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ON THE CAUCHY PROBLEM FOR A SINGULAR PARABOLIC EQUATION

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The existence of a renormalized solution is established for the Cauchy problem for the parabolic P-Laplacain equation in which p is allowed to be close to 1 and the initial data are only assumed to be locally integrable.

1. Introduction.

We shall be concerned with the existence of a solution to the following problem

$$\frac{\partial}{\partial t}u - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0 \quad \text{in } \Sigma_T \equiv \mathbf{R}^N \times (0,T), \tag{1.1a}$$

$$u(x,0) = u_0(x) \quad \text{on } \mathbf{R}^N \tag{1.1b}$$

in the case where $T > 0$, $1 < p < 2$, and $u_0 \in L^1_{\text{loc}}(\mathbf{R}^N)$. The restriction on p makes the equation (1.1a) singular because the term $|\nabla u|^{p-2}$, which measures the modulus of ellipticity of the principal part of (1.1a), is unbounded at points where $|\nabla u|$ is 0. Thus we are dealing with a singular parabolic problem.

It is observed in **[DH]** that in the generality considered here an estimate of the form

$$|\nabla u| \in L^q_{\text{loc}}(\Sigma_T), \qquad q \geq 1 \tag{1.2}$$

is no longer possible. This suggests that solutions of (1.1a) display new phenomena that cannot be incorporated into the classical weak formulation. To define our notion of a weak solution, we follow the approach adopted in **[X1]**. Let $\mathcal{A} = \{\theta \in C(\mathbf{R}) : \theta$ is a Lipschitz function whose derivative $\theta'(s)$ exists except at finitely many points and $\theta'(s) = 0$ for $|s|$ sufficiently large$\}$. If a measurable function v on Σ_T is such that $\theta(v) \in L^p\left(0,T; W^{1,p}_{\text{loc}}(\mathbf{R}^N)\right)$ for all $\theta \in \mathcal{A}$, then we can define a measurable function $g : \Sigma_T \to \mathbf{R}^N$ so that

$$g = \nabla P_M(v) \text{ almost everywhere on } \{|v| < M\}$$

for all $M > 0$, where $P_M(s) = \min\{|s|, M\}\operatorname{sign}(s)$. The function g is viewed as the spatial gradient of v, and is also denoted by ∇v. We are ready to present our definition of a solution.

Definition. A measurable function u on Σ_T is said to be a renormalized solution of (1.1) if:

1. $u \in C\left([0,T]; L^1_{\text{loc}}(\mathbf{R}^N)\right)$;

2. For each $\theta \in \mathcal{A}$, $\theta(u) \in L^p\left(0,T;W^{1,p}_{\text{loc}}(\mathbf{R}^N)\right)$ and $\nabla\theta(u) = \theta'(u)\nabla u$ almost everywhere on Σ_T, where $\theta'(u)$ is understood to be 0 if $u \in B_\theta \equiv \{s \in \mathbf{R} : \theta'(s) \text{ does not exist}\}$;
3. $|\nabla u|^{p-1} \in L^1(0,T;L^1_{\text{loc}}(\mathbf{R}^N))$ and

$$-\int_{\Sigma_T}\int_0^u \theta(s)ds\varphi_t dxdt + \int_{\Sigma_T}|\nabla u|^{p-2}\nabla u\left(\nabla\theta(u)\varphi + \theta(u)\nabla\varphi\right)dxdt$$
$$= \int_{\mathbf{R}^N}\varphi(x,0)\int_0^{u_0(x)}\theta(s)dsdx$$

for all $\theta \in \mathcal{A}$ and all $\varphi \in C_0^\infty(\mathbf{R}^N \times (-\infty,T))$.

The idea of a renormalized solution was originated in the study of the Boltzmann equation; see [**DL1**, **DL2**] for details. An elliptic version of this idea appears in [**BGDM**]. The definition here is a slight modification of that in [**X1**]; also see [**X2**] where it is evident that the notion of a renormalized solution is the correct notion of solution for p-Laplacian problems. The objective of this paper is to show that there exists a renormalized solution to (1.1).

If $u_0 \geq 0$, the existence and uniqueness of a solution to (1.1) are established in [**DH**]. In [**X1**], the sign restriction on u_0 is removed, but R^N is replaced with a bounded domain Ω. The stationary problem is considered in [**X2**] and references therein. The question of existence and uniqueness of a solution to (1.1) in the case where u_0 may change sign was proposed as an open problem in [**DH**]. In this paper, we solve the question of existence, while the question of uniqueness remains open.

It is interesting to note that we obtain a renormalized solution to (1.1) without imposing any growth condition on u_0. This is in sharp contrast with the case $p > 2$ [**D**]. Also, it is easy to infer from the argument in [**D**, p. 188-192] that if $u_0 \in L^s(\mathbf{R}^N)$, $s = N(2-p)/p$, $1 < p < 2N/(N+1)$, and $N \geq 2$, then the renormalized solution u constructed here will extinct in finite time, i.e., there exists a positive number T^* such that $u(x,t) = 0$ for all $t > T^*$.

The main gap between the case $u_0 \geq 0$, and the case where u_0 may change sign, is that in the latter case an estimate of the type

$$\int_s^T\int_{\{|x|<R\}}\frac{u_t^2}{(1+|u|)^{1+\varepsilon}}dxdt < \infty, \qquad s \in (0,T), \varepsilon > 0, R > 0$$

is no longer available. To overcome this difficulty, we develop an analysis that combines the best features of the arguments in [**DH**] and [**X1**] with a compactness theorem of Simon [**S**].

This work is organized as follows. In Section 2, we prove a comparison principle for classical weak solutions of (1.1a). This result is used in Section 3 to prove the existence of a renormalized solution.

