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CHARACTERS OF THE CENTRALIZER ALGEBRAS OF MIXED TENSOR REPRESENTATIONS OF $GL(r,\mathbb{C})$ AND THE QUANTUM GROUP $\mathcal{U}_q(g\ell(r,\mathbb{C}))$.

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We consider the tensor product representation of m copies of the natural representation with n copies of its dual representation for both the general linear group $GL(r,\mathbb{C})$ and the quantum group $U_q(g\ell(r,\mathbb{C}))$. These tensor spaces determine rational representations of GL_r and $\mathcal{U}_q(g\ell(r,\mathbb{C}))$. The centralizer algebras of these representations are, respectively, the complex algebra $\mathcal{B}^r_{m,n}$, which is a subalgebra of the Brauer algebra B^r_{m+n}, and the algebra $H^r_{m,n}(q)$ over the field of complex rational functions with indeterminate q, which is a generalization of the Iwahori-Hecke algebra. Upon setting $q = 1$, the algebra $H^r_{m,n}(q)$ specializes to $\mathcal{B}^r_{m,n}$. The algebra $\mathcal{B}^r_{m,n}$ contains as a subalgebra the group algebra $\mathbb{C}[S_m \times S_n]$ of the product of two symmetric groups, and the algebra $H^r_{m,n}(q)$ contains as a subalgebra the tensor product $H_m(q) \otimes H_n(q)$ of two Iwahori-Hecke algebras. In each centralizer, we find a distinguished basis and define an analog of conjugacy class. We then exploit Schur's double centralizer theory to derive a "Frobenius formula" which we use to compute their irreducible characters in terms of symmetric group characters and Iwahori-Hecke algebra characters. In the process, we obtain branching rules that give the decomposition of $\mathcal{B}^r_{m,n}$-modules into irreducible $\mathbb{C}[S_m \times S_n]$-modules and $H^r_{m,n}(q)$-modules into irreducible $H_m(q) \otimes H_n(q)$-modules.

0. Introduction.

A sequence of integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t) \in \mathbb{Z}^t$ is a partition if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t \geq 0$. The length $\ell(\lambda)$ of λ is the largest i such that $\lambda_i > 0$. If $\lambda_1 + \lambda_2 + \cdots + \lambda_t = f$, then λ is a partition of f which we denote by $\lambda \vdash f$. Let $\mathcal{S}_f$ denote the symmetric group on f letters. The irreducible representations of $\mathcal{S}_f$ and its conjugacy classes are indexed by the partitions λ of f. Frobenius [**F**] proved a remarkable formula

$$p_\alpha(x_1, \dots, x_r) = \sum_{\substack{\lambda \vdash f \\ \ell(\lambda) \leq r}} \chi^\lambda_{\mathcal{S}_f}(\alpha) s_\lambda(x_1, \dots, x_r) \tag{0.1}$$

relating the power symmetric function p_α labeled by $\alpha \vdash f$, the value of the irreducible $\mathcal{S}_f$-character $\chi^\lambda_{\mathcal{S}_f}(\alpha)$ on an element of the conjugacy class labeled by α, and the Schur function s_λ. This result, referred to as the Frobenius formula, has been used to derive the Murnaghan-Nakayama rule—a completely combinatorial method of computing symmetric group characters (see [**Sa**]).

Schur [**Sc1**, **Sc2**] gave the Frobenius formula a representation-theoretic interpretation by showing that it is a consequence of the connection between $\mathcal{S}_f$ and polynomial representations of the general linear group $GL(r, \mathbb{C})$. If $V = \mathbb{C}^r$ is the natural representation of $GL(r, \mathbb{C})$, then $T^f = \otimes^f V$ is both a $GL(r, \mathbb{C})$-module and a module for $\mathcal{S}_f$, which acts on T^f by place permutation. Schur proved that the group algebra $\mathbb{C}[\mathcal{S}_f]$ and the algebra generated by $GL(r, \mathbb{C})$ on T^f are full centralizers of each other when $r \geq f$ and that the Frobenius formula represents the matrix trace of $\mathcal{S}_f \times GL(r, \mathbb{C})$ on T^f.

We extend these results to the mixed tensor representations of $GL(r, \mathbb{C})$. Let V^* be the dual space to V. The mixed tensor space $T^{m,n} = (\otimes^m V) \otimes (\otimes^n V^*)$ is a completely reducible $GL(r, \mathbb{C})$-module whose irreducible summands are rational $GL(r, \mathbb{C})$-modules. Irreducible rational $GL(r, \mathbb{C})$-modules are indexed by r-staircases which are sequences of integers

$$\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_r) \in \mathbb{Z}^r \quad \text{such that} \quad \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_r.$$

The positive integers $\gamma_1, \gamma_2, \ldots, \gamma_i$ and the negative integers $\gamma_j, \gamma_{j+1}, \ldots, \gamma_r$ of γ determine partitions

$$\gamma^+ = (\gamma_1, \gamma_2, \ldots, \gamma_i) \quad \text{and} \quad \gamma^- = (-\gamma_r, -\gamma_{r-1}, \ldots, -\gamma_j).$$

We let $\Phi_r^{m,n}$ denote the set of r-staircases γ which satisfy $\gamma^+ \vdash (m - h(\gamma))$ and $\gamma^- \vdash (n - h(\gamma))$ for some integer $h(\gamma)$ with $0 \leq h(\gamma) \leq \min(m, n)$. Stembridge [**Ste**] proves that the irreducible $GL(r, \mathbb{C})$-summands of $T^{m,n}$ are indexed by the r-staircases in $\Phi_r^{m,n}$.

The centralizer algebra of the action of $GL(r, \mathbb{C})$ on $T^{m,n}$ has been described by [**Koi**] and [**BCHLLS**]. When $r \geq m+n$, [**BCHLLS**] proves that this centralizer is a semisimple subalgebra $\mathcal{B}^r_{m,n}$ of the Brauer algebra $\mathcal{B}^r_{m+n}$, which was introduced in [**Bra**] to describe the centralizer of the orthogonal group $O(r, \mathbb{C})$ on $T^{m,n}$. From the duality between $GL(r, \mathbb{C})$ and $\mathcal{B}^r_{m,n}$, one concludes that the irreducible $\mathcal{B}^r_{m,n}$-representations are also indexed by $\Phi_r^{m,n}$. The algebra $\mathcal{B}^r_{m,n}$ is not a group algebra, but it has a distinguished basis of diagrams. In a fashion similar to that of [**R2**] for the Brauer algebra, we partition this basis into classes on which characters are constant. Since the elements of $\mathcal{B}^r_{m,n}$ are not all invertible, the notion of conjugacy is not a priori natural, but these classes are an extension of the conjugacy classes in $\mathcal{S}_m \times \mathcal{S}_n$. Using the duality between $GL(r, \mathbb{C})$ and $\mathcal{B}^r_{m,n}$, we prove the rational Frobenius formula

$$(0.2) \qquad r^{h(\zeta)} p_\zeta(x_1, \ldots, x_r) = \sum_{\gamma \in \Phi^{m,n}_r} \chi^\gamma_{\mathcal{B}^r_{m,n}}(\zeta) s_\gamma(x_1, \ldots, x_r),$$

relating a generalization p_ζ of the power-symmetric function, the irreducible $\mathcal{B}^r_{m,n}$-character $\chi^\gamma_{\mathcal{B}^r_{m,n}}(\zeta)$ evaluated on the class labeled by ζ, and the rational Schur function s_γ. When $n = 0$, $\mathcal{B}^r_{m,n}$ becomes $\mathbb{C}[\mathcal{S}_m]$ and Equation (0.2) reduces to (0.1).

The Iwahori-Hecke algebra $H_f(q)$ is the q-deformation of $\mathbb{C}[\mathcal{S}_f]$ that is the centralizer of the action of the quantum general linear group $\mathcal{U}_q(g\ell(r, \mathbb{C}))$ on the f-fold tensor product of its natural representation $V_q = V \otimes \mathbb{C}(q)$. Recently, Ram [**R1**] proved the Frobenius formula for $H_f(q)$ which is a q-extension of (0.1). Kosuda [**Kos**] gives a two-parameter Iwahori-Hecke algebra $H^r_{m,n}(q)$ which, when $r \geq m + n$, is the q-deformation of $\mathcal{B}^r_{m,n}$ that is the centralizer of the action of $\mathcal{U}_q(g\ell(r, \mathbb{C}))$ on the mixed tensor space $T^{m,n}_q = (\otimes^m V_q) \otimes (\otimes^n V_q{}^*)$. We describe $H^r_{m,n}(q)$ as an algebra of q-diagrams based on Kauffman's [**Ka**] tangle monoid. We identify a basis of q-diagrams in $H^r_{m,n}(q)$ that specializes when $q = 1$ to the basis of $H^r_{m,n}(q)$, and we partition this basis into character conjugacy classes also indexed by $\Phi^{m,n}_r$. We then give a q-extension of (0.2) that is the Frobenius formula for $H^r_{m,n}(q)$.

The algebra $\mathcal{B}^r_{m,n}$ contains as a subalgebra the group algebra $\mathbb{C}[\mathcal{S}_m \times \mathcal{S}_n]$ of the product of two symmetric groups. The Kosuda algebra $H^r_{m,n}(q)$ contains as a subalgebra the tensor product of two Iwahori-Hecke algebras $H_m(q) \otimes H_n(q)$. We determine the branching rule for writing irreducible $H^r_{m,n}(q)$-modules in terms of irreducible $H_m(q) \otimes H_n(q)$-modules and for writing irreducible $\mathcal{B}^r_{m,n}$-modules in terms of $\mathbb{C}[\mathcal{S}_m \times \mathcal{S}_n]$-modules. Using the Frobenius formulas we derive a formula (Theorem 7.19) that gives $H^r_{m,n}(q)$-characters in terms of characters of $H_m(q) \otimes H_n(q)$ and $\mathcal{B}^x_{m,n}$-characters in terms of $\mathcal{S}_m \times \mathcal{S}_n$-characters.

This paper is organized as follows. Section 1 presents general results about centralizer algebras and q-deformations. Section 2 describes the rational representations of $GL(r, \mathbb{C})$, the decomposition of $T^{m,n}$ as a $GL(r, \mathbb{C})$-module, and the branching rules for $GL(r, \mathbb{C})$-modules. In Section 3, the Brauer subalgebra is defined, and we give the branching rule for $\mathbb{C}[\mathcal{S}_m \times \mathcal{S}_n] \subseteq \mathcal{B}^r_{m,n}$. Section 4 describes the two-parameter Iwahori-Hecke algebra as an algebra of q-diagrams. In Section 5, we distinguish a basis of $H^r_{m,n}(q)$ and partition it into character conjugacy classes. Setting $q = 1$ gives the character conjugacy classes for $\mathcal{B}^r_{m,n}$. In Section 6, we define the quantum general linear group, and note that, in the $r \geq m+n$ case, $H^r_{m,n}(q)$ is the centralizer of the quantum group for this representation. In Section 7, we prove the Frobenius formulas for $\mathcal{B}^r_{m,n}$ and $H^r_{m,n}(q)$. Using these we derive the character formulas, and as a consequence, obtain the branching rules for $\mathbb{C}[\mathcal{S}_m \times \mathcal{S}_n] \subseteq \mathcal{B}^x_{m,n}$ and $H_m(q) \otimes H_n(q) \subseteq H^r_{m,n}(q)$.

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1. Preliminaries.

Let $\mathbb{C}$ denote the field of complex numbers and $\mathfrak{M}_d(\mathbb{C})$ denote the algebra of all $d \times d$ matrices with entries in $\mathbb{C}$. We say that an associative $\mathbb{C}$-algebra A is (split) *semisimple* if

$$A \cong \bigoplus_{\lambda \in \Phi} \mathfrak{M}_{d_\lambda}(\mathbb{C}), \tag{1.1}$$

for some finite index set Φ and positive integers d_λ. Corresponding to each $\lambda \in \Phi$ there is, up to isomorphism, one irreducible A-module. Its dimension is d_λ, and we denote it by V^λ. If T is a finite-dimensional A-module, then the decomposition of T into irreducible A-modules is

$$T \cong \bigoplus_{\lambda \in \Phi} m_\lambda V^\lambda, \tag{1.2}$$

where m_λ is a non-negative integer called the *multiplicity* of V^λ.

A set of idempotents $\{p_i\}$ in A is orthogonal if $p_i p_j = p_j p_i = \delta_{i,j}\, p_i$ for all i and j ($\delta_{i,j}$ is the Kronecker delta). An idempotent p is minimal if it cannot be written as a sum $p = p_1 + p_2$ of orthogonal idempotents. A decomposition $1 = \sum_i p_i$ of 1 into minimal orthogonal idempotents is a *partition of unity*. If $\phi : A \longrightarrow \bigoplus_{\lambda\in\Phi} \mathfrak{M}_{d_\lambda}(\mathbb{C})$ is an isomorphism, then associated to each $\lambda \in \Phi$ is the *minimal central idempotent*, $z_\lambda = \phi^{-1}(I_\lambda)$, where I_λ is the identity matrix in $\mathfrak{M}_{d_\lambda}(\mathbb{C})$. To each minimal idempotent p, there corresponds exactly one $\lambda \in \Phi$ so that $pz_\mu = \delta_{\mu,\lambda} p$ for all $\mu \in \Phi$. The *character* χ_ϕ of the representation ϕ is the $\mathbb{C}$-linear functional $\chi_\phi : A \longrightarrow \mathbb{C}$ given by $\chi_\phi(a) = Tr(\phi(a))$, where Tr denotes the usual matrix trace, i.e., the sum of the diagonal entries. The character associated to the irreducible A-module V^λ is denoted χ_A^λ and is called an irreducible character of A.

If T is a finite-dimensional A-module with associated representation ϕ : $A \longrightarrow End_{\mathbb{C}}(T)$, then the *centralizer algebra of A on T*, denoted by $End_A(T)$ is the set of linear transformations on T commuting with $\phi(A)$, namely

$End_A(T) = \{X \in End_{\mathbb{C}}(T) \mid X\phi(a)t = \phi(a)Xt$ for all $a \in A$ and $t \in T\}$. If T is an irreducible A-module of dimension d, then it follows from Schur's double centralizer theory that $End_A(T) \cong \bigoplus_{\lambda\in\Phi}\mathfrak{M}_{m_\lambda}(\mathbb{C})$. In particular, $End_A(T)$ is semisimple, and T decomposes into irreducible $End_A(T)$-modules as $T \cong \bigoplus_{\lambda\in\Phi} d_\lambda M^\lambda$, where M^λ is an irreducible module for $End_A(T)$ with $\dim_{\mathbb{C}} M^\lambda = m_\lambda$.

We regard T as a bimodule for the algebra $End_A(T) \otimes A$, where the product on $End_A(T) \otimes A$ is componentwise, and the action is afforded by $(c \otimes a) \cdot t = c(a \cdot t) = a(c \cdot t)$ for $c \in End_A(T)$ and $a \in A$ and extended linearly to $End_A(T) \otimes A$. That this action is well-defined follows from the fact that the actions of A and $End_A(T)$ on T commute. The decomposition of T into irreducible $End_A(T) \otimes A$-bimodules is given by

$$T \cong \bigoplus_{\lambda\in\Phi} M^\lambda \otimes V^\lambda \tag{1.3}$$

(see [**R3**] for a proof). If T is a faithful A-module, then by switching the roles of A and $End_A(T)$ and comparing dimensions, we have $A = End_{End_A(T)}(T)$. That is, A and $End_A(T)$ are full centralizers of each other in $End_{\mathbb{C}}(T)$.

If $a \in A$ and $c \in End_A(T)$, then the trace $Tr(c \otimes a) = Tr(ca)$ of the action of $c \otimes a$ on T is called the *bicharacter* of $End_A(T) \otimes A$. From (1.3) we have

$$Tr(ca) = \sum_{\lambda\in\Phi} \chi^\lambda_{End_A(T)}(c)\chi^\lambda_A(a). \tag{1.4}$$

Since A and $End_A(T)$ commute, the bicharacter satisfies the trace property in each component. That is $Tr(c_1c_2a) = Tr(c_1ac_2) = Tr(c_2c_1a)$ and $Tr(ca_1a_2) = Tr(a_1ca_2) = Tr(ca_2a_1)$ for all $c_i \in End_A(T)$ and $a_i \in A$.

If A and B are semisimple algebras, and B is a subalgebra of A, then the irreducible A-module V^λ is also a B-module. If $\{\tilde{V}^\mu\}_{\mu\in\tilde{\Phi}}$ are the irreducible B-modules, then the decomposition of V^λ into irreducible B-modules is given by

$$V^\lambda\downarrow^A_B \cong \bigoplus_{\mu\in\tilde{\Phi}} g_{\lambda\mu}\tilde{V}^\mu, \tag{1.5}$$

for some non-negative integers $g_{\lambda,\mu}$ called the multiplicity of $\tilde{V}^\mu$ in V^λ. Equation (1.5) is called the *branching rule* for $B \subseteq A$. If $g_{\lambda\mu} \in \{0,1\}$ for all λ and μ, we say that the branching rule for $B \subseteq A$ is *multiplicity free*. It is clear that since $B \subseteq A$, their centralizers satisfy $End_A(T) \subseteq End_B(T)$, so if $\{\tilde{M}^\mu | \mu \in \tilde{\Phi}\}$ is the set of irreducible $End_B(T)$-modules, then we can consider the branching rule

$$\tilde{M}^\mu\downarrow^{End_B(T)}_{End_A(T)} \cong \bigoplus_{\lambda\in\Phi} M_\lambda. \tag{1.6}$$

The following presumably well-known theorem says that the branching rule for $B \subseteq A$ is the same as that for $End_A(T) \subseteq End_B(T)$. The proof we include is due to Ram [**R3**].

Theorem 1.7. *If* (1.5) *and* (1.6) *are the branching rules for $B \subseteq A$ and $End_A(T) \subseteq End_B(T)$ respectively, then $g_{\lambda\mu} = g'_{\mu\lambda}$ for all $\lambda \in \Phi$ and $\mu \in \tilde{\Phi}$.*

Proof. As an $End_B(T) \otimes B$-bimodule, we have $T \cong \bigoplus_{\mu \in \tilde{\Phi}} \tilde{M}^\mu \otimes \tilde{V}^\mu$. The algebra $End_A(T) \otimes B$ is a subalgebra of both $End_A(T) \otimes A$ and $End_B(T) \otimes B$, so consider the following branching rules

$$T\downarrow^{End_A(T)\otimes A}_{End_A(T)\otimes B} \cong \bigoplus_{\lambda,\mu} g_{\lambda\mu} M^\lambda \otimes \tilde{V}^\mu \quad \text{and} \quad T\downarrow^{End_B(T)\otimes B}_{End_A(T)\otimes B} \cong \bigoplus_{\mu,\lambda} g'_{\mu\lambda} M^\lambda \otimes \tilde{V}^\mu.$$

Comparing multiplicities gives $g_{\lambda\mu} = g'_{\mu\lambda}$. □

An important application of this theorem is the following. Let G be a group and $\{V_1, V_2, \dots, V_f\}$ be an ordered set of irreducible G-modules. Then for $k = 1, \dots, f$, the tensor product space $T^f = V_1 \otimes V_2 \otimes \cdots \otimes V_f$ is a G-module under the *diagonal action*

$$g \cdot (u_1 \otimes u_2 \otimes \cdots \otimes u_k) = g \cdot u_1 \otimes g \cdot u_2 \otimes \cdots \otimes g \cdot u_f, \tag{1.8}$$

for all $g \in G$ and $u_i \in V_i$. Let $C_k = End_G(T^k)$. Then we have $\mathbb{C} = C_1 \subseteq C_2 \subseteq \cdots \subseteq C_f$. Now let $(g,h) \in G \times G$ act on $T^f = T^{f-1} \otimes V$ with g acting diagonally on T^{f-1} and h acting on the copy of V in the fth tensor slot. If we consider $G \subseteq G \times G$ by the diagonal embedding $g \mapsto (g,g)$, then the centralizer of the action of $G \times G$ on T^f is $C_{f-1} \otimes \mathbb{C} \cong C_{f-1}$. Theorem 1.7 tells us that the branching rule for $C_{f-1} \subseteq C_f$ are the same as for $G \subseteq G \times G$. That is,

$$(V^\lambda \otimes V)\downarrow^{G\times G}_G \cong \bigoplus_\pi g_{\lambda,\pi} V^\pi \quad \Leftrightarrow \quad M^\pi\downarrow^{C_f}_{C_{f-1}} \cong \bigoplus_\lambda g_{\lambda,\pi} M^\lambda, \tag{1.9}$$

where V^λ and V^π are irreducible G-modules, M^π is an irreducible C_f-module, and M^λ is an irreducible C_{f-1}-module.

A tower of semisimple algebras is a sequence of semisimple algebras $\mathbb{C} \cong C_1 \subseteq C_2 \subseteq \cdots \subseteq C_m \subseteq \cdots$ such that C_i is a subalgebra of C_{i+1} for all i. The tower has multiplicity-free branching if the branching rule for each inclusion $C_i \subseteq C_{i+1}$ is multiplicity-free. Let $\{M^\lambda | \lambda \in \Phi_i\}$ denote the set of irreducible C_i-modules, and suppose $\dim_{\mathbb{C}}(M^\lambda) = m_\lambda$. Let $\Omega^i_\lambda = \{t^1_\lambda, \dots, t^{m_\lambda}_\lambda\}$ be a basis for M^λ. If $\pi \in \Phi_{i+1}$, and $\lambda \in \Phi_i$ such that M^λ appears in the decomposition of M^π into C_i-modules with multiplicity 1, then we write $\lambda \leq \pi$. The branching rule for $C_i \subseteq C_{i+1}$ is thus written as

$$M^\pi\downarrow^{C_{i+1}}_{C_i} \cong \bigoplus_{\lambda \leq \pi} M^\lambda. \tag{1.10}$$

Considering (1.10), we pick a bijection $\beta : \Omega_\lambda^m \longrightarrow \bigcup_{\mu\leq\lambda}\Omega_\mu^{m-1}$. For $t_\lambda^j \in \Omega_\lambda^m$, an idempotent $p_{t_\lambda^j} \in C_m$ is defined as follows

$$(1.11) \qquad p_{t_\lambda^j} = \begin{cases} 1, & \text{if } m = 1, \text{ and} \\ z_\lambda p_{\beta(t_\lambda^j)}, & \text{if } m > 1, \end{cases}$$

where z_λ is the minimal central idempotent in C_m associated to λ. This construction is a generalization of the work of Wenzl [**Wen1**], who defines these idempotents in the Iwahori-Hecke algebra.

