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COMPLETE INTERSECTION SUBVARIETIES OF GENERAL HYPERSURFACES

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In this paper we classify the nontrivial complete intersection curves on a general hypersurface of large enough degree. We prove, that in principle, one can classify nontrivial complete intersection curves on hypersurfaces with relatively small degree as well, and give a recipe for doing so. We also estimate the codimension of the components of the Noether-Lefschetz locus corresponding to complete intersection curves. Similar theorems hold for higher dimensional complete intersection subvarieties.

Introduction.

Let $X \subset \mathbb{P}^3$ be a very general hypersurface of degree at least four. The classical theorem of Noether–Lefschetz asserts that any curve on X is the complete intersection of X with some other surface. For hypersurfaces in higher dimensional projective spaces similar questions are poorly understood. Griffiths–Harris [**3**] posed a series of conjectures about curves on hypersurfaces. The strongest one turned out to be false (Voisin in [**1**]), the weaker ones have been proved in some cases ([**2**], [**7**], and Kollár-s example in Trento Examples in [**6**]). There is another generalization in [**5**].

The aim of this note is to look at a special case of the above problem: to try to understand those curves $C \subset X \subset \mathbb{P}^{n+1}$ which are complete intersections in $\mathbb{P}^{n+1}$ but not in X. Even in this special case the problem turns out to be surprisingly subtle. We give a complete answer in case $\deg X > \dim X + 2$ (Corollary A). We prove that if $\deg X > \epsilon \cdot \dim X$ for a fixed $\epsilon > 0$, then a complete description is possible in principle (Corollary B). This is somewhat surprising since other existence theorems about special curves on hypersurfaces seem to predict that there are lots of nontrivial curves if $\deg X < \dim X$.

Actually, we prove more in Corollary B. We prove that if $\deg X > \epsilon \cdot \dim X$ for a fixed $\epsilon > 0$, then (in principle) one can classify all complete intersection curves $C \subset \mathbb{P}^{n+1}$ such that the corresponding component of the Noether-Lefschetz locus has small codimension, say smaller than $C \deg(X)$ for some constant C. Moreover, this classification depends only on C and ϵ, but independent of $\deg(X)$ and $\dim(X)$.

We prove also that essentially the same results hold for r dimensional complete intersection subvarieties, just one has to replace $\deg(X)$ with $\deg(X)^r$ everywhere (Corollary B).

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Throughout this paper we shall work with projective varieties over a fixed algebraically closed field (of any characteristic). That a statement is true for a general point means that it is true in a dense open subset.

Let $f = 0$ be the equation of $X \subset \mathbb{P}^N$, and $g_1 = 0, \dots g_n = 0$ be the equations of a complete intersection variety $V \subset \mathbb{P}^N$ for some N. One way to ensure that $V \subset X$ is to take $g_i = f$ for some i. We are not interested in these kind of subvarieties.

Definition. A complete intersection subvariety V of a hypersurface X is called *nontrivial* if we cannot write V as the complete intersection of X and other hypersurfaces.

If we find a $V \subset X$ then we can find polynomials $h_1, \dots h_n$ such that $f = \sum_{i=1}^n g_i h_i$. For generic choice of g_i and h_i we can interchange some g_i with the corresponding h_i and get another complete intersection $\bar{V} \subset X$. The following definition reformulates this symmetry in terms of the multidegree of V. Also it is convenient to talk about lines, plane cubics, etc. without specifying the dimension of the ambient space.

Definition. We say that two sequences $l = (l_1, l_2, \dots l_m)$ and $L = (L_1, L_2, \dots L_n)$ are *equivalent* if one can get the first from the second by adding or deleting some 1 to or from it, and permuting the entries. We denote it by $l \sim L$. Fix a degree d. The above two sequences are *related* (with respect to d) if one can get the first from the second by replacing some of the L_i with $d - L_i$, adding or deleting some 1 or $d-1$ to or from it, and permuting the entries.

Let us see what we should expect. Let $X \subset \mathbb{P}^{n+1}$ be a general hypersurface of degree d. Let V be a complete intersection curve of multidegree $l \approx (l_1, \dots l_m)$ where $l_i > 1$. We shall fix m and l_i, and vary n and d. An easy dimension counting gives that the dimension of the family of complete intersection curves of multidegree l contained in X is at most $\alpha n - \beta d + \gamma$ with coefficients independent of n and d. This dimension estimation is done in Lemma Γ in a more general setup. In this formula one can easily calculate α and β, but it seems hard to give useful estimates for γ. From the calculation one sees that α/β decreases rapidly if we increase m or any of the l_i; hence if $d \geq \epsilon n$ for some $\epsilon > 0$ then one can try to list the possible multidegrees for V contained in a general X (and the list should not depend on d or n). In particular, for large enough ϵ, a general X should not contain any nontrivial complete intersection subvariety. Fortunately this picture is essentially correct, as we shall see in Corollary A, but the proof is long, because of the presence of the above γ. Main Theorem A reduces the problem to a finite number of multidegrees, and then the above estimate of α/β can be used to get the actual list of exceptions. The basic estimates for Main Theorem A are proved in the second half of the paper, namely in Theorem 3.

If we consider r-dimensional complete intersection subvarieties, we can get very similar estimates. The only change is that we have to replace the term βd with a degree r polynomial (coming from the Hilbert polynomial of V), and we can calculate the leading coefficient. We shall see also that when a general X does not contain subvarieties of multidegree l, then the same formula gives an estimate for the codimension of the loci of those X that contains one. One can hope for the same kind of picture as in dimension one, and indeed, one gets finite, easily calculable exceptional lists. This more general (but less explicit) result is contained in Main Theorem B and Corollary B, and this proof is based on the estimates given in Theorem 2.

Proposition C contains some result in the other direction. It gives some example when a general X contains curves of multidegree l. The proof is based on a construction given in Lemma C. I learned this construction from János Kollár. The result is far from being complete, but at least proves that Corollary A is sharp, and classifies the complete intersection subvarieties of a general quintic threefold.

Proposition D is an easy calculation for the next simplest case, for projectively normal curves.

Now we state the main results precisely.

Main Theorem A. *Let $d > n \geq 1$ be integers and assume that a general hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d contains a nontrivial complete intersection curve C of multidegree $l = (l_1, \dots l_n)$.*

If $d \geq 11$ then l is related to (1), (2), $(2,2)$ or (3).

If $d = 8, 9, 10$ then l is related to (1), (2), $(2,2)$, (3) or (4).

If $d = 7$ then l is related to (1), (2), $(2,2)$, (3) or $(2,3)$.

If $d = 6$ then l is related to (1), (2), $(2,2)$, $(2,2,2)$, (3), $(2,3)$, $(2,2,3)$ or $(3,3)$.

Moreover, if $d \geq 6$ then a general X does not contain any complete inter-

section surface.

Corollary A. *Let $X \subset \mathbb{P}^{n+1}$ be a general hypersurface of degree d.*
- *If $d > 2n - 1$ then there are no nontrivial complete intersection curves on X.*
- *If $d > \frac{3}{2}n + \frac{1}{2}$ then the nontrivial complete intersection curves on X have multidegree related to (1). (line).*
- *If $d > n + 2$ then the nontrivial complete intersection curves on X have multidegree related to (1), or (2). (line, or plane conic).*

Corollary B. *For any dimension r and any multidegree $\hat{l} = (\hat{l}_1, \dots \hat{l}_m)$ with all $\hat{l}_i \geq 2$ there is a constant γ with the following property:*

A general hypersurface $X \subset \mathbb{P}^{n+r}$ of degree d does not contain any nontrivial r-dimensional complete intersection subvariety of multidegree $l \sim \hat{l}$, whenever

$$d \geq \beta n^{1/r} + \gamma \qquad \textit{where} \qquad \beta = \left(\frac{(m+r+1)r!}{\prod_{i=1}^{m} \hat{l}_i} \right)^{1/r}.$$

For any real number $\epsilon > 0$ and any dimension r there are only finitely many values of m and finitely many multidegrees $\hat{l}$ such that $\beta \geq \epsilon$. Moreover, there is a constant $\tilde{D}$ such that for arbitrary $n \geq 1$ and any degree $d \geq \max(\tilde{D}, \epsilon n^{1/r})$ a general hypersurface $X \subset \mathbb{P}^{r+n}$ of degree d can have nontrivial r-dimensional complete intersection subvarieties only with multidegrees related to an $\hat{l}$ with $\beta \geq \epsilon$. For $\epsilon = 1/3$ and $r = 1$ we give the list of these $\hat{l}$. Each entry has the form $(l_1, \dots l_m)\beta$ where β is the coefficient defined above.

$(1)2$ $(2)\frac{3}{2}$ $(3)1$ $(2,2)1$ $(4)\frac{3}{4}$ $(2,3)\frac{2}{3}$ $(2,2,2)\frac{5}{8}$ $(5)\frac{3}{5}$
$(6)\frac{1}{2}$ $(2,4)\frac{1}{2}$ $(3,3)\frac{4}{9}$ $(7)\frac{3}{7}$ $(2,2,3)\frac{5}{12}$ $(2,5)\frac{2}{5}$ $(8)\frac{3}{8}$
$(2,2,2,2)\frac{3}{8}$ $(9)\frac{1}{3}$ $(2,6)\frac{1}{3}$ $(3,4)\frac{1}{3}$.