We conclude this section by making some remarks on notation. Let $R > 0$, and we denote by B_R the ball centered at the origin with radius R. Fix $R > r > 0$. We say that ξ is a cut-off function associated with R and r if $\xi \in C_0^\infty(B_R)$, $0 \le \xi \le 1$, $\xi = 1$ on B_r, and $|\nabla\xi| \le \dfrac{2}{R-r}$. Let E be a measurable set in $\mathbf{R}^{N+1}$. We use $|E|$ to denote the Lebesque measure of E.

2. Preliminaries.

In this section we consider the problem

$$\frac{\partial}{\partial t}u - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0 \quad \text{in } \Sigma_T, \tag{2.1a}$$

$$u(x,0) = u_0(x) \quad \text{on } \mathbf{R}^N \tag{2.1b}$$

in the case where $u_0 \in L^2_{\rm loc}(\mathbf{R}^N)$ and $1 < p < 2$. A function u on Σ_T is said to be a classical weak solution of (2.1) if:

(i) $u \in C\left([0,T]; L^2_{\rm loc}(\mathbf{R}^N)\right) \cap L^p\left(0,T; W^{1,p}_{\rm loc}(\mathbf{R}^N)\right)$;

(ii) $-\int_{\Sigma_T} u\varphi_t\,dxdt + \int_{\Sigma_T} |\nabla u|^{p-2}\nabla u\nabla\varphi\,dxdt = \int_{\mathbf{R}^N}\varphi(x,0)u_0(x)\,dx$ for all $\varphi \in C_0^\infty\left(\mathbf{R}^N \times (-\infty,T)\right)$.

Let u be a classical weak solution to (2.1). Then we can easily deduce from (ii) that for each $\rho > 0$,

$$u_t \in L^{p'}\left(0,T; W^{-1,p'}(B_\rho)\right) \tag{2.2}$$

(2.3)
$$u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0 \text{ in } W^{-1,p'}(B_\rho) \text{ for almost every } t \in (0,T).$$

Here and in what follows $p' = p/(p-1)$.

Lemma 2.1. *Let u be a classical weak solution of* (2.1). *Then $u_0 \in L^\infty_{\rm loc}(\mathbf{R}^N)$ implies $u \in L^\infty\left(0,T; L^\infty_{\rm loc}(\mathbf{R}^N)\right)$.*

Remark. If $u_0 \ge 0$, then this lemma is a direct consequence of Theorem III.6.2 in [**DH**].

Proof of Lemma 2.1. We modify a device in [**DH**]. Fix $R > 0$. For $n = 0, 1, 2, \dots$, define

$$\rho_n = R\left(1 + 2^{-n}\right), B_n = B_{\rho_n}, k_n = M\left(2 - 2^{-n}\right),$$

where $M \geq \|u_0\|_{L^\infty(B_{2R})}$ will be selected later. Let ξ_n be a cut-off function associated with ρ_n and ρ_{n+1}. Then we can derive from the chain rule [**X1**] that the function $t \to \frac{1}{2}\int_{B_n}\left[(u-k_n)^+\right]^2 \xi_n^p dx$ is absolutely continuous on $[0.T]$, and

$$\text{(2.4)} \quad \frac{d}{dt}\frac{1}{2}\int_{B_n}\left[(u-k_n)^+\right]^2 \xi_n^p dx = \left(u_t, (u-k_n)^+ \xi_n^p\right)$$

almost everywhere on $(0,T)$,

where $(\cdot,\cdot)$ denotes the duality pairing between $W^{-1,p'}(B_n)$ and $W_0^{1,p}(B_n)$. Keep this in mind, use $(u-k_n)^+ \xi_n^p$ as a test function in (2.3), thereby obtain

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_{B_n}\left[(u-k_n)^+\right]^2 \xi_n^p dx + \int_{B_n}\left|\nabla (u-k_n)^+\right|^p \xi_n^p dx \\
&= -\int_{B_n}\left|\nabla (u-k_n)^+\right|^{p-2} \nabla (u-k_n)^+ (u-k_n)^+ p\xi_n^{p-1}\nabla\xi_n dx \\
&\leq \frac{1}{2}\int_{B_n}\left|\nabla (u-k_n)^+\right| \xi_n^p dx + 2^{p-1}\left(\frac{p}{R}\right)^p 2^{p(n+1)}\int_{B_n}\left[(u-k_n)^+\right]^p dx.
\end{aligned}$$

Consequently,

$$\text{(2.5)} \quad \begin{aligned}
&\max_{0\leq t\leq T}\int_{B_n}\left[(u-k_n)^+\right]^2 \xi_n^p dx + \int_{B_n\times(0,T)}\left|\nabla (u-k_n)^+\right|^p \xi_n^p dxdt \\
&\leq \left(\frac{p}{R}\right)^p 2^{p(n+2)}\int_{B_n\times(0,T)}\left[(u-k_n)^+\right]^p dxdt.
\end{aligned}$$

This, in conjunction with the Gagliardo-Nirenberg-Sobolev inequality, implies

$$\begin{aligned}
&\int_{B_n\times(0,T)}\left[(u-k_n)^+ \xi_n\right]^{p\frac{N+2}{N}} dxdt \\
&\leq c_0\left(\sup_{0\leq t\leq T}\int_{B_n}\left[(u-k_n)^+ \xi_n\right]^2 dx\right)^{\frac{p}{N}} \\
&\quad \cdot \int_{B_n\times(0,T)}\left|\nabla\left((u-k_n)^+ \xi_n\right)\right|^p dxdt \\
&\leq c_1\frac{2^{\left(\frac{p(N+p)}{N}\right)n}}{R^{\frac{p(N+p)}{N}}}\left(\int_{B_n\times(0,T)}\left[(u-k_n)^+\right]^p dxdt\right)^{\frac{(N+p)}{N}}.
\end{aligned}$$

