PACIFIC JOURNAL OF MATHEMATICS
Vol. 175, No. 1, 1996

ON POLYNOMIALS ORTHOGONAL WITH RESPECT TO SOBOLEV INNER PRODUCT ON THE UNIT CIRCLE

XIN LI AND FRANCISCO MARCELLAN

We study polynomials orthogonal with respect to an indefinite Sobolev inner product on the unit circle. We establish the existence of such polynomials of large degree. Algebraic properties and asymptotic behavior of such polynomials are obtained.

0. Introduction.

The study of orthogonal polynomial with respect to standard inner product, i.e.

$$\langle f, g\rangle = \int_\Gamma f(z)\overline{g(z)}d\mu(z) \tag{1}$$

where Γ is a curve in the complex plane and μ is a positive definite Borel measure supported on Γ, constitutes an important subject of research in several areas such as approximation theory, numerical analysis as well as in other applied fields (signal processing, linear systems, ets.).

Recently, many people have been interested in the analysis of orthogonal polynomials with respect to some nonstandard inner products. One of the most important examples is related with Sobolev inner products.

In particular, much attention is now been paid to the case

$$\langle f, g\rangle = \int_I f(x)g(x)d\mu_0(x) + \int_J f'(x)g'(x)d\mu_1(x), \tag{2}$$

where μ_0 and μ_1 are positive definite Borel measures supported on I and J, subsets of the real line, respectively (see [**10**] for a survey of the subject matter). But very few results are known when the support of the measures is not contained in the real line.

A study of Borel measures supported on the unit circle has been initiated in [**3**], where the case when the inner product is given by

$$\langle f, g\rangle = \int_{|z|=1} f(z)\overline{g(z)}d\mu + \lambda f'(a)\overline{g'(a)},$$

where μ is a Borel measure on $|z| = 1$ and $|a| = 1$, is investigated and some questions such as representation formulas and relative asymptotics for the new sequence of orthogonal polynomial with respect to the standard one are provided. A comparison with the usual applications of the standard orthogonal polynomials, problems such as the location of zeroes, quadrature formulas, relation with continued fractions, and rational approximation to some integral transforms of measures need further studies.

Our present work can be focussed from two different points of view:

First, we perturb a standard inner product as in (1) on the unit circle using the first derivatives in several points off the circle instead of using higher order derivatives at only one point as in the direction pointed out in some recent research when the support of measure is contained in the real line (cf. [**1**, **8**, **11**]).

Secondly, we drop the requirement of positivity and Hermitian character of the inner product and establish the existence of orthogonal polynomials of large degree. When the measure μ belongs to a wide class N (the analogue of Nevai's class, see the definition in §3), we obtain the relative asymptotics of the two families of orthogonal polynomials associated to the measure μ and the nonstandard inner product, respectively. These extend the results in [**3**].

The organization of the paper is as follows: §1 is devoted to some definitions and basic facts; §2 collects some algebraic properties: §3 contains the statement of our main results on relative asymptotics whose proofs are then given in §4.

1. Notations.

Let $d\mu$ be a positive measure on the unit circle $|z| = 1$ with an infinite set of support. Let $z_1, z_2, \ldots, z_m$ be m fixed points in the complex plane $\mathbb{C}$. Additional assumptions on the location of $\{z_i\}$ will be made later. Throughout this paper, $\boldsymbol{Z}$ denotes the vector $(z_1, z_2, \ldots, z_m)$. We will use $\mathcal{P}_n$ to denote the set of polynomials of degree at most n with complex coefficients.

Let $\varphi_n(z)$ be the n-th orthonormal polynomial associated with $d\mu$, i.e., $\varphi_n(z) = \kappa_n z^n + \cdots \in \mathcal{P}_n$ with $\kappa_n > 0$ and

$$\int \varphi_n(z)\overline{z^k}d\mu = \frac{\delta_{n,k}}{\kappa_n}, \quad k = 0, 1, \ldots, n.$$

It is convenient to apply the following convention: For any function $F(z)$ of a single variable z, we write $F(\boldsymbol{Z})$ for the vector $(F(z_1), F(z_2), \ldots, F(z_m))$.

So we have, for example,

$$f'(\boldsymbol{Z}) = (f'(z_1), f'(z_2), \dots, f'(z_m)) \text{ and}$$
$$K(z, \boldsymbol{Z}) = (K(z, z_1), K(z, z_2), \dots, K(z, z_m)),$$

if $f(z)$ is differentiable and $K(z, w)$ is a function of two variables z and w.

Define an indefinite inner product as follows

$$\langle f, g\rangle := \int f\bar{g}d\mu + f'(\boldsymbol{Z})\boldsymbol{A}g'(\boldsymbol{Z})^H, \tag{3}$$

where $\boldsymbol{v}^H$ denotes the conjugate transpose of a vector $\boldsymbol{v}$, and $\boldsymbol{A}$ is a $m \times m$ complex matrix. Throughout this paper, matrices will always be denoted by bold-face letters.

We say a polynomial $\psi_n(z) \in \mathcal{P}_n$ is (*left-*) *orthonormal* with respect to the indefinite inner product $\langle \cdot, \cdot\rangle$ if

$$\langle \psi_n, z^k\rangle = 0, \ k = 0, 1, \dots, n-1,$$

and

$$|\langle \psi_n, \psi_n\rangle| = 1.$$

Cf. [**2**]. If $\boldsymbol{A}$ is a Hermitian positive-definite matrix, then the existence and uniqueness of such polynomials is always guaranteed. In general, we can see that if such a polynomial exists then $\deg \psi_n = n$, and in this case we will always assume the leading coefficient of ψ_n is positive and denoted by γ_n. Even with these conventions, the uniqueness of ψ_n is still unknown; nevertheless, under the assumption that $\boldsymbol{A}$ is non-singular, we will show that such polynomials exist for n sufficiently large. In the sequel, we will use ψ_n to denote one of such polynomials. The theory is a natural extention of the results for the real line (cf. [**8**]).

2. Algebraic Properties.

We list some useful relations between $\{\varphi_n(z)\}$ and $\{\psi_n(z)\}$. Let

$$K_n(z, \zeta) = \sum_{j=0}^{n-1} \varphi_j(z)\overline{\varphi_j(\zeta)}.$$

Then K_n is the reproducing kernel in $\mathcal{P}_{n-1}$: for $f \in \mathcal{P}_{n-1}$

$$\int f(\zeta) K_n(z, \zeta) d\mu(\zeta) = f(z).$$

Put

$$K_n^{(i,j)}(z,\zeta) = \sum_{k=0}^{n-1} \varphi_k^{(i)}(z)\overline{\varphi_k^{(j)}(\zeta)},\ i,j = 0,1,2,\ldots$$

Then for $f \in \mathcal{P}_{n-1}$,

$$\int f(\zeta)\, K_n^{(i,0)}(z,\zeta)\, d\mu(\zeta) = f^{(i)}(z).$$

Note also that $K_n^{(i,j)}(z,\zeta) = \overline{K_n^{(j,i)}(\zeta,z)}.$.

Formula 1. If ψ_n exists, then

$$\psi_n(z) = \frac{\gamma_n}{\kappa_n}\varphi_n(z) - \psi_n'(\boldsymbol{Z})\, \boldsymbol{A} K_n^{(0,1)}(z,\boldsymbol{Z})^t,$$

where $\boldsymbol{v}^t$ denotes the transpose of a vector $\boldsymbol{v}$.