Part One.

The proof of the above theorems can be divided into two parts. The first part is a geometric argument, reducing the problem to an inequality about the Hilbert function of a complete intersection variety. The second part is a rather long inductive proof of this inequality. We shall try to separate the two parts as much as possible. Although the first part uses the results of the second, for aesthetical reasons we prefer to keep this order.

In the second part, in order to make the induction work, we use only the (higher order) convexity properties of the Hilbert functions. For convenience we include the statements with all the necessary definitions, but we postpone the proofs for that part.

Definition 1. For $n \geq 0$ we define the functions G_n from the integers to the reals. Let $G_n(x) = 0$ for $x < 0$ and $G_n(x) = \binom{n+x}{n}$ for $x \geq 0$. Let define for $n \geq 0$ the family of functions

$$\mathcal{F}_n = \left\{ f : \mathbb{Z} \longrightarrow \mathbb{R} \;\middle|\; f(x) = \sum_{i=0}^{\infty} a_i G_n(x-i)\ ,\ \ 0 \leq a_i \in \mathbb{R}\ ,\ \ a_0 \geq 1 \right\}.$$

Remark 1. For all $m \geq 0$ the functions $f \in \mathcal{F}_m$ are nondecreasing and for $m > 0$ they are convex functions. $f(x) = 0$ for $x < 0$ and $f(y) \geq 1$ for $y \geq 0$. For any x we have $f(x) \geq G_m(x)$.

Definition 2. For an integer $l > 0$ and any function $f : \mathbb{Z} \longrightarrow \mathbb{R}$ let $\Delta_l f : \mathbb{Z} \longrightarrow \mathbb{R}$ be the difference $\Delta_l f(x) = f(x) - f(x-l)$. For any sequence $l_1, l_2, \dots l_k > 0$ of integers let $\Delta_{l_1 \dots l_k} f = \Delta_{l_1} \Delta_{l_2} \dots \Delta_{l_k} f$.

Remark 2. $\Delta_{l_1 \dots l_k} f$ does not depend on the order of the l_i. For $m > 0$ and $t > 0$ we get $\Delta_t G_m(x) = \sum_{i=0}^{t-1} G_{m-1}(x-i)$. This implies that:
- If $k \leq m$ and $l_1, \dots l_k$ are integers and $f \in \mathcal{F}_m$ then $\Delta_{l_1 \dots l_k} f \in \mathcal{F}_{m-k}$.
- If $k \leq m$ and $f \in \mathcal{F}_m$ then $\Delta_{l_1 \dots l_k} f$ is a nondecreasing function in l_i for all i.
- For $a < b$ we have $\mathcal{F}_b \subset \mathcal{F}_a$.
- For $k \leq m$, any $f \in \mathcal{F}_m$, any number x and any sequence $l_1, \dots l_k$ we have $\Delta_{l_1 \dots l_k} f(x) \geq \Delta_{l_1 \dots l_k} G_m(x)$.

Theorem 2. *For any real numbers $C \geq 0$, $\alpha > 0$ and any integer $r \geq 1$ there are integers $D, L, T > 0$ such that whenever we choose integers $n \geq 1$, $d \geq \max(D, \alpha n^{1/r})$, a sequence $1 \leq l_1 \leq l_2 \dots \leq l_n \leq d/2$ and a function $f \in \mathcal{F}_{n+r}$, we find either that $l_1 = l_2 = \dots = l_{n-T} = 1$ and $l_i < L$ for all i, or*

$$(**) \qquad Cd^r + \sum_{i=1}^{n} \Delta_{l_1 \dots l_n} f(l_i) < \Delta_{l_1 \dots l_n} f(d).$$

Theorem 3. *Choose arbitrary integers $n \geq 1$, $1 \leq l_1 \leq \dots \leq l_n$, $d \geq \max\{2l_n, n+1\}$, and any function $f \in \mathcal{F}_{n+1}$.*
- *If $d \geq 11$ and the sequence $l = (l_1, l_2 \dots l_n)$ is not equivalent to any of the following sequences: (1), (2), $(2,2)$, or (3), then the following inequality holds:*

$$(***) \qquad \sum_{i=1}^{n} \Delta_{l_1 \dots l_n} f(l_i) < \Delta_{l_1 \dots l_n} f(d).$$

- *If $d = 8, 9, 10$ and l is not equivalent to (1), (2), $(2,2)$, (3), (4), then again $(***)$ holds.*

• *If* $d = 7$ *and* l *is not equivalent to* (1), (2), (2,2), (3), (2,3), *then again* (***) *holds.*
• *For* $d = 6$ *and* l *is not equivalent to* (1), (2), (2,2), (2,2,2), (3), (2,3), (2,2,3), (3,3), *then again* (***) *holds.*

Now we are ready for the geometric part of the proof. We start with a definition.

Definition Γ. Let $V \subseteq \mathbb{P}^{n+r}$ be a subvariety of dimension r such that only one component of the Hilbert scheme of $\mathbb{P}^{n+r}$ passes through the point $[V]$. Let $\mathcal{V}$ be that component. Let $\mathcal{P}$ be the projective space parametrizing the hypersurfaces $X \subset \mathbb{P}^{n+r}$ of degree d. Let $\mathcal{P}_V$ be the locus of those hypersurfaces that contain any subvariety from $\mathcal{V}$. Then $\Gamma(n,d,V)$ denotes the codimension of $\mathcal{P}_V$ in $\mathcal{P}$.

For a hypersurface $X \subseteq \mathbb{P}^{n+r}$ of degree d let $\mathcal{H}_X$ denote the subscheme of the Hilbert scheme of X that parametrize schemes from $\mathcal{V}$. Let $\tilde{\Gamma}(n,d,V) = \Gamma(n,d,V)$ if it is positive, otherwise let $\tilde{\Gamma}(n,d,V) = -\dim(\mathcal{H}_X)$ for a general hypersurface X.

If V is a complete intersection variety of multidegree $l = (l_1, \dots l_n)$ (and dimension r) then we use also the notations $\Gamma(n,d,r,l) = \Gamma(n,d,V)$ and $\tilde{\Gamma}(n,d,r,l) = \tilde{\Gamma}(n,d,V)$

By an easy general position argument one can prove the following:

Observations. If we get the multidegree $\hat{l} = (\hat{l}_1, \dots \hat{l}_{\hat{n}})$ from the multidegree $l = (l_1, \dots l_n)$ by omitting some of the degrees l_i, and $\hat{r} = r + (n - \hat{n})$, then $\Gamma(n,d,r,l) \leq \Gamma(n,d,\hat{r},\hat{l})$ and they are equal if we drop only degrees $l_i \geq d$. If $l' = (l'_1, \dots l'_n)$ is another multidegree related to l with respect to d, then again we have $\Gamma(n,d,r,l) = \Gamma(n,d,r,l')$.

Lemma Γ. *Let* $V \subset \mathbb{P}^{n+r}$ *be an* r*-dimensional locally complete intersection variety. Assume that the natural map* $H^0\left(\mathbb{P}^{n+r}, \mathcal{O}_{\mathbb{P}^{n+r}}(d)\right) \longrightarrow H^0(V, \mathcal{O}_V(d))$ *is surjective and the Hilbert scheme of* $\mathbb{P}^{n+r}$ *is regular at* $[V]$. *Let* $\mathcal{N}_V$ *denote the normal bundle of* V *in* $\mathbb{P}^{n+r}$. *Then the following inequality holds:*

$$(\Gamma) \qquad \Gamma(n,d,V) \geq \tilde{\Gamma}(n,d,V) \geq \dim H^0(V, \mathcal{O}_V(d)) - \dim H^0(V, \mathcal{N}_V).$$

The second inequality becomes equality if the middle term is nonpositive.

Let I_V *be the ideal sheaf of* V *on* $\mathbb{P}^{n+r}$ *and assume now that the natural map* $H^0(\mathbb{P}^{n+r}, I_V) \longrightarrow H^0(V, I_V/I_V^2)$ *is surjective. Assume moreover that we can find a homomorphism* $g : \mathcal{N}_V \longrightarrow \mathcal{O}_V(d)$ *such that*

$$H^0(g) : H^0(V, \mathcal{N}_V) \longrightarrow H^0(V, \mathcal{O}_V(d))$$

is either injective or surjective. Then the second inequality in (Γ) *becomes again an equality. Moreover in this case the Hilbert scheme of* X *is regular at* $[V]$ *for a general* X *containing* V.