Here, and in what follows, $c_i, i \in \{0,1,2,\dots\}$, denote positive constants depending only upon p, N. We estimate

$$\int_{B_{n+1}\times(0,T)}\left[(u-k_{n+1})^+\right]^p dxdt$$

$$
\begin{aligned}
(2.6)\qquad &\le \int_{B_n\times(0,T)} \left[(u-k_{n+1})^+\xi_n\right]^p dxdt \\
&\le |B_n\times(0,T)\cap\{u>k_{n+1}\}|^{\frac{2}{N+2}} \\
&\quad\cdot\left(\int_{B_n\times(0,T)} \left[(u-k_{n+1})^+\xi_n\right]^{p\frac{N+2}{N}} dxdt\right)^{\frac{N}{N+2}} \\
&\le c_2\frac{2^{\left(\frac{p(N+p)}{(N+2)}\right)n}}{R^{\frac{p(N+p)}{(N+2)}}}|B_n\times(0,T)\cap\{u>k_{n+1}\}|^{\frac{2}{N+2}} \\
&\quad\cdot\left(\int_{B_n\times(0,T)} \left[(u-k_n)^+\right]^p dxdt\right)^{\frac{N+p}{N+2}}.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\int_{B_n\times(0,T)} \left[(u-k_n)^+\right]^p dxdt \\
&\ge \int_{B_n\times(0,T)\cap\{u>k_{n+1}\}} (k_{n+1}-k_n)^p\, dxdt \\
&= M^p 2^{-p(n+1)}\,|B_n\times(0,T)\cap\{u>k_{n+1}\}|.
\end{aligned}
$$

This, together with (2.6) shows that

$$
\begin{aligned}
&\int_{B_{n+1}\times(0,T)} \left[(u-k_{n+1})^+\right]^p dxdt \\
&\qquad\le c_3\frac{2^{\left[\frac{p(N+p)}{(N+2)}+\frac{2p}{(N+2)}\right]n}}{R^{\frac{p(N+p)}{(N+2)}}M^{\frac{2p}{(N+2)}}}\left(\int_{B_n\times(0,T)}\left[(u-k_n)^+\right]^p dxdt\right)^{1+\frac{p}{N+2}}.
\end{aligned}
$$

According to a result in [**LSU**, p. 95], $\lim_{n\to\infty}\int_{B_n\times(0,t)}\left[(u-k_n)^+\right]^p dxdt = 0$, provided we can select $M\ge\|u_0\|_{L^\infty(B_{2R})}$ so that

$$
\begin{aligned}
(2.7)\qquad \int_{B_{2R}\times(0,T)} [(u-M)^+]^p dxdt &\le \left(\frac{c_3}{R^{\frac{p(N+p)}{(N+2)}}M^{\frac{2p}{(N+2)}}}\right)^{-\frac{N+2}{p}} \\
&\quad\cdot\left(2^{\frac{(pN+p^2+2p)}{(N+2)}}\right)^{-\left(\frac{N+2}{p}\right)^2} \\
&\le c_4 R^{(N+p)}M^2.
\end{aligned}
$$

This can be easily done, and hence

$$
\int_{B_R\times(0,T)}[(u-2M)^+]^p dxdt \le \lim_{n\to\infty}\int_{B_R\times(0,T)}\left[(u-k_n)^+\right]^p dxdt = 0.
$$

To see that u is also bounded below, note that $v = -u$ is a classical weak solution of the following problem

$$\frac{\partial v}{\partial t} - \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) = 0 \quad \text{in } \Sigma_T,$$
$$v(x,0) = -u_0(x) \quad \text{in } \mathbf{R}^N.$$

This completes the proof of the lemma.

Before we continue, let us recall the following lemma from [**O**, pp. 145–147].

Lemma 2.2. *Let x, y be any two vectors in $\mathbf{R}^N$ and $p \in (1,2]$. Then,*

(a) $(|x|^{p-2}x - |y|^{p-2}y)(x-y) \geq (p-1)\frac{|x-y|^2}{(|x|+|y|)^{2-p}}$;

(b) $\left||x|^{p-2}x - |y|^{p-2}y\right| \leq \sqrt{5}|x-y|^{p-1}$.

Lemma 2.3. *Let u_0, v_0 be two functions in $L^\infty_{\text{loc}}(\mathbf{R}^N)$. Assume that u and v are classical weak solutions of* (2.1a) *with initial conditions u_0 and v_0, respectively. Then $u_0 \leq v_0$ implies $u \leq v$.*

Proof. Fix $R > r > 0$. Let ξ be a cut-off function associated with R and r. By Lemma 2.2, $u, v \in L^\infty(0,T; L^\infty_{\text{loc}}(\mathbf{R}^N))$. Thus for each $q > 1$, $[(u-v)^+]^q \xi^2 \in L^p\left(0,T; W_0^{1,p}(B_R)\right)$. We can conclude from (2.3) and the chain rule [**X1**] that

(2.8)
$$\begin{aligned}
&\frac{d}{dt}\frac{1}{q+1}\int_{B_R}\left[(u-v)^+\right]^{q+1}\xi^2 dx \\
&\quad + \int_{B_R}\left(|\nabla u|^{p-2}\nabla u - |\nabla u|^{p-2}\nabla v\right) q\left[(u-v)^+\right]^{q-1}\nabla(u-v)\xi^2 dx \\
&= -\int_{B_R}\left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\left[(u-v)^+\right]^q 2\xi\nabla\xi dx \\
&\leq \frac{2}{R-r}\int_{B_R}\left||\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right|\left[(u-v)^+\right]^q \xi dx.
\end{aligned}$$

Set

$$A_t = \left\{x : (u(x,t)-v(x,t))^+ \frac{2}{R-r} \leq \frac{1}{2}q\left|\nabla(u(x,t)-v(x,t))^+\right|\xi(x)\right\}.$$

We compute, with the aid of Lemma 2.2, that

$$\frac{2}{R-r}\int_{B_R}\left||\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right|\left[(u-v)^+\right]^q \xi dx$$

$$\leq \frac{1}{2}\int_{B_R\cap A_t} \left||\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right| |\nabla(u-v)^+| \, q\left[(u-v)^+\right]^{q-1}\xi^2 dx$$
$$+ \frac{2}{R-r}\int_{B_R\setminus A_t} \sqrt{5}|\nabla u - \nabla v|^{p-1}\left[(u-v)^+\right]^q \xi dx$$
$$\leq \frac{1}{2}\int_{B_R} \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\nabla(u-v) q\left[(u-v)^+\right]^{q-1}\xi^2 dx$$
$$+ \frac{2}{R-r}\int_{B_R} \sqrt{5}\left(\frac{4}{q(R-r)}(u-v)^+\right)^{p-1}\left[(u-v)^+\right]^q dx.$$

