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CHAOS OF CONTINUUM-WISE EXPANSIVE HOMEOMORPHISMS AND DYNAMICAL PROPERTIES OF SENSITIVE MAPS OF GRAPHS

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In this paper, we study several properties of chaos of maps of compacta. We show that if a homeomorphism $f : X \to X$ of a compactum X with $\dim X > 0$ is continuum-wise expansive, then there is an f-invariant closed subset Y of X with $\dim Y > 0$ such that f is (two-sided strongly) chaotic on Y in the sense of Ruelle-Takens. Also, we investigate dynamical properties of maps of graphs which are sensitive. In particular, we prove the decomposition theorem of sensitive maps of graphs as follows: If $f : G \to G$ is map of a graph G which is sensitive, then there exist finite subgraphs G_i $(1 \leq i \leq N)$ of G such that (a) each G_i is f-invariant and $G_i \cap G_j$ is empty or a finite set for $i \neq j$, (b) for each $1 \leq i \leq N$, f is (two-sided strongly) chaotic on G_i in the sense of Devaney and there exists a connected subgraph H_i of G_i and a natural number $n(i) \geq 1$ such that H_i is $f^{n(i)}$-invariant, $f^{n(i)}|f^k(H_i) : f^k(H_i) \to f^k(H_i)$ $(0 \leq k \leq n(i) - 1)$ is topologically mixing, $\cup_{k=0}^{n(i)-1} f^k(H_i) = G_i$, and $f^k(H_i) \cap f^{k'}(H_i)$ is empty or a finite set for $0 \leq k < k' \leq n(i) - 1$, and (c) $\dim F(f) \leq 0$, where

$$F(f) = \left\{x \in G | f^n(x) \in Cl\left(G - \cup_{i=1}^N G_i\right) \quad \text{for each} \quad n \geq 0\right\}.$$

As a corollary, we show that in case of maps of graphs, chaos in the sense of Ruelle-Takens is equal to (two-sided strongly) chaos in the sense of Devaney, and sensitive maps of graphs induce two-sided chaos in the sense of Li-Yorke.

1. Introduction.

All spaces under consideration are assumed to be metric. By a *compactum*, we mean a compact metric space. A *continuum* is a nondegenerate connected compactum. A *map* is a continuous function. By $\dim X$, we mean the *topological dimension* of X (e.g., see [**8**]). Note that for a compactum X, $\dim X > 0$ if and only if there is a nondegenerate subcontinuum of X.

Let $\mathbb{Z}$ be the set of all integers. Let X be a compactum with metric d. For a subset A of X, put $\operatorname{diam} A = \sup\{d(x,y) | x, y \in A\}$. A homeomorphism

$f : X \to X$ is called *expansive* ([**25**], [**7**]) (resp. *continuum-wise expansive* [**14**]) if there is a positive number $c > 0$ such that if $x, y \in X$ and $x \neq y$ (resp. if A is a nondegenerate subcontinuum of X), then there is an integer $n \in \mathbb{Z}$ such that

$$d(f^n(x), f^n(y)) \geq c \quad (\text{resp. } \operatorname{diam} f^n(A) \geq c).$$

Such a positive number c is called an *expansive constant for* f. Note that every expansive homeomorphism is continuum-wise expansive, but the converse assertion is not true (see [**14**]). There are many important examples of continuum-wise expansive homeomorphisms which are not expansive. Note that (continuum-wise) expansiveness does not depend on the choice of the metric d of the compactum X. These properties have frequent applications in topological dynamics, ergodic theory and continuum theory (e.g., see the references below). A map $f : X \to X$ of a compactum X is *sensitive* ($= f$ has *sensitive dependence on initial conditions*) [**5**] if there exists a positive number $\tau > 0$ such that for each $x \in X$ and each neighborhood U of x in X there exists a point y of U such that $d(f^n(x), f^n(y)) \geq \tau$ for some $n \geq 0$. Now, we introduce new notion which is stronger than sensitivity. A map $f : X \to X$ is *strongly sensitive* if there is a positive number $\tau > 0$ such that for each $x \in X$ and each nonempty open set U in X (U may not contain x), there exists a point y of U such that $\liminf_{n\to\infty} d(f^n(x), f^n(y)) \geq \tau$. Such $\tau > 0$ is also called an *expansive constant for* f.

In this paper, first, we study the relation between (continuum wise) expansive homeomorphisms and chaos in the sense of Ruelle-Takens. In fact, we prove that $f : X \to X$ is a continuum-wise expansive homeomorphism of a compactum X with $\dim X > 0$, then there is an f-invariant closed subset Y with $\dim Y > 0$ such that f is "two-sided strongly chaotic" on Y in the sense of Ruelle-Takens. Note that Y is not a minimal set of f, because that if $f : X \to X$ is a continuum-wise expansive homeomorphism of a compactum X, then each minimal set of f is 0-dimensional ([**14**], [**20**]). Next, we investigate dynamical properties of maps of graphs which are sensitive. We prove the decomposition theorem of sensitive maps of graphs: If a map $f : G \to G$ of a graph G is sensitive, then there exist finite subgraphs G_i ($1 \leq i \leq N$) of G such that (a) each G_i is f-invariant and $G_i \cap G_j$ is empty or a finite set for $i \neq j$, (b) for each $1 \leq i \leq N$, f is (two-sided strongly) chaotic on G_i in the sense of Devaney, and there exists a connected subgraph H_i of G_i and a natural number $n(i) \geq 1$ such that H_i is $f^{n(i)}$-invariant, $f^{n(i)}|f^k(H_i) : f^k(H_i) \to f^k(H_i)$ is topologically mixing for $0 \leq k \leq n(i) - 1$, $\cup_{k=0}^{n(i)-1} f^k(H_i) = G_i$, $f^k(H_i) \cap f^{k'}(H_i)$ is empty or a finite set for $0 \leq k < k' \leq n(i) - 1$, and (c) $\dim F(f) \leq 0$, where

$$F(f) = \left\{x \in G | f^n(x) \in Cl\,(G - \cup_{i=1}^{N} G_i) \quad \text{for each} \quad n \geq 0\right\}.$$

Also, as a corollary, we show that in case of maps of graphs, chaos in the sense of Ruelle-Takens is equal to (two-sided strong) chaos in the sense of Devaney, and sensitive maps of graphs induce two-sided chaos in the sense of Li-Yorke.

2. Preliminaries.

In this section, we give some definitions and results which will be needed in the sequel.

Let $f : X \to X$ be a map of a compactum X. Then f is called *one-sided topologically transitive* [**26**] if there exists some $x \in X$ with $\{f^n(x)|n \geq k\}$ dense in X for each $k \geq 0$. Note that if $f : X \to X$ is one-sided topologically transitive, then f is an onto map. Let Y be an *f-invariant closed subset* of X, i.e., $f(Y) = Y$. A map $f : X \to X$ is said to be *chaotic on Y in the sense of Ruelle-Takens* [**23**] if the following conditions (C_1) and (C_2) are satisfied; (C_1) the restriction $f|Y : Y \to Y$ of f to the set Y is one-sided topologically transitive and (C_2) $f|Y : Y \to Y$ is sensitive. A point p of X is a *periodic point of f* with period k if $f^k(p) = p$ and $f^i(p) \neq p$ for $1 \leq i \leq k-1$. If the conditions (C_1) and (C_2) are satisfied, and moreover (C_3) the set of periodic points of $f|Y$ is dense in Y, then f is said to be *chaotic on Y in the sense of Devaney* [**5**].

Let $f : X \to X$ be a map of a compactum X. A subset S of X is called a *scrambled set* of f if the following conditions are satisfied. For any $x, y \in S$ and $x \neq y$,

(1) $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$,
(2) $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$, and
(3) $\limsup_{n\to\infty} d(f^n(x), f^n(p)) > 0$ for any periodic point p of f.

If there is an uncountable scrambled set S of f, is said to be *chaotic (on S) in the sense of Li-Yorke* [**19**].

A map $f : X \to X$ of a compactum X is *topologically mixing* [**1**] if for any nonempty open sets U and V of X, there is a natural number $n_0 \geq 1$ such that if $n \geq n_0$, then $f^n(V) \cap U \neq \emptyset$. By definitions, we can easily see that topological mixing implies one-sided topological transitivity.

Let $f : X \to X$ be a map of a compactum X. A two-sided sequence $\{x_n\}_{n=0,\pm1,\pm2,\ldots,}$ of points of X is called a *bisequence of f* if $f(x_{n+1}) = x_n$ for each $-\infty < n < \infty$. A map $f : X \to X$ is *two-sided strongly sensitive* if there is $\tau > 0$ such that for each bisequence $\{x_n\}_{n=0,\pm1,\pm2,\ldots,}$ of f, and each nonempty open set U in X, there is a bisequence $\{y_n\}_{n=0,\pm1,\pm2,\ldots,}$ of f such that $y_0 \in U$ and

$$\liminf_{n\to\infty} d(x_n, y_n) \geq \tau \quad \text{and} \quad \liminf_{n\to\infty} d(x_{-n}, y_{-n}) \geq \tau.$$

Similarly, a map $f : X \to X$ is *two-sided topologically transitive* if there is a bisequence $\{x_n\}_{n=0,\pm1,\pm2,\dots,}$ of f such that $\{x_n | n \geq k\}$ and $\{x_{-n} | n \geq k\}$ are dense in X for each $k \geq 0$. But, in (2.5) we see that one-sided topological transitivity is equal to two-sided topological transitivity. It is easily seen that strong sensitivity implies sensitivity, but the converse assertion is not true. Also, continuum-wise expansiveness is different from sensitivity and (two-sided) strong sensitivity. Clearly, (strong) sensitivity does not mean two-sided (strong) sensitivity.