The conditions on V *in the first part are satisfied, for example, if* V *is projectively normal, locally a complete intersection and* $H^1(V, \mathcal{N}_V) = 0$. *They are also satisfied when* V *is a complete intersection variety of multidegree* $l = (l_1, \dots l_n)$. *Moreover, in this latter case the map* $H^0(\mathbb{P}^{n+r}, I_V(d)) \longrightarrow H^0(V, I_V/I_V^2(d))$ *is surjective and the inequality* (Γ) *becomes the following:*

$$\Gamma(n,d,r,l) \geq \tilde{\Gamma}(n,d,r,l) \geq \dim H^0(V, \mathcal{O}_V(d)) - \sum_{i=1}^{n} \dim H^0(V, \mathcal{O}_V(l_i)).$$

Remark. (Theorem 3.1, p. 172, [4]) is another result about the regularity of the Hilbert Scheme.

Proof of Lemma Γ. It is clear that $\Gamma(n,d,V) \geq \tilde{\Gamma}(n,d,V)$. Also, if V is a complete intersection variety of multidegree $l = (l_1, \dots l_n)$ then $\mathcal{N}_V = \bigoplus_{i=1}^n \mathcal{O}(l_i)$. This explains the last statement. We prove the rest. We note that the assumption that V is locally a complete intersection is needed only to be able to talk about the normal bundle. In general one can replace $H^0(V, \mathcal{N}_V)$ with $\mathrm{Hom}(I_V, \mathcal{O}_V)$.

Let V be a locally complete intersection variety such that the natural map $H^0\left(\mathbb{P}^{n+r}, \mathcal{O}_{\mathbb{P}^{n+r}}(d)\right) \longrightarrow H^0(V, \mathcal{O}_V(d))$ is surjective and the Hilbert scheme of $\mathbb{P}^{n+r}$ is regular at $[V]$. We shall use the notations introduced in the Definition Γ. The dimension of $\mathcal{V}$ is

$$A = \dim H^0(V, \mathcal{N}_V)$$

where $\mathcal{N}_V$ is the normal bundle of V in $\mathbb{P}^{n+r}$. Let $[W]$ be a general element of $\mathcal{V}$. Then semicontinuity gives us that

$$\dim H^0(V, \mathcal{O}_V(d)) \geq \dim H^0(W, \mathcal{O}_W(d)) \geq \dim H^0(\mathbb{P}^{n+r}, \mathcal{O}_{\mathbb{P}^{n+r}}(d)) -$$

$$- \dim H^0(\mathbb{P}^{n+r}, I_W(d)) \geq \dim H^0(\mathbb{P}^{n+r}, \mathcal{O}_{\mathbb{P}^{n+r}}(d)) - \dim H^0(\mathbb{P}^{n+r}, I_V(d))$$

where I_V and I_W denote the ideal sheaves of V and W on $\mathbb{P}^{n+r}$. By assumption, the first and the last terms are equal, hence all four terms are equal. So the dimension of the family of those hypersurfaces of degree d that are going through W is

$$\begin{aligned} B &= \dim H^0(\mathbb{P}^{n+r}, I_W(d)) - 1 \\ &= \dim H^0(\mathbb{P}^{n+r}, \mathcal{O}_{\mathbb{P}^{n+r}}(d)) - 1 - \dim H^0(V, \mathcal{O}_V(d)). \end{aligned}$$

The dimension of $\mathcal{P}$ (the projective space parametrizing all hypersurfaces of degree d) is

$$F = \dim H^0(\mathbb{P}^{n+r}, \mathcal{O}_{\mathbb{P}^{n+r}}(d)) - 1.$$

Let $\mathcal{I}$ denote the scheme parametrizing the pairs $(W \subset X)$ where $X \subset \mathbb{P}^{n+r}$ is a hypersurface of degree d, and $W \subset \mathbb{P}^{n+r}$ is a scheme with $[W] \in \mathcal{V}$. From the above calculation we see that $\dim \mathcal{I} = A + B$.

There is a projection $\Phi : \mathcal{I} \longrightarrow P$, and the fiber of Φ at a hypersurface X is exactly $\mathcal{H}_X$ (the intersection of $\mathcal{V}$ with the Hilbert scheme of X). Hence if Φ is not surjective then $\tilde{\Gamma}(n,d,V)$ is the codimension of the image. On the other hand, if Φ is surjective then $-\tilde{\Gamma}(n,d,V)$ is just the dimension of a general fiber of Φ. The codimension of the image is at least $F - A - B$, and if every fiber is nonempty then a general fiber has dimension exactly $A + B - F$. This proves the inequality (Γ). It is clear that if the middle term of (Γ) is nonpositive then Φ is surjective, and the second half of (Γ) becomes an equality.

Now we turn to the second part of the lemma. Let $\mathcal{N}_P$ and $\mathcal{N}_X$ denote the normal bundle of V relative to $\mathbb{P}^{n+r}$ and X. There is an exact sequence on X:

$$0 \longrightarrow \mathcal{N}_X \longrightarrow \mathcal{N}_P \stackrel{g}{\longrightarrow} \mathcal{O}_V(d).$$

From the cohomology long exact sequence of the above sequence we get

$$0 \longrightarrow H^0(\mathcal{N}_X) \longrightarrow H^0(\mathcal{N}_P) \stackrel{H^0(g)}{\longrightarrow} H^0(\mathcal{O}_V(d)).$$

$\mathcal{H}_X$ is the fiber at $[X]$ of the above map Φ and in the first part of the proof we actually got a lower estimate for the dimension of any nonempty fiber. On the other hand deformation theory gives an upper estimate for the dimension of the Hilbert scheme. Comparing the upper and the lower estimates we get

$$h^0(\mathcal{N}_X) \geq \dim \mathcal{H}_X \geq h^0(\mathcal{N}_P) - h^0(\mathcal{O}_V(d)).$$

So if we find an example of V and X were $H^0(g)$ is surjective or injective, then at least one (nonempty) fiber has dimension $h^0(\mathcal{N}_X)$, and therefore the second half of (Γ) becomes an equality. In this case $h^0(\mathcal{N}_X) = \dim_{[V]} \mathcal{H}_X$, hence $\mathcal{H}_X$ is regular at $[V]$. The map g is constructed in the following way. The homogeneous equation of X is a global section of $I_V(d)$, and g is just its image in $I_V(d)/{I_V}^2(d) \simeq \mathrm{Hom}_V(\mathcal{N}_P, \mathcal{O}_V(d))$. Since the map

$$H^0(\mathbb{P}^{n+r}, I_V(d)) \longrightarrow H^0(V, I_V(d)/{I_V}^2(d))$$

is surjective, for different choices of X we can get every possible homomorphism

$g : \mathcal{N}_{\mathbb{P}^{r+1}} \longrightarrow \mathcal{O}_V$. So in order to get equality in (Γ) we need only to construct a g with surjective or injective $H^0(g)$.

Together with Main Theorem A we shall prove the following higher dimensional version:

Main Theorem B. *For any real numbers $C \geq 0$, $\alpha > 0$ and any integer $r \geq 1$ there are integers $D, L, T > 0$ such that whenever we choose integers $n \geq 1$, $d \geq \max(D, \alpha n^{1/r})$, and a multidegree $l = (l_1, \dots l_n)$, then one of the following holds:*
either $\Gamma(n, d, r, l) > Cd^r$
or l is related (with respect to d) to some $(\hat{l}_1, \dots \hat{l}_T)$ with $1 \leq \hat{l}_i \leq L$.

Proof of the Main Theorems. We shall prove the last part of Main Theorem A at the end of this section. Now we concentrate only on Main Theorem B and the curve case of Main Theorem A. Let $V \subset \mathbb{P}^{n+r}$ be a complete intersection variety of multidegree $l = (l_1, \dots l_n)$.

First we prove the theorems under the extra assumption that $l_1 \leq l_2 \leq \dots l_n \leq d/2$. Main Theorem B follows immediately from Theorem 2 below once we realize that

$$H^0(V, \mathcal{O}_V(x)) = \Delta_{l_1, l_2 \dots l_n} G_{n+r}(x)$$

for all x. Similarly, Main Theorem A follows at once from Theorem 3.

Next, the second Observation above proves these theorems with the weaker extra hypothesis that all $l_i < d$.

Fix C, α as in Main Theorem B, and r (set r=1 for Main Theorem A). The above special case of Main Theorem B gives us constants L, D, T. Increasing D if necessary, we can assume $D > 2L + 1$. I claim, that both theorems are true without the extra hypothesis, and we can take these modified D, L, T in Main Theorem B. Moreover, I prove the last part of Main Theorem A.

Choose n, d as in the theorems, and let $l = (l_1, \dots l_m)$ be a multidegree which is a counterexample ($m = n$ except for the last part of Main Theorem A, where $m = n - 1$). Let $\hat{l} = (\hat{l}_1, \dots \hat{l}_{\hat{n}})$ be the multidegree obtained from l by omitting all $l_i \geq d$, and let $\hat{r} = r + n - \hat{n}$.

Let $l^* = (l_1^*, \dots l_{n^*}^*)$ be any sequence that we get from $\hat{l}$ by adding arbitrarily $n^* - \hat{n}$ new degrees such that $l_i^* < d$ for all i ($\hat{n} \leq n^* \leq m$). The above Observation said that

$$\Gamma(m, d, r, l) = \Gamma(\hat{n}, d, \hat{r}, \hat{l}) \geq \Gamma(n^*, d, r, l^*)$$

for all such l^*. However l^* satisfies the extra hypothesis, so both Main Theorem B and the first part of Main Theorem A apply to l^*.