Use this in (2.8) to obtain
(2.9)
$$\int_{B_r}\left[(u-v)^+\right]^{q+1} dx \leq \frac{\sqrt{5}(q+1)2^{2p-1}}{q^{p-1}(R-r)^p}\int_{B_R\times(0,t)}\left[(u-v)^+\right]^{q+p-1} dx d\tau.$$

Now we are ready to employ an argument in **[DH]**. Fix $\rho > 0$, and set

$$\rho_n = \left(\sum_{i=0}^{n} 2^{-i}\right)\rho, \quad B_n = B_{\rho_n},$$
$$\Lambda_n = \sup_{0\leq T\leq t}\int_{B_n}\left[(u-v)^+\right]^{q+1} dx \qquad (n = 0, 1, 2, \dots).$$

We can infer from (2.9) that

$$\Lambda_n \leq c\frac{2^{p(n+1)}}{\rho^p}\int_{B_{n+1}\times(0,t)}\left[(u-v)^+\right]^{q+p-1} dx d\tau$$
$$\leq c t^{\frac{2-p}{q+1}+1}(2p)^{N\frac{2-p}{q+1}}\Lambda_{n+1}^{\frac{q+p-1}{q+1}}\frac{2^{p(n+1)}}{\rho^p}$$
$$= c_1 t^{\frac{3-p+q}{q+1}}\frac{2^{pn}}{\rho^{p-\frac{(2-p)N}{(q+1)}}}\Lambda_{n+1}^{\frac{(q+p-1)}{(q+1)}}$$
$$\leq \delta\Lambda_{n+1} + \left(2^{p\frac{q+1}{2-p}}\right)^n c(\delta)\left(\frac{t^{\frac{(3-p+q)}{(q+1)}}}{\rho^{p-\frac{(2-p)N}{(q+1)}}}\right)^{\frac{q+1}{2-p}}.$$

Here $\delta > 0$ is arbitrary. This implies

$$\Lambda_0 \leq \delta^n\Lambda_n + \frac{1}{\delta}c(\delta)\left(\frac{t^{\frac{(3-p+q)}{(q+1)}}}{\rho^{p-\frac{(2-p)N}{(q+1)}}}\right)^{\frac{q+1}{2-p}}\sum_{i=0}^{n+1}\left(\delta 2^{p\frac{q+1}{2-p}}\right)^i. \tag{2.10}$$

Now we select $\delta > 0$ and $q > 0$ so that

$$\delta 2^{p\frac{q+1}{2-p}} = \frac{1}{2} \qquad \text{and} \qquad (q+1)p - (2-p)N > 0.$$

We conclude from (2.10) that

$$\sup_{0\le\tau\le t}\int_{B_\rho}\left[(u-v)^+\right]^{q+1}dx\le c\left(\frac{t^{\frac{(3-p+q)}{(q+1)}}}{\rho^{\frac{(q+1)p-(2-p)N}{q+1}}}\right)^{\frac{q+1}{2-p}}$$

$$\to 0 \qquad \text{as} \qquad \rho\to\infty.$$

This proves the lemma. □

An easy consequence of Lemma 2.1 and Lemma 2.3 is that

$$\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)}\le\|u_0\|_{L^\infty(\mathbf{R}^N)}$$

for each $t>0$.

3. Existence.

The main result of this section is:

Theorem 3.1. *Assume that* $u_0\in L^1_{\mathrm{loc}}(\mathbf{R}^N)$, *and* $1<p<2$. *Then there exists a renormalized solution to* (1.1).

Proof. If $k\in\{1,2,\dots\}$, define

$$f_k(x)=\min\left\{u_0^+(x),k\right\},\tag{3.1}$$

$$g_k(x)=\min\left\{u_0^-(x),k\right\}.\tag{3.2}$$

For each k, consider the approximating problem

$$\frac{\partial u_k}{\partial t}-\operatorname{div}\left(|\nabla u_k|^{p-2}\nabla u_k\right)=0 \qquad \text{on} \quad \Sigma_T,\tag{3.3a}$$

$$u(x,0)=u_{0k}(x)=f_k-g_k \qquad \text{in} \quad \mathbf{R}^N.\tag{3.3b}$$

The existence of a classical weak solution to (3.3) can be inferred from a result in [**DH, D**]. Since $u_{0k}\in L^\infty(\mathbf{R}^N)$, Lemma 2.3 asserts the uniqueness. The remaining proof is divided into several lemmas. □

Lemma 3.1. *For each* $\rho>0$, *there exists a* $c(\rho)>0$ *such that*

$$\max_{0\le t\le T}\int_{B_\rho}|u_k(x,t)|\,dx\le c(\rho),\tag{3.4}$$

$$\int_{B_\rho\times(0,T)}|\nabla u_k|^{p-1}\,dxdt\le c(\rho)\ (k=1,2,\dots).\tag{3.5}$$

Proof. For each k, let v_k be the classical weak solution of the following problem

$$\frac{\partial}{\partial t}v_k - \operatorname{div}\left(|\nabla v_k|^{p-2}\nabla v_k\right) = 0 \quad \text{in } \Sigma_T, \tag{3.6a}$$

$$v_k(x,0) = f_k(x) \quad \text{on } \mathbf{R}^N, \tag{3.6b}$$

and w_k be the classical weak solution of the following problem

$$\frac{\partial}{\partial t}w_k - \operatorname{div}\left(|\nabla w_k|^{p-2}\nabla w_k\right) = 0 \quad \text{in } \Sigma_T, \tag{3.7a}$$

$$w_k(x,0) = -g_k(x) \quad \text{on } \mathbf{R}^N. \tag{3.7b}$$

In light of Lemma 2.3, we have

$$w_k \le u_k \le v_k \quad \text{almost everywhere on } \Sigma_T \tag{3.8}$$

for all k. Since $f_k \ge 0$ on $\mathbf{R}^N$, we can invoke a result in [**DH**, p. 260] to obtain that there exists a $c_1(\rho) > 0$ such that

$$\max_{0\le t\le T}\int_{B_\rho} v_k(x,t)dx \le c_1(\rho) \qquad (k = 1,2,\dots). \tag{3.9}$$

