangle = 0$ and $\langle e(s), \hat{e}(s)\rangle = -1$ in the leaf at each $s \in (a, b)$. Then $\hat{e}$ and e span a Lorentzian plane in $T_x\tilde{H}_1^n$, $\forall x \in U \subset \tilde{H}_1^n$. The orthogonal complement of the plane is in the null space of the immersion. Since each leaf intersects the curve β once and only once, we can extend the vector fields $e(s)$ and $\hat{e}(s)$ to the entire U by parallelly displacing them along the radial geodesic rays staring at $\beta(s)$, $s \in (a,b)$ in each leaf passing through $\beta(s)$. We denote the resulting vector fields by the same letter e and $\hat{e}$, respectively. Denote by $\gamma_s = \gamma_s(t)$, $t \in (c,d)$ the maximal integral curve of the vector field $\hat{e}$ passing through $\beta(s)$, $s \in (a,b)$. As before, we denote by $e_j(s)$, $1 \leq j \leq n-3$, the parallel displacement along the curve s of an orthonormal basis for the orthogonal complement of the plane spanned by $e(s')$ and $\hat{e}(s')$ at a point $s' \in (a,b)$. The leaf passing through a point $\beta(s)$ is given as the intersection of H_1^n and the n-plane spanned by $e(s), e_j(s)$, $1 \leq j \leq n-2$, and the position vector $\beta(s)$. In particular, the plane $P(s)$ spanned by $e(s)$ and $\beta(s)$ gives a geodesic in the leaf. The geodesic has $e(s)$ as its velocity vector at $\beta(s)$; hence, it has to be a light-like geodesic. On the other hand, $P(s)$ is degenerate and contains the line spanned by $e(s)$ as a unique null line. This implies that the geodesic must be the straight null line itself. Let us consider the $(n-1)$-plane in R_2^{n+1} which is perpendicular to the 2-plane that contains the center circle. Call the $(n-1)$-plane P_V . A dimensional argument then yields that the the tangent plane to the leaf at $\beta(s)$ contains a unique $(n-2)$-plane in P_V. Call the unique $(n-2)$-plane Θ. Since P_V is pointwisely fixed under the rotation in the plane containing the center circle, the tangent spaces to the leaves passing through $\beta(s)$, which are parallel along β, contain Θ as a unique common subspace for all $s \in (a,b)$. Denote by $\Theta^\perp(s)$ the orthogonal complement of Θ in the tangent space $T_sH_1^n$ to H_1^n at $\beta(s)$. Since Θ is a space-like space, $\Theta^\perp(s)$ is a Lorentzian space of dimension 2. $e(s)$ is in $\Theta^\perp(s)$; hence , there is a unique light like vector $\hat{e}(s) \in \Theta^\perp(s)$; such that $\langle e(s), \hat{e}(s)\rangle = -1$. The vector field $\hat{e}(s)$ is parallel along $\beta = \beta(s)$. The vector $\hat{e}(s)$ generates a unique light-like geodesic, denoted by $\gamma = \gamma(t)$, $t \in R$. The geodesic γ meets all the leaves of the foliation transversally except possibly one leaf which is exactly at the antipodal position in the center circle. As is proved in Lemma 1, one can come up with a parallel null vector field $e(t)$, $t \in R$ along γ such that $\langle e(t), \hat{e}(t)\rangle = -1$, $t \in R$ and $e(t)$ determines the unique light-like direction in each leaf at $\gamma(t)$. The parallel displacement $\Theta^\perp(t)$ of $\Theta^\perp(s)$ along $\gamma(t)$ is tangential to the leaf at $t \in R$. Denote by $e_j(t)$, $1 \leq j \leq n-2$, the parallel displacement along $\gamma(t)$ of an orthonormal basis $e_j(s)$, $1 \leq j \leq n-2$, for $\Theta^\perp(s)$. Denote by $x = x(t) = f \circ \gamma(t)$, $t \in R$. Denote also $f_*(\hat{e}(t)), f_*(e(t)), f_*(e_j(t))$, $1 \leq j \leq n-2$, by $A(t), B(t), B_j(t)$, $1 \leq j \leq n-2$, respectively. Then the $(n+2)$-tuple of vector fields $\{\gamma(t), A(t), B(t), C(t), B_j(t),\ 1 \leq j \leq n-2\}$ form what might be called a generalized Cartan framed curve in H_1^{n+1}. Here $C(t)$ is a unit normal field to the immersion f along $x = x(t)$, $t \in R$. Each leaf passing through $\gamma(t)$ is mapped onto the totally geodesic submanifold given as the intersection of H_1^{n+1} and the span of $\{x(t), B(t), B_j(t),\ 1 \leq j \leq n-2\}$. It can be expressed as a mapping $\tilde{f} : R^n \to H_1^{n+1}$ in the following manner:

$$\tilde{f}(t, u_1, \cdots, u_{n-2}, u) = x(t) + \sum_{j=1}^{n-2} u_j B_j(t) + uB(t),$$

with

$$-u^2 + \sum_{j=1}^{n-2} (u_j)^2 = -1.$$

The induced metric in R^n under $\tilde{f}$ gives the same constant curvature -1 as H_1^n; hence, making R^n into $\tilde{H}_1^n$. The shape operator of the immersion $\tilde{f}$ is given in terms of the matrix representation relative to the basis $\{e(t), \hat{e}(t), e_j(t),\ 1 \leq j \leq n-2\}$ by

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad \text{where} \qquad A_{11} = \begin{bmatrix} 0 & -k(s) \\ 0 & 0 \end{bmatrix}$$

and the remaining $A_{ij}'s$ are the zero matrices of appropriate sizes, where $k = k(t)$ is a differentiable function of $t \in R$.

Conversely, suppose there is given a generalized Cartan framed curve $\{x(t), A(t), B(t), B_j(t),\ 1 \leq j \leq n-2\}$ in H_1^{n+1}. We have the following conditions satisfied:

$$\begin{array}{lll}
\langle x(t), x(t) \rangle = -1, & \langle A(t), A(t) \rangle = 0, & \langle B(t), B(t) \rangle = 0, \\
\langle C(t), C(t) \rangle = 1, & \langle A(t), B(t) \rangle = -1, & \langle A(t), B_j(t) \rangle = 0, \\
\langle B(t), B_j(t) \rangle = 0, & \langle A(t), C(t) \rangle = 0, & \langle B(t), C(t) \rangle = 0, \\
\langle B_i(t), B_j(t) \rangle = \delta_{ij}. & &
\end{array}$$

Furthermore, assume the following hold:

$$\begin{aligned}
&dx/dt = A, \\
&\bar{\nabla}_{A(t)} A(t) = k(t)C(t),\ k = k(t), \\
&\bar{\nabla}_{A(t)} B(t) = w(t)C(t),\ w = w(t), \\
&\bar{\nabla}_{A(t)} C(t) = w(t)A(t) + k(t)B(t), \\
&\bar{\nabla}_{A(t)} B_j(t) = 0,\ 1 \leq j \leq n-2.
\end{aligned}$$

Here $\bar{\nabla}$ is the connection in H_1^{n+1}. If, in general, $Y = Y(t)$ is a vector field tangent to H_1^{n+1} along a curve $x = x(t)$, then we have:

$$dY/dt = \bar{\nabla}_{dx/dt}Y(t) + \lambda(t)x(t),$$

where $\lambda(t) = \langle dY/dt, Y(t)\rangle = -\langle Y(t), A(t)\rangle$. In terms of the differentiation in R_2^{n+2}, the above equations can be written as follows:

$$\begin{aligned}
&dx/dt = A(t),\\
&dA(t)/dt = k(t)C(t),\\
&dB(t)/dt = x(t) + w(t)C(t),\\
&dC(t)/dt = w(t)A(t) + k(t)B(t),\\
&dB_j/dt = 0.
\end{aligned}$$