If l was a counterexample to Main Theorem B then at least one $l_i \geq d$. Choose $n^* = n$. We get for all i that either $l_i^* \leq L$ or $d - l_i^* \leq L$. However we can choose at least one l_i^* arbitrarily, we can set $l_i^* = L+1 > L$. This is a contradiction, because $d - l_i^* \geq D - (L+1) > L$.

If l was a counterexample to the last part of Main Theorem A then $m = n-1$, choose $n^* = n-1$. Then we get a new complete intersection surface on a general X, so we can assume at the beginning that each $l_i < d$. We repeat the previous construction with $n^* = n$, so we get $l^* = (l_1, l_2, \dots l_{n-1}, l_n^*)$ with l_n^* chosen arbitrarily. All these l^* have to be on the lists of Main Theorem A. So either $l_n^* < 5$ or $d - l_n^* < 5$. Therefore $d \leq 9$, and there are very few choices for l. One can check Lemma Γ for all of them (by computer), and one gets no counterexample.

If l was a counterexample to the rest of Main Theorem A then at least one $l_i \geq d$, so we can choose $n^* = n-1$. This gives a counterexample to the last part of Main Theorem A, and we proved that it is impossible. This proves the theorem.

Proof of the Corollaries. We shall fix a dimension r and a multidegree $\hat{l} = (\hat{l}_1, \dots \hat{l}_m)$ with $\hat{l}_i \geq 2$ for all i. We shall vary n, d. If $V \subset \mathbb{P}^{n+r}$ is a complete intersection variety of dimension r and multidegree $l \sim \hat{l}$, then clearly $\dim H^0(V, \mathcal{O}_V(d))$ depends only on d and $\hat{l}$, but not on n. Let $P(d)$ denote the Hilbert polynomial of V, $\deg(P) = r$ and the leading coefficient is just $\deg(V)/r! = (\prod_{i=1}^m \hat{l}_i)/r!$. There is a d_0 depending on m, r and $\hat{l}$ such that $\dim H^0(V, \mathcal{O}_V(d)) \geq P(d)$ for all $d \geq d_0$. In fact for odd r we can take $d_0 = 0$, for even r we can choose $d_0 = \sum_{i=1}^m \hat{l}_i - (m+r+2)$. It is clear also that $\dim H^0(V, \mathcal{O}_V(1)) = m+r+1$. Hence for $d > d_0$ the inequality (Γ) becomes

$$\begin{aligned} \Gamma(n,d,r,l) &\geq P(d) - (n-m)(m+r+1) - \sum_{i=0}^{m} H^0(V, \mathcal{O}_V(\hat{l}_i)) \\ &= P(d) - (m+r+1)n + W \end{aligned}$$

where W is a constant depending only on m, r and $\hat{l}$. Solving this inequality, for any constant $C < (1/r!) \prod_{i=1}^m \hat{l}_i$ one can find an integer γ such that for $d \geq d_0$ one gets

$$\Gamma(n,d,r,l) > Cd^r \qquad \text{whenever} \quad d \geq n^{1/r} \left(\frac{(m+r+1)r!}{\prod_{i=1}^m \hat{l}_i - Cr!} \right)^{1/r} + \gamma.$$

We shall use this inequality for $C = 0$. First we prove Corollary B. Increasing γ we can drop the condition $d \geq d_0$. This proves the first statement.

Main Theorem B gives us a constant D and a finite list of $\hat{l}$-s such that the multidegree of every r-dimensional nontrivial complete intersection subvariety of a general X of degree $d \geq \max(D, \epsilon n^{1/r})$ is related to one $\hat{l}$ on the list. For some $\hat{l}$ on this list, the above computation might give $\beta < \epsilon$. Choosing a $\tilde{D} \geq D$ we can achieve $\beta n^{1/r} + \gamma < \epsilon n^{1/r}$ for all such $\hat{l}$, whenever $\beta n^{1/r} + \gamma \geq \tilde{D}$, so we can exclude them from the list. So we have to find only those $\hat{l}$ that has $\beta \geq \epsilon$. We observe that $(\prod_{i=1}^{m} \hat{l}_i)/(m+2) \geq 2^m/(m+2) > 1/\epsilon$ for large enough m so there is only finitely many choice of $\hat{l}$ and β. Now set $r = 1$ and $\epsilon = 1/3$. It is enough to calculate the β for all $\hat{l}$ with $m < 5$, and it can be done easily. This proves Corollary B.

Now we prove that Main Theorem A implies Corollary A. We prove the last statement, then inequality (Γ) gives us immediately the first two. The Observations tell us that it is enough to exclude the counterexamples of the following kind. Let $l = (l_1, \dots l_m)$ be a sequence different from (2), with $2 \leq l_1 \leq \cdots \leq l_m \leq d/2$, such that for some $m \leq n < d-2$ a general $X \subset \mathbb{P}^{n+1}$ of degree d contains a curve with multidegree related to l. If $d \geq 6$ then Main Theorem A implies that l has to be on one of the lists in Main Theorem A. If $d \leq 5$ then $l_i = 2$, and $n \leq 3$, so again l is on the lists. If l is one of the sequences

(3), (2,2), (4), (2,3), (2,2,2), (3,3) or (2,2,3),
then the inequality (Γ) gives us these upper bounds for d:

$$(n+2)\,, (n+2), \left(\frac{3}{4}n + \frac{13}{4}\right), \left(\frac{2}{3}n + \frac{19}{6}\right), \left(\frac{5}{8}n + \frac{25}{8}\right), \left(\frac{4}{9}n + \frac{37}{9}\right), \left(\frac{5}{12}n + \frac{47}{12}\right).$$

Since $n \geq m$ and $d \geq 2l_i$, it is easy to deduce, that in all cases $d < n + 3$. This is a contradiction, so this proves Corollary A.

Now we turn to the question of existence. The following lemma will be very useful during the proof of Proposition C and Proposition D.

Lemma C. *Let C be a smooth curve of genus ρ and let $L_1, \dots L_n$, E be line bundles on C. Let define* $t_i = \min\{\deg L_i - \rho,\ \deg L_{i+1} - \rho - 1\}$, $T_i = \min\{t_i, \rho - 1\}$, *and* $\delta = (\deg E - \sum_{i=1}^{n} \deg L_i + (n-1)(\rho - 1))$. *In this lemma g will denote a homomorphism $g : \bigoplus_{i=1}^{n} L_i \longrightarrow E$. Assume that $\delta \leq 0$ and the following conditions hold:*

(a) $n \geq 3$, $\deg L_1 \geq 2\rho - 1$ *and* $\deg L_i \geq 2\rho$ *for* $2 \leq i \leq n$,

(b) $\deg E \geq \sum_{i=1}^{n} \deg L_i - \sum_{j=1}^{n-1} t_j$.

Then there is a homomorphism g such that $H^0(g) : \bigoplus_{i=1}^{n} H^0(L_i) \longrightarrow H^0(E)$ is surjective. We can find g with injective $H^0(g)$ if $n \geq 3$ and the following condition holds:

(c) $\deg E \geq \sum_{i=1}^{n} \deg L_i - \sum_{j=1}^{n-1} T_j$.

Condition (a) *and $\delta \geq 0$ together imply condition* (c), *hence the existence of* *a g with injective $H^0(g)$. If condition* (a) *holds and $\delta = 0$ then there exists a g such that $H^0(g)$ is an isomorphism. Assume now that $\rho = 0$ and the following condition holds:*

(d) $-1 \leq \deg L_i \leq \deg E$ *for* $1 \leq i \leq n$.

If $\delta > 0$, $\delta = 0$ or $\delta < 0$ then there exists a g such that $H^0(g)$ is injective, isomorphic or surjective.

Proof. We shall construct a subbundle $\bigoplus_{i=1}^{n-1} G_i \leq \bigoplus_{i=1}^{n} L_i$ such that the quotient bundle is just E. g will be the natural homomorphism. If we can find this with $H^1(G_i) = 0$ for all i then our $H^0(g)$ must be surjective, and if all $H^0(G_i) = 0$ then we find an injective $H^0(g)$.

We shall construct $G_i = L_i(-D_i)$ with some effective divisor D_i, and the inclusion homomorphism will map G_i into $L_i \bigoplus L_{i+1} \leq \bigoplus_{i=1}^{n} L_i$. The first component of this homomorphism will be the natural inclusion $L_i(-D_i) \leq L_i$, and we shall construct the other component $h_i : G_i \longrightarrow L_{i+1}$ later. This construction gives an exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^{n-1} G_i \longrightarrow \bigoplus_{i=1}^{n} L_i \longrightarrow Q \longrightarrow 0$$

where Q is the quotient. We shall choose h_i such that Q is locally free. Then it is easy to see that

$$Q = \bigotimes_{i=1}^{n} L_i \otimes \bigotimes_{j=1}^{n-1} G_j^{-1} = \mathcal{O}\left(\bigoplus_{i=1}^{n-1} D_i\right).$$

We shall choose D_i such that we get $Q \simeq E$.