Note that $z_k = -w_k$ is the classical weak solution of the problem

$$\frac{\partial}{\partial t}z_k - \operatorname{div}\left(|\nabla z_k|^{p-2}\nabla z_k\right) = 0 \quad \text{in } \Sigma_T,$$

$$z_k(x,0) = g_k(x) \quad \text{on } \mathbf{R}^N.$$

Thus, we can find $c_2(\rho) > 0$ with

$$\max_{0\le t\le T}\int_{B_\rho} |w_k(x,t)|\,dx \le c_2(\rho) \qquad (k = 1,2,\dots). \tag{3.10}$$

We see that (3.4) is a consequence of (3.8), (3.9), and (3.10). To see (3.5), for each $\varepsilon > 0$ define

$$\phi_\varepsilon(s) = \begin{cases} 1 - \frac{1}{(1+s)^\varepsilon} & \text{if } s \ge 0, \\ -\phi_\varepsilon(-s) & \text{if } s < 0. \end{cases} \tag{3.11}$$

Let ξ be a cut-off function associated with 2ρ and ρ. Then using $\phi_\varepsilon(u_k)\xi^p$ as a test function in (3.3a), we derive from a standard argument [**X1**] that

(3.12)

$$\frac{d}{dt}\int_{B_{2\rho}}\int_0^{u_k(x,t)} \phi_\varepsilon(s)ds\xi^p(x)dx + \int_{B_{2\rho}} \phi'_\varepsilon(u_k)|\nabla u_k|^p\xi^p dx$$

$$= -\int_{B_{2\rho}} |\nabla u_k|^{p-2} \nabla u_k \phi_\varepsilon (u_k) p\xi^{p-1} \nabla \xi dx.$$

Note that

$$\phi'_\varepsilon = \frac{\varepsilon}{(1+|s|)^{1+\varepsilon}} \quad \text{and} \quad |\phi_\varepsilon| \le 1$$

and that

$$ab \le \sigma a^p + \sigma^{-\frac{p'}{p}} b^{p'},\ a > 0,\ b > 0,\ \sigma > 0. \tag{3.13}$$

We deduce from (3.12) that

(3.14)
$$\begin{aligned} &\int_{B_{2\rho}} \int_0^{u_k(x,t)} \phi_\varepsilon(s) ds \xi^p(x) dx + \frac{\varepsilon}{2} \int_{B_{2\rho}\times(0,t)} \frac{|\nabla u_k|^p \xi^p}{(1+|u_k|)^{1+\varepsilon}} dx d\tau \\ &\le \int_{B_{2\rho}} \int_0^{u_{0k}(x)} \phi_\varepsilon(s) ds \xi^p(x) dx \\ &\quad + \left(\frac{\varepsilon}{2}\right)^{1-p} \left(\frac{p}{\rho}\right)^p \int_{B_{2\rho}\times(0,t)} (1+|u_k|)^{(1+\varepsilon)(p-1)} dx d\tau. \end{aligned}$$

Observe that $\int_0^{u_k(x,t)} \phi_\varepsilon(s) ds \ge 0$ on Σ_T. Then select $\varepsilon_0 > 0$ so that

$$(1+\varepsilon_0)(p-1) = 1.$$

It follows from (3.14) and (3.4) that there exists a $c(\rho) > 0$ with

$$\int_{B_\rho\times(0,T)} \frac{|\nabla u_k|^p}{(1+|u_k|)^{1+\varepsilon_0}} dx dt \le c(\rho).$$

We estimate that

$$\begin{aligned} \int_{B_\rho\times(0,T)} |\nabla u_k|^{p-1} dx dt &= \int_{B_\rho\times(0,T)} \frac{|\nabla u_k|^{p-1}}{(1+|u_k|)^{\frac{(1+\varepsilon_0)}{p'}}} (1+|u_k|)^{\frac{(1+\varepsilon_0)}{p'}} dx dt \\ &\le \frac{\varepsilon_0}{2} \int_{B_\rho\times(0,T)} \frac{|\nabla u_k|^p}{(1+|u_k|)^{1+\varepsilon_0}} dx dt \\ &\quad + \left(\frac{\varepsilon_0}{2}\right)^{1-p} \int_{B_\rho\times(0,T)} (1+|u_k|)^{(1+\varepsilon_0)(p-1)} dx dt. \end{aligned}$$

This implies (3.5). □

Lemma 3.2. *For $k \in \{1, 2, \dots\}$, there hold*

$$\int_{B_\rho\times(0,T)} \frac{1}{(1+|u_k|)^{1+\varepsilon}} |\nabla u_k|^p dx dt \le \frac{c(\rho)}{\varepsilon} \quad (\varepsilon > 0), \tag{3.15}$$

$$\int_{B_\rho \times (0,T) \cap \{|u_k| \le M\}} |\nabla u_k|^p \, dxdt \le Mc(\rho) \quad (M > 0) \tag{3.16}$$

for some $c(\rho) > 0$.