Let Y be an f-invariant closed set of a map $f : X \to X$. Then f is said to be *two-sided strongly chaotic on Y in the sense of Ruelle-Takens* if $(C_1)'$ $f|Y : Y \to Y$ is two-sided topologically transitive and $(C_2)'$ $f|Y$ is two-sided strongly sensitive. If the conditions $(C_1)'$ and $(C_2)'$ are satisfied and moreover (C_3) the set of periodic points of $f|Y$ is dense in Y, then f is said to be *two-sided strongly chaotic on Y in the sense of Devaney.*

Let $f : X \to X$ be a map of a compactum X and let $\mathcal{G}$ be an uncountable set of bisequences of f. Then f is said to be *two-sided chaotic on $\mathcal{G}$ in the sense of Li-Yorke* if the following conditions are satisfied: For any $\mathbb{X} = \{x_n\}_{n=0,\pm1,\dots}$, $\mathbb{Y} = \{y_n\}_{n=0,\pm1,\dots} \in \mathcal{G}$ with $\mathbb{X} \neq \mathbb{Y}$,

(1) $\limsup_{n\to\infty} d(x_n, y_n) > 0$, $\limsup_{n\to\infty} d(x_{-n}, y_{-n}) > 0$,
(2) $\liminf_{n\to\infty} d(x_n, y_n) = 0 = \liminf_{n\to\infty} d(x_{-n}, y_{-n})$,
(3) $\limsup_{n\to\infty} d(x_n, p_n) > 0$ and $\limsup_{n\to\infty} d(x_{-n}, p_{-n}) > 0$ for any $p = \{p_n\}_{n=0,\pm1,\dots}$, where p is a bisequence of f such that there is a periodic point p of f with period $k \geq 1$ such that $p_{k\cdot i} = p$ for each $i \in \mathbb{Z}$.

To clarify the differences between the above properties of maps, we give some examples in the Section 6.

Let X be a compactum with metric d. By the *hyperspace* of X, we mean $C(X) = \{A|\ A$ is a nonempty subcontinuum of $X\}$ with the *Hausdorff metric* d_H, i.e., $d_H(A, B) = \inf\{\varepsilon > 0|\ U_\varepsilon(A) \supset B$ and $U_\varepsilon(B) \supset A\}$, where $U_\varepsilon(A)$ denotes the ε-neighborhood of A in X. Then the space $C(X)$ is also a compactum (see **[21]**). Also, for any sets A and B, put $d(A, B) = \inf\{d(a, b)|\ a \in A$ and $b \in B\}$.

A subset E of a space X is a *G_δ-set* if E is a countable intersection of open sets E_n of X, i.e., $E = \cap_{n=1}^{\infty} E_n$. A subset F of a space X is an *F_δ-set* if F is a countable union of closed sets F_n of X, i.e., $F = \cup_{n=1}^{\infty} F_n$.

Let $f : X \to X$ be a homeomorphism of a compactum X. For each $x \in X$, the *stable set* $W^s(x)$ and the *unstable set* $W^u(x)$ of f are defined by

$$W^s(x) = \left\{ y \in X \;\middle|\; \lim_{n\to\infty} d(f^n(x), f^n(y)) = 0 \right\},$$
$$W^u(x) = \left\{ y \in X \;\middle|\; \lim_{n\to\infty} d(f^{-n}(x), f^{-n}(y)) = 0 \right\}.$$

Also, for each closed set Z of X and $x \in Z$, the *continuum-wise stable* and *unstable sets* $V^s(x;Z)$ and $V^u(x;Z)$ (see [**16**]) are defined by

$V^s(x;Z) = \{y \in Z |$ there is a subcontinuum A of Z such that $x, y \in A$ and $\lim_{n\to\infty} \operatorname{diam} f^n(A) = 0\}$,

$V^u(x;Z) = \{y \in Z |$ there is a subcontinuum A of Z such that $x, y \in A$ and $\lim_{n\to\infty} \operatorname{diam} f^{-n}(A) = 0\}$.

Note that $V^\sigma(x;Z)$ is a connected subset of Z.

A subcontinuum Z of X is called *a chaotic continuum of f with respect to s* (resp. u) [**16**] if

(i) for each $x \in Z$, $V^s(x;Z)$ (resp. $V^u(x;Z)$) is dense in Z, and

(ii) there is a positive number $\tau > 0$ such that for each $x \in Z$ and each neighborhood U of x in X there exists a point $y \in U \cap Z$ such that $\liminf_{n\to\infty} d(f^n(x), f^n(y)) \geq \tau$ (resp. $\liminf_{n\to\infty} d(f^{-n}(x), f^{-n}(y)) \geq \tau$).

From topology, we know that inverse limits spaces yield powerful techniques for constructing complicated spaces and maps from simple spaces and maps. Also, inverse limit spaces and shift maps are important in chaotic dynamical systems (e.g., see [**24**]).

Let $f : X \to X$ be a map of a compactum X. Consider the inverse limit (X, f) of f as follows:

$$(X, f) = \{ (x_n)_{n=0}^\infty |\; x_n \in X \quad \text{and} \quad f(x_{n+1}) = x_n \quad \text{for each} \quad n \geq 0\}.$$

For each $\tilde{x} = (x_n)_{n=0}^\infty$, $\tilde{y} = (y_n)_{n=0}^\infty \in (X, f)$, define a metric $\tilde{d}$ by

$$\tilde{d}(\tilde{x}, \tilde{y}) = \sum_{n=0}^{\infty} d(x_n, y_n)/2^n.$$

Then (X, f) is a compactum. Let $p_n : (X, f) \to X$ be the natural projection, i.e., $p_n((x_0, x_1, x_2, \dots)) = x_n$. Also, define a map $\tilde{f} : (X, f) \to (X, f)$ by $\tilde{f}((x_0, x_1, x_2, \dots)) = (f(x_0), x_0, x_1, \dots)$. Then $\tilde{f}$ is a homeomorphism of (X, f) which is called the *shift map* of f. Note that for any onto map $f : X \to X$ of a compactum X, f is two-sided topologically transitive (resp. two-sided strongly sensitive) if and only if the shift map $\tilde{f}$ of f is so. Also, a map $f : X \to X$ of a compactum X is two-sided chaotic on an uncountable set $\mathcal{G}$ of bisequences of f in the sense of Li-Yorke if and only if both $\tilde{f}$ and $\tilde{f}^{-1}$ is chaotic on $\mathcal{G}^+$ in the sense of Li-Yorke, where

$$\mathcal{G}^+ = \{\tilde{x} = (x_n)_{n\geq 0} \in \; (X, f) \mid (\dots, x_{-1}, x_0, x_1, x_2, \dots,) \in \mathcal{G}\}.$$

Note that $p_n\,|\mathcal{G}^+ : \mathcal{G}^+ \to p_n\,(\mathcal{G}^+)$ is a bijection for each $n \geq 0$.

The following theorem was not explicitly stated in [**16**], but its proof follows readily that of [**16**, (3.6)].

Theorem 2.1 [**16**, (3.6)]. *If $f : X \to X$ is a continuum-wise expansive homeomorphism of a compactum X with $\dim X > 0$, then there is a chaotic continuum Z of f with respect to $\sigma = s$ or u. Moreover, if there is a nondegenerate subcontinuum A of X such that $\lim_{n\to\infty} \operatorname{diam} f^n(A) = 0$ (resp. $\lim_{n\to\infty} \operatorname{diam} f^{-n}(A) = 0$), then there is a chaotic continuum Z of f with respect to $\sigma = s$ (resp. $\sigma = u$).*

The following lemmas will be used in the next sections.

Lemma 2.2 ([**14**, (2.1)]). *Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with an expansive constant $c > 0$ and let $0 < \varepsilon \leq c$. If $A \in C(X)$ and $\operatorname{diam} f^n(A) \leq \varepsilon$ for each $n \geq 0$ (resp. $\operatorname{diam} f^{-n}(A) \leq \varepsilon$ for each $n \geq 0$), then $\lim_{n\to\infty}$ onadiam$f^n(A) = 0$ (resp. $\lim_{n\to\infty} \operatorname{diam} f^{-n}(A) = 0$).*

Lemma 2.3 (see the proof of [**14**, (2.3)]). *Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with an expansive constant $c > 0$ and let $0 < \varepsilon \leq c/2$. Then there is $\delta > 0$ such that if A is any nondegenerate subcontinuum of X such that $\operatorname{diam} A \leq \delta$ and $\operatorname{diam} f^m(A) \geq \varepsilon$ for some integer $m \in \mathbb{Z}$, then one of the following conditions holds:*

(a) *If $m \geq 0$, then $\operatorname{diam} f^n(A) \geq \delta$ for each $n \geq m$. More precisely, for any $x \in f^n(A)$ there is a subcontinuum B of A such that $x \in f^n(B)$, $\operatorname{diam} f^j(B) \leq \varepsilon$ for $0 \leq j \leq n$ and $\operatorname{diam} f^n(B) = \delta$.*

(b) *If $m < 0$, then $\operatorname{diam} f^{-n}(A) \geq \delta$ for each $n \geq -m$. More precisely, for each $x \in f^{-n}(A)$ there is a subcontinuum B of A such that $x \in f^{-n}(B)$, $\operatorname{diam} f^{-j}(B) \leq \varepsilon$ for $0 \leq j \leq n$ and $\operatorname{diam} f^{-n}(B) = \delta$.*

In particular, for each $\eta > 0$ there is a natural number $N(\eta) \geq 0$ such that if $A \in C(X)$ and $\operatorname{diam} A \geq \eta$, then either $\operatorname{diam} f^n(A) \geq \delta$ for each $n \geq N(\eta)$ or $\operatorname{diam} f^{-n}(A) \geq \delta$ for each $n \geq N(\eta)$ holds.

Lemma 2.4 [**14**, (2.5)]. *If $f : X \to X$ is a continuum-wise expansive homeomorphism of a compactum X with $\dim X > 0$, then there is a nondegenerate subcontinuum A of X such that either $\lim_{n\to\infty} \operatorname{diam} f^n(A) = 0$ or $\lim_{n\to\infty} \operatorname{diam} f^{-n}(A) = 0$ holds.*

The next lemma is trivial and it may be known, but for completeness, we give the proof.