We note that for a general line bundle M if $\deg M \geq \rho - 1$ then $H^1(M) = 0$, and if $\deg M \leq \rho - 1$ then $H^0(M) = 0$. If $\deg M \geq \rho + 1$ and M is general, then the linear system $|M|$ is nonempty and it has no fixed points.

First we do the construction of a surjective $H^0(g)$. We want to find divisors D_i such that G_i is general, $H^1(G_i) = 0$, and $\mathcal{O}\left(\bigoplus_{i=1}^{n-1} D_i\right) \simeq E$. We can easily find them if $n - 1 \geq 2$ and we can choose a sequence $\deg G_i$, $i = 1 \ldots n-1$, such that $\deg L_i - \deg G_i \geq \rho$ (so we can choose general G_i), $\deg G_i \geq \rho - 1$ (so we shall get $H^1(G_i) = 0$), and $\deg E = \sum_{i=1}^{n} \deg L_i - \sum_{j=1}^{n-1} \deg G_j$ (so we get $Q \simeq E$ at the end).

h_i is a global section of $L_{i+1} \bigotimes G_i^{-1}$. So to choose h_i we have to find a divisor $H_i \in |L_{i+1} \bigotimes G_i^{-1}|$. If we have also $\deg H_i = \deg L_{i+1} - \deg G_i \geq \rho + 1$ then we can choose H_i such that their support is disjoint from all D_j and all other H_j. Then it is easy to show that the cokernel of the homomorphism

$$\bigoplus_{j=1}^{n-1} G_j \longrightarrow \bigoplus_{i=1}^{n} L_i$$

is locally free. Putting together the inequalities for $\deg G_i$ we get

$$\rho - 1 \leq \deg G_i \leq \min\{\deg L_i - \rho, \deg L_{i+1} - \rho - 1\} = t_i.$$

Comparing the upper and lower bounds for $\deg G_i$ we get condition (a) of the lemma. Substituting these bounds into the equality we get condition (b), and the condition $\delta \leq 0$. These are clearly sufficient conditions for the existence of $\deg G_i$, hence for the existence of g with surjective $H^0(g)$.

Next we construct g with injective $H^0(g)$. One can prove the same way as before, that this g exists, provided that $n \geq 3$ and there exists a sequence $\deg G_i$ such that

$$\deg G_i \leq \min\{\deg L_i - \rho, \deg L_{i+1} - \rho - 1, \rho - 1\} = T_i$$

and

$$\deg E = \sum_{i=1}^{n} \deg L_i - \sum_{j=1}^{n-1} \deg G_j.$$

The condition (c) is satisfied iff this system has a solution sequence $\deg G_i$.

Under condition (a) we get $T_i = \rho - 1$; hence condition (c) is equivalent to $\delta \geq 0$. If (a) holds and $\delta = 0$ then there is a g such that $H^0(g)$ is an injective homomorphism between vectorspaces of the same dimension; hence it is an isomorphism.

Assume now that $\rho = 0$ and (d) holds. If $n = 1$, then there is a natural inclusion $L_1 \leq E$, and this induces an injective homomorphism on the global sections, so the lemma is true in this case. On the other hand for $n \geq 2$ we number the L_i so that $\deg(L_i)$ is nondecreasing. We do the same construction as before, except that for every degree there is only one line bundle, so the construction works for $n = 2$ as well. Moreover, $T_i = -1$ in this case, so (c) is equivalent to $\delta \geq 0$. Therefore, by what we have proved earlier, $\delta \geq 0$ implies that there exists a g with injective $H^0(g)$. If $\delta = 0$, then this is an injective homomorphism between isomorphic vectorspaces, so it must be an isomorphism.

So we need to deal only with the case $\delta < 0$. We do again the same construction as before, and again, it works for $n \geq 2$. The conditions we need are the following: $\deg(L_i) \geq \deg(G_i)$ (so D_i is effective), $\deg(H_i) = \deg(L_{i+1}) - \deg(G_i) \geq 0$ (so H_i is effective, we can always find H_i disjoint from all D_j and all other H_j), $\deg(G_i) \geq -1$ (to make $H^1(G_i) = 0$), and $\deg E = \sum_{i=1}^{n} \deg L_i - \sum_{j=1}^{n-1} \deg G_j$. Putting this together, we need to find $\deg(G_i)$ satisfying

$$-1 \leq \deg(G_i) \leq \min(\deg(L_i), \deg(L_{i+1})) = \deg(L_i)$$

and the previous equation for $\deg(E)$. This is possible to find if and only if $-1 \leq \deg L_i$ for all i, $\deg(L_n) \leq \deg(E)$, and $\delta \leq 0$. This proves the lemma.

Proposition C. *Let $V \subset \mathbb{P}^{n+1}$ be a complete intersection curve of multidegree $l = (l_1, \dots l_n)$, and d a degree such that $d > l_i$ for all i and let*

$$\beta = \sum_{i=1}^{n} \dim H^0\left(V, \mathcal{O}_V(l_i)\right) - \dim H^0\left(V, \mathcal{O}_V(d)\right).$$

If V has genus 0, or if $n \geq 3$ and the genus is 1 then $-\tilde{\Gamma}(n, d, 1, l) = \beta$. In particular a general X contains interesting complete intersection subvarieties of multidegree l iff $\beta \geq 0$. This happens when l is associated to (1), (2), (3) or (2,2). This implies that Corollary A is sharp.

If $d = 5$ and l is related to $(2, 2, 2)$ then again $-\tilde{\Gamma}(n, d, 1, l) = \beta \geq 0$. In particular, a general quintic hypersurface contains a curve of multidegree l. On a general quintic threefold the nontrivial complete intersection subvarieties are curves, they have multidegree (l_1, l_2, l_3) with $1 \leq l_i \leq 4$, and all such multidegrees occur.

Proof of Proposition C. By Lemma Γ we need only to construct in all cases a g such that $H^0(g)$ is either surjective or injective.

First we solve the case where the genus $\rho \leq 1$. We claim that in this case we can find the above homomorphism g such that $H^0(g)$ is either injective or surjective. Set $L_i = \mathcal{O}(l_i)$ and $E = \mathcal{O}(d)$, clearly the normal bundle of V is $\mathcal{N}_V = \bigoplus_{i=1}^n L_i$.

If $\rho = 0$ then (d) of Lemma C is satisfied, hence our g exists. Assume now $\rho = 1$, $n \geq 3$, and let l_0 be $(2,2)$ or (3) such that $l \sim l_0$. Fix l_0, and vary n, d. It is enough to prove that for every value of d there is an $n = n_0$ such that we can find a g which gives an isomorphism $H^0(g)$. For other values of n the normal bundle has less or more $\mathcal{O}(1)$ components, one can restrict g, or extend it arbitrarily. For each value of d one can find an $n = n_0$ such that $\delta = 0$ in Lemma C. It is clear that (a) holds. Lemma C proves the existence of g for $n = n_0$, and our claim follows.

It is clear now that Corollary A is sharp for $n \geq 3$. If $d \leq 2n - 1$ then a general X contains a line, if $d \leq \frac{3}{2}n + \frac{1}{2}$ then a general X contains a plane conic. If $d \leq n + 2$ then a general X contains genus 1 curves of multidegree equivalent to (3) and (2,2). The case $n \leq 2$ is obvious from the Noether-Lefschetz theorem.

Now let $d = 5$, l equivalent to $(2, 2, 2)$ and V is a smooth complete intersection curve of multidegree l. Then V has genus 5 and degree 8. We need only to prove that there exist a homomorphism

$$g : \mathcal{O}_V(2) \bigoplus \mathcal{O}_V(2) \bigoplus \mathcal{O}_V(2) \longrightarrow \mathcal{O}_V(5)$$

such that $H^0(g)$ is surjective. One can check, that conditions (a) and (b) of Lemma C are satisfied, and $\delta = 0$ in this case. Hence Lemma C implies the map g exists; hence (Γ) becomes an equality. Then the Observations imply that (Γ) is an equality for all l related to $(2,2,2)$.

We proved that all curves listed in the last part of Proposition Cexist on a general quintic threefold. We see either from the inequality (Γ) or from the Noether-Lefschetz theorem that there are no nontrivial complete intersection surfaces on it. Hence there are no other curves on it (see the Observations). This completes proof of the proposition.

Proposition D. *Let $V \subset \mathbb{P}^{n+1}$ be a projectively normal curve of genus ρ and degree l. Assume that $H^1(\mathcal{N}_V) = 0$ where $\mathcal{N}_V$ is the normal bundle of V in $\mathbb{P}^{n+1}$. Let $\mathcal{V}$ be the component of the Hilbert scheme of $\mathbb{P}^{n+1}$ that contains $[V]$. In this case we have*

$$\Gamma(n,d,V) \geq dl - (n+2)l - (n-3)(1-\rho).$$

In particular, if

$$d > n + 2 + \frac{(n-3)(1-\rho)}{l}$$

then a general hypersurface X of degree d does not contain any subscheme that belongs to $\mathcal{V}$.