Assuming that $x = x(t)$, $t \in R$, is a null curve in H_1^{n+1}, we define a mapping $\tilde{f} : R^n \to H_1^{n+1}$ by

$$\tilde{f}(t, u_1, \cdots, u_{n-2}, u) = x(t) + \sum_{j=1}^{n-2} u_j B_j(t) + uB(t)$$

with

$$-u^2 + \sum_{j=1}^{n-2}(u_j)^2 = -1.$$

We have

$$\begin{aligned}
&\tilde{f}_*(\partial/\partial u) = B(t),\\
&\tilde{f}_*(\partial/\partial t) = dx/dt + u dB(t)/dt = A(t) + uw(t)C(t) + ux(t),\\
&\tilde{f}_*(\partial/\partial u_j) = B_j(t),\ 1 \le j \le n-2;
\end{aligned}$$

so that

$$\begin{aligned}
&\langle \tilde{f}_*(\partial/\partial u), \tilde{f}_*(\partial/\partial u)\rangle = 0,\\
&\langle \tilde{f}_*(\partial/\partial u), \tilde{f}_*(\partial/\partial t)\rangle = -1,\\
&\langle \tilde{f}_*(\partial/\partial t), \tilde{f}_*(\partial/\partial t)\rangle = u^2(w^2 - 1),\\
&\langle \tilde{f}_*(\partial/\partial u), \tilde{f}_*(\partial/\partial u_j)\rangle = \langle \tilde{f}_*(\partial/\partial t), \tilde{f}_*(\partial/\partial u_j)\rangle = 0,\\
&\langle \tilde{f}_*(\partial/\partial u_i), \tilde{f}_*(\partial/\partial u_j)\rangle = \delta_{ij},\ 1 \le i, j \le n-2.
\end{aligned}$$

Clearly, the metric in R^n induced by $\tilde{f}$ is Lorentzian and of signature 1. We may call the immersion $\tilde{f} : R^n \to H_1^{n+1}$ the generalized B-scoll of the generalized Cartan framed curve

$$\{x(t), A(t), B(t), C(t), B_j(t),\ 1 \le j \le n-2\}.$$

In order to compute the shape operator of the generalized B-scroll, we choose a space-like unit normal field $\xi = C + uwB$. Then,

$$\begin{aligned}\bar{\nabla}_{\partial/\partial u}\xi &= d\xi/du \\ &= wB \\ &= f_*(w(\partial/\partial u))\end{aligned}$$

and

$$\begin{aligned}\bar{\nabla}_{\partial/\partial t}\xi = d\xi/dt &= dC/dt + u(dw/dt)B + uw(dB/dt) \\ &= (k + udw/dt)B + w(A + uwC + ux) \\ &= (k + udw/dt)f_*(w\partial/\partial u) + wf_*(w\partial/\partial t).\end{aligned}$$

Furthermore,

$$\bar{\nabla}_{\partial/\partial u_j}\xi = 0,\ 1 \le j \le n-2.$$

Denoting by A_ξ the shape operator of the map in the direction of ξ relative to the basis $\{\partial/\partial u, \partial/\partial s, \partial/\partial u_j\}$, $1 \le j \le n-2$, we get

$$A_\xi = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Here,

$$A_{11} = \begin{bmatrix} -w & -k - udw/dt \\ 0 & -w \end{bmatrix}.$$

And the remaining A_{ij}'s are the zero matrices of appropriate sizes.