Lemma 2.5. *Let $f : X \to X$ be a map of a compactum X. If f is one-sided topologically transitive, then f is two-sided topologically transitive.*

Proof. Consider the shift map $\widetilde{f} : (X, f) \to (X, f)$ of f. Since f is one-sided topologically transitive and hence f is an onto map, we see that $\widetilde{f}$ is also one-sided topologically transitive. Note that $\widetilde{f}$ is one-sided topologically transitive if and only if whenever $\widetilde{U}, \widetilde{V}$ are nonempty open sets of (X, f) there exists $n \geq 1$ with $\widetilde{f}^{-n}\left(\widetilde{U}\right) \cap \widetilde{V} \neq \emptyset$ (see [**26**, Theorem 5.9]). Since $\widetilde{f}^{-n}\left(\widetilde{U}\right) \cap \widetilde{V} \neq \emptyset$, then $\widetilde{U} \cap \widetilde{f}^{n}\left(\widetilde{V}\right) = \widetilde{f}^{n}\left(f^{-n}\left(\widetilde{U}\right) \cap \widetilde{V}\right) \neq \emptyset$. Hence $\widetilde{f}^{-1}$ is also one-sided topologically transitive. Then the set $\widetilde{A}$ of $\widetilde{x} \in (X, f)$ with $\left\{\widetilde{f}^{n}(\widetilde{x}) |\, n \geq k\right\}$ dense in (X, f) for each $k \geq 0$ is a dense G_δ-set. Also, the set $\widetilde{B}$ of $\widetilde{x} \in (X, f)$ with $\left\{\widetilde{f}^{-n}(\widetilde{x}) |\, n \geq k\right\}$ dense in (X, f) is a dense G_δ-set (see also [**26**, Theorem 5.9]). By Baire's category theorem, $\widetilde{A} \cap \widetilde{B} \neq \emptyset$. This means that $\widetilde{f}$ is two-sided topologically transitive, which implies that f is two-sided topologically transitive. □

Lemma 2.6 (see the proof of [**14**, (3.2)] and [**14**, (3.8)]). *Let $f : G \to G$ be a map of a graph G which is sensitive. Then there is a positive number $\tau > 0$ such that if $\widetilde{A}$ is a subcontinuum of (G, f) with* $\operatorname{diam} \widetilde{A} \leq \tau$, *then* $\lim_{n\to\infty} \operatorname{diam} \widetilde{f}^{-n}\left(\widetilde{A}\right) = 0$.

3. Chaos in the sense of Ruelle-Takens of Continuum-wise Expansive Homeomorphisms.

Let $f : X \to X$ be a map of a compactum X with $\dim X > 0$. Consider the set $\mathbb{I}(f) = \{E |\, E$ is an f-invariant closed subset of X with $\dim E > 0\}$, and put $\mathbb{M}(f) = \{E \in \mathbb{I}(f) |\, E$ is minimal of $\mathbb{I}(f)$ with respect to the inclusion$\}$. In general, $\mathbb{M}(f)$ may be an empty set. Note that the family $\mathbb{M}(f)$ is not the family of minimal sets of f.

Now, we prove the following theorem.

Theorem 3.1. *Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with* $\dim X > 0$. *Then* $\mathbb{M}(f) \neq \emptyset$ *and if* $Y \in \mathbb{M}(f)$, *then f is two-sided strongly chaotic on Y in the sense of Ruelle-Takens.*

To prove (3.1), we need Brouwer's Reduction Theorem.

Brouwer's Reduction Theorem 3.2 (e.g. see [**8**, p. 161]). *In a space X with countable basis, let $\{K_\lambda\}$ be a family of closed sets with this property: if $K_1 \supset K_2 \supset \cdots$ is any decreasing sequence of members of $\{K_\lambda\}$, then $\cap_{i=1}^{\infty} K_i$*

is a member of $\{K_\lambda\}$. *Then there exists an irreducible set* ($=$ *minimal set*) *in* $\{K_\lambda\}$.

Proof of Theorem 3.1. Consider the set $\mathbb{I}(f)$. Note that $X \in \mathbb{I}(f)$. By (2.3), if $E \in \mathbb{I}(f)$, then there is a component C of E such that $\operatorname{diam} C \geq \delta$, where δ is a positive number as in (2.3). Suppose that $E_i \in \mathbb{I}(f)$ $(i = 1, 2, \dots,)$ and $E_1 \supset E_2 \supset \cdots,$. Let C_i be a component of E_i with $\operatorname{diam} C_i \geq \delta$. Since $C(X)$ is compact, we may assume that $\lim_{i\to\infty} C_i = C$. Then C is a subcontinuum of $\cap_{i=1}^{\infty} E_i$ and $\operatorname{diam} C \geq \delta$.

Hence $\cap_{i=1}^{\infty} E_i \in \mathbb{I}(f)$. By (3.2), there is a minimal element Y of $\mathbb{I}(f)$. Hence $\mathbb{M}(f) \neq \emptyset$. Let $Y \in \mathbb{M}(f)$. Then Y is f-invariant and $\dim Y > 0$. By (2.4), we may assume that there exists a nondegenerate subcontinuum A_0 of Y such that $\lim_{n\to\infty} \operatorname{diam} f^{-n}(A_0) = 0$. The other case is similar.

Let $y \in Y$ be any point of Y and B a nondegenerate subcontinuum of Y such that $\lim_{n\to\infty} \operatorname{diam} f^{-n}(B) = 0$. We shall show that the following condition $(*)$ is satisfied.

$(*)$ For each $\varepsilon > 0$ and each natural number $k \geq 0$, there is a natural number $N \geq k$ such that $d(y, f^N(B)) < \varepsilon$.

If we put $D = Cl(\cup_{n=k}^{\infty} f^n(B)) \subset Y$, then $f(D) \subset D$. Put $E = \cap_{n=1}^{\infty} f^n(D)$. Then $f(E) = E$. Since $\liminf_{n\to\infty} \operatorname{diam} f^n(B) \geq \delta$ (see (2.3)), we see that $f^n(D)$ has a component C_n with $\operatorname{diam} C_n \geq \delta$. Since $C(Y)$ is compact, we may assume that $\lim_{n\to\infty} C_n = C$. Then C is a nondegenerate subcontinuum and $C \subset E$. Hence $\dim E > 0$. This implies that $Y = E = D$, because that $Y \in \mathbb{M}(f)$. Hence we see that for any $\varepsilon > 0$, there is $N \geq k$ such that $d(y, f^N(B)) < \varepsilon$.

Let $\mathcal{B} = \{B_i\}_{i=1}^{\infty}$ be a countable base of Y. By using the condition $(*)$ and induction on i, we can choose a subcontinuum A_i $(i \geq 1)$ of Y such that

(1) $A_0 \supset A_1 \supset A_2 \supset \cdots$

(2) there is a sequence $n(1) < n(2) < \cdots$ of natural numbers such that $B_i \cap f^{n(i)}(A_i) \neq \emptyset$ and $f^{n(j)}(A_i) \subset B_j$ for each $1 \leq j \leq i-1$.

Choose a point $a \in \cap_{i=0}^{\infty} A_i$. By (2), we see that $\{f^n(a) |\ n \geq k\}$ is dense in Y for each $k \geq 0$. Hence $f|Y : Y \to Y$ is one-sided topologically transitive. By (2.5), we see that $f|Y : Y \to Y$ is two-sided topologically transitive.

Next, we shall prove that $f|Y : Y \to Y$ is two-sided strongly sensitive. Since there is a nondegenerate subcontinuum A_0 of Y such that $\lim_{n\to\infty} \operatorname{diam} f^{-n}(A_0) = 0$, by (2.1) there exists a chaotic continuum Z of $f|Y$ with respect to $\sigma = u$. Note that for each $n \geq 0$ $\operatorname{diam} f^{-n}(Z) > 4 \cdot \eta$ for some fixed positive number $\eta > 0$ (see (ii) of the definition of chaotic continuum). Let $x \in Y$ and let U be any nonempty open set of Y. By

(2.3), we can choose a natural number $N \geq 1$ such that if $D \in C(Y)$ and $\operatorname{diam} D \geq \eta$, then either $\operatorname{diam} f^n(D) \geq \delta$ for $n \geq N$ or $\operatorname{diam} f^{-n}(D) \geq \delta$ for $n \geq N$ holds. We may assume that $4 \cdot \eta < \delta$. Choose a positive number $\lambda > 0$ such that if $y, y' \in Y$ and $d(y, y') < \lambda$, then $d(f^{-j}(y), f^{-j}(y')) < \eta$ for each $0 \leq j \leq N$. Since $Y \in \mathbb{M}(f)$, $\operatorname{diam} f^{-n}(Z) > 4 \cdot \eta$ for each $n \geq 0$, by the same argument as the above, we see that $\cup_{n=1}^{\infty} f^{-n}(Z)$ is dense in Y. Since $f^{-n}(Z)$ is also a chaotic continuum of f with respect to $\sigma = u$, we may assume that the chaotic continuum Z intersects U, i.e., $U \cap Z \neq \emptyset$. If necessary, we may replace $f^{-n}(Z)$ by Z. Let $i = 1, 2, 3, \ldots$, be any natural number. Choose a subcontinuum $Z(i)$ of $f^{-N \cdot i}(Z)$ such that $\lim_{j \to \infty} \operatorname{diam} f^{-j}(Z(i)) = 0$ and $\operatorname{diam} Z(i) > 4 \cdot \eta$ (see (i) of the definition of chaotic continuum). Choose a subcontinuum $H_{i,0}$ of $Z(i)$ such that $\operatorname{diam} H_{i,0} > \eta$ and $d\left(f^{-N \cdot i}(x), H_{i,0}\right) > \eta$. Since $\operatorname{diam} f^N(H_{i,0}) \geq \delta > 4 \cdot \eta$, we can choose a subcontinuum $H_{i,1}$ of $f^N(H_{i,0})$ such that $\operatorname{diam} H_{i,1} > \eta$ and $d\left(f^{-N \cdot (i-1)}(x), H_{i,1}\right) > \eta$. If we continue this procedure, we obtain a finite sequence $H_{i,0}, H_{i,1}, \ldots, H_{i,i}$ of subcontinua of Y such that