Proof. Note that

$$\psi_n(z) - \frac{\gamma_n}{\kappa_n}\varphi_n(z) \in \mathcal{P}_{n-1},$$

so

$$\text{(4)}\qquad \int \left(\psi_n(\zeta) - \frac{\gamma_n}{\kappa_n}\varphi_n(\zeta)\right) K_n(z,\zeta)\, d\mu(\zeta) = \psi_n(z) - \frac{\gamma_n}{\kappa_n}\varphi_n(z),$$

by the reproducing property of K_n. Now using the orthogonality of φ_n, we can write

$$\int \left(\psi_n(\zeta) - \frac{\gamma_n}{\kappa_n}\varphi_n(\zeta)\right) K_n(z,\zeta)\, d\mu(\zeta) = \int \psi_n(\zeta)\, K_n(z,\zeta)\, d\mu(\zeta).$$

Further, on using the orthogonality of ψ_n with respect to th inner product $\langle\cdot,\cdot\rangle$ defined above, we can rewrite the right side of the above equation as follows:

$$\langle \psi_n, K_n(z,\cdot)\rangle - \psi_n'(\boldsymbol{Z})\, \boldsymbol{A} K_n^{(0,1)}(z,\boldsymbol{Z})^t = -\psi_n'(\boldsymbol{Z})\, \boldsymbol{A} K_n^{(0,1)}(z,\boldsymbol{Z})^t.$$

This together with (4), establishes Formula 1. □

Consequently, we have the following.

Formula 2. If ψ_n exists, then

$$\psi_n'(z) - \frac{\gamma_n}{\kappa_n}\varphi_n'(z) = -\psi_n'(\boldsymbol{Z})\, \boldsymbol{A} K_n^{(1,1)}(z,\boldsymbol{Z})^t.$$

Proof. This formula follows from differentiating Formula 1 with respect to z on both sides. □

Formula 3. If ψ_n exists, then

$$\frac{\gamma_n}{\kappa_n} = \langle \psi_n, \psi_n \rangle \frac{\kappa_n}{\gamma_n} - \psi_n'(\boldsymbol{Z})\, \boldsymbol{A} \varphi_n'(\boldsymbol{Z})^H.$$

Proof. Consider different ways to calculate

$$\int \psi_n(z)\overline{\varphi_n(z)}d\mu(z).$$

Writing $\psi_n(z) = (\gamma_n/\kappa_n)\,\varphi_n(z) +$ *lower degree terms*, and using the orthogonality of φ_n, we have

$$\int \psi_n(z)\overline{\varphi_n(z)}d\mu(z) = \frac{\gamma_n}{\kappa_n}.$$

On the other hand, writing $\varphi_n(z) = (\kappa_n/\gamma_n)\,\psi_n(z) +$ *lower degree terms*, and using the orthogonality of ψ_n, we have

$$\begin{aligned}\int \psi_n(z)\overline{\varphi_n(z)}d\mu(z) &= \langle \psi_n, \varphi_n \rangle - \psi_n'(\boldsymbol{Z})\, \boldsymbol{A} \varphi_n'(\boldsymbol{Z})^H \\ &= \langle \psi_n, \psi_n \rangle \frac{\kappa_n}{\gamma_n} - \psi_n'(\boldsymbol{Z})\, \boldsymbol{A} \varphi_n'(\boldsymbol{Z})^H,\end{aligned}$$

which implies Formula 3. □

Formula 4. Let $w_m(z) := \prod_{j=1}^{m}(1 - \overline{z_j}z)$, then there exist two polynomials $p(z) \in \mathcal{P}_{2m}$ and $q(z) \in \mathcal{P}_{2m-1}$, uniquely determined by ψ_n, such that

$$w_m^2(z)\psi_n(z) = \varphi_n(z)p(z) + \varphi_n^*(z)q(z).$$

Proof. From [**6**], we have

$$\mathcal{P}_{n+2m} = \varphi_n \mathcal{P}_{2m} + \varphi_n^* \mathcal{P}_{2m-1} + z^{2m}\mathcal{P}_{n-2m-1}.$$

So, there exist $p \in \mathcal{P}_{2m}$, $q \in \mathcal{P}_{2m-1}$ and $r \in \mathcal{P}_{n-2m-1}$ such that

$$w_m^2(z)\psi_n(z) = \varphi_n(z)p(z) + \varphi_n^*(z)q(z) + z^{2m}r(z).$$

It remains to show $r \equiv 0$. Multiplying $\overline{z^{2m}r(z)}$ and integrating with respect to $d\mu$ gives us

$$\begin{aligned}\int \psi_n(z)\overline{(w_m^*(z))^2\, r(z)}d\mu = \int \varphi_n(z)\overline{p^*(z)r(z)}d\mu &+ \\ \int \varphi_n^*(z)\overline{zq(z)r(z)}d\mu &+ \int |r(z)|^2\, d\mu,\end{aligned}$$

where we have used the fact that $h_k^*(z) = z^k\overline{h(z)}$ for $|z| = 1$. Now, by the orthogonality of φ_n and ψ_n it then follows that $\int |r(z)|^2 \, d\mu = 0$, which implies $r \equiv 0$. □

Remark. 1. We emphasize that the two polynomials $p(z)$ and $q(z)$ in Formula 4 depend on $w_m(z)$ and $\psi_n(z)$, although the dependence is not given explicitly in the notation.

2. Formula 4 can be used to produce reccurence relation and determinantal representation for ψ_n. The following integral representation is just one of various other possibilities.

Formula 5. If $|z_j| > 1,\ j = 1, 2, \ldots, m$, then

$$q(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\varphi_n(\zeta)\, p(\zeta)\, (w_m^2(z) - w_m^2(\zeta))}{\varphi_n^*(\zeta)\, w_m^2(\zeta)\, (\zeta - z)} d\zeta, \ |z| < 1,$$

and

$$\psi_n(z) = \frac{\varphi_n^*(z)}{2\pi i} \int_{|\zeta|=1} \frac{\varphi_n(\zeta)\, p(\zeta)}{\varphi_n^*(\zeta)\, w_m^2(\zeta)\, (\zeta - z)} d\zeta, \ |z| < 1.$$

Proof. Using Formula 4, on substituting z by $1/\overline{z_j}$, we have

$$q\left(\frac{1}{\overline{z_j}}\right) = -\left(\frac{\varphi_n}{\varphi_n^*} p\right)\left(\frac{1}{\overline{z_j}}\right), \ j - 1, 2, \ldots, m, \tag{5}$$

and

$$q'\left(\frac{1}{\overline{z_j}}\right) = -\left(\frac{\varphi_n}{\varphi_n^*} p\right)'\left(\frac{1}{\overline{z_j}}\right), \ j - 1, 2, \ldots, m. \tag{6}$$

Now, Formula 5 follows from an application of the Hermite formula of interpolation (see, for example, [**4**, p. 68]). □

3. Relative Asymptotics.

We now assume the matrix $\boldsymbol{A}$ in (3) is non-singular. Denote the leading coefficient of φ_n by $\kappa_n > 0$ as before. We need some assumptions on the measure $d\mu$. If

$$\lim_{n\to\infty} \frac{\varphi_n(0)}{\kappa_n} = 0,$$

we say that the measure $d\mu$ belongs to class N and write $d\mu \in N$. This is analogous to the Nevai's class of measures supported on the real line $\mathbb{R}$, (cf.

[**11**]). A well known result of Rakhmanov (cf. [**14**]) says that any measure $d\mu$ with $\mu' > 0$ a.e. on $|z| = 1$ belongs to class N.

We now state our main results on relative asymptotics, their proofs are presented in the next section.