Proposition 1.12. *The decomposition* $1 = \sum_{\lambda\in\Phi_m}\sum_{t\in\Omega_\lambda^m} p_t$ *is a partition of unity.*

Proof. The fact that p_t is an idempotent is immediate from its definition. To see that the p_t are orthogonal idempotents, observe that

$$p_t p_{\tilde{t}} = z_\lambda p_{\beta(t)} z_\gamma p_{\beta(\tilde{t})} = z_\lambda z_\gamma p_{\beta(t)} p_{\beta(\tilde{t})},$$

where $t \in \Omega_\lambda^m$ and $\tilde{t} \in \Omega_\gamma^m$. If $\gamma \neq \lambda$, then $z_\lambda z_\gamma = 0$. If $\gamma = \lambda$, then apply the argument to $p_{\beta(t)}p_{\beta(\tilde{t})}$. Since $\dim(C_1) = 1$, we eventually have $\beta(t) \in M^\lambda$ and $\beta(\tilde{t}) \in M^\gamma$ with $\gamma \neq \lambda$. Furthermore,

$$\begin{aligned} 1 = \sum_{\lambda\in\Phi_m} z_\lambda &= \sum_{\lambda\in\Phi_m}\sum_{\mu\in\Phi_{m-1}}\sum_{t'\in\Omega_\mu^{m-1}} z_\lambda p_{t'} \\ &= \sum_{\lambda\in\Phi_m}\sum_{\mu\leq\lambda}\sum_{t'\in\Omega_\mu^{m-1}} z_\lambda p_{t'} = \sum_{\lambda\in\Phi_m}\sum_{t\in\Omega_\lambda^m} p_t. \end{aligned}$$

□

For q an indeterminate over $\mathbb{C}$, we let $\mathbb{C}(q)$ be the field of rational functions in q with coefficients in $\mathbb{C}$, and let $A(q)$ be a finite-dimensional, semisimple algebra over $\mathbb{C}(q)$. Suppose that $\{b_1, \ldots, b_m\}$ is a basis for $A(q)$ having structure constants $f_{ij}^k(q)$. Then,

$$(1.13) \qquad b_i b_j = \sum_{k=1}^m f_{ij}^k(q) b_k.$$

For all but finitely many $q_0 \in \mathbb{C}$, all the values $f_{ij}^k(q_0)$ exist. For such a q_0, let $A = A(q_0)$ be the $\mathbb{C}$-algebra spanned by $\{b_1, \ldots, b_m\}$, with multiplication given by $b_i b_j = \sum_{k=1}^m f_{ij}^k(q_0) b_k$. If $a(q) \in A(q)$ is written in terms of the basis as $a(q) = \sum_{k=1}^m c_k(q) b_k$ and each rational function $c_k(q) \in \mathbb{C}(q)$ is defined at $q = q_0$, then we let the element $a(q_0) \in A$ be $a(q_0) = \sum_{k=1}^m c_k(q_0) b_k$.

This gives us a partially-defined, surjective homomorphism from $A(q)$ onto A. The algebra $A(q)$ is said to be a *q-deformation of A*. Note that we have $\dim_{\mathbb{C}(q)} A(q) \geq \dim_{\mathbb{C}} A$. We say that the semisimple $\mathbb{C}(q)$-algebra $A(q)$ and the semisimple $\mathbb{C}$-algebra A have *the same matrix decomposition* if

$$(1.14) \qquad A(q) \cong \bigoplus_{\lambda \in \Phi} \mathfrak{M}_{d_\lambda}(\mathbb{C}(q)) \qquad \text{and} \qquad A \cong \bigoplus_{\lambda \in \Phi} \mathfrak{M}_{d_\lambda}(\mathbb{C}).$$

One way of extending character results known for the algebra A to the algebra $A(q)$ is to use a partition unity of $A(q)$ which specializes when $q = q_0$ to a (well-defined) partition of unity of A. The proof of the following well-known result is due to [**Wen2**].

Proposition 1.15. *Suppose $A(q)$ is a semisimple $\mathbb{C}(q)$-algebra, and $q_0 \in \mathbb{C}$ such that $A = A(q_0)$ and $A(q)$ have the same matrix decomposition. If $\{z_\lambda(q)\}_{\lambda\in\Phi}$ are the minimal central idempotents of $A(q)$, then $z_\lambda = z_\lambda(q_0)$ is defined for all $\lambda \in \Phi$, and $\{z_\lambda\}_{\lambda\in\Phi}$ is the set of minimal central idempotents in A.*

Proof. If $z_\lambda(q_0)$ is undefined then there exists a positive integer s such that $\tilde{z}_\lambda(q) = (q-q_0)^s z_\lambda(q)$ is defined and not zero when $q = q_0$. Then $\tilde{z}_\lambda(q)^2 = (q-q_0)^{2s} z_\lambda(q)$ is zero when $q = q_0$, so $\tilde{z}_\lambda(q_0)$ is a central nilpotent element in the semisimple algebra A which is a contradiction. The proposition follows from the fact that the two algebras have the same matrix decomposition. □

Theorem 1.16. *Let $C_1(q) \subseteq \cdots \subseteq C_m(q)$ be a tower of semisimple $\mathbb{C}(q)$-algebras with multiplicity-free branching. Suppose $q_0 \in \mathbb{C}$ so that, for $1 \leq i \leq m$, $C_i(q_0)$ has the same matrix decomposition as $C_i(q)$, and $C_1(q_0) \subseteq \cdots \subseteq C_m(q_0)$ has the same branching rules as $C_1(q) \subseteq \cdots \subseteq C_m(q)$. Then there exists a partition of unity $1 = \sum_i p_i(q)$ in $C_m(q)$ which specializes to a partition of unity $1 = \sum_i p_i(q_0)$ in C_m.*

Proof. Let $1 = \sum_i p_i(q)$ be a partition of unity defined as in (1.11). Then each of the $p_i(q)$ is defined as a product of minimal central idempotents for some $C_i(q)$ and are thus well defined for $q = q_0$. When $q = q_0$, the construction of the idempotents $p_i(q_0)$ is exactly the same as the construction (1.11) in C_m. □

2. Tensor Product Representations of $GL(r, \mathbb{C})$.

Let GL_r denote the complex general linear group $GL(r, \mathbb{C})$ of all $r \times r$ invertible matrices with entries from $\mathbb{C}$. If $\phi : GL_r \longrightarrow GL(d, \mathbb{C})$ is a representation of GL_r, then for $g \in GL_r$, let g_{ij} and $\phi(g)_{ij}$ denote the (i,j)-entry of g and

$\phi(g)$, respectively. If there exist rational functions $f_{ij}(x_1, x_2, \ldots, x_{r^2})$ such that $\phi(g)_{ij} = f_{ij}(g_{11}, g_{12}, \ldots, g_{rr})$, then we say that ϕ is a *rational representation* of GL_r. If each f_{ij} is a polynomial, then ϕ is a *polynomial representation* of GL_r. Let H be the Cartan subgroup of diagonal matrices in GL_r, and let $\epsilon_i \in H^*$ denote the map which takes a matrix to its (i,i)-entry. Each irreducible rational GL_r-module can be indexed by its highest weight relative to H, which is an integral linear combination $\gamma_1\epsilon_1 + \gamma_2\epsilon_2 + \cdots \gamma_r\epsilon_r$, whose coefficients satisfy $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_r$. If the representation is polynomial, then the coefficients satisfy $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_r \geq 0$.

A sequence of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t) \in \mathbb{Z}^t$ is a partition if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t \geq 0$. The length $\ell(\lambda)$ of λ is the largest i such that $\lambda_i > 0$. If $\lambda_1 + \lambda_2 + \cdots + \lambda_t = f$, then λ is a partition of f which we denote by $\lambda \vdash f$. Following Stembridge [**Ste**] we say that a sequence of integers $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_r) \in \mathbb{Z}^r$ satisfying $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_r$ is an *r-staircase.* Thus the polynomial GL_r-representations are indexed by partitions whose length is less than or equal to r, and rational representations are indexed by r-staircases. We will denote by V^λ and V^γ the GL_r-module indexed by the partition λ and the r-staircase γ, respectively. The positive integers $\gamma_1, \gamma_2, \ldots, \gamma_i$ and the negative integers $\gamma_j, \gamma_{j+1}, \ldots, \gamma_r$ of an r-staircase γ determine partitions $\gamma^+ = (\gamma_1, \gamma_2, \ldots, \gamma_i)$ and $\gamma^- = (-\gamma_r, -\gamma_{r-1}, \ldots, -\gamma_j)$. Conversely, any pair of partitions $\mu = (\mu_1, \ldots, \mu_{\ell(\mu)})$ and $\nu = (\nu_1, \ldots, \nu_{\ell(\nu)})$ with $\ell(\mu) + \ell(\nu) \leq r$ determines the r-staircase

$$[\mu, \nu]_r \stackrel{\text{def}}{=} (\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)}, \underbrace{0, 0, \ldots, 0}_{r-\ell(\mu)-\ell(\nu)}, -\nu_{\ell(\nu)}, \ldots, -\nu_2, -\nu_1), \tag{2.1}$$

where the partitions have been separated by $r - \ell(\mu) + \ell(\nu)$ zeros.

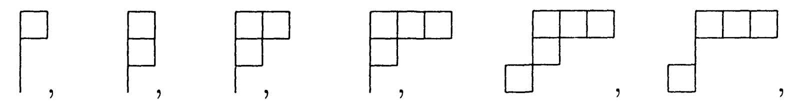
It is possible to realize all rational GL_r-modules as summands of tensor product representations. Let $V = \mathbb{C}^r$ viewed as $r \times 1$ matrices, and let $v_1, v_2, \ldots, v_r$ denote the canonical basis of V. Then GL_r acts naturally on V by matrix multiplication making V a GL_r-module. This representation is polynomial and is known as the "fundamental" representation of GL_r. The dual space V^* of V inherits a GL_r-module structure given by $(g \cdot u^*)v = u^*(g^{-1} \cdot v)$. Let $v_1^*, v_2^*, \ldots, v_r^*$ denote the dual basis to $v_1, v_2, \ldots, v_r$ in V^*. We identify v_i^* with the $1 \times r$ matrix having 1 in its ith column and 0 everywhere else. This is the contravariant representation of GL_r, and it is a rational GL_r-representation.

Fix integers $m, n \geq 0$ such that $m + n > 0$. Then the tensor product $T^{m,n} \stackrel{\text{def}}{=} (\otimes^m V) \otimes (\otimes^n V^*)$ becomes a GL_r-module under the diagonal action (1.8). Moreover, $T^{m,n}$ is a completely reducible GL_r-module, and its irreducible summands are rational GL_r-modules. When $n = 0$, its summands are polynomial GL_r-modules. To compute the multiplicity of V^γ in

$T^{m,n}$, Stembridge [**Ste**] defines up-down staircase tableaux. If γ and ρ are r-staircases, then we say that $\gamma \subseteq \rho$ if $\gamma_i \leq \rho_i$ for each $i = 1, \dots, r$. An *(m,n)-up-down staircase tableaux of shape γ* is a sequence of r-staircases

$$\emptyset = \gamma^{(0)} \subseteq \gamma^{(1)} \subseteq \gamma^{(2)} \subseteq \cdots \subseteq \gamma^{(m+n)} = \gamma \tag{2.2}$$

such that for $1 \leq i \leq m$, the r-staircase $\gamma^{(i)}$ is obtained from $\gamma^{(i-1)}$ by adding a box, and for $m+1 \leq i \leq m = n$, the r-staircase $\gamma^{(i)}$ is obtained from $\gamma^{(i-1)}$ either by removing a box from $\gamma^{(i-1)^+}$ or by adding a box to $\gamma^{(i-1)^-}$. For example

$\emptyset$, 

is a $(4,2)$-up-down staircase tableaux of shape $\gamma = (3, 0, -1)$. Since at step i with $i > m$, we either add a box to $\gamma^{(i-1)^-}$ or remove a box from $\gamma^{(i-1)^+}$, the final staircase γ will always satisfy $\gamma^+ \vdash (m-k)$ and $\gamma^- \vdash (n-k)$ for some non-negative integer k. Thus we let Φ_r be the set of all r-staircases, and we let $\Phi_r^{m,n}$ be the set

$$\Phi_r^{m,n} \stackrel{\text{def}}{=} \{\gamma \in \Phi_r \mid \gamma^+ \vdash (m-k), \gamma^- \vdash (n-k), 0 \leq k \leq \min(m,n)\}. \tag{2.3}$$

For $\gamma \in \Phi_r^{m,n}$, let m_γ denote the number of (m,n) up-down staircases of shape γ. Stembridge proves the following theorem which gives the decomposition of $T^{m,n}$ into irreducible rational GL_r-modules.

Theorem 2.4 [**Ste**]. *The decomposition of $T^{m,n}$ into irreducible GL_r-modules is*

$$T^{m,n} \cong \bigoplus_{\gamma \in \Phi_r^{m,n}} m_\gamma V^\gamma.$$

The decomposition of Theorem 2.4 "stabilizes" when r is large. To see this, let $\gamma \in \Phi_r^{m,n}$ with $\gamma^+ \vdash (m-k)$ and $\gamma^- \vdash (n-k)$. If $\ell(\gamma^+) + \ell(\gamma^-) + k \leq r$, then Stembridge [**Ste**] proves that the multiplicity m_γ is given by the formula

$$m_\gamma = \frac{m!n!}{k!h(\gamma^+)h(\gamma^-)}, \tag{2.5}$$

where $h(\gamma^+)$ and $h(\gamma^-)$ are the hook formulas for γ^+ and γ^-, respectively (see [**Sa**]). In particular, if $r \geq m+n$, then (2.5) holds for all $\gamma \in \Phi_r^{m,n}$. Moreover, by removing $r - (m+n)$ zero rows in each γ, we can index the irreducibles by $(m+n)$-staircases. That is, we let

(2.6)
$$\Phi^{m,n} \stackrel{\text{def}}{=} \{[\mu, \nu]_{m+n} \mid \mu \vdash (m-h), \nu \vdash (n-h), 0 \leq k \leq \min(m,n)\}.$$

Then for each $r \geq m+n$, there is a bijection $\pi : \Phi_r^{m,n} \longrightarrow \Phi^{m,n}$ given by $[\mu,\nu]_r \mapsto [\mu,\nu]_{m+n}$. The set $\Phi^{m,n}$ indexes the irreducibles for all $r \geq m+n$, and the multiplicity of these irreducibles is fixed for all $r \geq m+n$. However, the dimension $\dim V^\gamma$ depends on r (see [**EK**] for a dimension formula).

2.1. Schur Functions. Let $x_r = \{x_1, x_2, \ldots, x_r\}$ be independent, commuting variables. For each partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ define the *Schur function* $s_\lambda(x_r) \in \mathbb{Z}[x_r]$ as

$$s_\lambda(x_r) = s_\lambda(x_1, \ldots, x_r) = \frac{\det(x_i^{\lambda_j+n-j})}{\det(x_i^{n-j})}. \tag{2.7}$$

Then the set $\{s_\lambda(x_r) | \ell(\lambda) \leq r\}$ form a $\mathbb{Z}$-basis of the ring of symmetric functions $\mathbb{Z}[x_r]^{S_r}$ (see [**Mac**]). Furthermore, Schur [**Sc1, Sc2**] proved that if $g \in GL_r$ has eigenvalues $e_1, \ldots, e_r$, then the character of the irreducible polynomial GL_r-representation corresponding to λ evaluated at g is given by $s_\lambda(e_1, \ldots, e_r)$.

The irreducible rational representations were classified by Schur [**Sc1, Sc2**]. He showed that they are of the form

$$\phi(g) = \det(g)^s \phi_\lambda(g), \qquad \text{for all } g \in GL_r, \tag{2.8}$$

for some $s \in \mathbb{Z}$ and some irreducible polynomial representation ϕ_λ of GL_r indexed by the partition partition λ with $\ell(\lambda) \leq r$. Not all of these representations are distinct. In fact, $(\det)^s\phi_\lambda$ is equivalent to $(\det)^t\phi_\mu$ if and only if $\lambda_i + s = \mu_i + t$ for each $i = 1, 2, \ldots, r$. We associate to $\gamma \in \Phi^{m,n}$ the irreducible GL_r-representation ϕ_γ given by

$$\phi_\gamma \overset{\text{def}}{=} (\det)^{\gamma_r - 1}\phi_{\lambda(\gamma)}, \tag{2.9}$$

where $\lambda(\gamma) = (\lambda_1, \cdots, \lambda_r)$ is the partition of length r defined by $\lambda_i = \gamma_i - \gamma_r + 1$.

Motivated by (2.9), Stembridge [**Ste**] and King [**Ki**] defined rational Schur functions which specialize to the characters of rational GL_r-representations. For each r-staircase γ, the *rational Schur function* s_γ is given by

$$s_\gamma(x_r) \overset{\text{def}}{=} (x_1, \ldots, x_r)^{\gamma_r - 1} s_{\lambda(\gamma)}(x_r) = x_r^{\gamma_r - 1} s_{\lambda(\gamma)}(x_r). \tag{2.10}$$

It follows immediately that if $g \in GL_r$ has eigenvalues $e_1, \ldots, e_r$, then the character of the irreducible rational GL_r-module corresponding to the r-staircase γ evaluated at g is given by $s_\gamma(e_1, \ldots, e_r)$.

2.2. Branching Rules. Let μ and ν be partitions with $\ell(\mu) \leq r$ and $\ell(\nu) \leq r$. The *Littlewood-Richardson coefficient* $c^{\lambda}_{\mu\nu}$ is defined by the following expression in $\mathbb{Z}[x_r]^{S_r}$:

$$s_\mu(x_r)s_\nu(x_r) = \sum_{\lambda,\ell(\lambda)\leq r} c^{\lambda}_{\mu\nu} s_\lambda(x_r). \tag{2.11}$$

In other words, the Littlewood-Richardson coefficients are the structure constants of $\mathbb{Z}[x_r]^{S_r}$ with respect to the basis of Schur functions. Moreover, the Schur functions are the characters of irreducible GL_r-modules, so equation (2.11) is equivalent to the branching rule

$$(V^\mu \otimes V^\nu)\big\downarrow^{GL_r\times GL_r}_{GL_r} \cong \bigoplus_{\lambda,\ell(\lambda)\leq r} c^{\lambda}_{\mu\nu} V^\lambda \tag{2.12}$$

of the irreducible $GL_r \times GL_r$-module $V^\mu \otimes V^\nu$ into irreducible GL_r-modules V^λ. We are considering $GL_r \subseteq GL_r \times GL_r$ by the diagonal embedding $g \mapsto (g,g)$.

The following theorem is the rational analog of (2.11) and its corollary is the rational analog of (2.12). Theorem 2.13 holds in $\mathbb{Z}[x_r^{\pm}]^{S_r}$ and is due independently to King [**Ki**] and Koike [**Koi**]. Notice that part (b) is the case where $\eta = \emptyset$ and $\tau = \emptyset$ in (a). Stroomer [**Str**] gives a different albeit equivalent description of this product. The definition of $[\mu,\nu]_r$ is given in (2.1).

Theorem 2.13 ([**Ki**],[**Koi**]). *Let $[\lambda,\eta]_r, [\tau,\pi]_r \in \Phi_r$, and let λ and π be partitions with $\ell(\lambda) \leq r$ and $\ell(\pi) \leq r$. Then*

(a) $$s_{[\lambda,\eta]_r}(x_r)s_{[\tau,\pi]_r}(x_r) = \sum_{[\mu,\nu]_r\in\Phi_r}\left(\sum_{\rho,\zeta,\theta,\kappa,\delta,\epsilon} c^{\lambda}_{\delta\rho}c^{\pi}_{\delta\zeta}c^{\eta}_{\epsilon\theta}c^{\tau}_{\epsilon\kappa}c^{\mu}_{\rho\kappa}c^{\nu}_{\zeta\theta}\right) s_{[\mu,\nu]_r}(x_r)$$

(b) $$s_\lambda(x_r)s_\pi(x_r^{-1}) = \sum_{[\mu,\nu]_r\in\Phi_r}\left(\sum_{\delta\vdash k} c^{\lambda}_{\delta\mu}c^{\pi}_{\delta\nu}\right) s_{[\mu,\nu]_r}(x_r).$$

Corollary 2.14. *Let $[\lambda,\eta]_r, [\tau,\pi]_r \in \Phi_r$ and let λ and π be partitions with $\ell(\lambda) \leq r$ and $\ell(\pi) \leq r$. Then*

(a) $$(V^{[\lambda,\eta]_r} \otimes V^{[\tau,\pi]_r})\big\downarrow^{GL_r\times GL_r}_{GL_r} \cong \bigoplus_{[\mu,\nu]_r\in\Phi_r}\left(\sum_{\rho,\zeta,\theta,\kappa,\delta,\epsilon} c^{\lambda}_{\delta\rho}c^{\pi}_{\delta\zeta}c^{\eta}_{\epsilon\theta}c^{\tau}_{\epsilon\kappa}c^{\mu}_{\rho\kappa}c^{\nu}_{\zeta\theta}\right) V^{[\mu,\nu]_r},$$

(b) $$(V^\lambda \otimes (V^\pi)^*)\big\downarrow^{GL_r\times GL_r}_{GL_r} \cong \bigoplus_{[\mu,\nu]_r\in\Phi_r}\left(\sum_{\delta\vdash k} c^{\lambda}_{\delta\mu}c^{\pi}_{\delta\nu}\right) V^{[\mu,\nu]_r},$$

where $(V^\pi)^*$ is the dual space of V^π.

3. The Brauer Algebras.

We consider *symmetric group* $\mathcal{S}_f$ to be the group of permutations on the set $\{1, \ldots, f\}$ and identify the element s_i with the transposition $(i\ i+1)$ that switches i and $i+1$. Irreducible $\mathcal{S}_f$-modules are indexed by $\mathcal{S}_f$-conjugacy classes which are labeled by partitions $\lambda \vdash f$. We denote them by S^λ. These are the well-known *Specht modules* (see [**Sa**]). If $T = \otimes^f U$ is an f-fold tensor product of the vector space U, then $\mathcal{S}_f$ acts on T by place permutation. That is, for $\sigma \in \mathcal{S}_f$ we have

$$\sigma \cdot (u_1 \otimes u_2 \otimes \cdots \otimes u_f) = u_{\sigma^{-1}(1)} \otimes u_{\sigma^{-1}(2)} \otimes \cdots \otimes u_{\sigma^{-1}(f)} \tag{3.1}$$

where $u_i \in U$ for $i = 1, \ldots, f$. If $U = V = \mathbb{C}^r$, the natural representation for GL_r, then it is easy to check that the action of $\mathcal{S}_f$ and GL_r on T^f commute. Let $\mathcal{E}_{T^f}(GL_r)$ denote the algebra generated by GL_r in $End_{\mathbb{C}}(T^f)$.