The metric induced by the immersion $\tilde{f} : R^n \to H_1^{n+1}$ gives rise to the same constant curvature -1 in R^n if and only if $w = 0$. Now we have seen that

Theorem 3. *Let $\mathcal{F}$ be a complete totally geodesic foliation of codimension 1 by degenerate leaves in a connected open subset of $\tilde{H}_1^n$. Then $\mathcal{F}$ occurs as the relative nullity foliation of the generalized B-scroll of a generalized Cartan framed curve in H_1^{n+1} and vice versa. Such a foliation can have more than one immersions in the form of different B-scrolls, but they are completely determined by the curvature function $k = k(t)$, $t \in R$, of the null curve $x = x(t)$, $t \in R$.*

4. Foliations with non-degenerate leaves.

Let U be a connected open subset of $\tilde{H}_1^n$. Let $\mathcal{F}$ be a complete totally geodesic foliation of codimension 1 in U. Given a tangent vector Z at x to the leaf L_x of $\mathcal{F}$ passing through the point x. We denote by $c_Z(x)$ the conullity operator of $\mathcal{F}$ at x. Along the geodesic $\gamma = \gamma(t)$ determined by the initial conditions $\gamma(0) = x$ and $d\gamma/dt(0) = Z$ in L_x, $c_Z(t)$ satisfies the Ricatti type differential equation (4), i.e.,

$$dy/dt = y^2 - 1.$$

The only global solution is given by

$$c_Z(t) = -\frac{\sinh t - c_Z(0)\cosh t}{\cosh t - c_z(0)\sinh t}\,.$$

If we denote by $\sigma = \sigma(s)$, $s \in (a,b)$, the maximal integral curve through x of the unit orthogonal vector fields to the foliation $\mathcal{F}$, the global coordinate mapping $\Sigma : (a,b) \times R^{n-1} \to U \subset H_1^n \subset R_2^{n+1}$, which is given as F by (9) in the proof of Lemma 2 can be more specifically expressed as follows: Denote by T_xL_x the tangent space of the leaf L_x passing through $x \in U$. The induced inner product in T_xL_x is positive definite. Denote by S_x the unit $(n-2)$-sphere in the tangent space. Given a unit vector $Z(s) \in S_{\sigma(s)}$, define a map $\Sigma_Z : (a,b) \times R \to U \subset H_1^n \subset R_2^{n+1}$ by

$$\Sigma_Z(s,t) = \sigma(s)\cosh t + Z(s)\sinh t. \tag{10}$$

Our global coordinate map Σ coincides with Σ_Z in the plane

$$\{s\dot{\sigma}(s) + tZ(s)\}\,, \quad \text{where} \quad \dot{\sigma}(s) = \sigma_*(d/ds)(s).$$

We have

$$\begin{aligned}
\Sigma_*(\partial/\partial s)(s,t) &= \Sigma_{Z(s)_*}(\partial/\partial s)(s,t)\\
&= \dot{\sigma}(s)\cosh t + \dot{Z}(s)\sinh t\\
\Sigma_*(\partial/\partial t)(s,t) &= \Sigma_{Z(s)_*}(\partial/\partial t)(s,t)\\
&= \sigma(s)\sinh t + Z(s)\cosh t.
\end{aligned}$$

Hence, we have that

$$\begin{aligned}
\langle \Sigma_*(\partial/\partial s)(s,t), \Sigma_*(\partial/\partial t)(s,t)\rangle &= 0,\\
\langle \Sigma_*(\partial/\partial t)(s,t), \Sigma_*(\partial/\partial t)(s,t)\rangle &= 1,\\
\Sigma_*(\partial/\partial s)(s,t) &= (\cosh t - c(s)\sinh t)\dot{\sigma}(s).
\end{aligned}$$

Here, $c(s)$ is the curvature of $\sigma(s)$ in H_1^n, i.e., $\langle d^2/ds^2, Z(s)\rangle = c(s) = -\langle\dot\sigma(s), \nabla_{\dot\sigma}(s)Z(s)\rangle$. As $|c(s)| \leq 1$, $\Sigma_*(\partial/\partial s)$ never vanishes . This implies that Σ is an immersion. It is obvious that Σ is one-to one and onto.