$$H_{i,k} \subset f^N\left(H_{i,k-1}\right), \ \operatorname{diam} H_{i,k} > \eta$$

and

$$d\left(f^{-N \cdot (i-k)}(x), H_{i,k}\right) > \eta$$

for each $1 \leq k \leq i$. Note that $H_{i,i} \subset Z$. Choose a point $x_i \in H_{i,i}$. Then $d\left(f^{-N \cdot j}(x_i), f^{-N \cdot j}(x)\right) \geq \eta$ for each $0 \leq j \leq i$. Hence we see that $d(f^{-n}(x_i), f^{-n}(x)) \geq \lambda$ for each $0 \leq n \leq N \cdot i$. Since $x_i \in Z$ for each i, we may assume that $\lim_{i \to \infty} x_i = x' \in Z$. Then we can see that $d(f^{-n}(x),\ f^{-n}(x')) \geq \tau$ for each $n \geq 0$. Since $Z \cap U \neq \emptyset$ and $V^u(x'; Z)$ is dense in Z, we can choose a nondegenerate subcontinuum H contained in $U \cap V^u(x'; Z)$. Choose a natural number N' such that $\operatorname{diam} f^{N'}(H) \geq \delta > 4 \cdot \eta$. By the similar argument to the above, we can choose subcontinua $D_{N'+N \cdot i}$ $(i \geq 0)$ such that

$$D_{N'} \subset f^{N'}(H), \ D_{N'+N \cdot (i+1)} \subset f^N\left(D_{N'+N \cdot i}\right), \ \operatorname{diam} D_{N'+N \cdot i} \geq \eta$$

and

$$d\left(f^{N'+N \cdot i}(x),\ f^{N'+N \cdot i}\left(D_{N'+N \cdot i}\right)\right) \geq \eta.$$

Choose a point $y \in \cap_{i=1}^{\infty} f^{-(N'+N \cdot i)}\left(D_{N'+N \cdot i}\right) \subset H \subset U$. Then we can see that $\liminf_{n \to \infty} d(f^n(x), f^n(y)) \geq \lambda$ and $\liminf_{n \to \infty} d(f^{-n}(x), f^{-n}(y)) = \liminf_{n \to \infty} d(f^{-n}(x), f^{-n}(x')) \geq \lambda$, because $y \in V^u(x'; Z)$. Hence $f|Y$ is two-sided strongly sensitive, which implies that f is two-sided strongly chaotic on Y in the sense of Ruelle-Takens. This completes the proof. □

Remark 3.3. In (3.1), by the condition $(*)$ in the proof of (3.1), we see that there is a positive number $\delta > 0$ such that if $Y \in \mathbb{M}(f)$ and C is any component of Y, then $\operatorname{diam} C \geq \delta$.

4. Sensitive Maps of Graphs induce Chaos in the sense of Devaney.

In this section, we study some dynamical properties of sensitive maps of graphs. In particular, we show that sensitive maps of graphs induce two-sided chaos in the sense of Devaney.

A map $f : X \to X$ of a compactum X is *positively continuum-wise expansive* [**14**] if there is a positive number $c > 0$ such that if A is a nondegenerate subcontinuum of X, then there is a natural number $n \geq 0$ such that $\operatorname{diam} f^n(A) \geq c$.

Lemma 4.1 ([**14**, (3.1) and (5.7)]). *Suppose that $f : X \to X$ is a positively continuum-wise expansive map of a compactum X. Then the shift map $\widetilde{f}$ of f is a (positively) continuum-wise expansive homeomorphism. Moreover, if* $\dim X > 0$, *then* $\dim(X, f) > 0$.

First, we show the following proposition.

Proposition 4.2. *Suppose that a map $f : X \to X$ of a compactum X with $\dim X > 0$ is positively continuum-wise expansive. Then $\mathbb{M}(f) \neq \emptyset$ and if $Y \in \mathbb{M}(f)$, then f is two-sided strongly chaotic on Y in the sense of Ruelle-Takens.*

Proof. Consider the shift map $\widetilde{f} : (X, f) \to (X, f)$ of f. By (4.1), $\dim(X, f) > 0$ and the shift map $\widetilde{f}$ is (positively) continuum-wise expansive. By (3.1), $\mathbb{M}\left(\widetilde{f}\right) \neq \emptyset$. Note that $\mathbb{M}(f) = \left\{p_0\left(\widetilde{Y}\right) \middle| \ \widetilde{Y} \in \mathbb{M}\left(\widetilde{f}\right)\right\}$, because that $p_n \cdot \widetilde{f} = f \cdot p_n$ and $\widetilde{Y} = \left(p_0\left(\widetilde{Y}\right), f\left|p_0\left(\widetilde{Y}\right)\right.\right)$. Let $Y \in \mathbb{M}(f)$. Then $(Y, f|Y) \in \mathbb{M}\left(\widetilde{f}\right)$. By (3.1), $\widetilde{f}$ is two-sided strongly chaotic in the sense of Ruelle-Takens, and hence f is two-sided strongly chaotic on Y in the sense of Ruelle-Takens. $\square$

By a *graph*, we mean a compact 1-dimensional polyhedron which has no isolated point. We may not assume that it is connected. By a *subgraph* of G, we mean a 1-dimensional closed subset of the graph G which is homeomorphic to a graph.

Let $f : X \to X$ be a map of a compactum X. For each $x \in X$, put $\omega(x) = \left\{y \in X \mid \text{there is a subsequence } \{f^{n(i)}(x)\}_{i=1}^{\infty} \text{ of } \{f^n(x)\}_{n=1}^{\infty} \text{ such that } \lim_{i\to\infty} f^{n(i)}(x) = y\right\}$. Then $\omega(x)$ is called the *ω-limit set of x*. Note that if p is a periodic point of f, then $p \in \omega(p)$. A point $x \in X$ is called *non-wandering for* f if each neighborhood U of x, there exists $n \geq 1$ such that $f^{-n}(U) \cap U \neq \emptyset$. Note that a point $x \in X$ is non-wandering for f if and only if for each neighborhood U of x and every $N \geq 1$, there is $n \geq N$ such that $f^{-n}(U) \cap U \neq \emptyset$ (see [**26**, Theorem 5.7]).

The *non-wandering set* $\Omega(f)$ *for* f consists of all the points that are non-wandering. Note that $\omega(x)$ and $\Omega(x)$ are f-invariant closed subsets of $X, \Omega(f)$ contains all periodic points of f, $\omega(x) \subset \Omega(f)$ and if $f : X \to X$ is a homeomorphism, then $\Omega(f) = \Omega(f^{-1})$.

Lemma 4.3 ([**14**, (3.9)]). *Let $f : G \to G$ be a map of a graph G. Then the following are equivalent.*

(1) *f is sensitive.*

(2) *f is positively continuum-wise expansive.*

Now, we prove the following theorem which implies that sensitive maps of graphs induce two-sided strong chaos in the sense of Devaney.

Theorem 4.4. *Let $f : G \to G$ be a map of a graph G which is sensitive. Then $\mathbb{M}(f) \neq \emptyset$ and $\mathbb{M}(f) = \{G_i|\ 1 \leq i \leq N\}$ is a finite set of subgraphs of G satisfying the following properties:*

(a) *If $i \neq j$, then $G_i \cap G_j$ is empty or a finite set.*

(b) *For each i, f is two-sided stronglŷ chaotic on G_i in the sense of Devaney.*

(c) *If we put $L = Cl\,(G - \cup_{i=1}^{N} G_i)$ and $F(f) = \{x \in L|\ f^n(x) \in L$ for each $n \geq 0\}$, then $F(f)$ is a closed subset of L with $f(F(f)) \subset F(f)$, $\dim F(f) \leq 0$. If $x \in L - F(f)$, then there is a neighborhood U of x in G and a natural number $n(x) \geq 1$ such that $f^n(U) \subset \cup_{i=1}^{N} G_i$ for each $n \geq n(x)$, and hence $\Omega(f) \supset \cup_{i=1}^{N} G_i$ and $L \cap \Omega(f) \subset F(f)$.*

Proof. Consider the shift map $\widetilde{f} :\ (G, f) \to (G, f)$ and put $X = (G, f)$. Note that $\widetilde{f}$ is a positively continuum-wise expansive homeomorphism and $\dim X > 0$. By (4.2), we see that $\mathbb{M}(f) \neq \emptyset$ and if $Y \in \mathbb{M}(f)$, f is two-sided strongly chaotic in the sense of Ruelle-Takens. Note that $p_0\left(\mathbb{M}\left(\widetilde{f}\right)\right) = \mathbb{M}(f)$. By (3.3), there is a positive number $\delta > 0$ such that if $\widetilde{Y} \in \mathbb{M}\left(\widetilde{f}\right)$ and $\widetilde{C}$ is a component of $\widetilde{Y}$, then $\operatorname{diam} \widetilde{C} \geq \delta$. Let $\widetilde{Y}_1 \in \mathbb{M}\left(\widetilde{f}\right)$.

Now, we show that $\widetilde{Y}_1$ has finite components. Suppose, on the contrary, that $\widetilde{Y}_1$ has infinite components. Since G is a (finite) graph, for each $\lambda > 0$ there is a natural number $n(\lambda) \geq 1$ such that if a family $\{F_1, F_2, \dots, F_{n(\lambda)}\}$ of connected sets of G satisfies that $F_i \cap F_j$ is a finite set for $i \neq j$, then there is some i such that $\operatorname{diam} F_i < \lambda$. Choose $\lambda > 0$ and a natural number $k \geq 0$

such that if $\widetilde{D}$ is a subset of X and $\operatorname{diam} p_k\left(\widetilde{D}\right) < \lambda$, then $\operatorname{diam} \widetilde{D} < \delta$. Since $\widetilde{Y}_1$ has infinite components, we can choose a natural number $N \geq k$ and components $\widetilde{C}_1, \widetilde{C}_2, \ldots, \widetilde{C}_{n(\lambda)}$ of $\widetilde{Y}_1$ such that $p_N\left(\widetilde{C}_1\right), p_N\left(\widetilde{C}_2\right), \ldots, p_N\left(\widetilde{C}_{n(\lambda)}\right)$ are mutually disjoint. Hence there is some j such that $\operatorname{diam} p_N\left(\widetilde{C}_j\right) < \lambda$. Since $p_k \widetilde{f}^{-(N-k)}\left(\widetilde{C}_j\right) = p_N\left(\widetilde{C}_j\right)$ and $\widetilde{f}^{-(N-k)}\left(\widetilde{C}_j\right)$ is a component of $\widetilde{Y}_1$, then we see that $\operatorname{diam} \widetilde{f}^{-(N-k)}\left(\widetilde{C}_j\right) < \delta$, which is a contradiction. Hence $\widetilde{Y}_1$ has finite components.