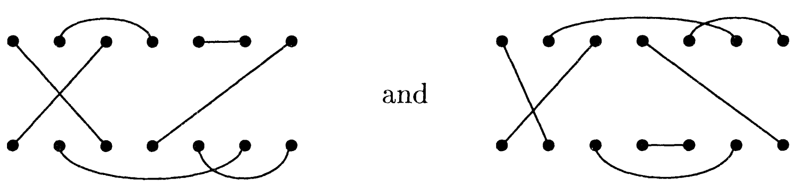
Theorem 3.2 [**Sc1, Sc2**]. *As a $\mathbb{C}[\mathcal{S}_f] \otimes \mathcal{E}_{T^f}(GL_r)$-bimodule, the decomposition of T^f into irreducibles is*

$$T^f \;\cong\; \bigoplus_{\substack{\lambda \vdash f \\ \ell(\lambda) \leq r}} S^\lambda \otimes V^\lambda.$$

Moreover, if $r \geq f$, then $\mathbb{C}[\mathcal{S}_f]$ and $\mathcal{E}_{T^f}(GL_r)$ are full centralizers of each other in $End_{\mathbb{C}}(T^f)$.

3.1. The Brauer Algebra.

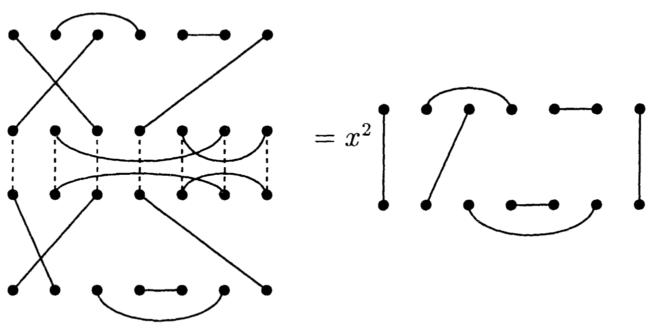
An *f-diagram* is a graph with $2f$ vertices and f edges such that each vertex is incident to precisely one edge. We view f-diagrams as having their vertices arranged in 2 rows of $m+n$ points, one above the other. We denote the set of vertices in the top row of diagram d by $t(d)$ and those in the bottom row of d by $b(d)$. An edge joining a vertex in $t(d)$ with a vertex in $b(d)$ is said to be *vertical*, while an edge connecting two vertices in the same row is said to be *horizontal*, and an edge that connects a vertex in $t(d)$ to the vertex immediately below it in $b(d)$ is said to be an *identity edge*. For example,



are 7-diagrams. We let $\mathcal{B}_f^x$ be the vector space spanned by the f-diagrams over the field of rational functions $\mathbb{C}(x)$ in the indeterminate x. To count the number of f-diagrams, observe that there are $2f-1$ possibilities for joining the first vertex to another, then $2f-3$ ways to join an unconnected vertex to another, and so forth. Thus

$$(3.3) \quad \dim_{\mathbb{C}(x)} \mathcal{B}_f^x = (2f-1)(2f-3)(2f-5)\cdots 5\cdot 3\cdot 1 \stackrel{\text{def}}{=} (2f)!!.$$

We multiply two f-diagrams d_1 and d_2 in the following way. Place d_1 directly above d_2 and connect the vertices in $b(d_1)$ to the corresponding vertices in $t(d_2)$. The resulting graph consists of f paths whose endpoints are in $t(d_1)\cup b(d_2)$ along with a certain number c of cycles which are adjacent to only vertices in the middle row. Let d be the f-diagram whose edges are the paths in this graph. Then the product of d_1 and d_2 is $d_1d_2 = x^c d$. For example the product of the 7-diagrams given above is



The product is extended linearly to $\mathcal{B}_f^x$. In general the product is not commutative, but $\mathcal{B}_f^x$ is an associative algebra whose identity is the diagram with only identity edges. The structure constants (1.14) for $\mathcal{B}_f^x$ are of the form x^c for non-negative integers c. Thus for each $\alpha\in\mathbb{C}$, we define the $\mathbb{C}$-algebra $\mathcal{B}_f^\alpha$ to be the $\mathbb{C}$ span of the f-diagrams with multiplication the same as in $\mathcal{B}_f^x$ except that each occurrence of x is replaced with α.

Richard Brauer [**Bra**] first introduced the Brauer algebra to study the centralizer of the action of the orthogonal group on tensor space. Let $O_r = O(r,\mathbb{C})$ denote the orthogonal group, which we view as a group of isometries with respect to a symmetric, nondegenerate bilinear form $b(.,.)$ on $V=\mathbb{C}^r$. That is,

$$(3.4) \qquad O_r = \{g\in GL_r \mid b(u,w) = b(g\cdot u, g\cdot w) \text{ for all } u,w\in V\}.$$

The action of O_r on V is precisely the restriction of the action of GL_r on V. Assume that $\{v_1^*,\ldots,v_f^*\}\subset V$ is the dual basis with respect to $b(.,.)$ so that $b(v_i,v_j^*)=\delta_{i,j}$. The action of GL_r on the dual space V^* when restricted to O_r is the same as the O_r-action if we identify v_i^* with v_i using the form

$b(.,.)$. For this reason, if we let $f = m + n$, then $T^{m,n} \cong T^{m+n} = T^f$ as a representation for O_r.

Brauer defined a representation $\phi : \mathcal{B}_f^r \longrightarrow End_{O_r}(T^f)$ of the Brauer algebra $\mathcal{B}_f^r$ onto the centralizer of O_r on T^f (see [**Bra**] or [**HW1**]). The homomorphism can be described explicitly on f-diagrams. Let d be an f-diagram, and define $\phi(d)$ to be the matrix whose $(\underline{i},\underline{j})$-entry for $\underline{i} = (i_1, i_2, \ldots, i_f)$ and $\underline{j} = (j_1, j_2, \ldots j_f)$ is determined by the following rules:

(1) Label the vertices in $t(d)$ from left to right with $v_{i_1}, v_{i_2}, \ldots, v_{i_f}$ and the vertices in $b(d)$ from left to right with $v^*_{j_1}, v^*_{j_2}, \ldots, v^*_{j_f}$.

(2) The $(\underline{i},\underline{j})$-entry of $\phi(d)$ is the product of the values of the bilinear form $b(u, w)$ over all the edges ϵ of d, where u and w are the labels on the vertices of ϵ.

Weyl [**Wey**] showed that ϕ is an isomorphism when $\lfloor r/2 \rfloor \geq f$, and Brown [**Bro1, Bro2**] proved that ϕ is an isomorphism whenever $r \geq f$. In particular, $\mathcal{B}_f^r$ is semisimple whenever $r \in \mathbb{Z}$ and $r \geq f$.

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ is a partition, then we define the conjugate partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_t)$ by $\lambda'_i = \mathrm{Card}\{j | \lambda_j \geq i\}$. Let $\mathcal{E}_{T^f}(O_r)$ denote the algebra generated by O_r in $End_{\mathbb{C}}(T^f)$. Weyl [**Wey**] proves that the irreducible O_r-modules are indexed by partitions μ with $\mu_1 + \mu_2 \leq r$.

Theorem 3.5 [**Wey**]. *The decomposition of T^f as a $\mathcal{B}_f^r \otimes \mathcal{E}_{T^f}(O_r)$-bimodule is*

$$T^f \cong \bigoplus_{k=0}^{\lfloor f/2 \rfloor} \bigoplus_{\substack{\mu \vdash (f-2k) \\ \lambda'_1 + \lambda'_2 \leq r}} \tilde{M}^\mu \otimes \tilde{V}^\mu, \tag{3.6}$$

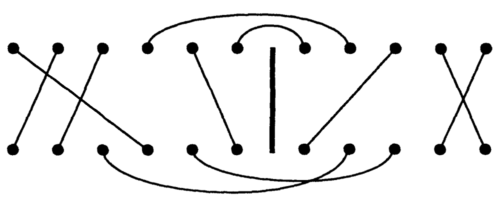
where $\tilde{M}^\mu$ is the irreducible $\mathcal{B}_f^r$-module and $\tilde{V}^\mu$ is the irreducible O_r-module corresponding to μ. Moreover, if $r \geq f$, then $\mathcal{B}_f^r$ and $\mathcal{E}_{T^f}(O_r)$ are full centralizers of each other in $End_{\mathbb{C}}(T^f)$.

In 1987, Hanlon and Wales [**HW1**] conjectured that $\mathcal{B}_f^\alpha$ is semisimple if α is not an integer. Wenzl [**Wen2**] proved in 1988 that $\mathcal{B}_f^\alpha$ is semisimple except for a finite number of $\alpha \in \mathbb{Z}$ with $-f + 1 \leq \alpha \leq f - 1$. It remains an open question to determine exactly which integral values of α cause $\mathcal{B}_f^\alpha$ to fail to be semisimple. Hanlon and Wales [**HW2**] give a tower construction of the radical of $\mathcal{B}_f^\alpha$ in low-rank cases.

3.2. The Brauer Subalgebra $\mathcal{B}_{m,n}^x$. Since $T^{m,n} \cong T^{m+n}$ as O_r-modules, we have $End_{GL_r}(T^{m,n}) \subseteq End_{O_r}(T^{m,n})$. Thus, we should find a copy of the centralizer algebra $End_{GL_r}(T^{m,n})$ inside the Brauer algebra $\mathcal{B}_{m+n}^r$. This observation motivated [**BCHLLS**] to define the subalgebra $\mathcal{B}_{m,n}^r$ of $\mathcal{B}_{m+n}^r$

which maps onto $End_{GL_r}(T^{m,n})$. Koike **[Koi]** independently described the centralizer of GL_r on $T^{m,n}$ in different terms.

An (m, n)-*diagram* is an $(m+n)$-diagram with a vertical wall between the mth and $(m+1)$st vertices such that vertical edges never cross the wall and horizontal edges always begin and end on opposite sides of the wall. We let $t_i^L(d)$ and $t_j^R(d)$ denote the ith and jth vertices in $t(d)$ on the right and left side of the wall, respectively, and $b_i^L(d)$ and $b_j^R(d)$ denote the ith and jth vertices in $b(d)$ on the right and left side of the wall respectively. We number the vertices on the left side of the wall from left to right with $1, \dots, m$ and those on the right side of the wall from left to right with $1, \dots, n$. The following is an example of a $(6, 5)$-diagram:



Let $\mathcal{D}_{m,n}$ be the set of all (m, n)-diagrams, and let $\mathcal{B}^x_{m,n}$ be the $\mathbb{C}(x)$-span of $\mathcal{D}_{m,n}$. It is not hard to check that $\mathcal{B}^x_{m,n}$ is closed under the multiplication of (m, n)-diagrams and is thus a subalgebra of $\mathcal{B}^x_{m+n}$. If $\alpha \in \mathbb{C}$, then $\mathcal{B}^\alpha_{m,n}$ is the subalgebra of the $\mathbb{C}$-algebra $\mathcal{B}^\alpha_{m+n}$.

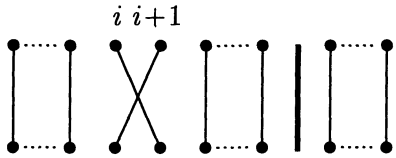
The dimension of $\mathcal{B}^x_{m,n}$ is obtained by counting the diagrams with k horizontal edges in each row and then summing over k. Thus,

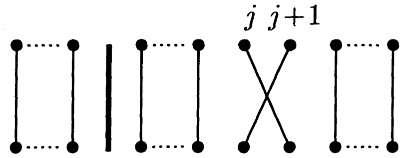
(3.7)

$$\begin{aligned}\dim_{\mathbb{C}(x)} \mathcal{B}^x_{m,n} \;=\; |\mathcal{D}_{m,n}| &= \sum_{k=0}^{min(m,n)} \left(\binom{m}{k}\binom{n}{k}k!\right)^2 (m-k)!(n-k)! \\ &= m!n! \sum_{k=0}^{min(m,n)} \binom{m}{k}\binom{n}{n-k} \\ &= m!n!\binom{m+n}{n} \;=\; (m+n)!.\end{aligned}$$

The fourth equality is proved by counting the occurrence of the monomial x^n in the product $(1+x)^{m+n} = (1+x)^m(1+x)^n$. Another way of counting the diagrams is to flip the part of the (m, n)-diagram d that is to the right of the wall over its horizontal axis without disconnecting any edges. Then each vertex of $t(d)$ is connected to a vertex in $b(d)$. Moreover, any $(m+n)$-diagram having no horizontal edges can be "flipped" in this way to obtain an (m, n)-diagram. There are clearly $(m+n)!$ such diagrams.

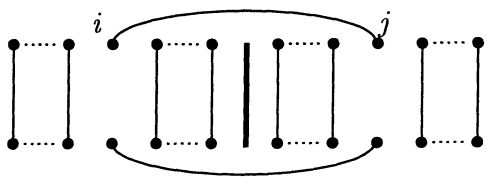
There is a natural embedding of the group algebra $\mathbb{C}[\mathcal{S}_m \times \mathcal{S}_n]$ in $\mathcal{B}^x_{m,n}$ where the simple transpositions $s_i \in \mathcal{S}_m$ and $s^*_j \in \mathcal{S}_n$ correspond to the diagrams

$s_i =$ 

$s_j =$ 

In general, the permutation $\pi \in \mathcal{S}_m \times \mathcal{S}_n$ is associated to the (m,n)-diagram d_π with the property that $b^L_i(d_\pi)$ is connected to $t^L_{\pi(i)}(d_\pi)$ and $b^R_j(d_\pi)$ is connected to $t^R_{\pi(j)}(d_\pi)$. Notice that it is exactly the diagrams in $\mathcal{B}^x_{m,n}$ with no horizontal edges that correspond to $\mathcal{S}_m \times \mathcal{S}_n$.

For $1 \leq i \leq m$ and $1 \leq j \leq n$ let $e_{i,j}$ denote the diagram

$e_{ij} =$ 

and let $e = e_{m,1}$. It is not hard to check (see [**Hal**]) that the set of diagrams $\{s_i, s^*_j \mid 1 \leq i \leq m-1, 1 \leq j \leq n-1\} \cup \{e\}$ generates all of $\mathcal{B}^x_{m,n}$. In Section 4, we give a presentation of $\mathcal{B}^r_{m,n}$ on these generators subject to a set of relations.

If we restrict the representation $\phi : \mathcal{B}^r_{m+n} \longrightarrow End_{O_r}(T^{m,n})$ to the subalgebra $\mathcal{B}^r_{m,n}$, we get a representation of $\mathcal{B}^r_{m,n}$ on $T^{m,n}$. Under this representation, the diagrams of $\mathcal{S}_m \times \mathcal{S}_n$ act on simple tensors of $T^{m,n}$ by place permutation. That is if $(\sigma,\tau) \in \mathcal{S}_m \times \mathcal{S}_n$, $t \in \otimes^m V$, and $u \in \otimes^n V^*$, then $(\sigma,\tau)\cdot t \otimes u = \sigma \cdot t \otimes \sigma \otimes u$ where $\sigma \otimes t$ and $\tau \otimes u$ are given by (3.2) The simple tensors $\underline{v} = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_m} \otimes v^*_{j_1} \otimes v^*_{j_2} \otimes \cdots \otimes v^*_{j_n}$ form a basis of $T^{m,n}$, and the action of the diagram e on $\underline{v}$ is

$$(3.8) \qquad e \cdot \underline{v} \;=\; \delta_{i_m,j_1} \sum_{k=1}^{r} v_{i_1} \otimes \cdots \otimes v_{i_{m-1}} \otimes v_k \otimes v^*_k \otimes v^*_{j_2} \cdots \otimes v^*_{j_n}.$$

The transformation $\underline{v} \mapsto e \cdot \underline{v}$ is called a *contraction map*. Koike [**Koi**] proves that the action of the group $\mathcal{S}_m \times \mathcal{S}_n$ together with the contraction map generate all of $End_{GL_r}(T^{m,n})$. Using this fact, [**BCHLLS**] obtains the following theorem. We let $\mathcal{E}_{T^{m,n}}(GL_r)$ denote the algebra generated by GL_r in $End_{\mathbb{C}}(T^{m,n})$.

Theorem 3.9 [**BCHLLS**]. *The map $\phi : \mathcal{B}^r_{m,n} \longrightarrow End_{\mathbb{C}}(T^{m,n})$ maps $\mathcal{B}^r_{m,n}$ onto $End_{GL_r}(T^{m,n})$ for all r and is an isomorphism when $r \geq m+n$. Thus, when $r \geq m+n$, the algebras $\mathcal{E}_{T^{m,n}}(GL_r)$ and $\mathcal{B}^r_{m,n}$ are full centralizers of each other in $End_{\mathbb{C}}(T^{m,n})$ and $\mathcal{B}^r_{m,n}$ is semisimple.*

Since $\mathcal{B}^r_{m,n}$ maps onto $End_{GL_r}(T^{m,n})$ for all r, the set $\Phi^{m,n}_r$ indexes irreducible $\mathcal{B}^r_{m,n}$-modules, and GL_r and $\mathcal{B}^r_{m,n}$ are in Schur-Weyl duality on $T^{m,n}$. That is

Theorem 3.10. *The decomposition of $T^{m,n}$ as a $\mathcal{B}^r_{m,n} \otimes \mathcal{E}_{T^{m,n}}(GL_r)$-bimodule is*

$$T^{m,n} \cong \bigoplus_{\gamma \in \Phi^{m,n}_r} M^\gamma \otimes V^\gamma,$$

where M^γ is the irreducible $\mathcal{B}^r_{m,n}$-module labeled by γ and $\dim M^\gamma = m_\gamma$.

It follows that the decomposition of $T^{m,n}$ as a module for $\mathcal{B}^r_{m,n}$ is given by

$$T^{m,n} \cong \bigoplus_{\gamma \in \Phi^{m,n}_r} d_\gamma M^\gamma \tag{3.11}$$

where $d_\gamma = \dim V^\gamma$, and $\dim M^\gamma = m_\gamma$ (see (2.5)). If $r \geq m+n$, then $T^{m,n}$ is a faithful $\mathcal{B}^r_{m,n}$-module, the set $\{M^\gamma \mid \gamma \in \Phi^{m,n}_r\}$ is a complete set of irreducible $\mathcal{B}^r_{m,n}$-modules, and

$$\mathcal{B}^r_{m,n} \cong \bigoplus_{\gamma \in \Phi^{m,n}_r} \mathfrak{M}_{m_\gamma}(\mathbb{C}), \tag{3.12}$$

is the matrix decomposition of $\mathcal{B}^r_{m,n}$. The modules M^γ are explicitly constructed in [**BCHLLS**]. When $r \geq m+n$, the irreducible $\mathcal{B}^r_{m,n}$-modules can be denoted by $(m+n)$-staircases in $\Phi^{m,n}$. Thus for $\gamma \in \Phi^{m,n}$, we let $M^\gamma(x) = M^\gamma \otimes \mathbb{C}(x)$. Then the set $\{M^\gamma(x) \mid \gamma \in \Phi^{m,n}\}$ is a complete set of irreducible $\mathcal{B}^x_{m,n}$-modules (see [**BCHLLS**]), and

$$\mathcal{B}^x_{m,n} \cong \bigoplus_{\gamma \in \Phi^{m,n}} \mathfrak{M}_{m_\gamma}(\mathbb{C}(x)) \tag{3.13}$$

is the matrix decomposition of $\mathcal{B}^x_{m,n}$. In particular, when $r \geq m+n$, $\mathcal{B}^r_{m,n}$ and $\mathcal{B}^x_{m,n}$ have the same matrix decomposition.

By Theorem 1.7, the branching rule for $\mathbb{C}[\mathcal{S}_m \times \mathcal{S}_n] \subseteq \mathcal{B}^r_{m,n}$ is the same as for $GL_r \subseteq GL_r \times GL_r$, given in Corollary 2.14(b). As a corollary of Theorem 7.19, we will extend the branching rules to $\mathbb{C}[\mathcal{S}_m \times \mathcal{S}_n] \subseteq \mathcal{B}^x_{m,n}$.

Theorem 3.14. *Let $\gamma \in \Phi^{m,n}$. Then*

$$M^\gamma \big\downarrow^{\mathcal{B}^r_{m,n}}_{\mathbb{C}[\mathcal{S}_m \times \mathcal{S}_n]} \cong \bigoplus_{\substack{\alpha \vdash m \\ \beta \vdash n}} \left(\sum_{\delta \vdash k} (c^\alpha_{\delta\gamma^+})(c^\beta_{\delta\gamma^-}) \right) S^\alpha \otimes S^\beta.$$

If λ and π are partitions, then we say that $\lambda/\pi = \Box$ if $\pi \subseteq \lambda$ and $|\lambda| - |\pi| = 1$. That is, the diagram of π is obtained from the diagram of λ by deleting one box in such a way that π remains a partition. We say that $[\mu,\nu]_r/[\alpha,\beta]_r = \Box$ if one of the following hold:

(a) $\mu/\alpha = \Box$ and $\nu = \beta$, or

(b) $\mu = \alpha$ and $\beta/\nu = \Box$.

In other words, $[\mu,\nu]_r/[\alpha,\beta]_r = \Box$ if the diagram of $[\alpha,\beta]_r$ is obtained from the diagram of $[\mu,\nu]_r$ either (a) by deleting a box from μ and fixing ν or (b) by adding a box to ν and fixing μ.