Lemma 4. *Let $\{a_i\}$ and $\{b_i\}$, $i = 1, 2, \cdots$, be two sequences of positive real numbers such that $(a_i, b_i) \subset (a_{1+1}, b_{1+1}) \subset (a, b)$ and $\lim_{i\to\infty} a_i = a$ and $\lim_{i\to\infty} b_i = b$. Then there exists a countable family*

$$\{g = g(s),\ g_i = g_i(s),\ i = 1, 2, \cdots\}$$

of c^∞-realvalued functions defined in R satisfying the following conditions:

i) $g(s) \geq 0,\ s \in (a, b)$ *and* 0 *elsewhere;*

ii) $g_i(s) \geq 0,\ s \in (a_i, b_i)$ *and* 0 *elsewhere;*

iii) $d^j g_i/ds^j \to d^j g/ds^j,\ i, j = 1, 2, \cdots$, *uniformly in any compact subset of* R.

Proof. It is well-known that there exists a c^∞-function h_i such that $0 < h_i \leq 1$ in (a_i, b_i) and $h_i \equiv 0$ outside. Each h_i has compact support $[a_i, b_i]$. Set $m_i = \max\{|d^j h_i/ds^j(s)|,\ s \in [a_i, b_i],\ 0 \leq j \leq i\}$. Then $0 < m_i < \infty$. Let ρ_i be a positive real number such that $m_i \leq \rho_i(1/2)^i,\ i = 1, 2, \cdots$. Define $g_i(s) = \sum_{l=1}^{i} h_l(s)/\rho_l,\ i = 1, 2, \cdots$. Clearly $g_i(s)$ is a c^∞-function and satisfies (ii). From the choice of ρ_i's, $d^j g_i/ds^j$ converges to $d^j g/ds^j$ uniformly in any compact subset of R, where $g(s) = \lim_{i\to\infty} g_i(s)$. Hence, $g = g(s)$ is a c^∞-function. With this choice of g and g_i's, (i), (ii) and (iii) are satisfied.

Lemma 5. *Let $\mathcal{F}$ be a codimension one complete totally geodesic foliation of a connected open subset $U \subset \tilde H_1^n$. Furthermore, assume that all the leaves are non-degenerate. Let $c(s)$ be the curvature of a maximal orthogonal trajectory of the foliation $\mathcal{F}$. Then there is a c^∞-function G defined in $\tilde H_1^n$ of the following form:*

$$G \circ \Sigma_{Z(s)}(s, t) = g(s)k(s)/(\cosh t - c(s)\sinh t),\ \forall (s, t) \in (a, b) \times R;$$
$$G(x) = 0,\ \forall x \in \tilde H_1^n - U.$$

Here $\Sigma_Z(s)(s, t)$ is the restriction of the global coordinate mapping to the span of $\sigma(s)$ and $Z(s)$, $s \in (a, b)$ and $k : (a, b) \to R$ is a c^∞-function.

Proof. Define G_i to be the c^∞-function defined in $\tilde H_1^n$ by

$$G_i \circ \Sigma_{Z(s)}(s, t) = g_i(s)k(s)/(\cosh t - c(s)\sinh t),\ (s, t) \in (a_i, b_i) \times R;$$
$$G_i(x) = 0,\ \forall x \in \tilde H_1^n - U_i.$$

The g_i's are given in Lemma 4 and $U_i = \Sigma((a_i, b_i) \times R^{n-1})$, $i = 1, 2, \cdots$. Since $\tilde{H}_1^n$ is diffeomorphic to R^n, there are global coordinate systems in $\tilde{H}_1^n$. Choose one of them and denote it by $(u_1, \cdots, u_n)$, $-\infty < u_i < \infty$. Indeed, we might as well assume that it is the Cartesian coordinate system in R^n. Then a function in $\tilde{H}_1^n$ is of c^∞ if all the partials in the u_i, $1 \leq i \leq n$, exist and continuous . Define a positive real number B_i by

$$B_i = \max\Big\{|\partial^j G_i/\partial u^\alpha \partial u^\beta(\Sigma(s, Z))| : \\ \alpha + \beta = j, 0 \leq j \leq i, |Z| \leq i, (s, Z) \in (a, b) \times R^{n-1}\Big\}.$$