Put $G_1 = p_0\left(\widetilde{Y}_1\right) \left(= p_n\left(\widetilde{Y}_1\right)\right)$. Then G_1 is a graph, $G_1 \in \mathbb{M}(f)$ and the diameters of components of G_1 are larger than some $\lambda_0 > 0$, because that the diameters of components of $\widetilde{Y}_1$ is $\geq \delta$. By (4.2), f is two-sided strongly chaotic on G_1 in the sense of Ruelle-Takens. Note that $\widetilde{Y}_1 = (G_1, f|G_1)$. If $\widetilde{Y}_2 \in \mathbb{M}\left(\widetilde{f}\right)$ and $\widetilde{Y}_2 \neq \widetilde{Y}_1$, then $\dim \widetilde{Y}_2 > 0$ and $\widetilde{f}$ is (two-sided strongly) chaotic on $\widetilde{Y}_2$ in the sense of Ruelle-Takens. By the same argument as above, we see that $G_2 = p_0\left(\widetilde{Y}_2\right)$ has finite components whose diameters are bigger than $\lambda_0 > 0$. Note that $G_1 \cap G_2$ does not contain a nonempty open set, because that $G_1 \neq G_2$ and $f|G_i : G_i \to G_i$ $(i = 1, 2)$ is one-sided topologically transitive. Hence $G_1 \cap G_2$ is empty or a finite set. By using this argument, we can see that $\mathbb{M}\left(\widetilde{f}\right)$ is finite. Let $\mathbb{M}\left(\widetilde{f}\right) = \left\{\widetilde{Y}_i \mid 1 \leq i \leq N\right\}$ and $p_0\left(\widetilde{Y}_i\right) = G_i$. Then $G_1, G_2, \ldots, G_N$ satisfies the property (a).

Next, for each $0 \leq i \leq N$, we show that the property (b) is satisfied. By (4.5) below, we can see that the property (b) is satisfied.

Finally, we shall show that property (c) is satisfied. Let $x \in L - F(f)$. Then there is a natural number $n(x) \geq 1$ such that $f^{n(x)}(x) \in G - L$. Choose a neighborhood U of x in G such that $f^{n(x)}(U) \subset G - L \subset \cup_{i=1}^N G_i$. Since $\cup_{i=1}^N G_i$ is f-invariant, we see that $f^n(U) \subset \cup_{i=1}^N G_i$ for each $n \geq n(x)$.

Next, we show $\dim F(f) \leq 0$. Let A be an arc in L. Then there is a natural number $n \geq 1$ such that $f^n(A) \cap (G - L) \neq \emptyset$. Suppose, on the contrary, that $f^n(A) \subset L$ for each $n \geq 1$. Put $\omega(A) = \cup\{B \in C(L) \mid$ there is a subsequence $\{f^{n(i)}(A)\}$ of $\{f^n(A)\}_{n=1}^\infty$ such that $\lim_{i\to\infty} f^{n(i)}(A) = B\}$. Then we see that $\omega(A)$ is a closed subset of L, $f(\omega(A)) = \omega(A)$ and $\dim \omega(A) > 0$. By (4.1), we can choose another element $\widetilde{Y}_{N+1} \in \mathbb{M}\left(\widetilde{f}\right)$ such that $\widetilde{Y}_{N+1} \subset (\omega(A), f|\omega(A))$. This implies that $\widetilde{Y}_{N+1} \neq \widetilde{Y}_i$ $(1 \leq i \leq N)$. This is a contradiction. Hence the set $F(f)$ contains no arc. Clearly, $\dim F(f) \leq 0$. By (b), the set $P(f|G_i)$ of periodic points of $f|G_i : G_i \to G_i$ is dense in G_i for each $i = 1, 2, \ldots, N$ (see (4.5)). Hence $\cup_{i=1}^N G_i = \cup_{i=1}^N Cl(P(f|G_i)) \subset \Omega(f)$. This completes the proof. □

Theorem 4.5. *Suppose that $f : G \to G$ is a map of a graph G. Then the* *following are equivalent.*

(1) f *is chaotic* (*on* G) *in the sense of Ruelle-Takens.*
(2) f *is chaotic* (*on* G) *in the sense of Devaney.*
(3) f *is two-sided strongly chaotic* (*on* G) *in the sense of Devaney.*

Proof. We show that (1) implies (2). Suppose that f is sensitive and one-sided topologically transitive. We must show that the set of periodic points of f is dense in G. Let U be a nonempty open set of G. Choose an arc $[a,b]$ in U such that $(a,b)\,(=[a,b]-\{a,b\})$ is an open set of G, because that G is a graph. Since f is two-sided topologically transitive, we choose a point $\widetilde{x}=(x_n)_{n=0}^{\infty}\in(G,f)$ such that $\{x_n|\,n\geq 0\}$ is dense in G. Since $\{x_n|\,n\geq i\}$ is dense in G for each $i\geq 0$, we may assume that $x_0\in(a,b)$. By (2.7), we can choose a small nondegenerate subcontinuum $\widetilde{A}$ of (G,f) containing $\widetilde{x}$ such that $\lim_{n\to\infty}\operatorname{diam}\widetilde{f}^{-n}\left(\widetilde{A}\right)=0$ and $p_0\left(\widetilde{A}\right)\subset(a,b)$. Note that $p_0\left(\widetilde{A}\right)$ is a nondegenerate arc, because that f is sensitive. Note that $\lim_{n\to\infty}\operatorname{diam}\widetilde{f}^{-n}\left(\widetilde{A}\right)=0$ implies that $\lim_{n\to\infty}\operatorname{diam}p_n\left(\widetilde{A}\right)=0$. Since $\{x_n|\,n\geq i\}$ is dense in G, we can choose a point x_N such that $x_N\in\operatorname{Int}\left(p_0\left(\widetilde{A}\right)\right)$. Also, moreover, we may assume that $p_N\left(\widetilde{A}\right)\subset\operatorname{Int}\left(p_0\left(\widetilde{A}\right)\right)$. Put $[a_0,b_0]=p_0\left(\widetilde{A}\right)$ and $[a_N,b_N]=p_N\left(\widetilde{A}\right)$. Hence we see that $f^N([a_N,b_N])=[a_0,b_0]$ and $[a_N,b_N]\subset[a_0,b_0]$. This implies that we can choose a point p in $[a_N,b_N]$ such that $f^N(p)=p$. Therefore the set of periodic points of f is dense in G, which implies that f is chaotic in the sense of Devaney. The rest of the proof follows from Proposition (4.2), because that $\mathbb{M}(f)=\{G\}$. □

Corollary 4.6. *Let* $f:G\to G$ *be a map of a graph* G *that is sensitive, and let* $\widetilde{f}:(G,f)\to(G,f)$ *be the shift map of* f *and* $X=(G,f)$. *Then* $\mathbb{M}\left(\widetilde{f}\right)=\left\{\widetilde{Y}_1,\widetilde{Y}_2,\dots,\widetilde{Y}_N\right\}$ *is nonempty and a finite set and the following conditions are satisfied:*

(a) *Each* $\widetilde{Y}_i$ *has finite nondegenerate components,* $\widetilde{Y}_i\cap\widetilde{Y}_j$ $(i\neq j)$ *is empty or a finite set whose elements are periodic points of* $\widetilde{f}$.

(b) $\widetilde{f}$ *is two-sided strongly chaotic on* $\widetilde{Y}_i\,(1\leq i\leq N)$ *in the sense of Devaney.*

(c) *There is a closed subset* $F\left(\widetilde{f}\right)$ *in* $Cl\left(X-\cup_{i=1}^{N}\widetilde{Y}_i\right)$ *such that* $\dim F\left(\widetilde{f}\right)\leq 0$ *and* $F\left(\widetilde{f}\right)$ *is* $\widetilde{f}$*-invariant and if any* $\widetilde{x}\in X-\left(\left(\cup_{i=1}^{N}\widetilde{Y}_i\right)\cup F\left(\widetilde{f}\right)\right)$ *and any* $\varepsilon>0$, *then there is a neighborhood* $\widetilde{U}$ *of* $\widetilde{x}$ *in* X *and a natural number* $n(\varepsilon)\geq 1$ *such that* $\widetilde{f}^n\left(\widetilde{U}\right)$ *is contained in the* ε*-neighborhood of* $\cup_{i=1}^{N}\widetilde{Y}_i$ *in* X *for each* $n\geq n(\varepsilon)$, *and hence* $\Omega\left(\widetilde{f}\right)\supset\cup_{i=1}^{N}\widetilde{Y}_i$ *and* $\Omega\left(\widetilde{f}\right)\cap$

$\left(Cl\left(X - \cup_{i=1}^{N}\widetilde{Y}_i\right)\right) \subset F\left(\widetilde{f}\right)$.