Proof. Let $\rho > 0$ and ξ be a cut-off function associated with 2ρ and ρ. Use $\phi_\varepsilon(u_k)\,\xi$ as a test function in (3.3a) to obtain

$$\begin{aligned}&\int_{B_\rho \times (0,T)} \frac{\varepsilon}{(1+|u_k|)^{1+\varepsilon}} |\nabla u_k|^p \, dxdt \\ &\le \int_{B_{2\rho}} |u_0(x)| \, dx + \frac{1}{\rho} \int_{B_{2\rho} \times (0,T)} |\nabla u_k|^{p-1} \, dxdt.\end{aligned}$$

This, together with (3.5) implies (3.15). To see (3.16), for $M > 0$ let $P_M(s)$ be given as before. Then use $P_M(u_k)\,\xi$ as a test function in (3.3a) to get

$$\int_{B_\rho \times (x,T)} P'_M(u_k) |\nabla u_k|^p \, dxdt \le M \int_{B_{2\rho}} |u_0| \, dx + \frac{M}{\rho} \int_{B_{2\rho} \times (0,T)} |\nabla u_k|^{p-1} \, dxdt.$$

This completes the proof. □

Lemma 3.3. *There exists a subsequence of* $\{u_k\}$*, still denoted by* $\{u_k\}$*, and a function* $u \in L^1_{\mathrm{loc}}(\mathbf{R}^N \times (0,T))$ *with*

$$u_k \to u \quad \textit{almost everywhere on} \quad \Sigma_T. \tag{3.17}$$

Proof. Fix $\rho > 0$, and let ξ be given as in the proof of Lemma 3.2. We conclude from (3.3a) that

(3.18)
$$\begin{aligned}\int_0^T \left(\frac{\partial}{\partial t} u_k, \frac{1}{1+u_k^2}\xi\varphi\right) dt &+ \int_{B_{2\rho} \times (0,T)} |\nabla u_k|^{p-2} \nabla u_k \nabla \xi \varphi dxdt \\ &+ \int_{B_{2\rho} \times (0,T)} \frac{1}{1+u_k^2} |\nabla u_k|^{p-2} \nabla u_k \xi \nabla \varphi dxdt \\ &- \int_{B_{2\rho} \times (0,T)} \frac{2u_k}{(1+u_k^2)^2} |\nabla u_k|^p \xi \varphi dxdt = 0\end{aligned}$$

for all $\varphi \in C_0^\infty(B_{2\rho} \times (0,T))$. Here, $(\cdot,\cdot)$ denotes the duality pairing between $W^{-1,p'}(B_{2\rho})$ and $W_0^{1,p}(B_{2\rho})$. We infer from an argument in **[X1]** that

$$\left(\frac{\partial}{\partial t} u_k, \frac{1}{1+u_k^2}\xi\varphi\right) = \left(\frac{\partial}{\partial t}(\xi \arctan u_k), \varphi\right) \quad \text{almost everywhere on} \quad (0,T).$$

This, combined with (3.18) indicates that

$$\frac{\partial}{\partial t}\left(\xi \arctan u_k\right) - \operatorname{div}\left(\frac{1}{1+u_k^2}\xi \left|\nabla u_k\right|^{p-2} \nabla u_k\right) + \left|\nabla u_k\right|^{p-2} \nabla u_k \nabla \xi - \frac{2u_k}{\left(1+u_k^2\right)^2}\xi \left|\nabla u_k\right|^p = 0 \quad \text{in} \quad \mathcal{D}'\left(B_{2\rho} \times (0,T)\right). \tag{3.19}$$

Now set

$$F_k = \operatorname{div}\left(\frac{1}{1+u_k^2}\xi \left|\nabla u_k\right|^{p-2} \nabla u_k\right),$$
$$G_k = -\left|\nabla u_k\right|^{p-2} \nabla u_k \nabla \xi - \frac{2u_k}{\left(1+u_k^2\right)^2}\xi \left|\nabla u_k\right|^p.$$

It is easy to verify from (3.5) and (3.15) that

$$\{G_k\} \text{ is bounded in } L^1\left(B_{2\rho} \times (0,T)\right),$$
$$\{F_k\} \text{ is bounded in } L^{p'}\left(0,T; W^{-1,p'}\left(B_{2\rho}\right)\right),$$
$$\{\xi \arctan u_k\} \text{ is bounded in } L^p\left(0,T; W_0^{1,p}\left(B_{2\rho}\right)\right).$$

This puts us in a position to invoke Lemma 4.2 in [**BM**] to conclude that

$$\{\xi \arctan u_k\} \text{ is precompact in } L^p_{\mathrm{loc}}\left(B_{2\rho} \times (0,T)\right).$$

In particular, we can extract a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, such that

$$\arctan u_k \text{ converges almost everywhere on } B_\rho \times (0,T).$$

Note that $u_k = \tan\left(\arctan u_k\right)$. We may define

$$u(x,t) = \lim_{k\to\infty} u_k(x,t) \qquad \text{for almost everywhere } (x,t) \in B_\rho \times (0,T).$$

To conclude that $\{u_k\}$ converges almost everywhere on $B_\rho \times (0,T)$, we must show that $|u| < \infty$ almost everywhere on $B_\rho \times (0,T)$. However, this is an easy consequence of Fatou's lemma and (3.4). Since $\rho > 0$ is arbitrary, we can appeal to the classical diagonal argument to conclude the proof. □

Lemma 3.4. *There exists a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, and a measurable function $F(x,t)$ on Σ_T such that*

$$\nabla u_k \to F \qquad \textit{almost everywhere on } \Sigma_T. \tag{3.20}$$

Proof. Fix $\rho > 0$, and let ξ be given as in the proof of Lemma 3.3. Assume (3.17) holds. According to Egorov's theorem, for each $\eta > 0$ there exists a measurable set $E_\eta \subset B_\rho \times (0,T)$ such that

$$|B_\rho \times (0,T)\backslash E_\eta| < \eta \quad \text{and} \quad u_k \to u \qquad \text{uniformly on } E_\eta.$$

We may assume that $\{u_k\}$ is bounded in $L^\infty(E_\eta)$, and thus by (3.16),

$$\int_{E_\eta} |\nabla u_k|^p \, dxdt \le c(\eta, \rho). \tag{3.21}$$

For $\delta > 0$, we can find a $K(\delta)$ with

$$|u_k - u_m| < \delta \text{ on } E_\eta \text{ for all } m, k > K(\delta). \tag{3.22}$$