These B_i's are bounded by positive constants. Let τ_i be positive number such that $B_i \leq \tau_i(1/2)^i$, $i = 1, 2, \cdots$. Then define G as follows:

$$G(x) = \sum_{i=1}^{\infty}(1/\tau_i)G_i(x), \ x \in \tilde{H}_1^n. \tag{11}$$

Expressing $g = \sum_{i=1}^{\infty}(1/\tau_i)g_i(x)$, G has the desired form.

Next let A_i be a symmetric tensor field of type (1,1) in $\tilde{H}_1^n$ defined as follows:

$$\begin{aligned} A_i(\Sigma(s,t))(\Sigma_*(Z)) &= 0, \\ A_i(\Sigma(s,t))(\Sigma_*(\partial/\partial s)) &= G_i(s, Z)\Sigma_*(\partial/\partial s), \ \Sigma(s, Z) \in U; \\ A_i(x) &= 0, \ x \in \tilde{H}_1^n - U. \end{aligned}$$

A_i is obviously a c^∞-symmetric tensor field of type (1,1). Let $\{e_1, \cdots, e_n\}$ be the orthonormal frame field in $\tilde{H}_1^n$ obtained from the global coordinate vector fields $\{\partial/\partial u_1, \cdots, \partial/\partial u_n\}$ through the Gram-Schmidt orthonormalization process. Then A_i can be expressed as a c^∞-matrix-valued function of $\tilde{H}_1^n$, which we denote by

$$A_i(x) = \left[a_{pq}^i(x)\right], \ 1 \leq p, q \leq n, \ x \in \tilde{H}_1^n. \tag{12}$$

The components $A_{pq}^i(x)$'s of A_i are c^∞-functions of $\tilde{H}_1^n$ and vanish outside U_i. Just as in the proof of Lemma 4, we make the infinite sum of these A_{pq}^i's with appropriate weights if necessary. The sum will be denoted by $A = A(x)$. A can be expressed in terms of the base $\Sigma_*(\partial/\partial s)$ and $\Sigma_*(Z_1), \cdots, \Sigma_*(Z_{n-1})$, where $\{Z_1, \cdots, Z_{n-1}\}$ forms an orthonormal basis for R^{n-1},

$$\begin{aligned} A(x)(\Sigma_*(\partial/\partial s)) &= \Psi(x)\Sigma_*(\partial/\partial s); \\ A(x)(Z_j) &= 0, \ \forall x = \Sigma(s, Z) \in U \ (1 \leq j \leq n-1); \end{aligned}$$

$$A(x) = 0, \ \forall x \in \tilde{H}_1^n - U.$$

Here, Ψ is a c^∞-function of the form given in (11) with an appropriate function ψ replacing g.

Lemma 6. *A satisfies the Gauss and Codazzi equations in $\tilde{H}_1^n$.*

Proof. We first show that each A_i satisfies these equations. Since rank $A_i \leq 1$, the Gauss equation is automatically satisfied. Since A_i is a tensor field, it suffices to show that in U

$$(\nabla_{\Sigma_*(\partial/\partial s)} A_i)(Z_j) = (\nabla_{\Sigma_*(Z_j)} A_i)(\partial/\partial s). \tag{13}$$