Theorem 3.16. *Let $[\mu,\nu]_r \in \Phi^{m,n}$. Then*

$$M^{[\mu,\nu]_r}\downarrow^{\mathcal{B}^r_{m,n}}_{\mathcal{B}^r_{m-1,n}} \cong \bigoplus_{\substack{[\alpha,\beta]_r \in \Phi^{m-1,n}_r \\ [\mu,\nu]_r/[\alpha,\beta]_r = \Box}} M^{[\alpha,\beta]_r},$$

and

$$M^{[\mu,\nu]_r}\downarrow^{\mathcal{B}^r_{m,n}}_{\mathcal{B}^r_{m,n-1}} \cong \bigoplus_{\substack{[\alpha,\beta]_r \in \Phi^{m,n-1}_r \\ [\alpha,\beta]_r/[\mu,\nu]_r = \Box}} M^{[\alpha,\beta]_r}.$$

Proof. By (1.9), the branching rules for $\mathcal{B}^r_{m-1,n} \subseteq \mathcal{B}^r_{m,n}$ are the same as those for the diagonal embedding $GL_r \subseteq GL_r \times GL_r$ on the tensor product space $T^{m-1,n} \otimes V$. Since V is the natural representation for GL_r, its highest weight is ϵ_1, and we denote it by $V^{[\omega_1,\emptyset]_r}$, where $\omega_1 = (1)$, the unique partition of 1. By Corollary 2.14(a), we have

$$(V^{[\alpha,\beta]_r} \otimes V^{[\omega_1,\emptyset]_r})\downarrow^{GL_r \times GL_r}_{GL_r} \cong \bigoplus_{[\mu,\nu] \in \Phi^{m,n}_r} \left(\sum_{\rho,\zeta,\theta,\kappa,\delta,\epsilon} c^{\alpha}_{\delta\rho} c^{\emptyset}_{\delta\zeta} c^{\beta}_{\epsilon\theta} c^{\omega_1}_{\epsilon\kappa} c^{\mu}_{\rho\kappa} c^{\nu}_{\zeta\theta} \right) V^{[\mu,\nu]_r}.$$

Now $c^{\lambda}_{\lambda\emptyset} = c^{\lambda}_{\emptyset\lambda} = 1$ and $c^{\lambda}_{\mu\nu} = 0$ unless $\mu \subseteq \lambda$ and $\nu \subseteq \lambda$ (see [**Sa**]), so we must have $\delta = \zeta = \emptyset$. This, in turn, forces $\rho = \alpha$, and so $c^{\alpha}_{\delta\rho} = 1$, $\theta = \nu$, and $c^{\nu}_{\zeta\theta} = 1$. The multiplicity of $V^{[\mu,\nu]_r}$ is then $\sum_{\kappa,\epsilon} c^{\beta}_{\epsilon\nu} c^{\omega_1}_{\epsilon\kappa} c^{\mu}_{\alpha\kappa}$. Moreover, $c^{\omega_1}_{\epsilon\kappa} = 0$ unless $\epsilon = \emptyset$ and $\kappa = \omega_1$ or $\epsilon = \omega_1$ and $\kappa = \emptyset$. In the first case, the multiplicity of $V^{[\mu,\nu]_r}$ becomes $c^{\beta}_{\emptyset\nu} c^{\mu}_{\alpha\omega_1}$, so we must have $\beta = \nu$ and $\alpha \subseteq \mu$ with $|\mu| = |\alpha| = 1$. In the second case, the multiplicity becomes $c^{\beta}_{\omega_1\nu} c^{\mu}_{\alpha\emptyset}$, forcing $\alpha = \mu$ and $\nu \subseteq \beta$ with $|\beta| = |\nu| = 1$. Thus the branching rule is proved for $\mathcal{B}^r_{m-1,n} \subseteq \mathcal{B}^r_{m,n}$. For $\mathcal{B}^r_{m,n-1} \subseteq \mathcal{B}^r_{m,n}$, consider the multiplicity of $V^{[\mu,\nu]_r}$ in $V^{[\alpha,\beta]_r} \otimes V^{[\emptyset,\omega_1]_r}$ and proceed similarly. $\Box$

4. The Two-Parameter Iwahori-Hecke Algebras.

Let q be an indeterminate, and let $\mathbb{C}(q)$ denote the field of rational functions. The Iwahori-Hecke algebra (of type A), denoted $H_f(q)$, is the $\mathbb{C}(q)$-algebra generated by $1, g_1, \dots, g_{f-1}$ subject to the relations

(B1) $\quad g_ig_j = g_jg_i, \quad$ if $|i-j| > 1$,

(B2) $\quad g_{i+1}g_ig_{i+1} = g_1g_{i+1}g_i$,

(IH) $\quad {g_i}^2 = (q-1)g_i + q$.

Upon specializing $q = 1$, relation (IH) becomes the Coxeter relation $g_i^2 = 1$ for the symmetric group, and we get $H_f(1) \cong \mathbb{C}[\mathcal{S}_f]$. The irreducible representations of $H_f(q)$ are indexed by partitions $\lambda \vdash m$ and denoted S_q^λ. The algebras $H_f(q)$ and $\mathbb{C}[\mathcal{S}_f]$ are semisimple and have the same decompositions into matrix algebras over $\mathbb{C}(q)$ and $\mathbb{C}$, respectively.

Let $r \in \mathbb{Z}^+ \cup \{0\}$, and let $[\![r]\!]_q$ be the Gauss polynomial given by $[\![0]\!]_q = 0$, $[\![1]\!]_q = 1$, and

$$[\![r]\!]_q = \frac{1-q^r}{1-q} = q^{r-1} + q^{r-2} + \cdots + q + 1. \tag{4.1}$$

Define $H^r_{m,n}(q)$ to be the $\mathbb{C}(q)$-algebra generated by

$$1, g_1, \dots, g_{m-1}, e, g_1^*, \dots, g_{n-1}^*$$

subject to the relations

(B1) $g_ig_j = g_jg_i$, if $|i-j| > 1$

(B2) $g_{i+1}g_ig_{i+1} = g_ig_{i+1}g_i$

(IH) ${g_i}^2 = (q-1)g_i + q$

(B1*) $g_i^*g_j^* = g_j^*g_i^*$, if $|i-j| > 1$

(B2*) $g_{i+1}^*g_i^*g_{i+1}^* = g_i^*g_{i+1}^*g_i^*$

(IH*) $g_i^{*2} = (q-1)g_i^* + q$

(HH) $g_ig_j^* = g_j^*g_i$

(K1) $eg_i = g_ie$, for $1 \le i \le m-2$

(K1*) $eg_j^* = g_j^*e$, for $2 \le j \le n-1$

(K2) $eg_{m-1}e = q^re$

(K2*) $eg_1^*e = q^re$

(K3) $e^2 = [\![r]\!]_qe$

(K4) $g_{m-1}g_1^{*-1}eg_{m-1}^{-1}g_1^*e = eg_{m-1}^{-1}g_1^*e = eg_{m-1}g_1^{*-1}eg_{m-1}^{-1}g_1^*$.

If $m = 0$ or $n = 0$, then we omit the generator e and its corresponding relations.

If $q_o \in \mathbb{C} \setminus \{0\}$, then let $[\![r]\!]_{q_o} \stackrel{\text{def}}{=} {q_o}^{r-1} + {q_o}^{r-2} + \cdots + q_o + 1$. There exists a basis for $H^r_{m,n}(q)$ consisting of monomials in the generators (see Proposition 5.4) whose structure constants are well-defined for $q_o \in \mathbb{C} \setminus \{0\}$, so we can define the $\mathbb{C}$-algebra $H^r_{m,n}(q_o)$. Multiplying relations (IH) and (IH*) by g_i^{-1} and g_j^{*-1}, respectively, gives

$$(4.2)\quad g_i^{-1} = q^{-1}g_i + (q^{-1}-1) \quad \text{and} \quad g_j^{*-1} = q^{-1}g_j^* + (q^{-1}-1).$$

The subalgebra of $H^r_{m,n}(q)$ generated by $1, g_1, \dots, g_{m-1}$ satisfies the Iwahori-Hecke algebra relations (B1), (B2), and (IH) and is isomorphic to $H_m(q)$. The subalgebra of $H^r_{m,n}(q)$ generated by $1, g_1^*, \dots, g_{n-1}^*$ satisfies the Iwahori-Hecke algebra relations (B1*), (B2*), and (IH*) and is isomorphic to $H_n(q)$. Moreover, g_i commutes with g_j^*, so we have the embedding

$$(4.3)\qquad H_m(q) \otimes H_n(q) \subseteq H^r_{m,n}(q).$$

In Corollary 7.23 we give the branching rule for this containment.

Kosuda [**Kos**] originally presented the $H^r_{m,n}(q)$ on the set of generators $1, T_1, \dots, T_{m-1}, E, T_1^*, \dots, T_{n-1}^*$, where $T_i = q^{-1/2}g_i$, $T_j^* = q^{-1/2}g_j^*$, and $E = q^{1/2(1-r)}e$. With these identifications, the relations of $H^r_{m,n}(q)$ become Kosuda's relations with parameter $q^{1/2}$, so the two presentations are equivalent. The algebra $H^r_{m,n}(q)$ has been generalized by Leduc [**Le**] to a two-parameter algebra $\mathcal{A}_{m,n}(z,q)$. Specializing $x = q^r$ in $\mathcal{A}_{m,n}(z,q)$ gives $H^r_{m,n}(q)$. Many of the results of this paper carry over immediately to $\mathcal{A}_{m,n}(z,q)$.

For $\gamma \in \Phi_r^{m,n}$, let $M_q^\gamma = M^\gamma \otimes \mathbb{C}(q)$, where M^γ is the irreducible $\mathcal{B}^r_{m,n}$-module of Section 3. For $r \geq m+n$, Kosuda defines an action of $H^r_{m,n}(q)$ on M_q^γ that is a "q-extension" of the action of $\mathcal{B}^r_{m,n}$ on M^γ in the sense that when $q=1$ we get the action of $\mathcal{B}^r_{m,n}$ on M^γ. The following theorem is due to Kosuda.

Theorem 4.4 [**Kos**].
(a) $\dim H^r_{m,n}(q) \leq (m+n)!$.
(b) *If $r \geq m+n$, then $\dim H^r_{m,n}(q) = (m+n)!$, and $\{M_q^\gamma \mid \gamma \in \Phi_r^{m,n}\}$ is a complete set of irreducible $H^r_{m,n}(q)$-modules. In particular, $H^r_{m,n}(q)$ is semisimple and its decomposition into full matrix algebras is given by*

$$H^r_{m,n}(q) \cong \bigoplus_{\gamma \in \Phi_r^{m,n}} \mathfrak{M}_{m_\gamma}(\mathbb{C}(q)).$$

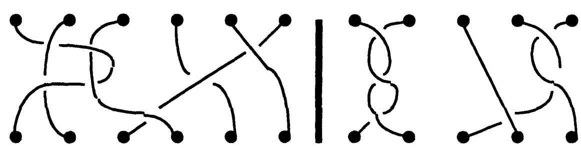
Comparing the matrix decompositions of $H^r_{m,n}(q)$ and $\mathcal{B}^r_{m,n}$ (see (3.12)) gives the following corollary.

Corollary 4.5. *If $r \geq m+n$, then the $\mathbb{C}$-algebras $H^r_{m,n}(1)$ and $\mathcal{B}^r_{m,n}$ are isomorphic.*

L.H. Kauffman [**Ka**] gives a diagrammatic context for the Birman-Wenzl algebra $BW_f(z,q)$—which is a q-deformation of the Brauer algebra $\mathcal{B}_f^r$ when

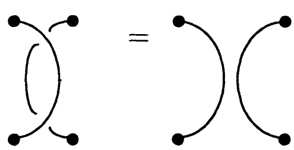
$z = q^r$ (see [**Wen3**]). We use his techniques to give a description of $H^r_{m,n}(q)$ in terms of q-diagrams.

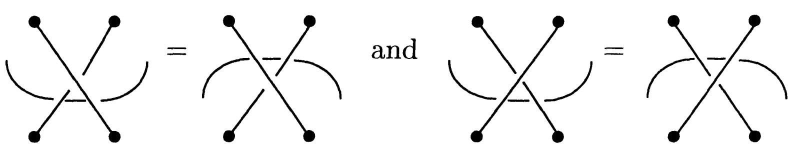
An f*-braid* is viewed as two rows of f vertices, one above the other, and f strands connecting each vertex in the top row with a vertex in the bottom row in such a way that each vertex is incident to precisely one strand. Strands cross over and under each other in three-space as they pass from the top row to the bottom row but are not allowed to cross themselves. An (m,n)*-braid* is an $(m+n)$-braid with a wall between the mth and $(m+1)$st vertices such that strands never cross the wall. We number the vertices from left to right in each row with $1, \ldots, m$ left of the wall and $1, \ldots, n$ right of the wall. For example



is a $(6,6)$-braid.

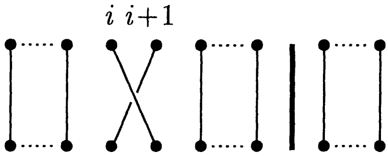
The Reidemeister moves of types II and III are (see [**Ka**] for details):

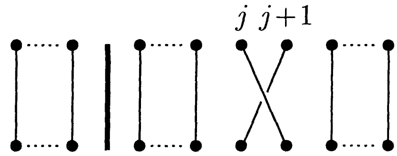
II. 

III. 

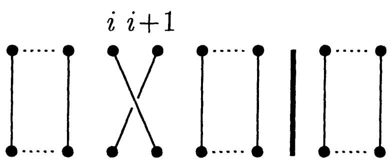
We can apply these "moves" to braids by isolating one of these pictures in an open disk in a braid diagram and applying the relations. When we apply these moves we always keep the strands connected to the vertices and keep the vertices fixed. The Reidemeister moves give an equivalence relation among braids known as regular isotopy. We take braids to be their equivalence classes up to regular isotopy and multiply braids b_1 and b_2 using the *concatenation product* given by identifying the top row of b_2 with the bottom row of b_1 and then re-scaling the result to obtain a new braid $b_1 b_2$. The concatenation product is associative and makes the set of all f-braids B_f a group called the braid group. The set of all (m,n)-braids with this product generates the subgroup $B_m \times B_n$ of B_{m+n}.

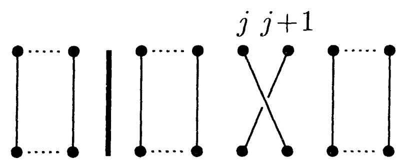
For $1 \le i \le m-1$ and $1 \le j \le n-1$, let σ_i and σ_j^* denote the (m,n)-braids given by

$\sigma_i =$ 

$\sigma_j =$ 

It follows from the second Reidemeister move that

$\sigma_i^{-1} =$ 

$\sigma_j^{-1} =$ 

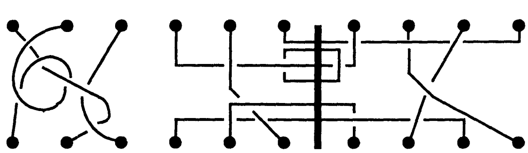
Moreover, it is well-known that the braid group $B_m \times B_n$ is generated by $1, \sigma_1, \dots, \sigma_{m-1}, \sigma_1^*, \dots, \sigma_{n-1}^*$ subject to the relations

(B1) $\sigma_i\sigma_j = \sigma_j\sigma_i$, if $|i-j| > 1$, (B1*) $\sigma_i^*\sigma_j^* = \sigma_j^*\sigma_i^*$, if $|i-j| > 1$,

(B2) $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$, (B2*) $\sigma_{i+1}^*\sigma_i^*\sigma_{i+1}^* = \sigma_i^*\sigma_{i+1}^*\sigma_i^*$.

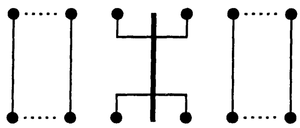
The braid relations (B2) and (B2*) follow from the third Reidemeister move.

To define a q-extension of $\mathcal{B}_{m,n}^r$ we need to include horizontal edges. Thus, we follow [**Ka**] and say that an *f-tangle* consists of two rows of f vertices one above the other, and f strands connecting each vertex with another in such a way that each vertex is incident to precisely one strand. We no longer insist that strands travel from top to bottom, but again, strands cross over and under each other as they pass from one vertex to another. Vertical strands are those that travel from top to bottom, and horizontal strands are those that connect vertices in the same row. An *(m,n)-tangle* is an $(m+n)$-tangle with a wall between the mth and $(m+1)$st vertices such that horizontal strands never cross the wall and vertical strands always connect vertices on opposite sides of the wall. Since the concatenation product of tangles with horizontal strands can create cycles and self-crossing edges, we allow a tangle to contain arbitrarily many closed cycles and curls. For example,



is a $(6,6)$-tangle. We take tangles to be their equivalence classes up to regular isotopy. The set of all f-tangles is denoted $\mathfrak{T}_f$ and the set of all (m,n)-tangles is denoted $\mathfrak{T}_{m,n}$. Tangles with horizontal edges are not invertible under the concatenation product, but the concatenation product makes $\mathfrak{T}_f$ and $\mathfrak{T}_{m,n}$ into monoids which we call the *tangle monoids*.

Let h denote the following (m,n)-tangle which Kauffman refers to as a *hook*:

$h =$ 

The *braid monoid* $M_{m,n}$ is the monoid generated in $\mathfrak{T}_{m,n}$ by

$$1, \sigma_1, \cdots, \sigma_{m-1}, h, \sigma_1^*, \ldots, \sigma_{n-1}^*.$$

For $m+n \geq 3$, the braid monoid $M_{m,n}$ does not contain all (m,n)-tangles (see [**Ka**]), and so when $m+n \geq 3$, we have the proper containments

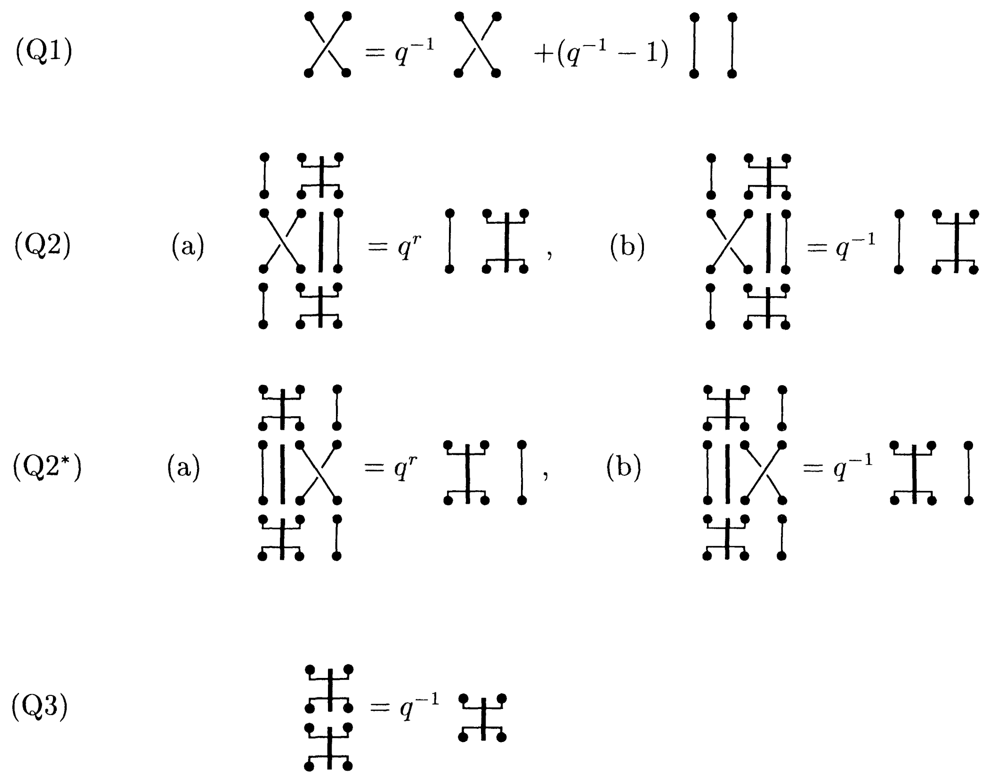
$$B_m \times B_n \subset M_{m,n} \subset \mathfrak{T}_{m,n}. \tag{4.6}$$

We associate to each of these monoids an algebra of diagrams as follows. Let $r \in \mathbb{Z}^+ \cup \{0\}$, and let $\mathcal{A}\mathfrak{T}_{m,n}$ denote the free associative algebra generated by $\mathfrak{T}_{m,n}$ over $\mathbb{C}(q)$ subject to the relations:

$$\begin{array}{llll}
\text{(Q1)} & {\sigma_i}^{-1} = q^{-1}\sigma_i + (q^{-1}-1)\mathrm{id}, & \text{(Q1}^*\text{)} & \sigma_j^{*-1} = q^{-1}\sigma_i^* + (q^{-1}-1)\mathrm{id}, \\
\text{(Q2)} & \text{(a) } h\sigma_{m-1}h = q^r h, & \text{(Q2}^*\text{)} & \text{(a) } h\sigma_1^* h = q^r h, \\
 & \text{(b) } h\sigma_{m-1}^{-1}h = q^{-1}h, & & \text{(b) } h\sigma_1^{*-1}h = q^{-1}h, \\
\text{(Q3)} & h^2 = [\![r]\!]_q h. & &
\end{array}$$

In terms of tangles, these relations give the tangle identities given in Figure 4.7.

Figure 4.7. Tangle Identities.



Like the Reidemeister moves, the tangle identities relate diagrams which differ in small open disks by the given relation and are the same outside the disk. Identity $(Q1)$ allows us to change over-crossings to under-crossings. The identities of $(Q2)$ allow us to remove *curls.* We will say that the curls in (Q2)(a) have positive orientation and are removed with a penalty of q^r. Those in (Q2)(b) have negative orientation and are removed with a penalty of q^{-1}. The *unknotted simple loop* in $(Q3)$ is removed with a penalty of $[\![r]\!]_q$.

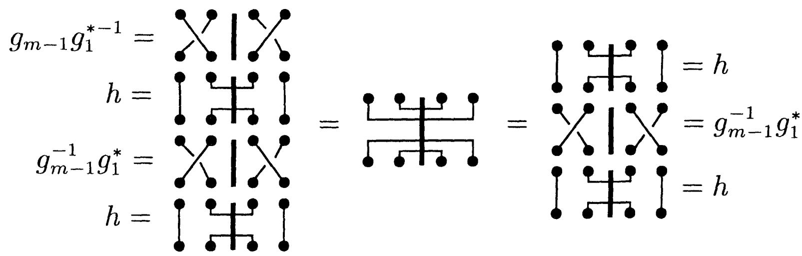
We refer to the images of the tangles in $\mathcal{A}\mathfrak{T}_{m,n}$ as *q-diagrams*, or sometimes, *$(m,n;q)$-diagrams*. Let $\mathcal{A}M_{m,n}$ denote the restriction of $\mathcal{A}\mathfrak{T}_{m,n}$ to tangles in the braid monoid $M_{m,n}$, and let $\mathcal{A}(B_m \times B_n)$ denote the restriction of $\mathcal{A}\mathfrak{T}_{m,n}$ to the braid group.

Theorem 4.8. *There exists a surjective homomorphism* $\pi : H^r_{m,n}(q) \longrightarrow \mathcal{A}M_{m,n}$ *given by* $\pi(g_i) = \sigma_i$, $\pi(g_j^*) = \sigma_j^*$, *and* $\pi(e) = h$.