Theorem 1. *If the points $z_1 \ldots, z_m$ ($z_j \neq z_k$ for $j \neq k$) all lie outside the unit circle, the matrix $\boldsymbol{A}$ in (3) is non-singular, and $d\mu \in N$, then there exists a positive integer n_0 such that for each $n \geq n_0$ the orthonormal polynomials with respect to the indefinite inner product $\langle\cdot,\cdot\rangle$, ψ_n, exists.*

Recall that $\gamma_n > 0$ denotes the leading coefficient of ψ_n and $|\langle\psi_n, \psi_n\rangle| = 1$.

Theorem 2. *Under the same assumptions as in Theorem 1, there hold*

$$\lim_{n\to\infty} \frac{\gamma_n}{\kappa_n} = \frac{1}{\prod_{i=1}^m |z_i|}, \tag{7}$$

and

$$\lim_{n\to\infty} \frac{\psi_n(z)}{\varphi_n(z)} = \lambda B(z), \quad |z| > 1, \tag{8}$$

where $\lambda = |B(0)|/B(0)$ and $B(z) = w_m^*(z)/w_m(z)$. (Recall that $w_m(z) = \prod_{j=1}^m (1 - \overline{z_j} z)$.) The convergence in (8) is uniform on every compact subset in $|z| > 1$.

Corollary 3. *Let $\rho := \min\{|z_j| \,|\, j = 1, 2, \ldots, m\}$. Then under the same assumption as in Theorem 1 we have*

$$\lim_{n\to\infty} \rho^{\epsilon n} \psi_n'(z_j) = 0,$$

for every $\epsilon \in (0,1)$ and $j = 1, 2, \ldots, m$.

Corollary 4. *Let $p(z)$ be defined as in Formula 4. Under the same assumption as in Theorem 1, we have*

$$\lim_{n\to\infty} p(z) = \lambda w_m(z) w_m^*(z) = \lambda w_m^2(z) B(z),$$

uniformly on every compact subset in the complex plane $\mathbb{C}$.

4. Proofs of the Relative Asymptotics.

We first establish some auxiliary lemmas.

Lemma 5. *For points $z_1, \ldots, z_m$ ($z_j \neq z_k$ for $j \neq k$) outside the unit circle, the matrix*

$$\boldsymbol{T}_m := \left(\frac{1}{\overline{z_j} z_k - 1}\right)_{j,k=1}^{m}$$

is non-singular.

Proof. This result is known and follows from Cauchy's result that

$$\det\left(\frac{1}{a_i + b_j}\right)_{i,j=1,2,\ldots,n} = \frac{\prod_{j>k}(a_j - a_k)(b_j - b_k)}{\prod_{j,k=1}^{n}(a_j + a_k)}$$

(see, for example, [**13**, Problem 3 p. 92]. See also [**7**, Lemma 4]. □

Lemma 6. *For points $z_1, \ldots, z_m$ ($z_j \neq z_k$ for $j \neq k$) outside the unit circle, let $B(z) = w_m^*(z)/w_m(z)$, then there exists a unique set of non-zero complex numbers $r_1, \ldots, r_m$ such that*

$$B(z) = \frac{1}{\overline{B(0)}} + \sum_{j=1}^{m} \frac{r_j}{1 - \overline{z_j} z}.$$

Proof. See [**7**, Lemma 5]. □

Lemma 7. *If $d\mu$ is a positive finite measure on the unit circle with infinitely many points in the support, then for every compact subset K in $|z| > 1$ there exists two positive constants $d = d(K)$ and $e = e(K)$ independent of n such that*

$$d \le \left|\frac{\varphi_n'(z)}{n\varphi_n(z)}\right| \le e$$

for all $z \in K$.

Proof. Denote $c := \min\{|z|;\ z \in K\}$ and $C := \max\{|z|;\ z \in K\}$. Then $1 < c < C$. To obtain the lower bound, write

$$\frac{\varphi_n'(z)}{n\varphi_n(z)} = \frac{1}{n}\sum_{j=1}^{n}\left\{\frac{\overline{z}}{|z-\zeta_j|^2} - \frac{\overline{\zeta_j}}{|z-\zeta_j|^2}\right\},$$

where ζ_j's denote the zeroes of φ_n which are all in $|z| < 1$, cf. [**15**, Th. 11.4.1.]. Then

$$\left|\frac{\varphi_n'(z)}{n\varphi_n(z)}\right| \ge \frac{|z|}{n}\sum_{j=1}^{n}\frac{1}{|z-\zeta_j|^2} - \frac{1}{n}\sum_{j=1}^{n}\frac{|\zeta_j|}{|z-\zeta_j|^2}$$
$$= \frac{1}{n}\sum_{j=1}^{n}\frac{|z| - |\zeta_j|}{|z-\zeta_j|^2} \ge \frac{c-1}{(C+1)^2}.$$

The proof of the upper bound is easier. □

Remark. Lemma 7 and its proof are parallel to those of Lemma 3 in [**5**].

Lemma 8. *If $d\mu \in N$, then*

$$\lim_{n\to\infty} \frac{\varphi'_{n+1}(z)}{\varphi'_n(z)} = z \tag{9}$$

uniformly on every compact subset of $|z| > 1$. Furthermore,

$$\lim_{n\to\infty} \frac{K_n^{(0,1)}(z,\zeta)}{\varphi'_n(z)\overline{\varphi_n(\zeta)}} = \frac{1}{z\bar{\zeta}-1}, \tag{10}$$

and

$$\lim_{n\to\infty} \frac{K_n^{(1,1)}(z,\zeta)}{\varphi'_n(z)\overline{\varphi'_n(\zeta)}} = \frac{1}{z\bar{\zeta}-1}, \tag{11}$$

uniformly for (z,ζ) on every compact subset of $|z| > 1$ and $|\zeta| > 1$.

Proof. If $d\mu \in N$, then by [**12, 14**]

$$\lim_{n\to\infty} \frac{\varphi_{n+1}(z)}{\varphi_n(z)} = z \tag{12}$$

uniformly on every compact subset of $|z| > 1$. As in [**5**, Lemma 4], note that

$$\frac{\varphi'_{n+1}(z)}{\varphi'_n(z)} = \frac{\varphi_n(z)}{\varphi'_n(z)}\left(\frac{\varphi_{n+1}}{\varphi_n}\right)'(z) + \frac{\varphi_{n+1}(z)}{\varphi_n(z)}. \tag{13}$$

Now (12) implies

$$\lim_{n\to\infty} \left(\frac{\varphi_{n+1}}{\varphi_n}\right)'(z) = 1,$$

uniformly on every compact subset of $|z| > 1$. On the other hand, by Lemma 7,

$$\lim_{n\to\infty} \frac{\varphi_n(z)}{\varphi'_n(z)} = 0, \tag{14}$$

uniformly on every compact subset of $|z| > 1$. Thus, on taking the limit as n tends to ∞ on both sides of (13), we obtain

$$\lim_{n\to\infty} \frac{\varphi'_{n+1}(z)}{\varphi'_n(z)} = z,$$

uniformly on every compact subset of $|z| > 1$. This proves (9).