Note that $A_i = 0$ outside U_i ; hence, the equations are trivially satisfied there. We calculate the terms in the above equation (12) by restricting them to the totally geodesic surfaces $\Sigma_{Z(s)}(s,t)$ of $\tilde{H}_1^n$ given by (10). Since $\Sigma_*(\partial/\partial s) = \Sigma_{Z(s)_*}(\partial/\partial s)$ and $\Sigma_*(\partial/\partial t) = \Sigma_{Z(s)_*}(\partial/\partial t)$ form a coordinate frame fields for R^2, the Lie bracket between them equals 0. This fact together with $A_i(\Sigma_*(\partial/\partial t)) = 0$ implies that the equation (12) is equivalent to the following:

$$(\Sigma_*(\partial/\partial t))G_i - G_i c(\Sigma(s,t)) = 0. \tag{14}$$

$c(\Sigma(s,t))$ is the conullity operator defined in (3) , which can be expressed as

$$c(\Sigma(s,t)) = c_Z(t) = -\frac{\sinh t - c(\Sigma(\sigma(s)))\cosh t}{\cosh t - c(\Sigma(\sigma(s)))\sinh t},$$

where the $\sigma(s)$ is the orthogonal trajectory of the foliation . Equation (13) implies that $dG_i/dt - G_i c(\Sigma(s,t)) = 0$. But

$$G_i c(\Sigma(s,t)) = \frac{g_i(s)k(s)}{\cosh t - c(\sigma(s))\sinh t}, \ (s,t) \in (a_i, b_i) \times R.$$

Hence,

$$(dG_i/dt)(\Sigma(s,t)) = \frac{g_i(s)k(s)(\sinh t - c(\sigma(s))\cosh t)}{(\cosh t - c(\sigma(s))\sinh t)^2}. \tag{15}$$

Substituting this expression (15) in (14) , we get the desired equation. Using the expression (12) for A_i, we see that the Codazzi equation (13) holds for A_i, $i = 1, 2, \cdots$. Since by the definition of A, the convergence of all partials is uniform in any compact subset of $\tilde{H}_1^n$, A also satisfies the Codazzi equation (13).

Theorem 4. *Given a c^∞-lamination $\{U, \mathcal{F}\}$, where U is a a connected open subset of $\tilde{H}_1^n$, there is an isometric immersion of $\tilde{H}_1^n$, into $\tilde{H}_1^{n+1}$*

so that the associated lamination is precisely $\{U, \mathcal{F}\}$. *In fact, there are infinitely many such immersions, which are parametrized by the space of differentiable functions* $k = k(s)$ *'s.*

Proof. Obvious from the proof of the previous lemmas and the fundamental theorem of hypersurfaces in the ambient space of constant curvature.

Now let $\{U, \mathcal{F}\}$ be a c^∞-lamination of $\tilde{H}_1^n$. This means that U is given as a disjoint union of countable connected open subset U_i, $i = 1, 2, \cdots$, of $\tilde{H}_1^n$ such that each U_i is endowed with a c^∞-complete totally geodesic foliation $\mathcal{F}_i$. By Lemma 6 the lamination gives rise to a c^∞-symmetric tensor field A_i of type (1,1), which satisfies the Gauss and Codazzi equations. Let us express each A_i as a matrix-valued function

$$.A_1(u_1, \cdots, u_n) = (A^i_{pq}(u_1, \cdots, u_n)),$$
$$\forall (u_1, \cdots, u_n) \in \tilde{H}_1^n, \ 1 \leq p, \ q \leq n$$

relative the orthogonal frame field $\{e_1, \cdots, e_n\}$ in $\tilde{H}_1^n$. Let M_i, $i = 1, 2, \cdots$, be the positive real number given by

$$M_i = \mathrm{Max}\left\{ \left| \frac{\partial^l a^i_{pq}}{\partial {u_j}^\alpha \partial {u_k}^\beta}(u_1(\Sigma_i(s, Z(s))), \cdots, u_n(\Sigma_i(s, Z(s)))) \right| : \right.$$
$$\left. 1 \leq p, q \leq n; \ 0 \leq l \leq i; \ \alpha + \beta = l; s \in [a_i, b_i]; \ \langle Z(s), Z(s) \rangle \leq i \right\}. \tag{16}$$