Proof. Let $F_{m,n}$ denote the free associative $\mathbb{C}(q)$-algebra generated by

$$1, g_1, \dots, g_{m-1}, e, g_1^*, \dots, g_{n-1}^*,$$

and define a homomorphism $\pi : F_{m,n} \longrightarrow \mathcal{A}M_{m,n}$ by $\pi(g_i) = \sigma_i$, $\pi(g_j^*) = \sigma_j^*$, and $\pi(e) = h$. Let $I_{m,n}$ be the ideal of $F_{m,n}$ generated by the $H_{m,n}^r(q)$-relations. Then the theorem is proved by showing that $I_{m,n}$ is in the kernel of π, or, equivalently, that σ_i, σ_j, and h satisfy the $H_{m,n}^r(q)$ relations. The commutativity relations (B1), (B1*), (HH), (K1), and (K1*) are topological moves in $\mathfrak{T}_{m,n}$ that trivially preserve the configuration of the crossings. The braid relations (B2), and (B2*) follow from the third Reidemeister move. Relations (IH) and (IH*) are equivalent to $(Q1)$. Relations (K2) and (K2*) are equivalent to $(Q2)$ and $(Q2^*)$, respectively, and relation (K3) is equivalent to $(Q3)$. One verifies that (K4) is satisfied by multiplying q-diagrams and using the second Reidemeister move. We show the first equality in (K4) here:



□

By specializing $q = 1$, we show that the homomorphism in Theorem 4.8 is an isomorphism. If we let $q = 1$ in the tangle identities of Figure 4.7, then $(Q1)$ identifies each braid generator with its inverse, $(Q2)$ removes curls, and $(Q3)$ becomes $h^2 = rh$. Kauffman [**Ka**] proves that modulo these relations, the tangles become Brauer diagrams (i.e., $(m+n)$-diagrams) and the multiplication is exactly the multiplication given in Section 3. It follows that upon setting $q = 1$, we have $\mathcal{A}M_{m,n} \cong \mathcal{B}_{m,n}^r$, as $\mathbb{C}$-algebras, and we conclude that

$$\dim_{\mathbb{C}(q)} \mathcal{A}M_{m,n} \geq \dim_{\mathbb{C}} \mathcal{B}_{m,n}^r = (m+n)! \tag{4.9}$$

Since, $\dim_{\mathbb{C}(q)} H_{m,n}^r(q) \leq (m+n)!$ and $H_{m,n}^r(q)$ maps onto $\mathcal{A}M_{m,n}$ we have $H_{m,n}^r(q) \cong \mathcal{A}M_{m,n}$. For the remainder of this work, we associate the generators of $H_{m,n}^r(q)$ with their q-diagrams, and we refer to the diagrams of g_i, g_j^*, and e as generator diagrams.

Recall that $\mathcal{D}_{m,n}$ is the set of (m,n)-diagrams and that $\mathcal{D}_{m,n}$ is a basis of $\mathcal{B}_{m,n}^r$. Since $H_{m,n}^r(q)$ is a q-deformation of $\mathcal{B}_{m,n}^r$, we have the following theorems:

Theorem 4.10. *For all $r \in \mathbb{Z}^+ \cup \{0\}$, $H^r_{m,n}(1) \cong \mathcal{B}^r_{m,n}$ as $\mathbb{C}$-algebras. In particular,* $\dim_{\mathbb{C}(q)}(H^r_{m,n}(q)) = (m+n)!$.

Theorem 4.11. *Any set $\tilde{D}$ of q-diagrams which specialize when $q = 1$ to $\mathcal{D}_{m,n}$ is a basis of $H^r_{m,n}(q)$.*

Proof. Since $\tilde{D}$ specializes when $q = 1$ to $\mathcal{D}_{m,n}$, which is a basis for $H^r_{m,n}(1)$, the set $\tilde{D}$ is $\mathbb{C}(q)$-independent. Since $|\tilde{D}| = (m+n)! = \dim_{\mathbb{C}(q)} H^r_{m,n}(q)$, the set $\tilde{D}$ spans $H^r_{m,n}(q)$. □

In (4.6) we remarked that not all (m,n)-tangles are in the monoid $M_{m,n}$, and thus not all $(m,n;q)$-diagrams are in $H^r_{m,n}(q)$. To identify the q-diagrams in $H^r_{m,n}(q)$, we say that a q-diagram d is in *standard form* if

(1) no strand of d crosses itself,

(2) no two strands of d cross more than once, and

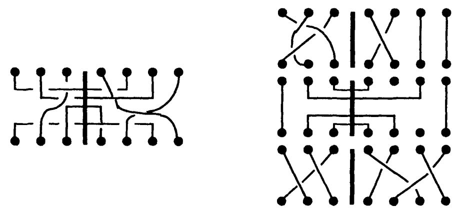
(3) d contains no cycles.

Notice that (2) precludes horizontal strands in $t(d)$ from crossing horizontal strands in $b(d)$.

Theorem 4.12. *Any q-diagram d in standard form can be written as a product of generator diagrams.*

Proof. First we assume that d has only vertical strands (i.e., d is a braid), and we induct on the number of crossings in d. If d has 0 crossings, then $d = 1$ and the theorem holds. Otherwise, if d has a crossing on the left side of the wall, then there exist two adjacent vertices, say i and $i+1$, such that the strands adjacent to them cross before, moving top to bottom, they cross any other strands. We can write $d = g_i^t d'$ where d' has only vertical strands and has one fewer crossing than d, and $t \in \{1, -1\}$. If d does not have crossings left of the wall, then it must have at least one crossing right of the wall, and the same argument holds with g_i^* in place of g_i. The result follows by induction.

As in Section 3, let $t_i^L(d)$ (respectively, $b_i^L(d)$) denote the ith vertex in the top (bottom) row of d to the left of the wall, and let $t_j^R(d)$ (respectively, $b_j^R(d)$) denote the jth vertex in the top (bottom) row of d to the right of the wall. If d has horizontal strands, then d can be written as $d = G_1 d' G_2$ where G_i contains only vertical strands and d' has horizontal strands that connect $t_m^L(d')$ to $t_1^R(d')$ and $b_m^L(d)$ to $b_i^R(d)$ and which do not cross any other strands in d'. To see this, consider the example



If we order the horizontal strands of d from top to bottom as they appear on the page, then it is the first and the last strand in d that become the distinguished strands in d'.

We now can write $d' = ed''$, where d'' has identity strands connecting $t^L_m(d)$ with $b^L_m(d)$ and $t^R_1(d)$ with $b^R_1(d)$. Continuing this way, we remove all horizontal strands from d, and the theorem follows from above. □

The product of two q-diagrams in standard form may not be in standard form. However, the tangle identities of Figure 4.7 allow us to "standardize" these diagrams inductively. If a diagram has edges that cross more than once, then we can use the third Reidemeister move to isolate a double crossing in a small disk. Applying either $(Q1)$ or the second Reidemeister move removes the double crossing without introducing any new crossings. Once we have removed all double crossings in a diagram, we then can remove all simple curls with $(Q2)$ and all simple cycles with $(Q3)$ without introducing any new crossings. In this way, we write the diagram as a $\mathbb{C}(q)$-linear combination of standard diagrams.

Proposition 4.13. *Any element of $H^r_{m,n}(q)$ can be written as a $\mathbb{C}(q)$-linear sum of q-diagrams in standard form.*

5. Character Classes.

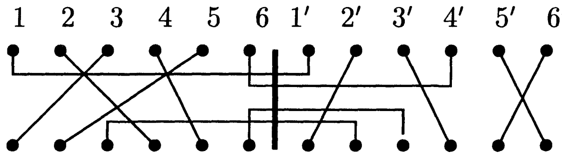
Recall that $t^L_i(d)$ (respectively, $b^L_i(d)$) denotes the ith vertex in the top (bottom) row of d to the left of the wall and $t^R_j(d)$ (respectively, $b^R_j(d)$) denotes the jth vertex in the top (bottom) row of d to the right of the wall. We define the cycle type of a diagram $d \in \mathcal{D}_{m,n}$ by *traversing* the diagram d as follows:

(1) Start with vertex $t^L_1(d)$, if it exists; otherwise, start with vertex $b^R_1(d)$.

(2) Follow the edge adjacent to this vertex. Upon reaching the opposite end of an edge, jump to the vertex directly above it if we are in $b(d)$ and to the vertex directly below it if we are in $t(d)$, and continue by following the edge adjacent to that vertex.

(3) Returning to the starting vertex completes a *cycle* in d. If not all of the edges of d have been traversed, we go to the first vertex in $t^L(d)$ or in $b^R(d)$ that has not been visited and repeat the process.

In this way, we decompose d into disjoint cycles. For example, the diagram

(5.1) $d =$ 

has 4 disjoint cycles. The first is on vertices $1, 1', 2', 3$, the second on vertices $2, 4, 5$ the third on vertices $6, 4', 3'$, and the fourth on vertices $5', 6'$.

For each cycle c in d, let *type*(c) denote the number of vertical edges in c on the left side of the wall minus the number of vertical edges in c on the right side of the wall. The integer *type*(c) is the *cycle type* of c, and we say that c is a *type*(c)-cycle. It is always possible to list the cycles of d in such a way, $c_1, c_2, \ldots, c_s$, that

$$type(c_1) \geq type(c_2) \geq \ldots \geq type(c_s), \tag{5.2}$$

where s is the number of cycles in d. In other words, the sequence (5.2) is an s-staircase. We associate with d the $(m+n)$-staircase $\zeta(d)$ obtained from (5.2) by inserting $m + n - s$ zeros into the sequence between the positive values and the negative values. The $(m+n)$-staircase $\zeta(d)$ is called the *cycle type* of d. The ordering on the cycles of d is not unique, but the $(m+n)$-staircase $\zeta(d)$ is uniquely defined. If $d \in \mathcal{S}_m \times \mathcal{S}_n$, then $(\zeta(d)^+, \zeta(d)^-)$ is exactly its cycle type when viewed as a pair of permutations. In example (5.1), the diagram d has $\zeta(d) = (3, 0^9, -1, -2)$, since $m = n = 6$.

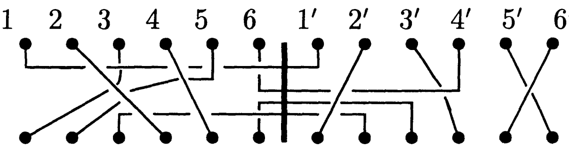
Zero cycles contain the same number of vertices on each side of the wall, and vertical edges in non-zero cycles do not get counted in the type of d only if they are paired with a vertical edge on the opposite side of the wall. Thus, there exists an integer $h(d)$ satisfying $1 \leq h(d) \leq min(m, n)$ and $\zeta(d)^+ \vdash (m - h(d))$ and $\zeta(d)^- \vdash (n - h(d))$. In our example $h(d) = 2$, $\zeta(d)^+ = (3)$, and $\zeta(d)^- = (2, 1)$.

To each $d \in \mathcal{D}_{m,n}$ we associate the q-diagram $d^{(q)}$ by *q-traversing* d as follows:

(1) Order the cycles of d by type as in (5.2) and traverse the cycles in this order.

(2) Traverse an individual cycle in $d^{(q)}$ just as we would traverse the cycle in d, only now, whenever we cross an edge, we go *under* it if that edge has already been traversed, and *over* it if that edge has not been traversed.

In this way, $d^{(q)}$ is in standard form, and the cycles of $d^{(q)}$ are layered from top to bottom according to their type. For our example (5.1), we have

$d^{(q)} =$ 

Remark 5.3. I would like to thank Arun Ram for suggesting that I use such a method of drawing the basis of $H^r_{m,n}(q)$.

It is clear from this construction that $d^{(q)}$ specializes when $q = 1$ to d, and it follows from Theorem 4.11 that

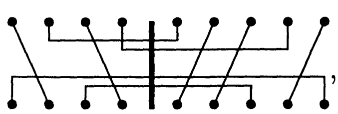
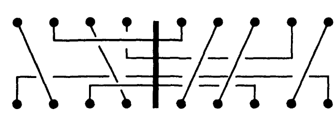
Proposition 5.4. *The set* $\mathcal{D}^q_{m,n} \overset{\text{def}}{=} \{d^{(q)} \mid d \in \mathcal{D}_{m,n}\}$ *is a* $\mathbb{C}(q)$*-basis for* $H^r_{m,n}(q)$.

We consider

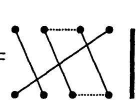
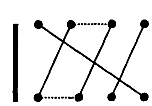
$$H^r_{k,\ell}(q) \otimes H^r_{m-k,n-\ell}(q) \subseteq H^r_{m,n}(q) \tag{5.5}$$

in the following way. If d_1 is a $(k, \ell; q)$-diagram and d_2 is an $(m-k, n-\ell; q)$-diagram, then $d_1 \otimes d_2$ is the $(m, n; q)$-diagram obtained by placing, in order, the first k dots of d_1, the first $m-k$ dots of d_2, the wall, the last ℓ dots of d_1, and the last $n-\ell$ dots of d_2. We then attach each strand to its original vertex while placing the strands of d_1 on top of the strands of d_2. Then $H^r_{k,\ell}(q)$ commutes with $H^r_{m-k,n-\ell}(q)$ inside of $H^r_{m,n}(q)$, so the tensor product is well defined.

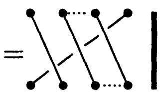
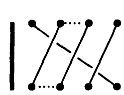
We say that a diagram $c \in \mathcal{D}_{m,n}$ is a *cycle diagram* if c consists of a single cycle and has the property that when we traverse it, we visit the columns on the left side of the wall in increasing order, and we visit the columns on the right side of the wall in increasing order. If c is a cycle diagram in $\mathcal{D}_{m,n}$, then we say that $c^{(q)}$ is a cycle diagram in $\mathcal{D}^q_{m,n}$. For example,

if $c =$ , then $c^{(q)} =$ 

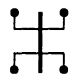
For each $k > 0$, we let d_k denote the cycle in $\mathcal{B}^r_{k,0} \cong \mathcal{S}_k$ given by $d_k = s_{k-1}s_{k-2}\cdots s_1$, and for $k < 0$, we let d_k denote the cycle in $\mathcal{B}^r_{0,-k}$ given by $d_k = s_1^* s_2^* \cdots s^*_{-k-1}$. They are drawn as

(5.6) $d_k =$  if $k > 0$, and $d_k =$  if $k < 0$

When $k = 0$, we let $d_k = \emptyset$. When $k > 0$, we lift d_k to the q-diagram $d_k^{(q)} = g_{k-1}g_{k-2}\cdots g_1 \in H_k(q)$, and when $k < 0$, we lift d_k to the q-diagram $d_k^{(q)} = g_1^{*-1}g_2^{*-1}\cdots {g_{-k-1}^{*}}^{-1} \in H_{-k}(q)$. Thus,

(5.7) $d_k^{(q)} =$  if $k > 0$, and $d_k^{(q)} =$  if $k < 0$.

The inverses are required when $k < 0$, since we traverse the cycle diagram starting with $b_1^R(d_k^{(q)})$. Recall that e is the diagram in $\mathcal{B}_{1,1}^r$ and in $H_{1,1}^r$ given by

(5.8) $e =$  ,

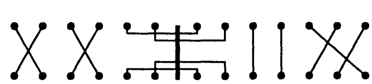
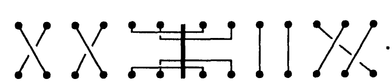
and let $e^{\otimes h} = e \otimes e \otimes \cdots \otimes e$ with h-factors. For $\zeta \in \Phi^{m,n}$, let $h(\zeta) \in \mathbb{Z}$ so that $\zeta^+ \vdash (m - h(\zeta))$ and $\zeta^- \vdash (n - h(\zeta))$, and assume that the lengths of the positive and negative parts of ζ are $\ell(\zeta^+) = i$ and $\ell(\zeta^-) = j$. Then $d_{\zeta^+}, d_{\zeta^-}, d_{\zeta^+}^{(q)}$, and $d_{\zeta^-}^{(q)}$ are the diagrams given by

(5.9) $d_{\zeta^+} = d_{\zeta_1} \otimes d_{\zeta_2} \otimes \cdots \otimes d_{\zeta_i}$ and $d_{\zeta^-} = d_{\zeta_{r-j}} \otimes \cdots \otimes d_{\zeta_{r-1}} \otimes d_{\zeta_r}$,
$d_{\zeta^+}^{(q)} = d_{\zeta_1}^{(q)} \otimes d_{\zeta_2}^{(q)} \otimes \cdots \otimes d_{\zeta_i}^{(q)}$ and $d_{\zeta^-}^{(q)} = d_{\zeta_{r-j}}^{(q)} \otimes \cdots \otimes d_{\zeta_{r-1}}^{(q)} \otimes d_{\zeta_r}^{(q)}$,

and $d_\zeta \in \mathcal{B}_{m,n}^r$ and $d_\zeta^{(q)} \in H_{m,n}^r(q)$ are the diagrams given by

(5.10) $$d_\zeta = d_{\zeta^+} \otimes e^{\otimes h(\zeta)} \otimes d_{\zeta^-} \qquad d_\zeta^{(q)} = d_{\zeta^+}^{(q)} \otimes e^{\otimes h(\zeta)} \otimes d_{\zeta^-}^{(q)}.$$

For example, if $\zeta = (2, 2, 0^8, (-1)^2, -3) \in \Phi^{6,7}$, then

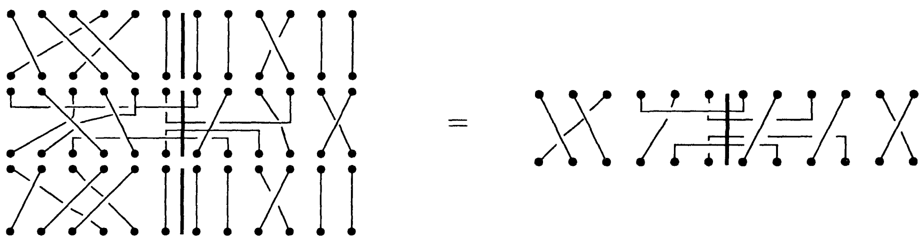
$d_\zeta =$  and $d_\zeta^{(q)} =$  .

Elements a and b of $H_{m,n}^r(q)$ (resp. $\mathcal{B}_{m,n}^x$) are said to be *conjugate*, written $a \sim b$, if there exists $h \in H_m(q) \otimes H_n(q)$ (resp. $h \in \mathcal{S}_m \times \mathcal{S}_n$) such that $hah^{-1} = b$. If $a \sim b$ and tr is any trace on $H_{m,n}^r(q)$ (resp. $\mathcal{B}_{m,n}^x$), then by the trace property, $tr(a) = tr(b)$.

Proposition 5.11.
(a) *If $d \in \mathcal{D}_{m,n}$, then $d \sim c_1 \otimes c_2 \otimes \cdots \otimes c_s$ and $d_q \sim c_1^{(q)} \otimes c_2^{(q)} \otimes \cdots \otimes c_k^{(q)}$, where $c_1, c_2, \ldots, c_s$ are the cycles of d ordered so that $type(c_1) \geq type(c_2) \geq \cdots \geq type(c_s)$.*

(b) *If d is any $(m-1, n-1; q)$-diagram, then $d \otimes e \sim e \otimes d$.*

Proof. For part (a), define $\pi = (\pi_1, \pi_2) \in \mathcal{S}_m \times \mathcal{S}_n$ and $h = h_1 \otimes h_2 \in H_m(q) \otimes H_n(q)$ as follows: if in q-traversing d_q as above, the ith column visited on the left side of the wall is j, then let $\pi_1(j) = i$ and connect the jth vertex in the bottom row of h_1 to the ith vertex in the top row of h_1 always passing under any edges that are already drawn. If the ith column visited on the right side of the wall is j, then let $\pi_2(j) = i$ and connect the jth vertex in the bottom row of h_2 to the ith vertex in the top row of h_2 always passing under any edges that are already drawn. Then $hd^{(q)}h^{-1}$ is equal to the tensor product of the cycles of $d^{(q)}$. For our example (5.1) we have



Notice that layering consecutive edges of h just as we did in $d^{(q)}$ allows us to pull the first cycle of $d^{(q)}$ to the front over the top of the other cycles, then pull the second cycle out from between the first and the third, etc. That $\pi d \pi^{-1}$ is the tensor product of its cycles follows from ignoring the over and under-crossings. For part (b), let $h = g_1 g_2 \cdots g_{m-1} g_1^{*-1} g_2^{*-2} \cdots g_{n-1}^{*-1}$, and then $h(d \otimes e)h^{-1} = e \otimes d$. The reader should note that in general $d_1 \otimes d_2 \sim d_2 \otimes d_1$ holds in $\mathcal{B}_{m,n}^r$ but not in $H_{m,n}^r(q)$. $\square$

Let

$$z(d) \stackrel{\text{def}}{=} (\text{number of 0-cycles in } d).$$

Since 0-cycles do not contribute to $\zeta(d)$ and since 0-cycles contain the same number of vertices on each side of the wall, we have $0 \le z(d) \le h(d)$. Let

$$v(d) \stackrel{\text{def}}{=} (\text{number of vertical edges of } d \text{ which do not get counted in } \zeta(d)),$$

$$w(d) \stackrel{\text{def}}{=} (\text{number of cycles } c \text{ in } d \text{ with } type(c) < 0 \text{ that have at least one horizontal edge}),$$

$$u(d) \stackrel{\text{def}}{=} [h(d) - v(d) - z(d) - w(d)]\, r - w(d),$$

and define functions $\xi : \mathcal{D}_{m,n} \longrightarrow \mathbb{C}(x)$ and $\xi_q : \mathcal{D}_{m,n} \longrightarrow \mathbb{C}(q)$ by

$$\xi(d) = x^{z(d)-h(d)} \quad \text{and} \quad \xi_q(d) = [\![r]\!]_q^{z(d)-h(d)} q^{u(d)}. \tag{5.12}$$

If $d \sim d_\zeta$ for some $\zeta \in \Phi^{m,n}$, then $z(d) = h(d)$, $v(d) = 0$, $w(d) = 0$, $\xi(d) = 1$, and $\xi_q(d) = 1$.

Theorem 5.13. *Let $d \in \mathcal{D}_{m,n}$ with $\zeta = \zeta(d)$ and $h = h(d)$. Then for any character χ_H of $H^r_{m,n}(q)$ and any character χ_B of $\mathcal{B}^x_{m,n}$, we have:*
(i) $\chi_B(d) = \xi(d)\,\chi_B(d_\zeta)$,

(ii) $\chi_H(d^{(q)}) = \xi_q(d)\,\chi_H(d^{(q)}_\zeta)$.