Proof of Proposition D. From Lemma Γ we know, that

$$\Gamma(n,d,V) \geq \dim H^0(V, \mathcal{O}_V(d)) - \dim H^0(V, \mathcal{N}_V) \geq \chi(\mathcal{O}_V(d)) - \chi(\mathcal{N}_V).$$

Clearly $\chi(\mathcal{O}_V(d)) = ld - \rho + 1$, and

$$\chi(\mathcal{N}_V) = \chi\left(T_{\mathbb{P}^{n+1}}|V\right) - \chi(T_V) =$$

$$= [(n+2)l - (n+1)(1-\rho)] + [3\rho - 3] = (n+2)l - (n-2)(\rho - 1).$$

Putting these together we get the required inequality.

Part Two.

In the rest of the paper we can forget about geometry. We shall study the classes of functions $\mathcal{F}_n$ defined in the previous part. The proof consists of a sequence of lemmas. Some of these lemmas have an a and a b version with different constants. In many case the proof of the lemmas is a strait forward computation, and we shall leave it to the reader. We shall prove only Lemma 7a and Lemma 7b because these are essential for the inductions.

Lemma 1. *For $m \geq 0$, $a \geq 0$, $l \geq k$ integers and $F \in \mathcal{F}_m$ we have*

$$F(l-a)G_m(k) \geq F(k-a)G_m(l).$$

In particular for $l \geq 0$ one gets

$$F(k) \leq F(l) \cdot \frac{G_m(k)}{G_m(l)}.$$

Lemma 2a. *For any choice of the integers $l \geq 3$, $m \geq 0$, $a \leq l$ and any function $f \in \mathcal{F}_m$ we have $f(a+l) \geq (2m-2)f(a)$.*

Lemma 2b. *For any choice of the integers $l \geq 4$, $m \geq 0$, $a \leq l$ and any function $f \in \mathcal{F}_m$ we have $f(a+l) \geq \frac{33}{28}(2m-2)f(a)$.*

Lemma 3a. *For any choice of the integers $l \geq 4$, $m \geq 0$, $a \leq l$ and any function $f \in \mathcal{F}_{m+1}$ we have $\frac{64}{65}\Delta_2 f(a+l) \geq m\Delta_2 f(a+1)$.*

Lemma 3b. *For any choice of the integers $l \geq 7$, $m \geq 0$, $a \leq l$ and any function $f \in \mathcal{F}_{m+1}$ we have $\frac{18}{25}\Delta_2 f(a+l) \geq m\Delta_2 f(a+1)$.*

Lemma 4a. *For any choice of the integers $l \geq 4$, $p \geq 0$, $q \geq 0$, $a \leq l$ and any function $f \in \mathcal{F}_{p+q+1}$ we have*

$$2p\Delta_1 f(a) + q\Delta_2 f(a+1) \leq \frac{64}{65}\Delta_2 f(a+l).$$

Lemma 4b. *For any choice of the integers $l \geq 7$, $p \geq 0$, $q \geq 0$, $a \leq l$ and any function $f \in \mathcal{F}_{p+q+1}$ we have*

$$2p\Delta_1 f(a) + q\Delta_2 f(a+1) \leq \frac{18}{25}\Delta_2 f(a+l).$$

Lemma 5. *For any integer $r \geq 1$ there is an integer $T > 0$ such that for any choice of the integers $d \geq 8$, $t \geq T$, for any sequence $2 \leq l_1 \leq l_2 \cdots \leq l_t$ of integers and for any function $f = \Delta_{l_1 \ldots l_t} F$, where $F \in \mathcal{F}_{t+r}$, we have $td^r f(3) < f(d)$.*

Lemma 6. *For any real number $A \geq 0$ and any integer $r \geq 0$ there are integers $L > 0$ depending on A, r and D depending on r, L such that for any choice of the integers $l \geq L$, $d \geq D$ and for any function $f = \Delta_l F$ where $F \in \mathcal{F}_{r+1}$ we have $Ad^r f(3) < f(d)$.*

Definition. The *type* of a sequence $1 \leq l_1 \leq \cdots \leq l_n$ is the pair (p, q) where p is the number of those l_i that are equal to l_n, and q is the number of those l_i that are equal to $l_n - 1$. If $l_n > 1$ then the *truncation* of this sequence is another sequence $1 \leq L_1 \leq \cdots \leq L_n$ with $L_i = \min(l_i, l_n - 1)$.

Clearly $1 \le l_1 \le l_2 \le \cdots \le l_{n-p-q} \le l_n - 2$, $l_{n-p-q+1} = l_{n-p-q+2} = \cdots = l_{n-p} = l_n - 1$ and $l_{n-p+1} = l_{n-p+2} = \cdots = l_n$. Also we have $L_i = l_i$ for $i \le n-p$ and $L_j = l_n - 1$ for $n-p+1 \le j \le n$.

Lemma 7a. *For any choice of the integer numbers $n \ge 1$, $l \ge 4$, $d \ge 2l$, any sequence $1 \le l_1 \le l_2 \le \cdots \le l_n = l$, and any function $f \in \mathcal{F}_{n+1}$ we denote by $(L_1, \dots L_n)$ the truncation of $(l_1, \dots l_n)$. Then we have*

$$\left[\sum_{i=1}^{n} \Delta_{l_1 l_2 \dots l_n} f(l_i)\right] - \left[\sum_{i=1}^{n} \Delta_{L_1 \dots L_n} f(L_i)\right] < \frac{64}{65}\left[\Delta_{l_1 \dots l_n} f(d) - \Delta_{L_1 \dots L_n} f(d-2)\right].$$

Proof. The right hand side is an increasing function of d because

$$\Delta_1\left[\Delta_{l_1 \dots l_n} f(d) - \Delta_{L_1 \dots L_n} f(d-2)\right] = \Delta_1 \Delta_{l_1 \dots l_n} f(d) - \Delta_1 \Delta_{L_1 \dots L_n} f(d-2) \ge$$
$$\ge \Delta_1 \Delta_{l_1 \dots l_n} f(d) - \Delta_1 \Delta_{l_1 \dots l_n} f(d-2) \ge 0.$$

Hence it is enough to prove it for $d = 2l$. Let (p,q) be the type of the sequence $(l_1, \dots l_n)$. Set $F = \Delta_{l_1 \dots l_{n-p-q}} f \in \mathcal{F}_{p+q+1}$. It is easy to see that $\Delta_{l_1 \dots l_n} f = \Delta_{l_{n-p-q+1} \dots l_n} F$; $\Delta_{L_1 \dots L_n} f = \Delta_{L_{n-p-q+1} \dots L_n} F$ and for $x < l-1$ we have

$$\Delta_{l_1 \dots l_n} f(x) = \Delta_{L_1 \dots L_n} f(x) = F(x).$$

In the sequence $(l_{n-p-q+1}, l_{n-p-q+2} \dots l_n)$ the number l occurs p times and the remaining q numbers are all $l-1$. Therefore we can easily calculate $\Delta_{l_{n-p-q+1} \dots l_n} F(x)$ and $\Delta_{L_{n-p-q+1} \dots L_n} F(x)$ for any $x \le 2l$ using the fact that $F(y) = 0$ for $y < 0$. Our inequality becomes

$$\sum_{i=1}^{n-p-q} F(l_i) + p\left[F(l) - qF(1) - pF(0)\right] + q\left[F(l-1) - qF(0)\right] -$$
$$- \sum_{i=1}^{n-p-q} F(l_i) - (p+q)\left[F(l-1) - (p+q)F(0)\right] <$$
$$< \frac{64}{65}\left[F(2l) - qF(l+1) - pF(l) + \binom{q}{2}F(2) + pqF(1) + \binom{p}{2}F(0) -\right.$$
$$\left. -F(2l-2) + (p+q)F(l-1) - \binom{p+q}{2}F(0)\right].$$

Using the fact that $\binom{p+q}{2} = \binom{p}{2} + \binom{q}{2} + pq$ we see that our inequality is the sum of the following two:

$$p\Delta_1 F(l) \le \frac{64}{65}\left[\Delta_2 F(2l) - p\Delta_1 F(l) - q\Delta_2 F(l+1)\right]$$

$$-pqF(1) + 2pqF(0) < \frac{64}{65}\left[\binom{q}{2}\Delta_2 F(2) + pq\Delta_1 F(1)\right].$$

Since $\frac{64}{65}\Big[-p\Delta_1 F(l) - q\Delta_2 F(l+1)\Big] \geq \Big[-p\Delta_1 F(l) - q\Delta_2 F(l+1)\Big]$, the first inequality follows from Lemma 4a if we set $a = l$. Lemma 1 implies that $F(1) \geq 3F(0)$; hence the left hand side of the second is negative, the right hand side is positive. Therefore the lemma is proved.