Let P_δ be defined as before. We can derive from (3.3a) that

$$\begin{aligned}
&\frac{d}{dt}\int_{B_{2\rho}}\int_0^{u_k(x,t)-u_m(x,t)} P_\delta(s)ds\xi(x)dx + \\
&\int_{B_{2\rho}} \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_m|^{p-2}\nabla u_m\right)(\nabla u_k - \nabla u_m)\xi(x)P'_\delta(u_k - u_m)\,dx \\
&= -\int_{B_{2\rho}} \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_m|^{p-2}\nabla u_m\right)\nabla\xi(x)P_\delta(u_k - u_m)\,dx \\
&\le \frac{\delta}{\rho}\int_{B_{2\rho}} \left(|\nabla u_k|^{p-1} + |\nabla u_m|^{p-1}\right)dx,
\end{aligned}$$

for k, m sufficiently large. Thus,

(3.23)
$$\begin{aligned}
&\int_{B_{2\rho}\times(0,T)} \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_m|^{p-2}\nabla u_m\right) \\
&\quad \cdot (\nabla u_k - \nabla u_m)\xi(x)P'_\delta(u_k - u_m)\,dxdt \\
&\le \int_{B_{2\rho}}\int_0^{u_{0k}-u_{0m}} P_\delta(s)dsdx + \frac{\delta}{\rho}\int_{B_{2\rho}\times(0,T)}\left(|\nabla u_k|^{p-1} + |\nabla u_m|^{p-1}\right)dxdt \\
&\le c(\rho)\delta
\end{aligned}$$

for k, m sufficiently large. We estimate, with the aid of (3.21), (3.22), and (3.23) that

(3.24)
$$\int_{E_\eta} |\nabla u_k - \nabla u_m|^p \, dxdt$$

$$
\begin{aligned}
&= \int_{E_\eta} \frac{|\nabla u_k - \nabla u_m|^p}{(|\nabla u_k| + |\nabla u_m|)^{\frac{(2-p)p}{2}}} (|\nabla u_k| + |\nabla u_m|)^{\frac{(2-p)p}{2}} \, dxdt \\
&\leq \left(\int_{E_\eta} \frac{|\nabla u_k - \nabla u_m|^2}{(|\nabla u_m| + |\nabla u_k|)^{2-p}} dxdt \right)^{\frac{p}{2}} \left(\int_{E_\eta} (|\nabla u_m| + |\nabla u_k|)^p \, dxdt \right)^{\frac{(2-p)}{2}} \\
&\leq c(\eta, \rho) \Bigg(\int_{B_{2\rho} \times (0,T)} \left(|\nabla u_k|^{p-2} \nabla u_k - |\nabla u_m|^{p-2} \nabla u_m \right) \\
&\quad \cdot (\nabla u_k - \nabla u_m) \, \xi(x) P_\delta' (u_k - u_m) \, dxdt \Bigg)^{\frac{p}{2}} \\
&\leq c_1(\eta, \rho) \delta^{\frac{p}{2}}
\end{aligned}
$$

for k, m sufficiently large. We see that $\{\nabla u_k\}$ is a Cauchy sequence in $(L^p(E_\eta))^N$. In particular, we can select a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, so that

$$\nabla u_k \qquad \text{converges almost everywhere on } E_\eta.$$

This is true for each $\eta > 0$, and so $\{\nabla u_k\}$ converges almost everywhere on $B_\rho \times (0, T)$. The lemma follows from the classical diagonal argument. □

Lemma 3.5. $\left\{ |\nabla u_k|^{p-2} \nabla u_k \right\}$ *is precompact in* $L^1(B_\rho \times (0,T))$ *for each* $\rho > 0$.

Proof. Note that the function $G(x) \equiv |x|^{p-2}x$ is continuous because $\lim_{|x|\to 0} |x|^{p-2}x = 0 \equiv G(0)$. Thus, we may assume that

$$\left\{ |\nabla u_k|^{p-2} \nabla u_k \right\} \qquad \text{converges almost everywhere on } B_\rho \times (0,T). \tag{3.25}$$

Now for each $q \in \left(0, \frac{p}{2}\right)$, we can choose $\varepsilon_0 > 0$ so that $q = \frac{1}{2+\varepsilon_0} p$. We deduce from (3.4) and (3.15) that

(3.26)

$$
\begin{aligned}
&\int_{B_\rho \times (0,T)} |\nabla u_k|^q \, dxdt \\
&= \int_{B_\rho \times (0,T)} \frac{1}{(1 + |u_k|)^{(1+\varepsilon_0)\frac{q}{p}}} |\nabla u_k|^q (1 + |u_k|)^{(1+\varepsilon_0)\frac{q}{p}} \, dxdt \\
&\leq \left(\int_{B_\rho \times (0,T)} \frac{1}{(1 + |u_k|)^{1+\varepsilon_0}} |\nabla u_k|^p \, dxdt \right)^{\frac{q}{p}}
\end{aligned}
$$

$$\cdot \left(\int_{B_\rho \times (0,T)} (1 + |u_k|)^{(1+\varepsilon_0)\frac{q}{(p-q)}} \, dxdt \right)^{\frac{(p-q)}{p}}$$

$$\leq c(\rho) \left(\int_{B_\rho \times (0,T)} (1 + |u_k|) \, dxdt \right)^{\frac{(p-q)}{p}} \leq c(\rho).$$

Since $0 < p - 1 < \frac{p}{2}$, there exists a $q \in \left(p - 1, \frac{p}{2}\right)$ such that

$$\int_{B_\rho \times (0,T)} |\nabla u_k|^q \, dxdt \leq c(\rho),$$

at least for k large enough. This implies that $\left\{ |\nabla u_k|^{p-2} \nabla u_k \right\}$ is uniformly integrable. This, in conjunction with (3.32) and Vitali's theorem, yields the lemma. □