In order to prove (11), we need the following limit relations:

$$\lim_{n\to\infty} \frac{\varphi_n^*(z)}{\varphi_n(z)} = 0, \tag{15}$$

and

$$\lim_{n\to\infty} \frac{\varphi_n^{*\prime}(z)}{\varphi_n'(z)} = 0, \tag{16}$$

uniformly on every compact subset of $|z| > 1$. Relation (15) is established in the proof of Theorem 4 in [**12**], while relation (16) folows from (14), (15) and the identity

$$\frac{\varphi_n^{*\prime}(z)}{\varphi_n'(z)} = \frac{\varphi_n(z)}{\varphi_n'(z)} \left(\frac{\varphi_n^*}{\varphi_n}\right)'(z) + \frac{\varphi_n^*(z)}{\varphi_n(z)}.$$

Now, using the Christoffel-Darboux formula (cf. [**15**, Th. 11.4.2.]) and by straightforward calculations, we have

$$\begin{aligned}
K_n^{(1,1)}(z;\zeta) &= \overline{\frac{\partial}{\partial \zeta}\frac{\partial}{\partial z}\left(\frac{\varphi_n^*(z)\overline{\varphi_n^*(\zeta)} - \varphi_n(z)\overline{\varphi_n(\zeta)}}{1 - z\overline{\zeta}}\right)} \\
&= \frac{\varphi_n^{*\prime}(z)\overline{\varphi_n^{*\prime}(\zeta)} - \varphi_n'(z)\overline{\varphi_n'(\zeta)}}{1 - z\overline{\zeta}} + \frac{\varphi_n^{*\prime}(z)\overline{\varphi_n^*(\zeta)} - \varphi_n'(z)\overline{\varphi_n(\zeta)}}{\left(1 - z\overline{\zeta}\right)^2} z \\
&\quad \frac{\varphi_n^*(z)\overline{\varphi_n^{*\prime}(\zeta)} - \varphi_n(z)\overline{\varphi_n'(\zeta)}}{\left(1 - z\overline{\zeta}\right)^2}\overline{\zeta} + \frac{\varphi_n^*(z)\overline{\varphi_n^*(\zeta)} - \varphi_n(z)\overline{\varphi_n(\zeta)}}{\left(1 - z\overline{\zeta}\right)^3}(1 + \overline{z}\zeta) \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

It is easy to verify that

$$\frac{I_1}{\varphi_n'(z)\overline{\varphi_n'(\zeta)}} = \frac{1}{1 - z\overline{\zeta}}\left(\frac{\varphi_n^{*\prime}(z)}{\varphi_n'(z)}\overline{\left(\frac{\varphi_n^{*\prime}(\zeta)}{\varphi_n'(\zeta)}\right)} - 1\right) \to \frac{1}{z\overline{\zeta} - 1},$$

and

$$\frac{I_4}{\varphi_n'(z)\overline{\varphi_n'(\zeta)}} = \frac{1 + \overline{z}\zeta}{\left(1 - z\overline{\zeta}\right)^3}\left(\frac{\varphi_n^*(z)}{\varphi_n(z)}\overline{\left(\frac{\varphi_n^*(\zeta)}{\varphi_n(\zeta)}\right)} - 1\right)\frac{\varphi_n(z)}{\varphi_n'(z)}\overline{\left(\frac{\varphi_n(\zeta)}{\varphi_n'(\zeta)}\right)} \to 0,$$

as $n \to \infty$, by (14), (15) and (16). Also, on writing

$$\frac{I_2}{\varphi_n'(z)\overline{\varphi_n'(\zeta)}} = \frac{z}{\left(1 - z\overline{\zeta}\right)^2}\left(\frac{\varphi_n^{*\prime}(z)}{\varphi_n'(z)}\overline{\left(\frac{\varphi_n^*(\zeta)\,\varphi_n(\zeta)}{\varphi_n(\zeta)\,\varphi_n'(\zeta)}\right)} - \overline{\left(\frac{\varphi_n(\zeta)}{\varphi_n'(\zeta)}\right)}\right),$$

we get, by (14), (15) and (16)

$$\frac{I_2}{\varphi_n'(z)\overline{\varphi_n'(\zeta)}} \to 0,$$

as $n \to \infty$. Similarly, we can obtain

$$\frac{I_3}{\varphi_n'(z)\overline{\varphi_n'(\zeta)}} \to 0,$$

as $n \to \infty$.

Note that all above limit processes are uniform for compact subsets in $|z| > 1$ and $|\zeta| > 1$. So (11) holds.

The proof of (10) is similar to that of (11), so we omit it. □

Proof of Theorem 1. We first claim that there exists a positive integer n_0 such that

$$\det\left(\boldsymbol{A}^{-1} + \boldsymbol{K}_n^{(1,1)}\right) \neq 0,\ n \geq n_0, \tag{17}$$

where

$$\boldsymbol{K}_n^{(1,1)} := \begin{pmatrix} K_n^{(1,1)}(z_1, z_1) & K_n^{(1,1)}(z_2, z_1) & \dots & K_n^{(1,1)}(z_m, z_1) \\ K_n^{(1,1)}(z_1, z_2) & K_n^{(1,1)}(z_2, z_2) & \dots & K_n^{(1,1)}(z_m, z_2) \\ \dots & \dots & & \dots \\ K_n^{(1,1)}(z_1, z_m) & K_n^{(1,1)}(z_2, z_m) & \dots & K_n^{(1,1)}(z_m, z_m) \end{pmatrix}.$$

Assume the validity of this claim for the moment. Define

$$\phi_n(z) := \varphi_n(z) - \varphi_n'(\boldsymbol{Z})\left(\boldsymbol{I} + \boldsymbol{A}\boldsymbol{K}_n^{(1,1)}\right)^{-1} \boldsymbol{A}\boldsymbol{K}_n^{(0,1)}(z, \boldsymbol{Z})^t,\ n \geq n_0.$$

It then follows that

$$\phi_n'(\boldsymbol{Z}) = \varphi_n'(\boldsymbol{Z})\left(\boldsymbol{I} + \boldsymbol{A}\boldsymbol{K}_n^{(1,1)}\right)^{-1}.$$

Thus, for $k < n$,

$$\begin{aligned} \langle \phi_n, \varphi_k \rangle &= \int \phi_n(z)\overline{\varphi_k(z)}d\mu + \phi_n'(\boldsymbol{Z})\boldsymbol{A}\varphi_k'(\boldsymbol{Z})^H \\ &= \int \varphi_n(z)\overline{\varphi_k(z)}d\mu - \varphi_n'(\boldsymbol{Z})\left(\boldsymbol{I} + \boldsymbol{A}\boldsymbol{K}_n^{(1,1)}\right)^{-1} \boldsymbol{A} \int K_n^{(0,1)}(z, \boldsymbol{Z})^t\,\overline{\varphi_k(z)}d\mu \\ &\quad + \varphi_n'(\boldsymbol{Z})\left(\boldsymbol{I} + \boldsymbol{A}\boldsymbol{K}_n^{(1,1)}\right)^{-1} \boldsymbol{A}\varphi_k'(\boldsymbol{Z})^H \\ &= 0. \end{aligned}$$

Similarly,

$$\langle \phi_n, \varphi_n \rangle = \int \phi_n(z)\overline{\varphi_n(z)}d\mu + \phi_n'(\boldsymbol{Z})\,\boldsymbol{A}\varphi_n'(\boldsymbol{Z})^H \tag{18}$$
$$= 1 + \varphi_n'(\boldsymbol{Z})\left(\boldsymbol{I} + \boldsymbol{A}\boldsymbol{K}_n^{(1,1)}\right)^{-1}\boldsymbol{A}\varphi_n'(\boldsymbol{Z})^H .$$