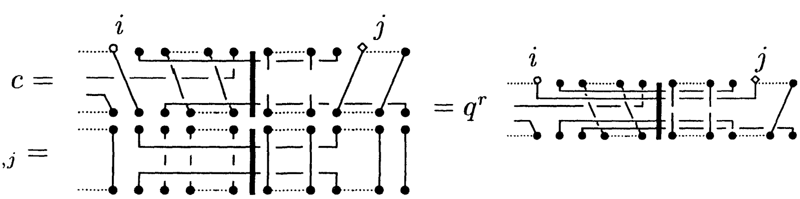
Proof. Part (i) follows from (ii) by setting $q = 1$, so we prove (ii). From Proposition 5.11(a), we have

$$d^{(q)} \sim d^{(q)\prime} = c_1^{(q)} \otimes c_2^{(q)} \otimes \cdots \otimes c_s^{(q)} \tag{5.14}$$

where $type(c_1^{(q)}) \geq type(c_2^{(q)}) \geq \cdots \geq type(c_s^{(q)})$. If each 0-cycle is e and each nonzero cycle has no horizontal edges, then $d^{(q)\prime} = d^{(q)}_\zeta$, and we are done. Otherwise, there exists a cycle $c \neq e$ in (5.14) that has a horizontal edge. Assume that c is an $(m', n'; q)$-cycle diagram and that the last horizontal edge encountered in $t(c)$ while q-traversing c connects $t_i^L(c)$ to $t_j^R(c)$. Then, $\varepsilon_{ij} c = [\![r]\!]_q c$, where we are considering $\varepsilon_{ij} \in H^r_{m',n'}(q)$. Moreover, if E is the embedding of $\varepsilon_{i,j}$ in $H^r_{m,n}(q)$, then

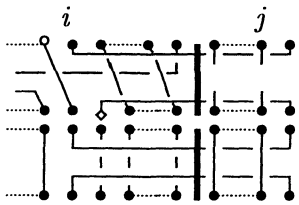
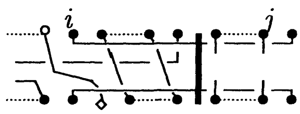
$$\chi_H(d^{(q)\prime}) = \frac{1}{[\![r]\!]_q}\chi_H(Ed^{(q)\prime}) = \frac{1}{[\![r]\!]_q}\chi_H(d^{(q)\prime}E). \tag{5.15}$$

Therefore, we are interested in the product $d^{(q)\prime}E$, or more particularly, $c\varepsilon_{i,j}$. If v and v' are vertices in c, we write $v \leftrightarrow v'$ if they are connected by an edge. Let $\circ$ and $\diamond$ be the vertices in c such that $b_i^L(c) \leftrightarrow \circ$ and $b_1^R(c) \leftrightarrow \diamond$, and consider the four possible locations of $\circ$ and $\diamond$.
Case (i): $\circ \in t^L(c)$ *and* $\diamond \in t^R(c)$. Then we have

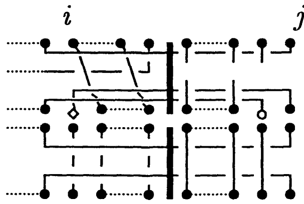
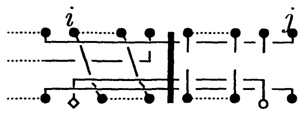


where only the edges of interest are included and where edges which come before these in the cycle pass over the edges shown here. Notice that $c\varepsilon_{i,j} \sim e \otimes c'$ where $c' \in \mathcal{D}^{(q)}_{m'-1,n'-1}$ is a cycle with the same type as c. If $i = 1$, then, since we are using the last horizontal edge encountered in $t(c)$, there is only one horizontal edge in each row of c, and we must have $\circ = t^L_m(c)$. Conjugating c by $h = g_1^{-1} g_2^{-1} \cdots g_{m'}^{-1}$, moves $t_1^L(c)$ to $t_2^L(c)$ and $\circ$ to $t_1^L(c)$, so we are safe to assume that $\circ = t^L_{i-1}(c)$ as pictured above.

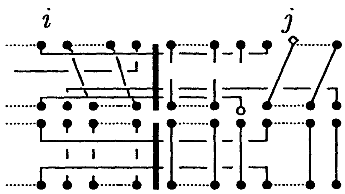
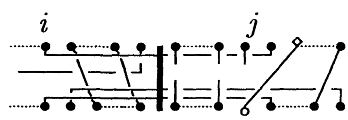
Case (ii): $\circ \in t^L(c)$ *and* $\diamond \in b^L(c)$. Here $\diamond = b^L_{i+1}(c)$ is forced by the definition of cycle diagram, and the picture is

$c =$, $\epsilon_{i,j} =$  $= q^r$ 

We remove the positively-oriented simple curl in $\circ \leftrightarrow \diamond$ with penalty q^r and have $c\varepsilon_{i,j} \sim q^r e \otimes c'$ where $c' \in \mathcal{D}^{(q)}_{m'-1,n'-1}$ is a cycle with the same type as c. As in case (i), we assume that $\circ = t^L_{i-1}(c)$.
Case (iii): $\circ \in b^R(c)$ *and* $\diamond \in b^L(c)$. We must have $j = n'$ and $\circ = b^R_{n'-1}(c)$, so the picture is

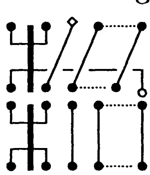
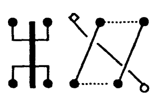
$c =$, $\epsilon_{i,j} =$  $= q^r$ 

We remove the curl and have $c\varepsilon_{i,n'} \sim q^r e \otimes c'$, where $c' \in \mathcal{D}^{(q)}_{m'-1,n'-1}$ is a cycle with the same type as c.
Case (iv): $\circ \in b^R(c)$ *and* $\diamond \in t^R(c)$. First we assume that c has more than one horizontal edge in each row. Then the picture is

$c =$, $\epsilon_{i,j} =$  $= q^r$ 

The curl is removed with a penalty of q^r and $c\varepsilon_{i,j} \sim q^r c' \otimes e$ where $c' \in \mathcal{D}^{(q)}_{m'-1,n'-1}$ has the same type as c.

If c has only one horizontal edge in each row, then c is a right cycle, and

$c =$, $\epsilon_{i,j} =$  $= q^{-1}$ 

Here, the curl is removed with a penalty of q^{-1} (see Figure 4.7. $(Q2^*)$(b)), since it has negative orientation. We have $c\varepsilon_{1,1} \sim q^{-1}c' \otimes e$ where $c' \in$

$\mathcal{D}^{(q)}_{m'-1,n'-1}$.

We repeat the process with c' in place of c until either $c' = e$ or c' has no horizontal edges. If c is a 0-cycle, then $z(c) = 1$, and after $h(c) - z(c)$ reductions we get $c' = e$. If c is a k-cycle with $k \neq 0$, then after $h(c)$ reductions we get $c' = d^{(q)}_k$. At each reduction, we multiply by $[\![r]\!]_q$, so in the end we pick up the scalar $[\![r]\!]_q^{h(c)-z(c)}$. Each reduction also introduces the constant q^r except for reductions in which we cancel vertical edges and for the last reduction in the case that $type(c) < 0$. In this event, we introduce q^{-1}. Note that in this last case $w(c) = 1$; otherwise, $w(c) = 0$. There are a total of $h(c) - z(c) - v(c) - w(c)$ reductions in which we multiply by q^r and $w(c)$ reductions in which we multiply by q^{-1}. If we do this for each cycle in (5.14) that has a horizontal edge and is not e, then by 5.11(b), we can conjugate the resulting diagram to be in the form of $d^{(q)}_\zeta$. To see that we have picked up the constant $\xi_q(d)$, observe that $h(d) = \sum_{i=1}^{\ell} h(c_i)$ and $u(d,r) = \sum_{i=1}^{\ell} u(c_i, r)$. $\square$

We rescale the bases of $H^r_{m,n}(q)$ and $\mathcal{B}^r_{m,n}$ as follows:

(5.16)
$$\tilde{\mathcal{D}}^q_{m,n} = \{\xi_q(d)^{-1} d_q \mid d \in \mathcal{D}_{m,n}\}, \qquad \tilde{\mathcal{D}}_{m,n} = \{\xi(d)^{-1} d \mid d \in \mathcal{D}_{m,n}\}.$$

Then $\tilde{\mathcal{D}}_{m,n}$ and $\tilde{\mathcal{D}}^q_{m,n}$ are bases for $\mathcal{B}^x_{m,n}$ and $H^r_{m,n}(q)$, respectively, that divides into classes labeled by $\Phi^{m,n}$ on which characters are constant. For this reason we call the classes indexed by $\Phi^{m,n}$ *character classes.* Note that we have shown, also, that d_ζ and $d^{(q)}_\zeta$ are representatives of the class $\zeta \in \Phi^{m,n}$ in their respective algebras.

The idea of diagram type is generalized from [**R2**]. There, the type of $(m+n)$-diagrams is given, and the $(m+n)$-diagrams are partitioned into $\mathcal{B}^r_{m+n}$-character classes.

6. The Quantum General Linear Group.

In this chapter we describe a q-deformation $\mathcal{U}_q(g\ell(r,\mathbb{C}))$ of the universal enveloping algebra $\mathcal{U}(g\ell(r,\mathbb{C}))$ of the Lie algebra $g\ell(r,\mathbb{C})$. Such deformations are known as quantum groups, although they are not groups but are Hopf algebras over the field $\mathbb{C}(q)$ of rational functions. Let $\mathcal{H}$ be the Cartan subalgebra of diagonal matrices in th Lie algebra $g\ell(r,\mathbb{C})$. For $1 \leq i \leq r$, let ϵ_i be the basis element of $\mathcal{H}^*$ that projects a matrix onto its i,i-entry. There exists a non-degenerate bilinear form on $\mathcal{H}^*$ given by $(\epsilon_i, \epsilon_j) = \delta_{i,j}$. If we let $\alpha_i = \epsilon_i - \epsilon_{i+1}$, then $\alpha_1, \ldots, \alpha_{r-1}$ is a base of simple roots for the subalgebra $s\ell(r,\mathbb{C})$ with respect to its Cartan subalgebra of diagonal matrices. Relative to this form $(\alpha_i, \alpha_i) = 1$, and the Cartan matrix (a_{ij}) of $s\ell(r, \mathbb{C})$ satisfies $a_{i,j} = 2(\alpha_i, \alpha_j)$. Thus $a_{ii} = 2$, $a_{ij} = -1$, if $j = i \pm 1$, and $a_{ij} = 0$, otherwise.

Let $\mathcal{U}_q = \mathcal{U}_q(g\ell(r, \mathbb{C}))$ be the associative $\mathbb{C}(q)$ algebra with generators $\{X_i^\pm \mid 1 \le i \le r-1\} \cup \{t_i^{\pm 1} \mid 1 \le i \le r\}$ subject to the relations:

(1) $t_i t_i^{-1} = 1 = t_i^{-1} t_i$, and $t_i t_j = t_j t_i$, if $i \neq j$,

(2) $t_i X_j^\pm t_i^{-1} = q^{\pm \frac{1}{2}(\epsilon_i, \alpha_j)} X_j^\pm$,

(3) $[X_i^+, X_j^-] = \delta_{ij} \dfrac{k_i^2 - k_i^{-2}}{q^{1/2} - q^{-1/2}}$, where $k_i = t_i t_{i+1}^{-1}$ for $1 \le i \le r-1$,

(4) $(X_i^\pm)^2 X_{i\pm 1}^\pm - (q^{1/2} + q^{-1/2}) X_i^\pm X_{i\pm 1}^\pm X_i^\pm + X_{i\pm 1}^\pm (X_i^\pm)^2 = 0$. Upon letting $q \to 1$, one obtains the classical Serre relations for the universal enveloping algebra of $g\ell(r, \mathbb{C})$. The element $t_1 t_2 \cdots t_r$ commutes with $\mathcal{U}_q$, and the subalgebra generated by $\{X_i^\pm, k_i^{\pm 1} \mid 1 \le i \le r-1\}$ is the quantum group $\mathcal{U}_q(s\ell(r, \mathbb{C}))$. Moreover, the algebra $\mathcal{U}_q$ is a Hopf algebra whose structure is given by

(1) Comultiplication $\Delta : \mathcal{U}_q \longrightarrow \mathcal{U}_q \otimes \mathcal{U}_q$, where

$$\Delta(X_i^\pm) = X_i^\pm \otimes k_i^{-1} + k_i \otimes X_i^\pm, \quad \text{and} \quad \Delta(t_i) = t_i \otimes t_i.$$

(2) Antipode $S : \mathcal{U}_q \longrightarrow \mathcal{U}_q$, where $S(X_i^\pm) = -q^{\mp 1/2} X_i^\pm$ and $S(t_i) = t_i^{-1}$.

(3) Counit $u : \mathcal{U}_q \longrightarrow \mathbb{C}(q)$, where $u(X_i^\pm) = 0$, and $u(t_i) = 1$.

Comultiplication Δ is co-associative, so we can define $\Delta^{(f)} : \mathcal{U}_q \longrightarrow \otimes^f \mathcal{U}_q$ by

$$\text{(6.1)} \qquad \Delta^{(f)}(X_i^\pm) = \sum_{i=1}^{f} \underbrace{k_i \otimes \cdots \otimes k_i}_{i-1} \otimes X_i^\pm \otimes \underbrace{k_i^{-1} \otimes \cdots \otimes k_i^{-1}}_{f-(i+1)},$$
$$\Delta^{(f)}(t_i) = t_i \otimes t_i \otimes \cdots \otimes t_i.$$

The fundamental representation of $g\ell(r, \mathbb{C})$ on $V = \mathbb{C}^r$ is extended to the natural representation $\phi_q : \mathcal{U}_q \longrightarrow End_{\mathbb{C}(q)}(V_q)$ of $\mathcal{U}_q$ on $V_q = V \otimes \mathbb{C}(q)$ by

(6.2)
$$\phi_q(X_i^+) = E_{i,i+1}, \quad \phi_q(X_i^-) = E_{i+1,i}, \quad \phi_q(t_i) = q^{1/4} E_{i,i} + \sum_{j \neq i} E_{jj},$$

where $E_{i,j}$ denotes the matrix unit that has a 1 in the (i, j)-position and 0 elsewhere. It is straightforward to check that ϕ_q is indeed a representation of $\mathcal{U}_q$ and that

$$\text{(6.3)} \qquad \phi_q(k_i) = q^{1/4} E_{ii} + q^{-1/4} E_{i+1,i+1} + \sum_{j \neq i, i+1} E_{jj}.$$

Let V^* be the dual space to V and let ${V_q}^* = V^* \otimes \mathbb{C}(q)$. The contragradient representation $\phi_q^* : \mathcal{U}_q \longrightarrow End({V_q}^*)$ of $\mathcal{U}_q$ on ${V_q}^*$ is given by $\phi_q^* = {}^t(\phi_q S)$, where t denotes matrix transpose. Thus

$$\begin{aligned} \phi_q^*(X_i^+) &= -q^{-1/2}E_{i+1,i}, \quad \phi_q^*(X_i^-) \;=\; -q^{1/2}E_{i,i+1}, \\ \phi_q^*(t_i) &= q^{-1/4}E_{i,i} + \sum_{j\neq i} E_{jj}. \end{aligned} \tag{6.4}$$

We induce a representation $\Phi : \mathcal{U}_q \longrightarrow End_{\mathbb{C}(q)}(T_q^{m,n})$ of $\mathcal{U}_q$ on $T_q^{m,n} = (\otimes^m V_q) \otimes (\otimes^n {V_q}^*)$ by

$$\Phi = ((\otimes^m \phi_q) \otimes (\otimes^n \phi_q^*))\Delta^{m+n}. \tag{6.5}$$

Lusztig [**Lu**] proves that every irreducible $\mathcal{U}_q$-module specializes when $q \to 1$ to an irreducible GL_r-module (see Leduc [**Le**] for details). Thus for $\gamma \in \Phi_r^{m,n}$, we let V_q^γ be the irreducible $\mathcal{U}_q$-module that specializes to V^γ. Lusztig [**Lu**] proves further that $T_q^{m,n}$ is a completely reducible $\mathcal{U}_q$-module. By letting $q \to 1$, we see that the decomposition of $T_q^{m,n}$ as a $\mathcal{U}_q$-module is

$$T_q^{m,n} \cong \bigoplus_{\gamma\in\Phi_r^{m,n}} m_\gamma V_q^\gamma, \tag{6.6}$$

where m_γ is the number of (m,n)-up-sown staircases of shape γ (see Section 2).

For any invertible element $\mathcal{R} \in \mathcal{U}_q \otimes \mathcal{U}_q$ given by $\mathcal{R} = \sum a_i \otimes b_i$, define $\mathcal{R}_{12}, \mathcal{R}_{13}, \mathcal{R}_{23} \in {\mathcal{U}_q}^{\otimes 3}$ to be the elements

$$\mathcal{R}_{12} = \sum a_i \otimes b_i \otimes 1, \qquad \mathcal{R}_{13} = \sum a_i \otimes 1 \otimes b_i, \qquad \mathcal{R}_{12} = \sum 1 \otimes a_i \otimes b_i.$$

Then we say that $\mathcal{R}$ satisfies the quantum Yang-Baxter equation (QYBE) if

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} \;=\; \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{6.7}$$

Let $T : \mathcal{U}_q \otimes \mathcal{U}_q \to \mathcal{U}_q \otimes \mathcal{U}_q$ be given by $T(a \otimes b) = b \otimes a$ for all $a, b \in \mathcal{U}_q$. Then $\mathcal{R} \in \mathcal{U}_q \otimes \mathcal{U}_q$ is a universal R-matrix if it satisfies the relations

$$\begin{aligned} T\Delta(a) &= \mathcal{R}\Delta(a)\mathcal{R}^{-1} \qquad \text{for all } a \in \mathcal{U}_q, \\ (\Delta \otimes \mathrm{id})(\mathcal{R}) &= \mathcal{R}_{13}\mathcal{R}_{23}, \\ (\mathrm{id} \otimes \Delta)(\mathcal{R}) &= \mathcal{R}_{13}\mathcal{R}_{12}. \end{aligned} \tag{6.8}$$

If $\mathcal{R}$ is a universal R-matrix, $\mathcal{R}$ satisfies (5.8). If ρ is a representation of $\mathcal{U}_q$, then $\check{R}$ is the matrix given by $\check{R} = \rho(T\mathcal{R})$. Jimbo [**J**] extracts the $\check{R}$

matrices for the representations $V_q \otimes V_q$ and $V_q{}^* \otimes V_q{}^*$, and shows that they have the form:

$$(6.9)\quad \begin{aligned} \check{R} &= q\sum_{j=1}^{r} E_{jj}\otimes E_{jj} + q^{1/2}\sum_{j\neq k}^{r} E_{jk}\otimes E_{kj} + (q-1)\sum_{j<k}^{r} E_{jj}\otimes E_{kk}, \\ \check{R}^* &= q\sum_{j=1}^{r} E_{jj}\otimes E_{jj} + q^{1/2}\sum_{j\neq k}^{r} E_{jk}\otimes E_{kj} + (q-1)\sum_{j>k}^{r} E_{jj}\otimes E_{kk}. \end{aligned}$$

In $End_{\mathbb{C}(q)}(T_q^{m,n})$ define the following matrices:

$$R_i = \underbrace{1\otimes\cdots\otimes 1}_{m-(i+1)}\otimes\check{R}\otimes\underbrace{1\otimes\cdots\otimes 1}_{n+(i-1)} \quad\text{and}\quad R_j^* = \underbrace{1\otimes\cdots\otimes 1}_{m+(j-1)}\otimes\check{R}^*\otimes\underbrace{1\otimes\cdots\otimes 1}_{n-(j+1)}.$$

It follows from (QYBE) (6.6) that

$$(6.10)\quad R_iR_{i+1}R_i = R_{i+1}R_iR_{i+1} \quad\text{and}\quad R_j^*R_{j+1}^*R_j^* = R_{j+1}^*R_j^*R_{j+1}^*,$$

and thus the R_i and the R_j^* satisfy the braid relations (B1) and (B2) and (B1*) and (B2*), respectively. It was this observation that led Jimbo [**J**] to define a representation $\pi : H_f(q) \longrightarrow End_{\mathbb{C}(q)}(T^f \otimes \mathbb{C}(q))$ given by $\pi(g_i) = R_i$.

Define the matrix $F \in End_{\mathbb{C}(q)}(T_q^{m,n})$ by

$$(6.11)\quad F = \underbrace{1\otimes\cdots\otimes 1}_{m-1}\otimes\left(\sum_{k=1}^{r} q^{(k-1)}\sum_{j=1}^{r} E_{jk}\otimes E_{jk}\right)\otimes\underbrace{1\otimes\cdots\otimes 1}_{n-1}.$$

Using the fact that $\sum_{i=1}^{r} q^{i-1} = [\![r]\!]_q$ (see 4.1) we get $F^2 = [\![r]\!]_q F$. One can directly verify the other $H_{m,n}^r(q)$-relations to see that the map $\pi : H_{m,n}^r(q) \longrightarrow End_{\mathbb{C}(q)}(T_q^{m,n})$ given on the generators by

$$(6.12)\qquad \pi(g_i) = R_i, \qquad \pi(g_j^*) = R_j^*, \qquad\text{and}\quad \pi(e) = F$$

is a representation of $H_{m,n}^r(q)$. Moreover, this representation is well-defined independent of r. It is straightforward, but quite tedious, to check that $\Phi(\mathcal{U}_q)$ commutes with R_i, R_j^*, and F, and, therefore, $\pi : H_{m,n}^r(q) \longrightarrow End_{\mathcal{U}_q}(T_q^{m,n})$. Kosuda proves the following theorem when $r \geq m+n$, and we extend it to all $r \geq 0$.

Theorem 6.13. *The map $\pi : H_{m,n}^r(q) \longrightarrow End_{\mathcal{U}_q}(T_q^{m,n})$ is onto for $r \in \mathbb{Z}^+ \cup \{0\}$.*

Proof. Since $H_{m,n}^r(q)$ and $\mathcal{U}_q$ commute, we know that π maps $H_{m,n}^r(q)$ into $End_{\mathcal{U}_q}(T_q^{m,n})$. By (6.6), $\dim_{\mathbb{C}(q)} End_{\mathcal{U}_q}(T_q^{m,n}) = \sum_{\gamma\in\Phi_r^{m,n}} m_\gamma^2$. We know that

$\dim_{\mathbb{C}(q)} H^r_{m,n}(q) \geq \dim_{\mathbb{C}} H^r_{m,n}(1) = \dim_{\mathbb{C}} \mathcal{B}^r_{m,n} = \sum_{\gamma\in\Phi^{m,n}_r} = m^2_\gamma$, and the result follows by comparing dimensions. $\square$

Remark 6.14. Since $H^r_{m,n}(q)$ maps onto $End_{\mathcal{U}_q}(T^{m,n}_q)$, there exists an irreducible $H^r_{m,n}(q)$-module M^γ_q for each $\gamma \in \Phi^{m,n}_r$. However, when $r < m+n$ the set $\{M^\gamma_q | \gamma \in \Phi^{m,n}_r\}$ does not form a complete set of irreducibles for $H^r_{m,n}(q)$. In fact, $H^r_{m,n}(q)$ is not necessarily semisimple when $r < m+n$ (see Leduc [**Le**]). $\square$

Let $\mathcal{C}^r_{m,n}(q) = End_{\mathcal{U}_q}(T^{m,n}_q)$ and $\mathcal{C}^r_{m,n} = End_{GL(r,\mathbb{C})}(T^{m,n})$. Then if $r \geq m+n$, we have $H^r_{m,n}(q) \cong \mathcal{C}^r_{m,n}(q)$ and $\mathcal{B}^r_{m,n} \cong \mathcal{C}^r_{m,n}$. Recall the definition of $[\mu,\nu]_r/[\alpha,\beta]_r = \square$ from (3.15). Kosuda proves the following theorem for $r \geq m+n$ and Leduc extends it to all $r \geq 0$.