Lemma 3.6. *$\{u_k\}$ is precompact in $C\left([0,T]; L^1\left(B_\rho\right)\right)$ for each $\rho > 0$.*

Proof. For $\delta > 0$ let

$$\theta_\delta(s) = \begin{cases} 1 & \text{if } s > \delta \\ s & \text{if } |s| < \delta \\ -1 & \text{if } s < -\delta \end{cases},$$

and ξ be given as in the proof of Lemma 3.2. We can conclude from (3.3a) that

(3.27)

$$\begin{aligned} &\int_{B_{2\rho}} \int_0^{u_k(x,t) - u_m(x,t)} \theta_\delta(s) ds \xi(x) dx \\ &\quad + \int_{B_{2\rho} \times (0,t)} \left(|\nabla u_k|^{p-2} \nabla u_k - |\nabla u_m|^{p-2} \nabla u_m \right) \\ &\quad \cdot \left(\nabla u_k - \nabla u_m \right) \xi(x) \theta'_\delta \left(u_k - u_m \right) dxd\tau \\ &= \int_{B_{2\rho}} \int_0^{u_{0k}(x) - u_{0m}(x)} \theta_\delta(s) ds \xi(x) dx \\ &\quad - \int_{B_{2\rho} \times (0,T)} \left(|\nabla u_k|^{p-2} \nabla u_k - |\nabla u_m|^{p-2} \nabla u_m \right) \theta_\delta \left(u_k - u_m \right) \nabla \xi dxd\tau. \end{aligned}$$

Observe that the second integral in (3.27) is nonnegative. Hence, we obtain

$$\int_{B_\rho} |u_k(x,t) - u_m(x,t)| \, dx$$

$$\leq \int_{B_{2\rho}} |u_{0k} - u_{0m}|\, dx + \frac{1}{\rho} \int_{B_{2\rho}\times(0,T)} \left| |\nabla u_k|^{p-2} \nabla u_k - |\nabla u_m|^{p-2} \nabla u_m \right| dxdt.$$

Then the lemma follows from Lemma 3.5. □

Lemma 3.7. *Let $E \subset \mathbf{R}^N \times (0,T)$ be bounded and measurable. Assume that there exists an $M > 0$ such that*

$$|u_k| \leq M \qquad \textit{almost everyshere on} \quad E \quad \textit{for } k \textit{ sufficiently large.}$$

Then $\{\nabla u_k\}$ is precompact in $(L^p(E))^N$.

Proof. Let $\rho > 0$ be such that

$$B_\rho \times (0,T) \supset E,$$

and let ξ be given as in the proof of Lemma 3.2. We conclude from (3.39) that

$$\begin{aligned}
&\int_{B_{2\rho}} \xi(x) \int_0^{u_k(x,t)-u_m(x,t)} P_{2M}(s)dsdx \\
&\quad + \int_{B_{2\rho}\times(0,T)} \left(|\nabla u_k|^{p-2} \nabla u_k - |\nabla u_m|^{p-2} \nabla u_m\right) \\
&\qquad \cdot (\nabla u_k - \nabla u_m) P'_{2M}(u_k - u_m)\xi(x)dxd\tau \\
&= \int_{B_{2\rho}} \xi(x) \int^{u_{0k}-u_{0m}} P_{2M} sdsdx \\
&\quad - \int_{B_{2\rho}\times(0,T)} \left(|\nabla u_k|^{p-2} \nabla u_k - |\nabla u_m|^{p-2} \nabla u_m\right) P_{2M}(u_k - u_m) \nabla\xi(x)dxd\tau.
\end{aligned}$$

Subsequently,

$$\begin{aligned}
&\int_E \left(|\nabla u_k|^{p-2} \nabla u_k - |\nabla u_m|^{p-2} \nabla u_m\right)(\nabla u_k - \nabla u_m)\, dxd\tau \\
&\leq 2M \int_{B_{2\rho}} |u_{0k} - u_{0m}|\, dx \\
&\quad + \frac{2M}{\rho} \int_{B_{2\rho}\times(0,T)} \left| |\nabla u_k|^{p-2} \nabla u_k - |\nabla u_m|^{p-2} \nabla u_m \right| dxdt.
\end{aligned}$$

A calculation similar to (3.24) yields

$$\begin{aligned}
&\int_E |\nabla u_k - \nabla u_m|^p\, dxdt \\
&\leq c(M,\rho) \left(\int_{B_{2\rho}} |u_{ok} - u_{0m}|\, dx \right.
\end{aligned}$$

$$+\int_{B_{2\rho}} \left| |\nabla u_k|^{p-2} \nabla u_k - |\nabla u_m|^{p-2} \nabla u_m \right| dxdt \Bigg)^{\frac{p}{2}}.$$

This implies the desired result. □

Now we are ready to conclude the proof of Theorem 3.1. Let $\{v_k\}$, $\{u_k\}$ be given as before. Note from Lemma 2.3 that

$$v_k \le v_{k+1} \quad \text{on} \quad \Sigma_T \quad \text{for all } k,$$
$$w_k \ge w_{k+1} \quad \text{on} \quad \Sigma_T \quad \text{for all } k.$$

Define

$$v(x,t) = \lim_{k\to\infty} v_k(x,t),$$
$$w(x,t) = \lim_{k\to\infty} w_k(x,t).$$

Consequently,

$$w \le u_k \le v \qquad \text{almost everywhere.} \tag{3.28}$$

By a result in [**DH**], there holds

$$\int_s^T \int_{B_\rho} \frac{(z_t)^2}{(|z|+1)^{1+\varepsilon}} dxdt \le c(\varepsilon, s, p), T > s > 0, \varepsilon > 0, \rho > 0,$$

where $z = w$ or v. The remaining proof is entirely similar to that in [**X1**]. The only difference is that in (3.23) of [**X1**] we require

$$\varphi \in C_0^\infty \left(\mathbf{R}^N \times (-\infty, T)\right).$$

This completes the proof.

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