Now, from the matrix identities

$$\begin{pmatrix} \boldsymbol{I}_{m\times m} & \varphi_n'(\boldsymbol{Z})^H \\ \boldsymbol{0}_{1\times m} & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{A}^{-1} + \boldsymbol{K}_n^{(1,1)} & -\varphi_n'(\boldsymbol{Z})^H \\ \varphi_n'(\boldsymbol{Z}) & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \boldsymbol{A}^{-1} + \boldsymbol{K}_n^{(1,1)} + \varphi_n'(\boldsymbol{Z})^H\varphi_n'(\boldsymbol{Z}) & \boldsymbol{0}_{1\times m} \\ \varphi_n'(\boldsymbol{Z}) & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} \boldsymbol{A}^{-1} + \boldsymbol{K}_n^{(1,1)} & -\varphi_n'(\boldsymbol{Z})^H \\ \varphi_n'(\boldsymbol{Z}) & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{I}_{m\times m} & \left(\boldsymbol{A}^{-1} + \boldsymbol{K}_n^{(1,1)}\right)^{-1}\varphi_n'(\boldsymbol{Z})^H \\ \boldsymbol{0}_{1\times m} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \boldsymbol{A}^{-1} + \boldsymbol{K}_n^{(1,1)} & \boldsymbol{0}_{m\times 1} \\ \varphi_n'(\boldsymbol{Z}) & 1 + \varphi_n'(\boldsymbol{Z})\left(\boldsymbol{A}^{-1} + \boldsymbol{K}_n^{(1,1)}\right)^{-1}\varphi_n'(\boldsymbol{Z})^H \end{pmatrix},$$

we get the following determinant identity

$$\det\left(\boldsymbol{A}^{-1} + \boldsymbol{K}_n^{(1,1)} + \varphi_n'(\boldsymbol{Z})^H\varphi_n'(\boldsymbol{Z})\right) =$$
$$\det\left(\boldsymbol{A}^{-1} + \boldsymbol{K}_n^{(1,1)}\right)\left(1 + \varphi_n'(\boldsymbol{Z})\left(\boldsymbol{A}^{-1} + \boldsymbol{K}_n^{(1,1)}\right)^{-1}\varphi_n'(\boldsymbol{Z})^H\right).$$

Note also that

$$\boldsymbol{K}_{n+1}^{(1,1)} = \boldsymbol{K}_n^{(1,1)} + \varphi_n'(\boldsymbol{Z})^H\varphi_n'(\boldsymbol{Z}),$$

so there holds

$$1 + \varphi_n'(\boldsymbol{Z})\left(\boldsymbol{A}^{-1} + \boldsymbol{K}_n^{(1,1)}\right)^{-1}\varphi_n'(\boldsymbol{Z})^H = \frac{\det\left(\boldsymbol{A}^{-1} + \boldsymbol{K}_{n+1}^{(1,1)}\right)}{\det\left(\boldsymbol{A}^{-1} + \boldsymbol{K}_n^{(1,1)}\right)}.$$

This together with (18), gives us

$$\langle \phi_n, \varphi_n \rangle = \frac{\det\left(\boldsymbol{A}^{-1} + \boldsymbol{K}_{n+1}^{(1,1)}\right)}{\det\left(\boldsymbol{A}^{-1} + \boldsymbol{K}_n^{(1,1)}\right)} \neq 0, \ n \geq n_0.$$

Therefore, a (left-) orthonormal polynomial ψ_n exists and equals

$$\psi_n(z) = \eta_n \frac{\phi_n(z)}{\langle \phi_n, \phi_n \rangle},$$

where η_n is a number of modulus 1 such that the above polynomial has a positive leading coefficient.

It remains to prove our claim (17). Assume, to the contrary, that there is an infinite subsequence of positive integers, say $\mathcal{L}$, such that

$$\det\left(\boldsymbol{A}^{-1} + \boldsymbol{K}_n^{(1,1)}\right) = 0, \text{ for all } n \in \mathcal{L}. \tag{19}$$

Let

$$\boldsymbol{\Lambda}_n := \begin{pmatrix} \frac{1}{\varphi_n'(z_1)} & 0 & \dots & 0 \\ 0 & \frac{1}{\varphi_n'(z_2)} & \dots & 0 \\ \dots & \dots & & \dots \\ 0 & 0 & \dots & \frac{1}{\varphi_n'(z_m)} \end{pmatrix},$$

and

$$\boldsymbol{T}_{m,n} := \begin{pmatrix} \frac{K_n^{(1,1)}(z_1,z_1)}{\varphi_n'(z_1)\overline{\varphi_n'(z_1)}} & \frac{K_n^{(1,1)}(z_2,z_1)}{\varphi_n'(z_2)\overline{\varphi_n'(z_1)}} & \dots & \frac{K_n^{(1,1)}(z_m,z_1)}{\varphi_n'(z_m)\overline{\varphi_n'(z_1)}} \\ \frac{K_n^{(1,1)}(z_1,z_2)}{\varphi_n'(z_1)\overline{\varphi_n'(z_2)}} & \frac{K_n^{(1,1)}(z_2,z_2)}{\varphi_n'(z_2)\overline{\varphi_n'(z_2)}} & \dots & \frac{K_n^{(1,1)}(z_m,z_2)}{\varphi_n'(z_m)\overline{\varphi_n'(z_2)}} \\ \dots & \dots & & \dots \\ \frac{K_n^{(1,1)}(z_1,z_m)}{\varphi_n'(z_1)\overline{\varphi_n'(z_m)}} & \frac{K_n^{(1,1)}(z_2,z_m)}{\varphi_n'(z_2)\overline{\varphi_n'(z_m)}} & \dots & \frac{K_n^{(1,1)}(z_m,z_m)}{\varphi_n'(z_m)\overline{\varphi_n'(z_m)}} \end{pmatrix}.$$

Then we can write

$$\boldsymbol{A}^{-1} + \boldsymbol{K}_n^{(1,1)} = \overline{\boldsymbol{\Lambda}_n^{-1}}\left(\overline{\boldsymbol{\Lambda}_n}\boldsymbol{A}^{-1}\boldsymbol{\Lambda}_n + \boldsymbol{T}_{m,n}\right)\boldsymbol{\Lambda}_n^{-1}.$$

Taking the determinant on both sides yields

$$\det\left(\overline{\boldsymbol{\Lambda}_n}\boldsymbol{A}^{-1}\boldsymbol{\Lambda}_n + \boldsymbol{T}_{m,n}\right) = 0, \text{ for all } n \in \mathcal{L}, \tag{20}$$

according to (19). Now, using (9) in Lemma 8, we have

$$\lim_{n\to\infty} \frac{1}{\varphi_n'(z_j)} = 0, \ j = 1, 2, \dots, m,$$

and thus

$$\lim_{n\to\infty} \boldsymbol{\Lambda}_n = 0; \tag{21}$$

while using (11), we get

$$\lim_{n\to\infty} \boldsymbol{T}_{m,n} = \boldsymbol{T}_m. \tag{22}$$

Thus, if we let $n \in \mathcal{L}$ and $n \to \infty$ in (20), we would have $\det \boldsymbol{T}_m = 0$, contradicting Lemma 5. So, the claim (17) must be true. □

Proof of Theorem 2. Let n_0 be defined as in Theorem 1. Assume $n \geq n_0$ in this proof. Our proof exploits the information obtained in the proof of Theorem 1. Let $z = z_j$, $j = 1, 2, \ldots, m$, in Formula 2,

$$\psi_n'(z_j) - \frac{\gamma_n}{\kappa_n}\varphi_n'(z_j) = -\psi_n'(\boldsymbol{Z})\,\boldsymbol{A}K_n^{(1,1)}(z_j, \boldsymbol{Z})^t, \; j = 1, 2, \ldots, m.$$