Proof. Let $G_i\,(1 \le i \le N)$ be the subgraphs as in (4.4). Put $\widetilde{Y}_i = (G_i, f|G_i)$ for each i. If $i \neq j$, then $\widetilde{Y}_i \cap \widetilde{Y}_j = (G_i \cap G_j, f|G_i \cap G_j)$ and $G_i \cap G_j$ is empty or a finite set. Note that $f(G_i \cap G_j) \subset G_i \cap G_j$. If $G_i \cap G_j \neq \emptyset$, we can find a natural number k such that for each $n \geq k$, $f^n(G_i \cap G_j) = f^k(G_i \cap G_j)$. Note that $(G_i \cap G_j, f|G_i \cap G_j) = (f^k(G_i \cap G_j), f|f^k(G_i \cap G_j))$ and $f|f^k(G_i \cap G_j)$ is bijective. Hence we see that each point of $\widetilde{Y}_i \cap \widetilde{Y}_j$ is a periodic point of $\widetilde{f}$. Put $F\left(\widetilde{f}\right) = (F(f), f|(F(f)))$, where $F(f)$ is the set as in (4.4). Then the set $F\left(\widetilde{f}\right)$ satisfies the desired properties. This completes the proof. □

Corollary 4.7. *Suppose that $f: G \to G$ is a map of a graph G which is sensitive. Then the following are equivalent.*

(1) *The set of periodic points of f is dense in G.*
(2) $\Omega(f) = G$.
(3) $G = \cup \{G_i |\, G_i \in \mathbb{M}(f)\}$.

Hence f is chaotic in the sense of Devaney if and only if $\mathbb{M}(f) = \{G\}$.

Proof. By the definitions, we can easily see that (1) implies (2). We show that (2) implies (3). Let $G_i\,(1 \le i \le N)$ be the subgraphs as in (4.4). Let $L = Cl\,(G - \cup_{i=1}^{N} G_i)$ and $F(f)$ be as in (c) of (4.4). Suppose that $L \neq \emptyset$. Note that $\dim F(f) \le 0$, hence $L - F(f) \neq \emptyset$. By (c) of (4.4), we see that each point of $L - F(f)$ is wandering for f. This is a contradiction. Hence $G = \cup_{i=1}^{N} G_i$. The rest of the proof follows from (4.4). □

Corollary 4.8. *Let $f: G \to G$ be a map of a graph G which is sensitive. Then*

(1) $\dim \omega(x) > 0$ *if and only if* $\omega(x) \in \mathbb{M}(f)$,
(2) *the set $W = \{x \in G|\ \dim \omega(x) > 0\}$ is G_δ-dense G, and*
(3) *the set $W' = \{x \in G|$ the orbit $O(x) = \{f^n(x) |\, n \geq 0\}$ is a finite set$\}$ is F_σ-dense in G.*

Proof. Suppose that $\mathbb{M}(f) = \{G_i|\ 1 \le i \le N\}$. Note that $\omega(x)$ is an f-invariant closed set.

We prove (1). Suppose that $\dim \omega(x) > 0$. If $x \in G_i$ $(1 \le i \le N)$, then $\omega(x) \subset G_i$. Since we can choose a subgraph H in $\omega(x)$ such that $H \in \mathbb{M}(f)$ (see the proof of (4.4)), we see that $\omega(x) = G_i$. If $x \in L$, then x is not contained in $F(f)$, because that $\dim F(f) \le 0$. Hence there is $n \geq 1$ such that $f^n(x) \in G_i$ for some i. Hence $\omega(x) = \omega(f^n(x)) = G_i$. The converse assertion is trivial.

Next, we prove (2). Let $\{B_k^i\}_{k=1}^{\infty}$ be an open base of G_i. Put $W_i = \cap_{k=1}^{\infty} \cup_{j=1}^{\infty} f^{-j}\left(B_k^i\right)$ for each i. Then each W_i is G_δ. Then $W = \cup_{i=1}^{N} W_i$ is also G_δ. We show that W is dense in G. Note that $\{x \in G_i|\ \dim \omega(x) > 0$, i.e., $\{f^n(x)|\ n \geq 0\}$ is dense in $G_i\}$ is dense in G_i. If U is a nonempty open set of L, then we can choose an arc A in U such that $A \cap F(f) = \emptyset$, because that $\dim F(f) \leq 0$. By (c) of (4.4), we can choose a point y in A such that $\dim \omega(y) > 0$, because that $f^n(A)$ has nonempty interior. Hence W is G_δ-dense in G.

Finally, we shall prove (3). Let $P_n = \{x \in G|$ the cardinal number $|O(x)|$ of the set $O(x)$ is $\leq n\}$. Clearly, P_n is a closed subset of G and $\cup_{n=1}^{\infty} P_n = W'$. Hence W' is F_σ. Let U be a nonempty open set of G. If $U \cap \cup_{i=1}^{N} G_i \neq \emptyset$, U contains a periodic point p of f, and hence $p \in W'$. If $U \subset L$, by (4.4) there is a nonempty open set V in U such that $f^n(V) \subset \cup_{i=1}^{N} G_i$ for some n. Since $f^n(V)$ contains a periodic point of f, there is a point $q \in V$ such that $q \in W'$. Hence W' is F_σ-dense in G. □

Related to transitivity, sensitivity and the property that periodic points are dense, the following are known.

Theorem 4.9 ([**2**]). *If $f : X \to X$ is a map of a metric space X such that X is an infinite set, f is one-sided topologically transitive and the set of periodic points of f is dense, then f is sensitive, and hence f is chaotic in the sense of Devaney.*

Theorem 4.10 ([**3**]). *If $f : I \to I$ is a map of the unit interval $I = [0, 1]$ such that f is one-sided topologically transitive, then f is chaotic on I in the sense of Devaney.*

5. Topological Mixing and Sensitive Maps of Graphs.

In this section, furthermore we investigate more detailed dynamical properties of sensitive maps of graphs.

First, we prove the following.

Theorem 5.1. *Suppose that $f : G \to G$ is a map of a graph G which is sensitive and one-sided topologically transitive, i.e., f is chaotic in the sense of Ruelle-Takens. Then there is a connected subgraph H of G and a natural number N such that $f^N(H) = H$, $\cup_{n=0}^{N-1} f^n(H) = G$, $f^i(H) \cap f^j(H)$ is empty or a finite set for $0 \leq i < j \leq N-1$, and $f^N|f^n(H) : f^n(H) \to f^n(H)$ is topologically mixing.*

Proof. Note that for a sensitive map $g : K \to K$ of a graph K, a subgraph K' (i.e., $\dim K' > 0$) of K is a member of $\mathbb{M}(g)$ if and only if $g(K') = K'$ and $g|K' : K' \to K'$ is one-sided topologically transitive (see the proof of

(4.4)). Also, by (4.4), if $g : K \to K$ is a sensitive map of a graph K, then there is a subgraph K' of $\mathbb{M}(g)$.

Put $K_1 = G$ and $f_1 = f$. Since f is one-sided topologically transitive, $K_1 \in \mathbb{M}(f_1)$. Suppose that there is a natural number $n(1) \geq 2$ and a proper subgraph K of K_1 (i.e., $K_1 \supsetneqq K$) such that $f_1^{n(1)}(K) = K$. Then we can choose a subgraph K_2 of $\mathbb{M}\left(f_1^{n(1)}\right)$. Put $f_2 = f_1^{n(1)}|K_2$. Since K_2 is a member of $\mathbb{M}\left(f_1^{n(1)}\right)$ and K_1 is a member of $\mathbb{M}(f_1)$, we see that

(1) for each $0 \leq i < j \leq n(1) - 1$, $f_1^i(K_2) \cap f_1^j(K_2)$ is empty or a finite set and $\cup_{n=0}^{n(1)-1} f_1^n(K_2) = K_1$, because that the set $\cup_{n=0}^{n(1)-1} f_1^n(K_2)$ is f_1-invariant. If there is a natural number $n(2) \geq 2$ and a proper subgraph K of K_2 such that $f_2^{n(2)}(K) = K$, we can continue this procedure. Hence we obtain a sequence $n(1), n(2), \ldots$, of natural numbers, a sequence $f_1, f_2, \ldots$, of maps and a sequence $K_1 \supsetneqq K_2 \supsetneqq, \ldots$, of subgraphs of G such that

(2) for each $k = 1, 2, \ldots$, K_k is a member of $\mathbb{M}(f_k)$,

(3) for each $0 \leq i < j \leq n(k) - 1$, $f_k^i(K_{k+1}) \cap f_k^j(K_{k+1})$ is empty or a finite set and $\cup_{n=0}^{n(k)-1} f_k^n(K_{k+1}) = K_k$.

Since (K_k, f_k) is $\widetilde{f}_k$-invariant and $\widetilde{f} : (G, f) \to (G, f)$ is a continuum-wise expansive homeomorphism , by (2.3) we can see that there is a positive number $\lambda > 0$ such that for each k, $\operatorname{diam} K_k \geq \lambda$. Hence by (3), we see that the above sequence $K_1 \supsetneqq K_2 \supsetneqq, \ldots$, is a finite sequence. Therefore we must reach the situation that there is a natural number $r \geq 1$ such that K_r is a member of $\mathbb{M}(f_r^n)$ for each $n = 1, 2, \ldots$, i.e., $f_r^n(K) \neq K$ if n is any natural number $n \geq 1$ and K is any proper subgraph of K_r. By using this fact, we shall show that K_r is connected. If K_r is not connected, K_r has finite components $C_1, C_2, \ldots, C_t$. Then $f_r^t(C_1) = C_1$. This is a contradiction. Put $H = K_r$ and $g = f_r : H \to H$. Note that $g^n : H \to H$ is one-sided topologically transitive for each $n \geq 1$. Since $f_r = f^N|H$ for some natural number $N \geq 1$, $f^N(H) = g(H) = H$, $f^i(H) \neq f^j(H)$ $(0 \leq i < j \leq N - 1)$. Since $f : G \to G$ is one sided topologically transitive, we see that $f^i(H) \cap f^j(H)$ is empty or a finite set for $0 \leq i < j \leq N - 1$ and $\cup_{n=0}^{N-1} f^n(H) = G$. Next, we shall show that $g : H \to H$ is topologically mixing. Consider the shift map $\widetilde{g} : (H, g) \to (H, g)$ of g. Since g is sensitive, $\widetilde{g}$ is a continuum-wise expansive homeomorphism of the continuum (H, g). By (4.5), we see that the set of periodic points of $\widetilde{g}$ is dense in (H, g). Also, $\widetilde{g}$ satisfies the following condition:

(#) For each natural number $n \geq 1$ and for any proper closed subset E of (H, g) with $\dim E > 0$, $\widetilde{g}^n(E) \neq E$. □

By the following proposition (5.2) below, we see that $\widetilde{g}$ is topologically mixing, which implies that g is topologically mixing.