Here $\Sigma_i : (a_i, b_i) \times R^{n-1} \to U_i$ is the global parametrization of U_i in (9). Now set η_i to be a positive number such that $M_i \leq \eta_i (1/2)^i$. Define A by the following series:

$$A = \sum_{i=1}^{\infty} (1/\eta_i) A_i \, . \tag{17}$$

The series converges uniformly in any compact subset of $\tilde{H}_1^n$. The components of A are c^∞-functions. Moreover, it is easy to see that A never vanishes in U but vanishes everywhere outside U. Since the A_i's satisfy the Gauss and Codazzi equations, so must A. Hence, we have reached

Theorem 5. *Given a* c^∞*-lamination* $\{U, \mathcal{F}\}$ *there exist infinitely many isometric immersions of* $\tilde{H}_1^n$ *into* $\tilde{H}_1^{n+1}$ *so that the associated laminations are precisely the given lamination. The space of such immersions is parametrized by the set of countably many* c^∞*-functions* $k_i = k_i(s) : I_i \to$

R, where I_i is an open interval of real numbers determined by the maximal integral curves of the orthogonal trajectory to the foliation in $\mathcal{F}_i$ of U_i, $i = 1, 2, \cdots$.

Proof. From Lemma 6, we have a c^∞-symmetric tensor field A of type (1,1) defined in $\tilde{H}_1^n$. A satisfies the Gauss (7) and Codazzi (8) equations. By the fundamental theorem of hypersurfaces in H_1^{n+1}, there is an isometric immersion whose shape operator is precisely A. Rank $A = 1$ in U and $A \equiv 0$ elsewhere. This completes the proof.

Remark. Let $R = R_1^1$ be the preimage of the center circle under the covering map in (2). Given a totally geodesic lamination $\{U, \mathcal{F}\}$, let $U = \cup_{j=1}^{\infty} U_j$, where U_j's are mutually disjoint connected open subsets, each of which is endowed with a complete totally geodesic foliation of codimension 1. Then $U_j \cap R$ is an open interval I_j, $1 \leq j$, in R. If the totally geodesic foliation of U_j is non-degenerate, we may consider the curvature function $k_j = k_j(s)$ defined in a maximal integral curve of the orthogonal distribution to the foliation as a differentiable function defined in I_j in a natural way. If the foliation in U_j happens to be degenerate, then the curvature function k_j defined in the complementary null curve can be regarded as a differentiable function in I_j. Thus, we may state that the space of isometric immersions $f : \tilde{H}_1^n \to \tilde{H}_1^{n+1}$ is parametrized by the family of triples $\{U_j, \mathcal{F}_j, k_j = k_j(s)\}$, $1 \leq j$.

References

[1] K. Abe and M. Magid, *Relative nullity foliations and indefinite isometric immersions*, Pacific J. Math., **124** (1986), 1-20.

[2] ———, *Indefinite Kähler submanifolds with positive index of relative nullity*, Ann. Global Anal. Geom., **6 (3)** (1988), 231-258.

[3] K. Abe and A. Haas, *Isometric immersions of H^n into H^{n+1}*, Proc. of Symposia in Pure Mathematics, **54** (1993), Part 3.

[4] W. Bonner, *Null curves in a Minkowski space-time*, Tensor (N.S.), **20** (1969), 229-242.

[5] L. Graves, *Codimension one isometric immersions between Lorentzian spaces*, Trans. Amer. Math. Soc., **252** (1979), 367-392.

[6] L. Graves and K. Nomizu, *Isometric immersions of Lorentzian space forms*, Math. Ann., **233** (1978), 125-136.

[7] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, (1983).

[8] A.Z. Petrov, *Einstein space*, Pergamon Press, Hungary, Oxford and New York, (1969).

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