Lemma 7b. *For any choice of the integer numbers $n \geq 1$, $l \geq 7$, $d \geq 2l$, any sequence $1 \leq l_1 \leq l_2 \leq \cdots \leq l_n = l$ and any function $f \in \mathcal{F}_{n+1}$ we denote by $(L_1, \ldots L_n)$ the truncation of $(l_1, \ldots l_n)$. Then we have*

$$\left[\sum_{i=1}^{n} \Delta_{l_1 l_2 \ldots l_n} f(l_i)\right] - \left[\sum_{i=1}^{n} \Delta_{L_1 \ldots L_n} f(L_i)\right] < \frac{18}{25}\Big[\Delta_{l_1 \ldots l_n} f(d) - \Delta_{L_1 \ldots L_n} f(d-2)\Big].$$

Proof. We can copy the proof of Lemma 7a with the following changes. We have to write $\frac{18}{25}$ everywhere in place of $\frac{64}{65}$, and we have to use Lemma 4b instead of Lemma 4a.

Theorem 1. *For any choice of the integers $m \geq 1$, $3 \leq l_1 \leq \cdots \leq l_m$, $d \geq 2l_m$, and any function $f \in \mathcal{F}_{m+1}$ we have*

$$\sum_{i=1}^{m} \Delta_{l_1 \ldots l_m} f(l_i) < \Delta_{l_1 \ldots l_m} f(d).$$

If we have $4 \leq l_1 \leq \cdots \leq l_m$ together with the above assumptions then we have

$$\sum_{i=1}^{m} \Delta_{l_1 \ldots l_m} f(l_i) < \frac{64}{65}\Delta_{l_1 \ldots l_m} f(d).$$

Proof. First we note that the left hand side is independent of d and the right hand side is monotonic. Hence we may assume that $d = 2l_m$. We shall prove the theorem by induction on l_m. If $l_1 = l_2 = \cdots = l_m = l$ then the inequality becomes

$$mf(l) - m^2 f(0) < f(2l) - mf(l) + \binom{m}{2} f(0).$$

Lemma 2a tells us that $2mf(l) = (2(m+1) - 2)\, f(l) \leq f(2l)$, and obviously $(m^2 + \binom{m}{2})f(0) \geq 0$, so the first inequality is true in this case. If $l \geq 4$ then Lemma 2b tells us that $2mf(l) \leq \frac{28}{33} f(2l) < \frac{64}{65} f(2l)$, so the second inequality is true as well. In particular we proved the first inequality for $l_m = 3$, and the second for $l_m = 4$. Let $l \geq 4$, assume that the theorem is true whenever $l_m < l$, and we shall prove it for $l_m = l$. Lemma 7a tells us that

$$\left[\sum_{i=1}^{m} \Delta_{l_1 l_2 \ldots l_m} f(l_i)\right] - \left[\sum_{i=1}^{m} \Delta_{L_1 \ldots L_m} f(L_i)\right] <$$

$$< \frac{64}{65}\left[\Delta_{l_1 \ldots l_m} f(2l) - \Delta_{L_1 \ldots L_m} f(2l-2)\right]$$

with a sequence $l_1 \leq L_1 \leq \cdots \leq L_m \leq l-1$. By our assumption we know that

$$\sum_{i=1}^{m} \Delta_{L_1 \ldots L_m} f(L_i) < \Delta_{L_1 \ldots L_m} f(2l-2)$$

and the sum of the last two inequalities gives us the first inequality ofthe theorem. If $4 \leq l_1$ then we have

$$\sum_{i=1}^{m} \Delta_{L_1 \ldots L_m} f(L_i) < \frac{64}{65}\left[\Delta_{L_1 \ldots L_m} f(2l-2)\right]$$

and adding it to the previous one we get the second inequality of the theorem.

Theorem 2. *For any real numbers $C \geq 0$, $\alpha > 0$ and any integer $r \geq 1$ there are integers $D, L, T > 0$ such that whenever we choose integers $n \geq 1$, $d \geq \max(D, \alpha n^{1/r})$, a sequence $1 \leq l_1 \leq l_2 \cdots \leq l_n \leq d/2$ and a function $f \in \mathcal{F}_{n+r}$, we find either that $l_1 = l_2 = \cdots = l_{n-T} = 1$ and $l_i < L$ for all i, or*

$$(**) \qquad Cd^r + \sum_{i=1}^{n} \Delta_{l_1 \ldots l_n} f(l_i) < \Delta_{l_1 \ldots l_n} f(d).$$

Proof. We set $A = 65(C + \alpha^{-r})$, then Lemma 6 gives us integers D, L depending on A, r. We can assume that $D \geq 8$. Lemma 5 gives us an integer T, we can assume that $T \geq A$. We shall prove the theorem with this D, L, T.

Let $n, d, l_1 \ldots l_n, f$ be as in the theorem, and assume, that we are not in the first case, so either $l_{n-T} \geq 2$ or $l_n \geq L$. Let j be the largest index such that $l_j \leq 3$. If $j < n$ then applying Theorem 1 to the function $\Delta_{l_1 \ldots l_j} f$ we get

$$\sum_{i=j+1}^{n} \Delta_{l_1 \ldots l_n} f(l_i) < \frac{64}{65} f(d).$$

If $j = n$, then the left hand side is 0; hence this inequality remains true. It is easy to see that

$$Cd^r + \sum_{i=1}^{n} \Delta_{l_1 \dots l_n} f(l_i) \leq Cd^r f(3) + nf(3) + \sum_{i=j+1}^{n} \Delta_{l_1 \dots l_n} f(l_i)$$
$$< \frac{1}{65} Ad^r f(3) + \frac{64}{65} f(d)$$

hence in order to prove (**) we need only to establish

$$Ad^r f(3) \leq f(d).$$

This follows from Lemma 5 if $l_n \geq l_{n-1} \geq \cdots \geq l_{n-T} \geq 2$, and follows from Lemma 6 if $l_n \geq L$. So the theorem is proved.

Lemma 8. *If $f = \Delta_t F$ where $F \in \mathcal{F}_2$ and $t \geq 7$ then $f(d) > df(1) + f(t)$ for all $d \geq 2t$.*

Lemma 9. *Let $F \in \mathcal{F}_3$ and choose integers $a \geq 7$, $b \geq 2$, $c \geq 2$ and $e \geq 2$. Then for all $d \geq 8$ we have*

$$2\Delta_{a,b} F(1) < \frac{7}{25} \left[\Delta_{a,b} F(d) - \Delta_{a-1,b} F(d-2) \right].$$

If in addition $F \in \mathcal{F}_5$ then for all $d \geq 14$ we have

$$2\Delta_{a,b,c,e} F(2) < \frac{7}{25} \left[\Delta_{a,b,c,e} F(d) - \Delta_{a-1,b,c,e} F(d-2) \right].$$

Lemma 10a. *If $f = \Delta_{2,2} F$ for some $F \in \mathcal{F}_3$ or $f = \Delta_t F$ for some $F \in \mathcal{F}_2$ and $t \geq 3$, then $f(1) \leq \Delta_1 f(d)$ for all $d \geq 2$.*

Lemma 10b. *If $f = \Delta_{t_1,t_2,t_3,t_4,t_5} F$ for some $F \in \mathcal{F}_6$ and every $t_i \geq 2$ then $f(2) > \Delta_1 f(d)$ for all $d \geq 4$.*

Theorem 3. *Choose arbitrary integers $n \geq 1$, $1 \leq l_1 \leq \cdots \leq l_n$, $d \geq \max\{2l_n, n+1\}$, and any function $f \in \mathcal{F}_{n+1}$.*

- *If $d \geq 11$ and the sequence $l = (l_1, l_2 \dots l_n)$ is not equivalent to any of the following sequences: (1), (2), (2,2), or (3), then the following inequality holds:*

$$(***) \qquad \sum_{i=1}^{n} \Delta_{l_1 \dots l_n} f(l_i) < \Delta_{l_1 \dots l_n} f(d).$$

- *If $d = 8, 9, 10$ and l is not equivalent to (1), (2), (2,2), (3), (4), then again (***) holds.*

• *If* $d = 7$ *and* l *is not equivalent to* (1), (2), (2,2), (3), (2,3), *then again* (***) *holds.*
• *For* $d = 6$ *and* l *is not equivalent to* (1), (2), (2,2), (2,2,2), (3), (2,3), (2,2,3), (3,3), *then again* (***) *holds.*

Proof. We shall prove the theorem in three steps. To start the inductions I verified by computer the first few cases. Since (***) is linear in f it is enough to check it for $f(x) = G_{n+1}(x-T)$ for all $0 \le T \le l_n$. I checked all cases with $d = n+1 \le 11$ and all cases with $d = n+1 = 2l_n = 12$.

If l is a sequence that is not equivalent to (1), (2), (2,2) and (3) then the sequence of the k largest element of l is not equivalent to these for all $k > 2$. If $l_n > 4$ then the truncation of l is also not equivalent to the above sequences. Let l denote the increasing sequence $(l_1, l_2 \dots l_n)$ and assume that l is not equivalent to (1), (2), (2,2) and (3). If $l_1 \ge 3$ then Theorem 1 proves (***). So we assume $l_1 \le 2$ in the entire proof.