Rearranging the terms in the above equations, and then putting them into a matrix form, we get

$$\frac{\gamma_n}{\kappa_n}(1, 1, \ldots, 1) = \psi_n'(\boldsymbol{Z})\,\boldsymbol{A}\boldsymbol{K}_n^{(1,1)}\boldsymbol{\Lambda_n} + \psi_n'(\boldsymbol{Z})\,\boldsymbol{\Lambda_n},$$

where $\boldsymbol{\Lambda_n}$ is the same as in the proof of Theorem 1, Rewrite the above equation further as

$$(1, 1, \ldots, 1) = \frac{\kappa_n}{\gamma_n}\psi_n'(\boldsymbol{Z})\left[\boldsymbol{A}\overline{\boldsymbol{\Lambda}_n^{-1}}\boldsymbol{T}_{m,n} + \boldsymbol{\Lambda_n}\right],$$

where $\boldsymbol{T}_{m,n}$ is as in the proof of Theorem 1. Using (20), (21) and (22), we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\kappa_n}{\gamma_n}\psi_n'(\boldsymbol{Z})\,\boldsymbol{A}\overline{\boldsymbol{\Lambda}_n^{-1}} \\
&= \lim_{n\to\infty}\frac{\kappa_n}{\gamma_n}\psi_n'(\boldsymbol{Z})\,\boldsymbol{A}\overline{\boldsymbol{\Lambda}_n^{-1}}\left[\boldsymbol{T}_{m,n} + \overline{\boldsymbol{\Lambda}_n}\boldsymbol{A}^{-1}\boldsymbol{\Lambda_n}\right]\left[\boldsymbol{T}_{m,n} + \overline{\boldsymbol{\Lambda}_n}\boldsymbol{A}^{-1}\boldsymbol{\Lambda_n}\right]^{-1} \\
&= \lim_{n\to\infty}(1, 1, \ldots, 1)\left[\boldsymbol{T}_{m,n} + \overline{\boldsymbol{\Lambda}_n}\boldsymbol{A}^{-1}\boldsymbol{\Lambda_n}\right]^{-1} \\
&=(1, 1, \ldots, 1)\boldsymbol{T}_{\boldsymbol{m}}^{-1} = \overline{B(0)}\,(r_1, r_2, \ldots, r_m),
\end{aligned}$$

where the last equality is according to Lemma 6 when $z = 0$.

Write $\boldsymbol{A} = (\boldsymbol{a}_1^t, \boldsymbol{a}_2^t, \ldots, \boldsymbol{a}_m^t)$ with $\boldsymbol{a}_j = (a_{1j}, a_{2j}, \ldots, a_{mj})$ $(j = 1, 2, \ldots, m)$, then we have

$$\lim_{n\to\infty}\frac{\kappa_n}{\gamma_n}\psi_n'(\boldsymbol{Z})\,\boldsymbol{a}_j^t\overline{\varphi_n(z_j)} = \overline{B(0)}r_j, \;, j = 1, 2, \ldots, m. \tag{23}$$

Now, by Formula 3,

$$\langle\psi_n, \psi_n\rangle\left(\frac{\kappa_n}{\gamma_n}\right)^2 - 1 = \frac{\kappa_n}{\gamma_n}\psi_n'(\boldsymbol{Z})\,(\boldsymbol{a}_1^t, \boldsymbol{a}_2^t, \ldots, \boldsymbol{a}_m^t)\,\varphi_n'(\boldsymbol{Z})^H,$$

thus

$$\begin{aligned}
\lim_{n\to\infty}\langle\psi_n, \psi_n\rangle\left(\frac{\kappa_n}{\gamma_n}\right)^2 &= 1 + \lim_{n\to\infty}\sum_{j=1}^m\frac{\kappa_n}{\gamma_n}\psi_n'(\boldsymbol{Z})\,\boldsymbol{a}_j^t\overline{\varphi_n'(z_j)} \\
&= 1 + \overline{B(0)}\sum_{j=1}^m r_j = 1 + \overline{B(0)}\left[B(0) - \frac{1}{\overline{B(0)}}\right] \\
&= |B(0)|^2.
\end{aligned}$$

Therefore $\lim_{n\to\infty} \langle \psi_n, \psi_n \rangle = 1$ and

$$\lim_{n\to\infty} \frac{\kappa_n}{\gamma_n} = |B(0)| = \frac{1}{\prod_{j=1}^{m} |z_j|}.$$

This completes the proof of (7).

We now prove (8). From Formula 1,

$$\frac{\psi_n(z)}{\varphi_n(z)} = \frac{\gamma_n}{\kappa_n} - \psi_n'(\boldsymbol{Z})\, \boldsymbol{A} \frac{K_n^{(0.1)}(z, \boldsymbol{Z})^t}{\varphi_n(z)}.$$

Recall that $K_n^{(0,1)}(z, \boldsymbol{Z}) = \left(K_n^{(0,1)}(z, z_1), \ldots, K_n^{(0,1)}(z, z_m)\right)$. Using $\boldsymbol{A} = (\boldsymbol{a}_1^t, \ldots, \boldsymbol{a}_m^t)$ and expanding the product on the right-hand side in the above equation, we have

$$\frac{\psi_n(z)}{\varphi_n(z)} = \frac{\gamma_n}{\kappa_n} - \sum_{j=1}^{m} \psi_n'(\boldsymbol{Z})\, \boldsymbol{a}_j^t \overline{\varphi_n'(z_j)} \frac{K_n^{(0,1)}(z, z_j)}{\varphi_n(z)\overline{\varphi_n'(z_j)}}.$$

Letting $n \to \infty$ on both sides in the above equation yields

$$\lim_{n\to\infty} \frac{\psi_n(z)}{\varphi_n(z)} = \frac{1}{|B(0)|} - \sum_{j=1}^{m} \overline{B(0)} r_j \frac{1}{z\overline{z_j} - 1},$$

by (7), (23) and (10). It follows from Lemma 6 and the definition of λ that the right-hand side in the above equation is $\lambda B(z)$. So (8) holds, and the proof of the theorem is completed. □

Proof of Corollary 3. From (23) and (7),

$$\lim_{n\to\infty} \psi_n'(\boldsymbol{Z})\, \boldsymbol{a}_j^t \overline{\varphi_n(z_j)} = \frac{\overline{B(0)}}{|B(0)|} \boldsymbol{r},$$

where $\boldsymbol{r} := (r_1, r_2, \ldots, r_m)$. So

$$\psi_n'(\boldsymbol{Z})\, \boldsymbol{A} \overline{\boldsymbol{\Lambda}_n^{-1}} = \frac{\overline{B(0)}}{|B(0)|} \boldsymbol{r} + \boldsymbol{f}_m,$$

where $\lim_{n\to\infty} \boldsymbol{f}_m = \boldsymbol{0}_m$. Thus

$$\psi_n'(\boldsymbol{Z}) = \frac{\overline{B(0)}}{|B(0)|} \boldsymbol{r} \overline{\boldsymbol{\Lambda}_n} \boldsymbol{A}^{-1} + \boldsymbol{f}_m \overline{\boldsymbol{\Lambda}_n} \boldsymbol{A}^{-1},$$

and so $\lim_{n\to\infty} \rho^{\epsilon n} \psi_n'(\boldsymbol{Z}) = \boldsymbol{0}_m$, according to (9). □

Proof of Corollary 4. Denote

$$\alpha_{n,j} := p\left(\frac{1}{\overline{z_j}}\right), \text{ and } \beta_{n,j} := p'\left(\frac{1}{\overline{z_j}}\right),$$

for $j = 1, 2, \ldots, m$. Choose λ_n such that $p(z) - \lambda_n w_m^2(z) \in \mathcal{P}_{2m-1}$. Since $\deg \psi_n = n$ for $n \geq n_0$, by comparing the leading coefficients in Formula 4, we find that $\lambda_n = \frac{\gamma_n}{\kappa_n}$.