Theorem 6.15 [Kos], [Le]. *Let $[\mu,\nu]_r \in \Phi^{m,n}$. Then*

$$M_q^{[\mu,\nu]_r}\big\downarrow^{\mathcal{C}^r_{m,n}(q)}_{\mathcal{C}^r_{m-1,n}(q)} \cong \bigoplus_{\substack{[\alpha,\beta]_r\in\Phi^{m-1,n}_r \\ [\mu,\nu]_r/[\alpha,\beta]_r=\,\square}} M_q^{[\alpha,\beta]_r}$$

and

$$M_q^{[\mu,\nu]_r}\big\downarrow^{\mathcal{C}^r_{m,n}(q)}_{\mathcal{C}^r_{m,n-1}(q)} \cong \bigoplus_{\substack{[\alpha,\beta]_r\in\Phi^{m,n-1}_r \\ [\alpha,\beta]_r/[\mu,\nu]_r=\,\square}} M_q^{[\alpha,\beta]_r}.$$

Thus, $\mathbb{C}(q) \cong \mathcal{C}^r_{1,0}(q) \subseteq \cdots \subseteq \mathcal{C}^r_{m,0}(q) \subseteq \mathcal{C}^r_{m,1}(q) \subseteq \cdots \subseteq \mathcal{C}^r_{m,n}(q)$ and $\mathbb{C} \cong \mathcal{C}^r_{1,0} \subseteq \cdots \subseteq \mathcal{C}^r_{m,0} \subseteq \mathcal{C}^r_{m,1} \subseteq \cdots \subseteq \mathcal{C}^r_{m,n}$ are towers of semisimple algebras with multiplicity free branching such that $\mathcal{C}^r_{i,j}(q)$ and $\mathcal{C}^r_{i,j}$ have the same matrix decomposition. The next proposition follows immediately from Theorem 1.16.

Proposition 6.16. *There exists a partition of unity in $\mathcal{C}^r_{m,n}(q)$ which specializes when $q=1$ to a partition of unity in $\mathcal{C}^r_{m,n}$. In particular, if $r \geq m+n$ there exists a partition of unity in $H^r_{m,n}(q)$ that specializes when $q=1$ to a partition of unity in $\mathcal{B}^r_{m,n}$.*

As in Section 2, let $\{v_1,\dots,v_r\}$ be the standard basis for V, and let $\{v_1^*,\dots,v_r^*\}$ be its dual. Then the simple tensors $\underline{v} = v_{i_1}\otimes\cdots\otimes v_{i_m}\otimes v^*_{j_1}\otimes\cdots\otimes v^*_{j_n}$ form a basis of $T^{m,n}_q$. In this section we give the action of $H^r_{m,n}(q)$ on this basis. If $1 \leq k \leq m-1$ and $1 \leq \ell \leq n-1$, then the action of g_k,

g_ℓ^{*-1}, and e on $\underline{v}$ is given by

(6.17)

$$g_k \cdot \underline{v} = \begin{cases} q\underline{v} & \text{if } i_k = i_{k+1}, \\ q^{1/2}s_k \cdot \underline{v} + (q-1)\underline{v} & \text{if } i_k < i_{k+1}, \\ q^{1/2}s_k \cdot \underline{v} & \text{if } i_k > i_{k+1}, \end{cases}$$

$$g_\ell^{*-1} \cdot \underline{v} = \begin{cases} q^{-1}\underline{v} & \text{if } j_\ell = j_{\ell+1}, \\ q^{-1/2}s_\ell^* \cdot \underline{v} + (q^{-1}-1)\underline{v} & \text{if } j_\ell < j_{\ell+1}, \\ q^{-1/2}s_\ell^* \cdot \underline{v} & \text{if } j_\ell > j_{\ell+1}, \end{cases}$$

$$e \cdot \underline{v} = \delta_{i_m, j_1} q^{(i_m-1)} \sum_{k=1}^{r} v_{i_1} \otimes \cdots \otimes v_{i_{m-1}} \otimes v_k \otimes v_k^* \otimes v_{j_2}^* \otimes \cdots \otimes v_{j_n}^*,$$

where the transpositions s_k and s_ℓ^* act on $T_q^{m,n}$ by place permutations (3.1). One can check directly that this action of $H_{m,n}^r(q)$ on $T_q^{m,n}$ is well-defined. Notice that if $q = 1$, then the action of $H_{m,n}^r(q)$ specializes exactly to the action of $\mathcal{B}_{m,n}^r$.

7. Characters.

Denote by $\chi_{H_{m,n}^r}^\gamma(\zeta)$ the value of the $H_{m,n}^r(q)$-character $\chi_{H_{m,n}^r(q)}^\gamma(d_\zeta^{(q)})$ and by $\chi_{\mathcal{B}_{m,n}^r}^\gamma(\zeta)$ the value of the $\mathcal{B}_{m,n}^r$-character $\chi_{\mathcal{B}_{m,n}^r}^\gamma(d_\zeta)$. We derive the Frobenius formulas for $\mathcal{B}_{m,n}^r$ and $H_{m,n}^r(q)$ and use them to give a character formula for $\chi_{\mathcal{B}_{m,n}^r}^\gamma(\zeta)$ and $\chi_{H_{m,n}^r}^\gamma(\zeta)$ in terms of $\mathcal{S}_m \times \mathcal{S}_n$ and $H_m(q) \otimes H_n(q)$-characters, respectively.

7.1. Rational Frobenius Formulas. Let $x_1, x_2, \ldots, x_r$ be commuting, independent variables. Let $\{v_1, \ldots, v_r\}$ be the canonical basis of $V = \mathbb{C}^r$ and $V_q = V \otimes \mathbb{C}(q)$, and let $\{v_1^*, \ldots, v_r^*\}$ be its dual. Then the set $\{v_{i_1} \otimes \cdots \otimes v_{i_m} \otimes v_{j_1}^* \otimes \cdots \otimes v_{j_n}^* \,|\, 1 \le i_k, j_\ell \le r\}$ is a basis of simple tensors of for both the $\mathbb{C}$-vector space $T^{m,n} = (\otimes^m V) \otimes (\otimes^n V^*)$ and the $\mathbb{C}(q)$-vector space $T_q^{m,n} = (\otimes^m V_q) \otimes (\otimes^n V_q{}^*)$. Define the weight of each simple tensor $\underline{v} = v_{i_1} \otimes \cdots \otimes v_{i_m} \otimes v_{j_1}^* \otimes \cdots \otimes v_{j_n}^*$ to be

$$wt(\underline{v}) = x_{i_1} x_{i_2} \cdots x_{i_m} x_{j_1}^{-1} x_{j_2}^{-1} \cdots x_{j_n}^{-1},$$

and for $h \in H_{m,n}^r(q)$ define a *weighted trace* $wtr(h)$ of h acting on $T_q^{m,n}$ by

$$wtr(h) = \sum_{\underline{v}} h \cdot \underline{v}|_{\underline{v}} wt(\underline{v}) \tag{7.1}$$

where the sum is over all simple tensors $\underline{v} \in T^{m,n}$, and where $h \cdot \underline{v}|_{\underline{v}}$ is the coefficient of $\underline{v}$ in $h \cdot \underline{v}$. The set up here is analogous to that of [**HR**].

Let Γ denote the set of all r-tuples $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ of integers with the property that $|\alpha_1| + \cdots + |\alpha_r| = m + n - 2k$ for some k satisfying $0 \leq k \leq \min(m, n)$. For $\alpha \in \Gamma$, define

$$x_r^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_r^{\alpha_r}.$$

Then $\{x_r^\alpha | \alpha \in \Gamma\}$ is the set of all possible weights of simple tensors in $T_q^{m,n}$. For each $\alpha \in \Gamma$, define the weight space $T_{q,\alpha}^{m,n}$ to be the span of the simple tensors of weight α. That is,

$$T_{q,\alpha}^{m,n} \stackrel{\text{def}}{=} \mathbb{C}(q)\text{-span}\{\underline{v} = v_{i_1} \otimes \cdots \otimes v_{i_m} \otimes v_{j_1}^* \otimes \cdots \otimes v_{j_n}^* \mid wt(\underline{v}) = x_r^\alpha\}.$$

The action of $H_{m,n}^r(q)$ on simple tensors (6.13) preserves weight, so $T_{q,\alpha}^{m,n}$ is an $H_{m,n}^r(q)$-module. Thus, we can re-write the weighted trace by summing over the weight spaces as follows

$$wtr(h) = \sum_{\alpha \in \Gamma} \sum_{\underline{v} \in T_\alpha^{m,n}} h \cdot \underline{v}|_{\underline{v}} x_r^\alpha = \sum_{\alpha \in \Gamma} tr_\alpha(h) x_r^\alpha, \tag{7.2}$$

where $tr_\alpha(h)$ is the trace of the action of h on $T_\alpha^{m,n}$. Notice that if $h_1, h_2 \in H_{m,n}^r(q)$, then

$$wtr(h_1 h_2) = \sum_{\alpha \in \Gamma} tr_\alpha(h_1 h_2) x_r^\alpha = \sum_{\alpha \in \Gamma} tr_\alpha(h_2 h_1) x_r^\alpha = wtr(h_2 h_1),$$

so wtr satisfies the trace property.

Theorem 7.3. *If $b \in \mathcal{B}_{m,n}^r$, then* $wtr(b) = \sum_{\gamma \in \Phi_r^{m,n}} \chi_{\mathcal{B}_{m,n}^r}^\gamma(b) s_\gamma(x_1, \ldots, x_r)$.

Proof. Let $g \in GL_r$ be diagonal with eigenvalues $e_1, e_2, \ldots, e_r$. Let $e_r = (e_1, e_2, \ldots, e_r)$ and $x_r = (x_1, x_2, \ldots, x_r)$, and denote by $wtr(b)|_{x_r = e_r}$ and $wt(\underline{v})|_{x_r = e_r}$ the specializations of $wtr(b)$ and $wt(\underline{v})$ given by setting $x_i = e_i$ for each $i = 1, 2, \ldots, r$. Since $g \cdot v_i = e_i v_i$ and $g \cdot v_i^* = e_i^{-1} v_i^*$, we have $g \cdot \underline{v} = wt(\underline{v})|_{x_r = e_r} \underline{v}$ for each simple tensor $\underline{v}$. Thus, the bicharacter of $b \otimes g$ on $T^{m,n}$ satisfies

$$Tr(b \otimes g) = \sum_{\underline{v}} (b \otimes g) \cdot \underline{v}|_{\underline{v}} = \sum_{\underline{v}} b \cdot \underline{v}|_{\underline{v}} wt(\underline{v})|_{x_r = e_r} = wtr(b)|_{x_r = e_r}.$$

Moreover, by Theorem 3.10, we have

$$Tr(b \otimes g) = \sum_{\gamma \in \Phi_r^{m,n}} \chi_{\mathcal{B}_{m,n}^r}^\gamma(b) s_\gamma(e_1, \ldots, e_r),$$

so $wtr(b)$ and $\sum_{\gamma\in\Phi_r^{m,n}} \chi^\gamma_{\mathcal{B}^r_{m,n}}(b)s_\gamma(x_r)$ are rational functions in x_r that agree at infinitely many specializations $x_r = e_r \in \mathbb{C}^r$. Thus they are equal, and the theorem is proved. □

Corollary 7.4. *Let $p^\gamma \in \mathcal{B}^r_{m,n}$ be a minimal idempotent such that $p^\gamma z^\gamma = p^\gamma$, where z^γ is the minimal central idempotent of $\mathcal{B}^r_{m,n}$ associated to $\gamma \in \Phi_r^{m,n}$. Then $wtr(p^\gamma) = s_\gamma(x_r)$.*

Proof. If p^γ is a minimal idempotent corresponding to γ, then $\chi^\rho_{\mathcal{B}^r_{m,n}}(p^\gamma) = \delta_{\rho,\gamma}$, so

$$wtr(p^\gamma) = \sum_{\rho\in\Phi_r^{m,n}} \chi^\rho_{\mathcal{B}^r_{m,n}}(p^\gamma)s_\rho(x_r) = s_\gamma(x_r).$$

□

Lemma 7.5. *If $p \in H^r_{m,n}(q)$ is any idempotent, then the weighted trace $wtr(p)$ is independent of q.*

Proof. If $p \in H^r_{m,n}(q)$ is an idempotent, then we can view p acting on $T^{m,n}_{q,\alpha}$ as a projection from $T^{m,n}_\alpha$ to $pT^{m,n}_{q,\alpha}$. We choose a basis of $pT^{m,n}_\alpha$ and extend it to $T^{m,n}_\alpha$. Relative to this basis, the trace of the matrix of p is its rank, and therefore $tr_\alpha(p) \in \mathbb{Z}$. Moreover, we know that $tr_\alpha(p)$ is a rational function in q, so it must be a constant. Thus $wtr(p) = \sum_{\alpha\in\Gamma} tr_\alpha(p)x_r^\alpha$ does not depend on q. □

As in Section 6, let $\mathcal{C}^r_{m,n}(q) = End_{\mathcal{U}_q}(T_q^{m,n})$ and $\mathcal{C}^r_{m,n} = End_{GL_r}(T^{m,n})$. Then $H^r_{m,n}(q)$ (respectively, $\mathcal{B}^r_{m,n}$) maps onto $\mathcal{C}^r_{m,n}(q)$ ($\mathcal{C}^r_{m,n}$) and is isomorphic to $\mathcal{C}^r_{m,n}(q)$ ($\mathcal{C}^r_{m,n}$) when $r \geq m+n$. Moreover, from Proposition 6.17 there is a partition of unity

$$1 = \sum_{\gamma\in\Phi_r^{m,n}} \sum_{i=1}^{m_\gamma} p_i^\gamma \tag{7.6}$$

in $\mathcal{C}^r_{m,n}(q)$ that specializes to a partition of unity in $\mathcal{C}^r_{m,n}$.

Theorem 7.7. *If $h \in H^r_{m,n}(q)$, then $wtr(h) = \sum_{\gamma\in\Phi_r^{m,n}} \chi^\gamma_{H^r_{m,n}(q)}(h)\, s_\gamma(x_r)$.*

Proof. For each p_i^γ in (7.6) let $h^\gamma_{ii} \in \mathbb{C}(q)$ be the constant that satisfies $p_i^\gamma h p_i^\gamma = h^\gamma_{ii}p_i^\gamma$. That is, h^γ_{ii} is the (i,i)-entry of the matrix of h in the irreducible representation indexed by γ with respect to the partition of unity. Therefore,

$$\chi^\gamma_{H^r_{m,n}(q)}(h) = \sum_{i=1}^{m_\gamma} h^\gamma_{ii}, \qquad \text{for each } \gamma \in \Phi_r^{m,n}.$$

Using the trace property of wtr, we have

$$\begin{aligned} wtr(p_i^\gamma h p_j^\rho) &= wtr(p_j^\rho p_i^\gamma h) \\ &= \delta_{\gamma,\rho}\delta_{i,j} wtr(p_i^\gamma h p_i^\gamma) \\ &= \delta_{\gamma,\rho}\delta_{i,j} h_{ii}^\gamma wtr(p_i^\gamma) \\ &= \delta_{\gamma,\rho}\delta_{i,j} h_{ii}^\gamma s_\gamma(x_r), \end{aligned}$$

where the last equality is proved by using Lemma 7.5 to say that $wtr(p_i^\gamma)$ is independent of q and using Corollary 7.4 to say that it equals $s_\gamma(x_r)$ when $q = 1$. Thus

$$\begin{aligned} wtr(h) &= \sum_{\gamma\in\Phi_r^{m,n}} \sum_{\rho\in\Phi_r^{m,n}} \sum_{i=1}^{m_\gamma}\sum_{j=1}^{m_\rho} wtr(p_i^\gamma h p_j^\rho) \\ &= \sum_{\gamma\in\Phi_r^{m,n}} \left(\sum_{i=1}^{m_\gamma} h_{ii}^\gamma\right) s_\gamma(x_r), \\ &= \sum_{\gamma\in\Phi_r^{m,n}} \chi^\gamma_{H^r_{m,n}(q)}(h) s_\gamma(x_r). \end{aligned}$$

□

The trick to computing the Frobenius formula is to now compute the weighted trace in another way. To do this requires the following property of the weighted trace.

Proposition 7.8. *For $i = 1, \dots, t$, let $h_i \in H^r_{m_i,n_i}(q)$. Then the weighted trace of $h_1 \otimes h_2 \otimes \cdots h_t \in H^r_{m_1,n_1}(q) \otimes H^r_{m_2,n_2}(q) \otimes \cdots \otimes H^r_{m_t,n_t}(q)$ satisfies*

$$wtr(h_1 \otimes h_2 \otimes \cdots \otimes h_t) = wtr(h_1)wtr(h_2)\cdots wtr(h_t).$$

Proof. It suffices to prove the result for $t = 2$. Let $h = h_1 \otimes h_2 \in H^r_{m_1,n_1}(q) \otimes H^r_{m_2,n_2}(q) \subseteq H^r_{m,n}(q)$ where $m_1 + m_2 = m$ and $n_1 + n_2 = n$. For each simple tensor $\underline{v} = v_{i_1} \otimes \cdots \otimes v_{i_m} \otimes v^*_{j_1} \otimes \cdots \otimes v^*_{j_n}$, let $\underline{v}' = v_{i_1} \otimes \cdots \otimes v_{i_{m_1}}$, $\underline{v}'' = v_{i_{m_1+1}} \otimes \cdots \otimes v_{i_{m_2}}$, $\underline{v}^{*'} = v^*_{j_1} \otimes \cdots \otimes v^*_{j_{n_1}}$, and $\underline{v}^{*''} = v^*_{j_{n_1+1}} \otimes \cdots \otimes v^*_{j_{n_2}}$. Then, since h_1 only acts on v' and $v^{*'}$ and h_2 only acts on v'' and $v^{*''}$, we have

$$\sum_{\underline{v}}(h_1 \otimes h_2) \cdot \underline{v}|_{\underline{v}} wt(\underline{v})$$
$$= \sum_{\substack{\underline{v}', \underline{v}^{*'} \\ \underline{v}'', \underline{v}^{*''}}} (h_1 \otimes h_2) \cdot (\underline{v}' \otimes \underline{v}'' \otimes \underline{v}^{*'} \otimes \underline{v}^{*''})|_{\underline{v}' \otimes \underline{v}'' \otimes \underline{v}^{*'} \otimes \underline{v}^{*''}} wt(\underline{v}' \otimes \underline{v}'' \otimes \underline{v}^{*'} \otimes \underline{v}^{*''})$$
$$= \left(\sum_{\underline{v}', \underline{v}^{*'}} h_1 \cdot (\underline{v}' \otimes \underline{v}^{*'})|_{\underline{v}' \otimes \underline{v}^{*'}} wt(\underline{v}' \otimes \underline{v}^{*'}) \right)$$
$$\cdot \left(\sum_{\underline{v}'', \underline{v}^{*''}} h_2 \cdot (\underline{v}'' \otimes \underline{v}^{*''})|_{\underline{v}'' \otimes \underline{v}^{*''}} wt(\underline{v}'' \otimes \underline{v}^{*''}) \right).$$

Therefore, $wtr(h_1 \otimes h_2) = wtr(h_1)wtr(h_2)$. Note this is essentially a proof of the fact that the trace of the action of $H^r_{m_1,n_1}(q) \otimes H^r_{m_2,n_2}(q)$ on $T_q^{m,n}$ is the product of the traces of the action of $H^r_{m_i,n_i}(q)$ on $T_q^{m_i,n_i}$. □

Let $\zeta \in \Phi_r^{m,n}$. Then by Proposition 7.8 the weighted trace of the character class representatives $d_\zeta^{(q)}$ and d_ζ (5.10) satisfies

$$wtr(d_\zeta^{(q)}) \;=\; wtr(d_{\zeta^+}^{(q)})wtr(e)^{h(\zeta)}wtr(d_{\zeta^-}^{(q)}). \tag{7.9}$$

Moreover, if the lengths of ζ^+ and ζ^- are $\ell(\zeta^+) = i$ and $\ell(\zeta^-) = j$, respectively, then

$$\begin{aligned} wtr(d_{\zeta^+}^{(q)}) &= wtr(d_{\zeta_1}^{(q)})wtr(d_{\zeta_2}^{(q)}) \cdots wtr(d_{\zeta_i}^{(q)}), \\ wtr(d_{\zeta^-}^{(q)}) &= wtr(d_{\zeta_{r-j}}^{(q)}) \cdots wtr(d_{\zeta_{r-1}}^{(q)})wtr(d_{\zeta_r}^{(q)}). \end{aligned} \tag{7.10}$$

Thus, we directly compute the weighted traces $wtr(e)$ and $wtr(d_k^{(q)})$ for each $k \in \mathbb{Z}$. To do this requires a q-extension of the power symmetric function given in [**R1**]. It is defined on the integer $k > 0$ by

$$\overline{p}_k(q; x_r) \;=\; \sum_{I=(i_1,\dots,i_k)} q^{E(I)}(q-1)^{L(I)} x_{i_1} x_{i_2} \cdots x_{i_k}, \tag{7.11}$$

where the sum is over all weakly increasing sequences $I = \{1 \le i_1 \le i_2 \le \cdots \le i_k \le r\}$ and $E(I) = |\{1 \le j < k \mid i_j = i_{j+1}\}|$ and $L(I) = |\{1 \le j < k \mid i_j < i_{j+1}\}|$. For the partition α let $\overline{p}_\alpha = \overline{p}_{\alpha_1}\overline{p}_{\alpha_2} \cdots \overline{p}_{\alpha_t}$. Notice that when $q = 1$, $\overline{p}_k = p_k$ and $\overline{p}_\alpha = p_\alpha$. The next theorem is due to Schur for $q = 1$ and was generalized to generic q by Ram.