First step: We prove the theorem under the assumption that $d = n+1 = 2l_n \ge 6$. The computer verified it for $d = n+1 = 2l_n \le 12$, we shall prove the rest by induction on $d = n+1 = 2l_n$. So we assume $l_n \ge 7$ and $d \ge 14$. Let L be the truncation of l. We can apply Lemma 7b to n,d, to our sequences l and L, and our function f, and we get

$$(2)\quad \sum_{i=1}^{n} \Delta_{l_1\dots l_n} f(l_i) - \sum_{i=1}^{n} \Delta_{L_1\dots L_n} f(L_i) < \frac{18}{25}\Big[\Delta_{l_1\dots l_n} f(d) - \Delta_{L_1\dots L_n} f(d-2)\Big].$$

If $l_{n-1} = 1$ then Lemma 8 proves (***). So we assume that $l_{n-1} \ge 2$.
Case A: If $l_2 \le 2$ then from the induction hypothesis for $n-2$, $d-2$, the sequence $(L_3, L_4 \dots L_n)$ and function $\Delta_{l_1 l_2} f$ we get

$$\sum_{i=3}^{n} \Delta_{l_1 l_2} \Delta_{L_3\dots L_n} f(L_i) < \Delta_{l_1 l_2} \Delta_{L_3\dots L_n} f(d-2).$$

We add (2) to this inequality, and use that $L_1 = l_1$, $L_2 = l_2$. Then we get

$$(3)\quad \begin{aligned} &\sum_{i=1}^{n} \Delta_{l_1 l_2} \Delta_{l_3\dots l_n} f(l_i) - \Delta_{l_1 l_2} \Delta_{L_3\dots L_n} f(l_1) - \Delta_{l_1 l_2} \Delta_{L_3\dots L_n} f(l_2) + \\ &+ \frac{7}{25}\Big[\Delta_{l_1 l_2} \Delta_{l_3\dots l_n} f(d) - \Delta_{l_1 l_2} \Delta_{L_3\dots L_n} f(d-2)\Big] < \Delta_{l_1 l_2} \Delta_{l_3\dots l_n} f(d). \end{aligned}$$

On one hand, if $l_1 = l_2 = 1$ then we apply the first half of Lemma 9 to the function $F = \Delta_{l_1 \dots l_{n-2}} f$ and get

$$\begin{aligned} \Delta_{l_1 l_2} \Delta_{L_3 \dots L_n} f(l_1) + \Delta_{l_1 l_2} \Delta_{L_3 \dots L_n} f(l_2) & \\ \leq 2\Delta_{l_n, l_{n-1}} F(1) & \\ < \frac{7}{25}\Big[\Delta_{l_n, l_{n-1}} F(d) - \Delta_{l_n - 1, l_{n-1}} F(d-2)\Big] & \\ \leq \frac{7}{25}\Big[\Delta_{l_1 l_2} \Delta_{l_3 \dots l_n} f(d) - \Delta_{l_1 l_2} \Delta_{L_3 \dots L_n} f(d-2)\Big]. & \end{aligned}$$

On the other hand, if $l_1 \leq l_2 = 2$ then we apply the second half of Lemma 9 to the function $F = \Delta_{l_1 \dots l_{n-4}} f$ and get

$$\begin{aligned} \Delta_{l_1 l_2} \Delta_{L_3 \dots L_n} f(l_1) + \Delta_{l_1 l_2} \Delta_{L_3 \dots L_n} f(l_2) & \\ \leq 2\Delta_{l_n, l_{n-1} l_{n-2} l_{n-3}} F(2) & \\ < \frac{7}{25}\Big[\Delta_{l_n, l_{n-1} l_{n-2} l_{n-3}} F(d) - \Delta_{l_n - 1, l_{n-1} l_{n-2} l_{n-3}} F(d-2)\Big] & \\ \leq \frac{7}{25}\Big[\Delta_{l_1 l_2} \Delta_{l_3 \dots l_n} f(d) - \Delta_{l_1 l_2} \Delta_{L_3 \dots L_n} f(d-2)\Big]. & \end{aligned}$$

Hence in both cases we get

$$\Delta_{l_1 l_2} \Delta_{L_3 \dots L_n} f(l_1) + \Delta_{l_1 l_2} \Delta_{L_3 \dots L_n} f(l_2) \leq \frac{7}{25}\Big[\Delta_{l_1 l_2} \Delta_{l_3 \dots l_n} f(d) - \Delta_{l_1 l_2} \Delta_{L_3 \dots L_n} f(d-2)\Big].$$

If we add (3) to this inequality, we get (***).

<u>Case B</u>: If $l_2 \geq 3$ then we use Theorem 1 for $n-1$, $d-2$, the sequence $(L_2, L_3 \dots L_n)$ and function $\Delta_{l_1} f$. This gives us the following inequality:

$$\sum_{i=2}^{n} \Delta_{l_1} \Delta_{L_2 \dots L_n} f(L_i) < \Delta_{l_1} \Delta_{L_2 \dots L_n} f(d-2).$$

If we add (2) to this inequality, we get

$$\begin{aligned} & \sum_{i=1}^{n} \Delta_{l_1} \Delta_{l_2 \dots l_n} f(l_i) - \Delta_{l_1} \Delta_{L_2 \dots L_n} f(l_1) + \\ & + \frac{7}{25}\Big[\Delta_{l_1} \Delta_{l_2 \dots l_n} f(d) - \Delta_{l_1} \Delta_{L_2 \dots L_n} f(d-2)\Big] \leq \Delta_{l_1} \Delta_{l_2 \dots l_n} f(d). \end{aligned} \tag{4}$$

We apply the second half of Lemma 9 to $F = \Delta_{l_1 \dots l_{n-4}} f$ and we get

$$\begin{aligned} \Delta_{l_1} \Delta_{L_2 \dots L_n} f(l_1) & \leq \Delta_{l_n, l_{n-1} l_{n-2} l_{n-3}} F(2) \\ & < 2\Delta_{l_n, l_{n-1} l_{n-2} l_{n-3}} F(2) \\ & < \frac{7}{25}\Big[\Delta_{l_n, l_{n-1} l_{n-2} l_{n-3}} F(d) - \Delta_{l_n - 1, l_{n-1} l_{n-2} l_{n-3}} F(d-2)\Big] \\ & \leq \frac{7}{25}\Big[\Delta_{l_1} \Delta_{l_2 \dots l_n} f(d) - \Delta_{l_1} \Delta_{L_2 \dots L_n} f(d-2)\Big]. \end{aligned}$$

Adding (4) to this inequality we get (***). We proved (***) in Case A and Case B; hence the First step is completed.

Second step: we prove the theorem under the assumption that $d = n+1 \geq 6$. We use induction on d. In the First step we proved it for $d = 2l_n$, the computer verified this for $d \leq 11$ so now we assume that $d > 2l_n$ and $d = n+1 \geq 12$. From the induction hypothesis for $n-1$, $d-1$, and $\Delta_{l_1} f$ we get

$$(5) \qquad \sum_{i=2}^{n} \Delta_{l_1} \Delta_{l_2 \dots l_n} f(l_i) \leq \Delta_{l_1} \Delta_{l_2 \dots l_n} f(d).$$

On one hand, if $l_1 = 2$ then we apply Lemma 10b to the function $F = \Delta_{l_1 \dots l_{n-5}} f$. Then we get

$$\begin{aligned} \Delta_{l_1 \dots l_n} f(l_1) &= \Delta_{l_n -1, l_{n-1}, l_{n-2}, l_{n-3}} F(2) \\ &< \Delta_1 \Delta_{l_n -1, l_{n-1}, l_{n-2}, l_{n-3}} F(d) \\ &= \Delta_{l_1 \dots l_n} f(d) - \Delta_{l_1} \Delta_{l_2 \dots l_n} f(d-1). \end{aligned}$$

On the other hand, if $l_1 = 1$ then the sequence l is not equivalent to (2). Hence either $l_n \geq 3$ or $l_n - 1 = l_n = 2$. We can apply Lemma 10a and we get again that

$$\Delta_{l_1 \dots l_n} f(l_1) \leq \Delta_{l_1 \dots l_n} f(d) - \Delta_{l_1} \Delta_{l_2 \dots l_n} f(d-1).$$

Adding (5) to this inequality we get (***), so the Second step is completed.

Last step: finally we prove the theorem with no extra assumption.
Let $L = (L_1 \dots L_{d-1})$ be the following sequence. $L_i = l_{i+n-d+1}$ for $d - n \leq i \leq d-1$ and $L_j = 1$ for $1 \leq j < d-n$. We choose a function $F \in \mathcal{F}_d$ such that $f = \Delta_{L_1 \dots L_{d-n-1}} F = \Delta_{1,1,\dots 1} F$. By the Second step we can apply (***) to F, d, and the sequence L . Then we get that

$$(d-n-1) \Delta_{L_1 \dots L_{d-1}} F(1) + \sum_{i=d-n}^{d-1} \Delta_{L_1 \dots L_{d-1}} F(L_i) < \Delta_{L_1 \dots L_{d-1}} F(d).$$

If we drop the first term we get (***) for f, d and l since $\Delta_{L_1 \dots L_{d-1}} F = \Delta_{l_1 \dots l_n} f$ and $l_i = L_{d-n+i}$ for all i. So we proved the theorem.

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