Recall the formula of the $(0,1)$ polynomial Hermite interpolation (cf. [**9**, p. 300]): for a differentiable function f,

$$H_{2m-1}(z, f) = \sum_{j=1}^{m} f\left(\frac{1}{\overline{z_j}}\right) s_j(z) + \sum_{j=1}^{m} f'\left(\frac{1}{\overline{z_j}}\right) t_j(z) \tag{24}$$

is the interpolating polynomial of the degree at most $2m - 1$ satisfying

$$H_{2m-1}\left(\frac{1}{\overline{z_j}}\right) = f\left(\frac{1}{\overline{z_j}}\right), \quad H'_{2m-1}\left(\frac{1}{\overline{z_j}}\right) = f'\left(\frac{1}{\overline{z_j}}\right),$$

for $j = 1, 2, \ldots, m$, where the polynomials $s_j(z)$ and $t_j(z)$ are given by

$$s_j(z) = v_j(z) L_j(z)^2, \quad t_j(z) = \left(z - \frac{1}{\overline{z_j}}\right) L_j(z)^2,$$

$$v_j(z) = 1 - \frac{w''_m\left(\frac{1}{\overline{z_j}}\right)}{w'_m\left(\frac{1}{\overline{z_j}}\right)}\left(z - \frac{1}{\overline{z_j}}\right), \quad L_j(z) = \frac{w_m(z)}{\left(z - \frac{1}{\overline{z_j}}\right) w'_m\left(\frac{1}{\overline{z_j}}\right)},$$

for $j = 1, 2, \ldots, m$. When f is a polynomial of degree $\leq 2m-1$, formula (24) reproduces f. In particular, for $f(z) = p(z) - \lambda_n w_m^2(z)$,

$$p(z) - \lambda_n w_m^2(z) = \sum_{j=1}^{m} \alpha_{n,j} s_j(z) + \sum_{j=1}^{m} \beta_{n,j} t_j(z). \tag{25}$$

On the other hand, using (5), (6) and (25), we see that

$$\begin{aligned} q(z) = &-\sum_{j=1}^{m} \frac{\varphi_n}{\varphi_n^*}\left(\frac{1}{\overline{z_j}}\right) \alpha_{n,j} s_j(z) \\ &-\sum_{j=1}^{m} \left\{\left(\frac{\varphi_n}{\varphi_n^*}\right)'\left(\frac{1}{\overline{z_j}}\right) \alpha_{n,j} + \frac{\varphi_n}{\varphi_n^*}\left(\frac{1}{\overline{z_j}}\right) \beta_{n,j}\right\} t_j(z). \end{aligned} \tag{26}$$

Substituting (25) and (26) into Formula 4 yields

$$w_m^2(z)\psi_n(z) = \lambda_n w_m^2(z)\varphi_n(z) + \sum_{j=1}^{m}\left[\varphi_n(z) - \varphi_n^*(z)\frac{\varphi_n}{\varphi_n^*}\left(\frac{1}{\overline{z_j}}\right)\right]\alpha_{n,j} s_j(z)$$
$$+\sum_{j=1}^{m}\left[\varphi_n(z) - \varphi_n^*(z)\frac{\varphi_n}{\varphi_n^*}\left(\frac{1}{\overline{z_j}}\right)\right]\beta_{n,j} t_j(z) - \sum_{j=1}^{m} \varphi_n^*(z)\left(\frac{\varphi_n}{\varphi_n^*}\right)'\left(\frac{1}{\overline{z_j}}\right)\alpha_{n,j} t_j(z).$$

Note that, by the Christoffel-Darboux formula,

$$\varphi_n(z) - \varphi_n^*(z)\frac{\varphi_n}{\varphi_n^*}\left(\frac{1}{\overline{z_j}}\right) = \frac{K_n(z,z_j)(\overline{z_j}z-1)}{\overline{\varphi_n(z_j)}}.$$

So

$$\frac{w_m^2\psi_n}{\varphi_n}(z) = \lambda_n w_m^2(z) + \sum_{j=1}^{m}\frac{K_n(z,z_j)(\overline{z_j}z-1)}{\varphi_n(z)\overline{\varphi_n(z_j)}}\alpha_{n,j}s_j(z)$$

$$+\sum_{j=1}^{m}\frac{K_n(z,z_j)(\overline{z_j}z-1)}{\varphi_n(z)\overline{\varphi_n(z_j)}}\beta_{n,j}t_j(z) - \sum_{j=1}^{m}\frac{\varphi_n^*}{\varphi_n}(z)\left(\frac{\varphi_n}{\varphi_n^*}\right)'\left(\frac{1}{\overline{z_j}}\right)\alpha_{n,j}t_j(z).$$

Taking arbitrary $2m$ distinct points in $|z|>1$, say $\zeta_1,\zeta_2,\ldots,\zeta_{2m}$, and then letting $z=\zeta_k$, $k=1,2,\ldots,2m$, in the above equation, and finally putting the obtained relations into the matrix form give us

$$\boldsymbol{v_n} = (\boldsymbol{\alpha_n},\boldsymbol{\beta_n})\,\boldsymbol{V_n},$$

where $\boldsymbol{\alpha_n} := (\alpha_{n,1},\alpha_{n,2},\ldots,\alpha_{n,m})$, $\boldsymbol{\beta_n} := (\beta_{n,1},\beta_{n,2},\ldots,\beta_{n,m})$,

$$\boldsymbol{v_n} := \left(\frac{w_m^2\psi_n}{\varphi_n}(\zeta_1) - \lambda_n w_m^2(\zeta_1),\ldots,\frac{w_m^2\psi_n}{\varphi_n}(\zeta_{2m}) - \lambda_n w_m^2(\zeta_{2m})\right),$$

and

$$\boldsymbol{V_n} := \begin{pmatrix} \hat{s}_1(\zeta_1) & \hat{s}_1(\zeta_2) & \ldots & \hat{s}_1(\zeta_{2m}) \\ \hat{s}_2(\zeta_1) & \hat{s}_2(\zeta_2) & \ldots & \hat{s}_2(\zeta_{2m}) \\ \ldots & \ldots & & \ldots \\ \hat{s}_m(\zeta_1) & \hat{s}_m(\zeta_2) & \ldots & \hat{s}_m(\zeta_{2m}) \\ \hat{t}_1(\zeta_1) & \hat{t}_1(\zeta_2) & \ldots & \hat{t}_1(\zeta_{2m}) \\ \hat{t}_2(\zeta_1) & \hat{t}_2(\zeta_2) & \ldots & \hat{t}_2(\zeta_{2m}) \\ \ldots & \ldots & & \ldots \\ \hat{t}_m(\zeta_1) & \hat{t}_m(\zeta_2) & \ldots & \hat{t}_m(\zeta_{2m}) \end{pmatrix}$$

with

$$\hat{s}_j(z) := \frac{K_n(z,z_j)(\overline{z_j}z-1)}{\varphi_n(z)\overline{\varphi_n(z_j)}}s_j(z) - \frac{\varphi_n^*}{\varphi_n}(z)\left(\frac{\varphi_n}{\varphi_n^*}\right)'\left(\frac{1}{\overline{z_j}}\right)t_j(z), \tag{27}$$

and

$$\hat{t}_j(z) := \frac{K_n(z,z_j)(\overline{z_j}z-1)}{\varphi_n(z)\overline{\varphi_n(z_j)}}t_j(z). \tag{28}$$