Proposition 5.2. *Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a continuum X. Suppose that the set of periodic points of f is dense in X and for each natural number $n \geq 1$ and each proper closed subset E of X with $\dim E > 0$, $f^n(E) \neq E$. Then f is topologically mixing.*

Proof. Let U and V be nonempty open sets of X. Let p be a periodic point of f. By (2.4), there is a nondegenerate subcontinuum A of X such that $\lim_{n\to\infty} \operatorname{diam} f^n(A) = 0$ or $\lim_{n\to\infty} \operatorname{diam} f^{-n}(A) = 0$. We may assume that $\lim_{n\to\infty} \operatorname{diam} f^{-n}(A) = 0$. The other case is similarly proved.

Now, we shall prove that for each $x \in X$, there is nondegenerate subcontinuum A_x of X such that $x \in A_x$ and $\lim_{n\to\infty} \operatorname{diam} f^{-n}(A_x) = 0$. Let ε and δ be positive numbers as in (2.3). We may assume that $\operatorname{diam} A \leq \delta/2$. Let x be any point of X. Since $X \in \mathbb{M}(f)$, by the same way as in the proof of (3.1), we see that for each $k \geq 0$, $\cup_{n=k}^{\infty} f^n(A)$ is dense in X. By (2.3), we can choose a sequence $n(1) < n(2) < \cdots$, of natural numbers and subcontinua B_i $(i = 1, 2, 3, \dots)$ of $f^{n(i)}(A)$ such that diam $B_i = \delta$, $\lim_{i\to\infty} d(x, B_i) = 0$ and diam $f^{-j}(B_i) \leq \varepsilon$ for each $0 \leq j \leq n(i)$. We may assume that $\lim_{i\to\infty} B_i = A_x$. Then $x \in A_x$ and $\lim_{n\to\infty} \operatorname{diam} f^{-n}(A_x) = 0$ (see (2.2)). Hence, we see that $V^u(x; X)$ is a nondegenerate connected set. Choose a periodic point $p \in V$ and let n_0 be the period of p. Since $f^j(V^u(p; X)) = V^u(f^j(p); X)$ for $0 \leq j \leq n_0 - 1$, $Cl(V^u(f^j(p); X))$ is an f^{n_0}-invariant set for $0 \leq j \leq n_0 - 1$. By the hypothesis, we see that $X = Cl(V^u(f^j(p); X))$ for each $0 \leq j \leq n_0 - 1$. For each $0 \leq j \leq n_0 - 1$, choose a point y_j of $U \cap V^u(f^j(p); X)$. Then $\lim_{n\to\infty} f^{-n_0 \cdot n}(y_j) = f^j(p) \in f^j(V)$. Hence, for each $0 \leq j \leq n_0 - 1$, we can choose a natural number $N_j \geq 1$ such that if $n \geq N_j$, then $f^{-n_0 \cdot n}(y_j) \in f^j(V)$, i.e., $f^{(-n_0 \cdot n) - j}(y_j) \in V$. Put $N' = \operatorname{Max}\{N_j |\ 0 \leq j \leq n_0 - 1\}$ and $N'' = n_0 \cdot N'$. Suppose that $n > N''$. Then put $n = s \cdot n_0 + j$, where $0 \leq j \leq n_0 - 1$. Then $s \cdot n_0 + j > n_0 \cdot N'$ implies that $s \geq N'$. Hence $f^{-n}(y_j) = f^{(-n_0 \cdot s) - j}(y_j) \in V$, which implies that $f^n(V) \cap U \neq \emptyset$. Hence $f : X \to X$ is topologically mixing. □

Combining (4.4) with (5.1), we obtain the following theorem.

Theorem 5.3 (Decomposition theorem of sensitive maps of graphs). *Let $f : G \to G$ be a map of a graph G which is sensitive. Then $\mathbb{M}(f) \neq \emptyset$ and $\mathbb{M}(f) = \{G |\ 1 \leq i \leq N\}$ is a finite set of subgraphs of G satisfying the following properties:*

(a) *If $i \neq j$, then $G_i \cap G_j$ is empty or a finite set.*

(b) *For each i, f is two-sided strongly chaotic on G_i in the sense of Devaney, and there exists a connected subgraph H_i of G_i and a natural number $n(i) \geq 1$ such that H_i is $f^{n(i)}$-invariant, $f^{n(i)}|f^k(H_i) : f^k(H_i) \to f^k(H_i)$ is topologically mixing for $0 \leq k \leq n(i) - 1$ and $G_i = \cup_{k=0}^{n(i)-1} f^k(H_i)$, and*

$f^k(H_i) \cap f^{k'}(H_i)$ $(0 \leq k < k' \leq n(i) - 1)$ *is empty or a finite set.*

(c) *If we put* $L = Cl\,(G - \cup_{i=1}^N G_i)$ *and* $F(f) = \{x \in L|\ f^n(x) \in L$ *for each* $n \geq 0\}$, *then* $F(f)$ *is a closed subset of* L *with* $f(F(f)) \subset F(f)$ *and* $\dim F(f) \leq 0$. *If* $x \in L - F(f)$, *then there is a neighborhood* U *of* x *in* G *and a natural number* $n(x) \geq 1$ *such that* $f^n(U) \subset \cup_{i=1}^N G_i$ *for each* $n \geq n(x)$. *In particular,* $\Omega(f) \supset \cup_{i=1}^N G_i$ *and* $L \cap \Omega(f) \subset F(f)$.

By (4.6), (5.3) and [**16**, (3.15) and (5.4)], we obtain the following theorem.

Theorem 5.4. *Suppose that* $f : G \to G$ *is a map of a graph* G *which is sensitive. Let* $\widetilde{f} : (G, f) \to (G, f)$ *be the shift map of* f *and* $X = (G, f)$. *Then* $\mathbb{M}\left(\widetilde{f}\right) = \left\{\widetilde{Y}_1, \widetilde{Y}_2, \dots, \widetilde{Y}_N\right\}$ *is nonempty and a finite set, and the following conditions are satified:*

(a) $\widetilde{Y}_i \cap \widetilde{Y}_j$ $(i \neq j)$ *is empty or a finite set of periodic points of* $\widetilde{f}$.

(b) $\widetilde{f}$ *is two-sided strongly chaotic on* $\widetilde{Y}_i$ *in the sense of Devaney, and there is a continuum* $\widetilde{Z}_i$ *in* $\widetilde{Y}_i$ *and a natural number* $n(i) \geq 1$ *such that* $f^{n(i)}\left(\widetilde{Z}_i\right) = \widetilde{Z}_i$, $\cup_{k=0}^{n(i)-1} \widetilde{f}^k\left(\widetilde{Z}_i\right) = \widetilde{Y}_i$, $\widetilde{f}^k\left(\widetilde{Z}_i\right) \cap \widetilde{f}^{k'}\left(\widetilde{Z}_i\right)$ *is empty or a finite set of periodic points of* $\widetilde{f}$ *for* $0 \leq k < k' \leq n(i) - 1$, *and* $\widetilde{f}^{n(i)}\Big|\widetilde{f}^k\left(\widetilde{Z}_i\right) : \widetilde{f}^k\left(\widetilde{Z}_i\right) \to \widetilde{f}^k\left(\widetilde{Z}_i\right)$ *is topologically mixing. In particular,* $\widetilde{Z}_i$ *is a chaotic continuum of* $\widetilde{f}$ *and hence* $\widetilde{Z}_i$ *is an indecomposable continuum.*

(c) *There is a closed subset* $F\left(\widetilde{f}\right)$ *in* $Cl\left(X - \cup_{i=1}^N \widetilde{Y}_i\right)$ *such that* $F\left(\widetilde{f}\right)$ *is* $\widetilde{f}$*-invariant,* $\dim F\left(\widetilde{f}\right) \leq 0$ *and if* $\widetilde{x} \in X - \left(\left(\cup_{i=1}^N \widetilde{Y}_i\right) \cup F\left(\widetilde{f}\right)\right)$ *and any* $\varepsilon > 0$, *there is a neighborhood* $\widetilde{U}$ *of* $\widetilde{x}$ *in* X *and a natural number* $n(\varepsilon) \geq 1$ *such that* $\widetilde{f}^n\left(\widetilde{U}\right)$ *is contained in the* ε*-neighborhood of* $\cup_{i=1}^N \widetilde{Y}_i$ *in* X *for each* $n \geq n(\varepsilon)$, *and hence* $\Omega\left(\widetilde{f}\right) \supset \cup_{i=1}^N \widetilde{Y}_i$ *and* $\Omega\left(\widetilde{f}\right) \cap \left(Cl\left(X - \cup_{i=1}^N \widetilde{Y}_i\right)\right) \subset F\left(\widetilde{f}\right)$.

Corollary 5.5. *Let* $f : G \to G$ *be a map of a connected graph* G. *Then* f *is topologically mixing if and only if* f *is sensitive and one-sided topologically transitive and the inverse limit* (G, f) *of* f *is indecomposable.*

Theorem 5.6. *Let* $f : G \to G$ *be a map of a graph* G *which is sensitive. Then there is a connected subgraph* H *of* G *and a natural number* $s \geq 1$ *such that* $f^s(H) = H$ *and the shift map* $\widetilde{f}$ *of* f *is two-sided chaotic on almost all Cantor sets of* Z *in the sense of Li-Yorke, where* $Z = (H, f^s|H) \subset (G, f^s) \cong (G, f)$. *Hence there exists an uncountable set* $\mathcal{G}$ *of bisequence of* f *such that* $p_n\left(\mathcal{G}^+\right)$ *is a Cantor set of* G *and* f *is two-sided chaotic on* $\mathcal{G}$ *in the sense of*

Li-Yorke.