Theorem 7.12 **[Sc1, Sc2]**, **[R1]**. *If $k > 0$, then the weighted trace of $d_k^{(q)}$ on $\otimes^k V_q$ is $wtr(d_k^{(q)}) = \overline{p}_\alpha(q; x_r)$.*

Proof. Let $\underline{v} = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_k$, $\underline{v}' = v_{i_2} \otimes \cdots \otimes v_k$, and $\underline{v}'' = v_{i_3} \otimes \cdots \otimes v_k$. Then, recalling the action of g_i on $T_q^{m,n}$ (see (6.18)), we have three cases to consider.
Case 1: $i_1 > i_2$.

$$(g_{k-1} \cdots g_2 g_1) \cdot \underline{v}|_{\underline{v}} wt(\underline{v}) \;=\; q^{1/2}(g_{k-1} \cdots g_2) \cdot (v_{i_2} \otimes v_{i_1} \otimes \underline{v}'')|_{\underline{v}} wt(\underline{v}) \;=\; 0,$$

since $v_{i_2} \neq v_{i_1}$ and $g_{k-1} \cdots g_2$ acts only on $v_{i_1} \otimes \underline{v}''$.
Case 2: $i_1 = i_2$.

$$\begin{aligned}(g_{k-1} \cdots g_2 g_1) \cdot \underline{v}|_{\underline{v}} wt(\underline{v}) &= q(g_{k-1} \cdots g_2) \cdot \underline{v}|_{\underline{v}} wt(\underline{v}) \\ &= q x_{i_1} (g_{k-1} \cdots g_2) \cdot \underline{v}'|_{\underline{v}'} wt(\underline{v}').\end{aligned}$$

Case 3: $i_1 < i_2$.

$$\begin{aligned}&(g_{k-1} \cdots g_2 g_1)\underline{v}|_{\underline{v}} wt(\underline{v}) \\ &= q^{1/2}(g_{k-1} \cdots g_2) \cdot (v_{i_2} \otimes v_{i_1} \otimes \underline{v}'')|_{\underline{v}} wt(\underline{v}) + (q-1)(g_{k-1} \cdots g_2) \cdot \underline{v}|_{\underline{v}} wt(\underline{v}) \\ &= 0 + (q-1) x_{i_1} (g_{k-1} \cdots g_2) \cdot \underline{v}'|_{\underline{v}'} wt(\underline{v}').\end{aligned}$$

The theorem follows by induction on k. □

We extend the definition of $\overline{p}_k$ to $k \leq 0$ by letting $\overline{p}_0 = 1$, and for $k < 0$ letting

$$\overline{p}_k(q; x_r) \;=\; \overline{p}_{-k}(q^{-1}; x_r^{-1}). \tag{7.13}$$

Then if $\zeta \in \Phi_r^{m,n}$, we let

$$\overline{p}_\zeta = \overline{p}_{\zeta_1} \overline{p}_{\zeta_2} \cdots \overline{p}_{\zeta_r},$$

and we immediately have the identity

$$\overline{p}_\zeta(q; x_r) = \overline{p}_{\zeta^+}(q; x_r) \overline{p}_{\zeta^-}(q^{-1}; x_r^{-1}). \tag{7.14}$$

When $q = 1$ we get the corresponding extension of the power symmetric functions to the r staircase $\zeta \in \Phi_r^{m,n}$ given by letting $p_0(x_r) = 1$, for $k < 0$ letting

$$p_k(x_r) = p_{-k}(x_r^{-1}) = x_1^k + x_2^k + \cdots + x_r^k, \tag{7.15}$$

and for $\zeta \in \Phi_r^{m,n}$ letting

$$p_\zeta = p_{\zeta_1} p_{\zeta_2} \cdots p_{\zeta_r}.$$

Proposition 7.16. *If $k \le 0$, then the weighted trace of $d_k^{(q)}$ on $\otimes^{-k} {V_q}^*$ is $wtr(d_k^{(q)}) = \overline{p}_k(q; x_r) = \overline{p}_{-k}(q^{-1}; x_r^{-1})$.*

Proof. If $k = 0$ the result holds trivially. If $k < 0$, then considering the action of $d_k^q = g_1^{*-1} \cdots g_{-k-1}^{*-1}$ on $\otimes^{-k} {V_q}^*$ (see (6.13)), the proof of the previous theorem holds with q^{-1} in place of q and x_i^{-1} in place of x_i. □

Proposition 7.17. *The weighted trace of e on $V_q \otimes {V_q}^*$ is $wtr(e) = [\![r]\!]_q$.*

Proof. Since $e \cdot (v_i \otimes v_j^*) = \delta_{i,j} q^{i-1} \sum_{i=1}^r v_k \otimes v_k^*$, we have

$$\begin{aligned} wtr(e) &= \sum_{i,j} x_i x_j^{-1} e \cdot (v_i \otimes v_j)|_{v_i \otimes v_j} \\ &= \sum_{i,j} x_i x_j^{-1} \delta_{i,j} \sum_{k=1}^r q^{k-1} (v_k \otimes v_k^*)|_{v_i \otimes v_j^*} \\ &= \sum_{i=1}^r q^{i-1} \; = \; [\![r]\!]_q. \end{aligned}$$

□

We conclude that if $\zeta \in \Phi_r^{m,n}$ with $\zeta^+ \vdash (m-h)$ and $\zeta^- \vdash (n-h)$, then

$$wtr(d_\zeta^{(q)}) \; = \; [\![r]\!]_q^h \overline{p}_\zeta(q; x_r) \quad \text{and} \quad wtr(d_\zeta) \; = \; r^h p_\zeta(x_r). \tag{7.18}$$

As an immediate consequence of (7.18) and Theorems 7.3 and 7.8 we get the Frobenius formulas:

Theorem 7.19 (Frobenius Formula). *If $\zeta \in \Phi_r^{m,n}$ with $\zeta^+ \vdash (m-h)$ and $\zeta^- \vdash (n-h)$, then*

(i) $$r^h p_\zeta(x_r) \; = \; \sum_{\gamma \in \Phi_r^{m,n}} \chi_{H_{m,n}^r}^\gamma(\zeta)(d_\zeta^{(q)}) s_\gamma(x_r),$$

(ii) $$[\![r]\!]_q^h \overline{p}_\zeta(q; x_r) \; = \; \sum_{\gamma \in \Phi_r^{m,n}} \chi_{H_{m,n}^r}^\gamma(\zeta)(d_\zeta^{(q)}) s_\gamma(x_r).$$

7.2. Character Formulas. We now use the Frobenius formulas to write the characters of $H_{m,n}^r(q)$ and $\mathcal{B}_{m,n}^x$ in terms of their subalgebras:

$$\begin{gathered} \mathbb{C}[\mathcal{S}_m \times \mathcal{S}_n] \subseteq \mathcal{B}_{m,n}^x, \\ H_m(q) \otimes H_n(q) \subseteq H_{m,n}^r(q). \end{gathered}$$

For $\alpha, \lambda \vdash f$, let $\chi_{\mathcal{S}_f}^\lambda(\alpha)$ denote the irreducible $\mathcal{S}_f$-character labeled by λ evaluated on the conjugacy class determined by α, and let $\chi_{H_f(q)}^\lambda(\alpha)$ denote the irreducible $H_f(q)$-character labeled by λ evaluated on the conjugacy class determined by α.

Theorem 7.20. *Let $\gamma, \zeta \in \Phi_r^{m,n}$ with $\gamma^+ \vdash (m-k)$, $\gamma^- \vdash (n-k)$, $\zeta^+ \vdash m' = (m-h)$, and $\zeta^- \vdash n' = (n-h)$. Then*

(i) $$\chi^\gamma_{B^x_{m,n}}(\zeta) = x^h \sum_{\substack{\lambda \vdash m' \\ \pi \vdash n'}} \left(\sum_{\delta \vdash (k-h)} (c^\lambda_{\delta\gamma^+})(c^\pi_{\delta\gamma^-}) \right) \chi^\lambda_{S_{m'}}(\zeta^+)\chi^\pi_{S_{n'}}(\zeta^-),$$

(ii) $$\chi^\gamma_{H^r_{m,n}}(\zeta) = [\![r]\!]_q^h \sum_{\substack{\lambda \vdash m' \\ \pi \vdash n'}} \left(\sum_{\delta \vdash (k-h)} (c^\lambda_{\delta\gamma^+})(c^\pi_{\delta\gamma^-}) \right) \chi^\lambda_{H_{m'}(q)}(\zeta^+)\chi^\pi_{H_{n'}(q^{-1})}(\zeta^-).$$

Proof. To prove (ii), we view the Frobenius formula (Theorem 7.19) first with $m = m'$ and $n = 0$ and then with $n = n'$ and $m = 0$ to obtain

$$\overline{p}_{\zeta^+}(q; x_r) = \sum_{\lambda \vdash m'} \chi^\lambda_{H_{m'}(q)}(\zeta^+) s_\lambda(x_r)$$

and

$$\overline{p}_{\zeta^-}(q^{-1}; x_r^{-1}) = \sum_{\pi \vdash n'} \chi^\pi_{H_{n'}(q^{-1})}(\zeta^-) s_\pi(x_r^{-1}).$$

Note that these are the Frobenius formulas for $H_{m'}(q)$ on $\otimes^{m'} V_q$ and $H_{n'}(q^{-1})$ on $\otimes^{n'} V_q^*$, respectively. Substituting into (7.15) gives

$$\overline{p}_\zeta(q; x_r) \;=\; [\![r]\!]_q^h \sum_{\substack{\lambda \vdash m' \\ \pi \vdash n'}} \chi^\lambda_{H_{m'}(q)}(\zeta^+)\chi^\pi_{H_{n'}(q^{-1})}(\zeta^-) s_\lambda(x_r) s_\pi(x_r^{-1}).$$

Using the branching rule of Theorem 2.13(b) to expand the product $s_\lambda(x_r)s_\pi(x_r^{-1})$ in terms of rational Schur functions gives

$$\begin{aligned} &\overline{p}_\zeta(q; x_r) \\ &= [\![r]\!]_q^h \sum_{\substack{\lambda \vdash m' \\ \pi \vdash n'}} \sum_{\gamma \in \Phi_r^{m,n}} \left(\sum_\delta (c^\lambda_{\delta\gamma^+})(c^\pi_{\delta\gamma^+}) \right) \chi^\lambda_{H_{m'}(q)}(\zeta^+)\chi^\pi_{H_{n'}(q^{-1})}(\zeta^-) s_\gamma(x_r). \end{aligned}$$

Since the rational Schur functions $s_\gamma(x_r)$ are linearly independent (see [**Koi**] or [**Hal**]), we can equate the coefficient of $s_\gamma(x_r)$ above with the coefficient of $s_\gamma(x_r)$ in Theorem 7.19 and obtain

$$\chi^\gamma_{H^r_{m,n}}(\zeta) \;=\; [\![r]\!]_q^h \sum_{\substack{\lambda \vdash m' \\ \pi \vdash n'}} \left(\sum_\delta (c^\lambda_{\delta\gamma^+})(c^\pi_{\delta\gamma^-}) \right) \chi^\lambda_{H_{m'}(q)}(\zeta^+)\chi^\pi_{H_{n'}(q^{-1})}(\zeta^-).$$

This proves (ii). Setting $q = 1$ proves (i) in the case where $x = r \geq m+n$. To extend to the indeterminate x, let

$$c(x) = x^h \sum_{\substack{\lambda \vdash m' \\ \pi \vdash n'}} \left(\sum_{\delta} \left(c^{\lambda}_{\delta\gamma^+}\right) \left(c^{\pi}_{\delta\gamma^-}\right)\right) \chi^{\lambda}_{S_{m'}}(\zeta^+)\chi^{\pi}_{S_{n'}}(\zeta^-).$$

Then for all integers $r \geq m+n$, we have $c(r) = \chi^{\gamma}_{\mathcal{B}^r_{m,n}}(\zeta)$, so $c(x)$ and $\chi^{\gamma}_{\mathcal{B}^x_{m,n}}(\zeta)$ are rational functions in x that agree at an infinite number of points and thus are equal. □

Corollary 7.21. *Let $\gamma, \zeta \in \Phi^{m,n}$ with $\gamma^+ \vdash (m-k)$ and $\gamma^- \vdash (n-k)$, and $\zeta^+ \vdash m' = (m-h)$ and $\beta \vdash n' = (n-h)$. Then*

(a) *If $h > k$, then $\chi^{\gamma}_{H^r_{m,n}}(\zeta) = 0$,*

(b) *If $h = k$, then $\chi^{\gamma}_{H^r_{m,n}}(\zeta) = [\![r]\!]^h_q \chi^{\lambda}_{H_{m'}(q)}(\zeta^+)\chi^{\pi}_{H_{n'}(q^{-1})}(\zeta^-)$,*

(c) *If $k \geq h > 0$, then $\chi^{\gamma}_{H^r_{m,n}}(\zeta) = [\![r]\!]_q \chi^{\gamma}_{H^r_{m-1,n-1}}(\zeta)$.*

Proof. To have $(c^{\lambda}_{\delta\gamma^+})(c^{\pi}_{\delta\gamma^-}) \neq 0$ requires that $\gamma^+ \subseteq \lambda$ and $\gamma^- \subseteq \pi$. Therefore, we must have $m-k \leq m-h$, which proves (i). If $h = k$, then to have $(c^{\lambda}_{\delta\gamma^+})(c^{\pi}_{\delta\gamma^-}) \neq 0$ requires that we have $\gamma^+ = \lambda, \gamma^- = \pi$, and $\delta = \emptyset$. In this case, $(c^{\lambda}_{\delta\gamma^+})(c^{\pi}_{\delta\gamma^-}) = 1$, and (ii) follows. If $k \geq h > 0$, then let $\tilde{d}^q_{\zeta} = d^q_{\zeta^+} \otimes e^{\otimes(h-1)} \otimes d^q_{\zeta^-}$, and

$$\begin{aligned}
\chi^{\gamma}_{H^r_{m,n}}(\zeta) &= [\![r]\!]_q [\![r]\!]^{h-1}_q \sum_{\substack{\lambda \vdash m' \\ \pi \vdash n'}} \left(\sum_{\delta} (c^{\lambda}_{\delta\gamma^+})(c^{\pi}_{\delta\gamma^-})\right) \chi^{\lambda}_{H_{m'}(q)}(\zeta^+)\chi^{\pi}_{H_{n'}(q^{-1})}(\zeta^-) \\
&= [\![r]\!]_q \chi^{\gamma}_{H^r_{m-1,n-1}(q)}(\zeta).
\end{aligned}$$

□

The *character table* for $H^r_{m,n}(q)$, denoted $\Xi^q_{m,n}$, is the matrix whose rows and columns are indexed by elements of $\Phi^{m,n}_r$ and whose (γ, ζ)-entry is given by $\chi^{\gamma}_{H^r_{m,n}}(\zeta)$. From Corollary 7.22, we see that if we put an order on $\Phi^{m,n}$ so that h and k are increasing, then $\Xi^q_{m,n}$ has the form Here Ξ^q_m is character

$\Xi^q_m \otimes \Xi^{q^{-1}}_n$	0
$\mathfrak{B}$	$[\![r]\!]_q \Xi^q_{m-1,n-1}$

Table 7.22: Character Table for $H^r_{m,n}(q)$.

table for $H_m(q)$, and $\Xi^{q^{-1}}_n$ is character table for $H_n(q^{-1})$. The matrix $\mathfrak{B}$

depends both on Iwahori-Hecke algebra characters and on the branching rule. The indices for the columns of the branching matrix $\mathfrak{B}$ have $h=0$ and the indices for the rows have $k>0$. Thus the (γ,ζ)-entry of $\mathfrak{B}$ is

$$\sum_{\substack{\lambda\vdash m\\ \pi\vdash n}}\left(\sum_{\delta\vdash k} c^{\lambda}_{\delta\gamma^+}c^{\pi}_{\delta\nu}\right)\chi^{\lambda}_{H_m(q)}(d^{(q)}_{\alpha})\chi^{\pi}_{H_n(q)}(d^{(q)^*}_{\beta}). \tag{7.23}$$

The results of Corollary 7.20 hold when $q=1$, and thus the character table for $\mathcal{B}^r_{m,n}$ is $\Xi^1_{m,n}$. We are using the fact that $\Xi^1_m\otimes\Xi^1_n$ is the character table for $\mathcal{S}_m\times\mathcal{S}_n$. Setting $q=1$ and replacing r with x in (7.23) gives the character table for $\mathcal{B}^x_{m,n}$.

As a final corollary of Theorem 7.19, we obtain the branching rules for the inclusions $\mathbb{C}[\mathcal{S}_m\times\mathcal{S}_n]\subseteq\mathcal{B}^x_{m,n}$ and $H_m(q)\otimes H_n(q)\subseteq H^r_{m,n}(q)$.

Corollary 7.24. *If $r\geq m+n$ and $\gamma\in\Phi^{m,n}_r$, then*
(i) *the decomposition of the $\mathcal{B}^x_{m,n}$-module M^γ into irreducible $\mathbb{C}[\mathcal{S}_m\times\mathcal{S}_n]$-modules is given by*

$$M^\gamma\downarrow^{\mathcal{B}^x_{m,n}}_{\mathbb{C}[\mathcal{S}_m\times\mathcal{S}_n]}\cong\bigoplus_{\substack{\alpha\vdash m\\ \beta\vdash n}}\left(\sum_{\delta}(c^{\alpha}_{\delta,\gamma^+}c^{\beta}_{\delta,\gamma^-})\right)S^\alpha\otimes S^\beta,$$

(ii) *the decomposition of the $H^r_{m,n}(q)$-module M^γ_q into irreducible $H_m(q)\otimes H_n(q)$-modules is given by*

$$M^\gamma_q\downarrow^{H^r_{m,n}(q)}_{H_m(q)\otimes H_n(q)}\cong\bigoplus_{\substack{\alpha\vdash m\\ \beta\vdash n}}\left(\sum_{\delta}(c^{\alpha}_{\delta,\gamma^+}c^{\beta}_{\delta,\gamma^-})\right)S^\alpha_q\otimes S^\beta_q.$$

Concluding Remarks.

1. Leduc [**Le**] has recently given a $\mathbb{C}(z,q)$-algebra $\mathcal{A}_{m,n}(z,q)$ which specializes to $\mathcal{B}^x_{m,n}$ when $q\to1$ and to $H^r_{m,n}(q)$ when $x\to q^r$. The algebra $\mathcal{A}_{m,n}(z,q)$ is analogous to the Birman-Wenzl algebra $BW_f(z,q)$ (see [**Wen3**]) which is isomorphic to the Brauer algebra $\mathcal{B}^x_f$ when $q\to1$. If, in our work here, we replace q^r with z and let $x=\frac{1-z}{1-q}$, then the basis $\mathcal{D}^q_{m,n}$ is a basis for $\mathcal{A}_{m,n}(z,q)$, and if we replace $[\![r]\!]_q$ in $\Xi^q_{m,n}$ with x, we get the character table for $\mathcal{A}_{m,n}(z,q)$.

2. Leduc [**Le**] constructs a Markov trace on $H^r_{m,n}(q)$ such that $H^r_{m+1,n+1}(q)$ is isomorphic to a direct sum of $H_m(q)\otimes H_n(q)$ and a Jones basic construction for $H^r_{m-1,n-1}(q)\subseteq H^r_{m,n}(q)$. A recent paper by Halverson and Ram [**HR**] studies the characters of algebras containing a Jones basic construction. It follows from the results in this paper that the character table of $H^r_{m,n}(q)$ should take the form of Table 7.23. However, the work done in this paper is necessary to give an explicit $H^r_{m,n}(q)$-basis on which to compute characters, to give the Frobenius formulas, and to give the character formulas.

3. Since $T^{m,n} \cong T^{m+n}$ as modules for the orthogonal group $O(r,\mathbb{C})$ (see (3.4)), the branching rule for $\mathcal{B}^r_{m,n} \subseteq \mathcal{B}^r_{m+n}$ is the same as for $O_r \subseteq GL_r$. This rule was given by Littlewood [**Li**] for irreducible polynomial GL_r-modules V^λ with $\ell(\lambda) \le \lfloor r/2 \rfloor$. It is

$$V^\lambda \downarrow^{GL_r}_{O_r} \cong \sum_{\mu} \left(\sum_{\beta \ even} c^\lambda_{\beta\mu} \tilde{V}^\mu \right), \tag{7.25}$$

where V^μ is the irreducible O_r-module labeled by μ, and β even meaning that β has even parts (i.e., rows). Since $\det(g) = \pm 1$ for $g \in O_r$, and the irreducible rational GL_r-module V^γ indexed by γ is given by the representation $\phi_\gamma = \det^{\gamma_r - 1} \phi_{\lambda(\gamma)}$ (see (2.9)), the restriction rule $V^\gamma \downarrow^{GL_r}_{O_r}$ is the same as $V^{\lambda(\gamma)} \downarrow^{GL_r}_{O_r}$ when $\ell(\lambda(\gamma)) \le \lfloor r/2 \rfloor$. In the case when $\ell(\lambda(\gamma)) > \lfloor r/2 \rfloor$, one must use the modification rules of King [**Ki**] and Koike and Terada [**KT**] to decompose $V^{\lambda(\gamma)}$ into irreducible O_r-modules. Thus, it remains an open question to determine in closed-form the multiplicity of the irreducible $\mathcal{B}^r_{m,n}$-module M^γ in the irreducible $\mathcal{B}^r_{m+n}$-module $\tilde{M}^\mu$.

4. A natural question to ask is whether $H^r_{m,n}(q)$ can be embedded in the Birman-Wenzl algebra $BW_{m+n}(z,q)$, which is a the q-deformation of the Brauer algebra (see [**Wen3**]), in the same way that $\mathcal{B}^x_{m,n}$ is embedded as a subalgebra of $\mathcal{B}^x_{m+n}$. It turns out that such an embedding is impossible, as it would force the containment of the quantum orthogonal group $\mathcal{U}_q(o(r,\mathbb{C}))$ in $\mathcal{U}_q(g\ell(r,\mathbb{C}))$. Such a containment does not hold. See [**HR**] for example.

5. Theorem 7.19 provides a completely algebraic proof that if $r \ge m+n$, then as an $H^r_{m,n}(q) \otimes \mathcal{U}_q(g\ell(r,\mathbb{C}))$-bimodule,

$$T^{m,n}_q \cong \bigoplus_{\gamma \in \Phi^{m,n}_r} M^\gamma_q \otimes V^\gamma_q,$$

where V^γ_q is an irreducible $\mathcal{U}_q(g\ell(r,\mathbb{C}))$-module.

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