Note the limit relations (cf. [**7**, Lemma 1])

$$\lim_{n\to\infty}\frac{K_n(z,z_j)(\overline{z_j}z-1)}{\varphi_n(z)\overline{\varphi_n(z_j)}} = 1 \tag{29}$$

for $|z| > 1$, and (cf. [**12**, (3.10)])

$$\lim_{n\to\infty} \frac{\varphi_n}{\varphi_n^*}(w) = 0$$

and consequently

$$\lim_{n\to\infty} \frac{\varphi_n^*}{\varphi_n}\left(\frac{1}{\overline{w}}\right) = 0 \text{ and } \lim_{n\to\infty}\left(\frac{\varphi_n}{\varphi_n^*}\right)'(w) = 0, \tag{30}$$

for $|w| < 1$. So, using (29) and (30) in (27) and (28), we have

$$\lim_{n\to\infty} \hat{s}_j(z) = s_j(z) \text{ and } \lim_{n\to\infty} \hat{t}_j(z) = t_j(z).$$

Therefore,

$$\lim_{n\to\infty} \boldsymbol{V_n} = \boldsymbol{V}$$

with

$$\boldsymbol{V} := \begin{pmatrix} s_1(\zeta_1) & s_1(\zeta_2) & \dots & s_1(\zeta_{2m}) \\ s_2(\zeta_1) & s_2(\zeta_2) & \dots & s_2(\zeta_{2m}) \\ \dots & \dots & & \dots \\ s_m(\zeta_1) & s_m(\zeta_2) & \dots & s_m(\zeta_{2m}) \\ t_1(\zeta_1) & t_1(\zeta_2) & \dots & t_1(\zeta_{2m}) \\ t_2(\zeta_1) & t_2(\zeta_2) & \dots & t_2(\zeta_{2m}) \\ \dots & \dots & & \dots \\ t_m(\zeta_1) & t_m(\zeta_2) & \dots & t_m(\zeta_{2m}) \end{pmatrix}.$$

It is easy to see that $\det \boldsymbol{V} \neq 0$ since $\{s_j\}$ and $\{t_j\}$ are linearly independent and form a basis of $\mathcal{P}_{2m-1}$. Also, using Theorem 2, we have $\lim_{n\to\infty} \boldsymbol{v_n} = \boldsymbol{v}$ with $\boldsymbol{v} :=$

$$\left(\lambda w_m(\zeta_1)\, w_m^*(\zeta_1) - \frac{\lambda}{\overline{B(0)}} w_m^2(\zeta_1), \dots, \lambda w_m(\zeta_{2m})\, w_m^*(\zeta_{2m}) - \frac{\lambda}{\overline{B(0)}} w_m^2(\zeta_{2m})\right).$$

So, letting $n \to \infty$ in $(\boldsymbol{\alpha_n}, \boldsymbol{\beta_n}) = \boldsymbol{v_n} \boldsymbol{V_n^{-1}}$, we get

$$\lim_{n\to\infty} (\boldsymbol{\alpha_n}, \boldsymbol{\beta_n}) = \boldsymbol{v}\boldsymbol{V^{-1}}. \tag{31}$$

Now, note that $\lambda w_m w_m^* - \lambda w_m^2/\overline{B(0)} \in \mathcal{P}_{2m-1}$. Thus, (24) implies

$$\lambda w_m(z)\, w_m^*(z) - \frac{\lambda}{\overline{B(0)}} w_m^2(z) = \sum_{j=1}^m \alpha_j s_j(z) + \sum_{j=1}^m \beta_j t_j(z).$$

(Actually, $\alpha_j = 0$, and $\beta_j = \lambda w_m' \left(1/\overline{z_j}\right) w_m^* \left(1/\overline{z_j}\right)$, $j = 1, \ldots, m$.) Substituting z by ζ_k, $k = 1, 2, \ldots, 2m$, we can obtain

$$\boldsymbol{v} = (\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m)\, \boldsymbol{V}.$$

This and (31) give

$$\lim_{n\to\infty} (\boldsymbol{\alpha_n}, \boldsymbol{\beta_n}) = (\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m).$$

Using this limit relation together with (25) and (29) we get

$$\lim_{n\to\infty} p(z) = \frac{\lambda}{\overline{B(0)}} w_m^2(z) + \sum_{j=1}^{m} \alpha_j s_j(z) + \sum_{j=1}^{m} \beta_j t_j(z) = \lambda w_m(z) w_m^*(z).$$

This completes the proof of the corollary. □

Acknoledgement. The authors thank the referee for helpful comments and suggestions.

References

[1] M. Alfaro, F. Marcellán, M.L. Rezola, and A. Ronveax, *Sobolev type orthogonal polynomials: The non diagonal case*, submitted,1993

[2] A. Atzmon, *n-Orthonormal operator polynomials*, in "Orthogonal Matrix-valued Polynomials and Applications", (Gohberg, I. Ed.), Birkhauser Verlag, Basel, 1988, 47-63.

[3] A. Cachafeiro and F. Marcellán, *Orthogonal polynomials of Sobolev type on the unit circle*, J. Approx. Theory, to appear.

[4] P.J. Davis, *"Interpolation and Approximation"*, Blaisdell Publishing Co., Waltham, Mass., 1963.

[5] A.A. Gonchar, *On the convergence of Padé approximants for some classes of meromorphic functions*, Math. USSR Sbornik, **26** (1975), 555-575.

[6] X. Li, *Representations of orthogonal polynomials for modified weight functions*, submitted, 1994.

[7] X. Li and K. Pan, *Asymptotic behavior of orthogonal polynomials corresponding to measure with discrete part off the unit circle*, J. Approx. Theory, to appear.

[8] G. López, F. Marcellán, and W. Van Asshe, *Relative assymptotics for polynomials orthogonal with repect to a discrete Sobolev inner product*, Constr. Approx., to appear.

[9] G.G. Lorentz, K. Jetter, and S.D. Riemenschneider, *Birkhoff Interpolation*, Addison-Wesley, London, 1983.

[10] F. Marcellán, M. Alfaro, and M.L. Rezola, *Orthogonal polynomials on Sobolev spaces: old and new directions*, J. Comp. Appl. Math., **48** (1993), 113-131.

[11] F. Marcellán and W. Van Asshe, *Relative assymptotics for orthogonal polynomials with Sobolev inner product*, J. Approx. Theory, **72** (1993), 193-209.

[12] A. Máté, P. Nevai, and V. Totik, *Extentions of Szegö theory of orthogonal polynomials*, II, Constr. Approx., **3** (1987), 51-72.

[13] G. Pólya and G. Szegö, *Problems and Theorems in Analysis* II, Springer-Verlag, New York, 1976.

[14] E.A. Rakhmanov, *On the asymptotics of the ratio of orthogonal polynomials*, II, USSR Sbornik, **53** (1983), 105-117.

[15] G. Szegö, *"Orthogonal Polynomials"*, 4th ed., Vol. **23**, Amer. Math. Soc. Colloquium Publ., Providence, RI, 1975.

Received December 7, 1993 and revised June 24, 1994. The research of the second author was supported by Comision interministerial de Ciencia y Tecnologia of Spain (CICYT) project PB-89/0181/C02/01.

UNIVERSITY OF CENTRAL FLORIDA
ORLANDO, FL 32816
E-mail address: xli@pegasus.cc.ucf.edu

AND

DEPARTAMENTO DE INGENIERIA
ESCUELA POLITÉCNICA SUPERIOR
UNIVERSIDAD CARLOS III DE MADRID
AVDA. MEDITERRÁNEO S/N
28913 LEGANÉS, MADRID
SPAIN