To prove (5.6), we need the following notations. A subset of X is *of the first category* if there are subset E_n of X such that $E = \cup_{n=1}^{\infty} E_n$, and E_n is nowhere dense, i.e., $\mathrm{Int}_X\, Cl(E_n) = \emptyset$. A subset F of a space X is said to be *independent in* $R \subset X^n\, (n \geq 1)$, if for every system $x_1, x_2, \ldots, x_n$ of different points of F, the point $(x_1, x_2, \ldots, x_n) \in F^n$ never belongs to R. In [**18**, Main theorem and Corolary 3], Kuratowski proved the following theorem.

Theorem 5.7 (Kuratowski's Independent Theorem). *If X is a complete space and $R \subset X^n$ is an F_σ-set of the first category, then the set of $J(R)$ of all compact subsets F of X independent in R is a dense G_δ-set in 2^X of all compact subsets of X with the Hausdorff metric. Moreover, if X has no isolated points, then almost all Cantor sets of X are independent in R.*

A homeomorphism $f : X \to X$ of a continuum X is *continuum-wise fully expansive* [**17**] if for any $\varepsilon > 0$ and any $\eta > 0$, there is a natural number $N = N(\varepsilon, \eta) \geq 1$ such that if A is a nondegenerate subcontinuum of X with diam $A \geq \eta$, then either $d_H(f^n(A), X) < \varepsilon$ for each $n \geq N$ or $d_H(f^{-n}(A), X) < \varepsilon$ for each $n \geq N$ holds. Note that for a map $f : G \to G$ of a connected graph G, f is topologically mixing if and only if the shift map $\widetilde{f}$ of f is *continuum-wise fully expansive* (see [**17**, (3.11)]).

Proposition 5.8. *Let $f : X \to X$ be a homeomorphism of a compactum X and Z an f-invariant nondegenerate subcontinuum of X. If $f|Z : Z \to Z$ is continuum-wise fully expansive, then f is two-sided chaotic on almost all Cantor sets of Z in the sense of Li-Yorke.*

Proof. We may assume that there is a subcontinuum A of Z such that $\lim_{n\to\infty} \operatorname{diam} f^{-n}(A) = 0$ (see (2.5)). By [**16**, (3.15)], Z is a chaotic continuum of $f|Z$ with respect to $\sigma = u$. Let $\tau > 0$ be a positive number as in the definition of chaotic continuum. Consider the following sets.

$R_1^u = \{(x, y) \in Z \times Z |\, \limsup_{n\to\infty} d(f^{-n}(x), f^{-n}(y)) < \tau/2\}$,
$R_2^u = \{(x, y) \in Z \times Z |\, \liminf_{n\to\infty} d(f^{-n}(x), f^{-n}(y)) > 0\}$,
$P^u = \{x \in Z|$ there is a periodic point of f such that
$\limsup_{n\to\infty} d(f^{-n}(x), f^{-n}(p)) \leq \tau/5\}$.

Since Z is a chaotic continuum of $f|Z$ with respect to u, by the proof of [**16**, (4.1)], we see that $R_1^u, R_2^u, Z \times P^u$ and $P^u \times Z$ are F_σ-sets of the first category.

Put diam $Z = \tau' > 0$. Similarly, consider the sets:

$R_1^s = \{(x, y) \in Z \times Z |\, \limsup_{n\to\infty} d(f^n(x), f^n(y)) < \tau'/3\}$,

$R_2^s = \{(x,y) \in Z \times Z | \liminf_{n\to\infty} d(f^n(x), f^n(y)) > 0\}$,
$P^s = \{x \in Z|$ there is a periodic point p of f such that $\limsup_{n\to\infty} d(f^n(x), f^n(p)) \leq \tau'/5\}$.

By the same way as in the proof of [**16**, (4.1)], we see that $R_1^s, R_2^s, P^s \times Z$ and $Z \times P^s$ are F_σ-sets. Since Z is a chaotic continuum of f with respect to $\sigma = u$ and by the condition (i) of the definition of chaotic continuum, we see that for each $z \in Z$, there is a nondegenerate subcontinuum A_z of Z such that $z \in A_z$ and $\lim_{n\to\infty} \operatorname{diam} f^{-n}(A_z) = 0$. Since $\lim_{n\to\infty} f^n(A_z) = Z$, by the argument as before, for each $x \in X$ and each nonempty open set U of Z, there exists a point $y \in U$ such that $\liminf_{n\to\infty} d(f^n(x), f^n(y)) \geq \tau'/3$, i.e., $f|Z : Z \to Z$ is strongly sensitive. This implies that R_1^s is of the first category in $Z \times Z$. Next, we shall show that R_2^s is of the first category in $Z \times Z$. Let $(x,y) \in Z \times Z$. Choose a small nondegenerate subcontinuum A_y of Z such that $y \in A_y$ and $\lim_{n\to\infty} \operatorname{diam} f^{-n}(A_y) = 0$. Note that for each nondegenerate subcontinuum B of A_y, $\lim_{n\to\infty} d(f^n(y), f^n(B)) = 0$, because that $\lim_{n\to\infty} f^n(B) = Z$. Hence we can choose a point y' in A_y such that $\liminf_{n\to\infty} d(f^n(x), f^n(y')) = 0$. This implies that R_2^s is of the first category in $Z \times Z$. Also, by the similar way as in the proof of [**16**, (4.1)], we see that $P^s \times Z$ and $Z \times P^s$ are of the first category in $Z \times Z$.

Consider the set in $Z \times Z$:

$$R = R_1^u \cup R_2^u \cup (P^u \times Z) \cup (Z \times P^u) \cup R_1^s \cup R_2^s \cup (P^s \times Z) \cup (Z \times P^s).$$

Then R is an F_σ-set of the first category in $Z \times Z$. Note that if $(x,y) \notin R$, then

(1) $\limsup_{n\to\infty} d(f^{-n}(x), f^{-n}(y)) \geq \tau/2$, and
$\limsup_{n\to\infty} d(f^n(x), f^n(y)) \geq \tau'/3$,
(2) $\liminf_{n\to\infty} d(f^{-n}(x), f^{-n}(y)) = 0 = \liminf_{n\to\infty} d(f^n(x), f^n(y))$,
(3) $\limsup_{n\to\infty} d(f^{-n}(x), f^{-n}(p)) \geq \tau/5$, and
$\limsup_{n\to\infty} d(f^n(x), f^n(y)) \geq \tau'/5$ for each periodic point p of f.

By Kuratowski's independent theorem, f is two-sided chaotic on almost all Cantor sets of Z in the sense of Li-Yorke. This completes the proof. □

Proof of Theorem 5.6. By (5.3), there is a connected subgraph H of G and a natural number $s \geq 1$ such that $f^s(H) = H$ and $f^s|H : H \to H$ is topologically mixing. Put $g = f^s$. Then $(H, g|H) \subset (G,g) \cong (G,f)$. Put $X = (G,g)$ and $Z = (H, g|H)$, and $\widetilde{g}|Z : Z \to Z$ is a continuum-wise fully expansive. By (5.8), we see that $\widetilde{g}$ is two-sided chaotic on almost all Cantor sets of Z in the sense of Li-Yorke. We may assume that $Z \subset (G,f)$. Then we can easily see that $\widetilde{f}$ is two-sided chaotic on almost all Cantor sets on Z in the sense of Li-Yorke. Also, if $\widetilde{f}$ is two-sided chaotic on a Cantor set C

in the sense of Li-Yorke, then $p_n|C : C \to p_n(C)$ is a homeomorphism. This completes the proof. $\square$

Now, we consider the case that G is the unit interval $I = [0,1]$. By (5.3), we obtain the following.

Corollary 5.9. *If a map $f : I \to I$ is sesitive, then $\mathbb{M}(f) \neq \emptyset$ and $\mathbb{M}(f) = \{G_1, \dots, G_N\}$ is a finite set satisfying the following properties:*

(a) *If $i \neq j$, then $G_i \cap G_j$ is empty or a finite set of periodic points of f.*

(b) *For each $1 \leq i \leq N$, f is two-sided strongly chaotic on G_i in the sense of Devaney, and if the cardinality of the set of components of G_i is $N(i)$, then one of the following two conditions holds:*

(1) *For each component C of G_i, $f^{N(i)}|C : C \to C$ is topologically mixing.*

(2) *For each component C of G_i, there is a subinterval J of C such that $J \cup f^{N(i)}(J) = C$, $J \cap f^{N(i)}(J)$ is a one point set of a periodic point of f whose period is $N(i)$, $f^{2 \cdot N(i)}(J) = J$, and*

$$f^{2 \cdot N(i)} \Big| f^{j \cdot N(i)}(J) : f^{j \cdot N(i)}(J) \to f^{j \cdot N(i)}(J) \ (j = 0, 1)$$

is topologically mixing.

(c) *There is a closed subset $F(f)$ of $L = Cl\,(I - \cup_{i=1}^N G_i)$ with $f(F(f)) \subset F(f)$ and $\dim F(f) \leq 0$ such that if $x \in L - F(f)$, then there is a neighborhood U of x in I and a natural number $n(x)$ such that $f^n(U) \in \cup_{i=1}^N G_i$ for each $n \geq n(x)$.*

Proof. Let $G_i\,(1 \leq i \leq N)$ be the subgraphs as in (5.3). If $G_i \cap G_j\,(i \neq j)$ is nonempty, then $G_i \cap G_j$ is a set of periodic points of f, because that the total space I is an arc and hence $f|G_i \cap G_j : G_i \cap G_j \to G_i \cap G_j$ is injective and hence bijective. For each i, there is a connected subgraph $H_i \subset C$ and a natural number $n(i) \geq 1$ as in (5.3). Consider the map $f^{N(i)}|C : C \to C$. Since C is an interval, there is a fixed point p of $f^{N(i)}$, i.e., p is a periodic point of f with period $N(i)$. Since $C = \cup_{k=0}^{m(i)} f^{k \cdot N(i)}(H_i)$, we may assume that $p \in H_i \subset C$. Put $H_i = J = [a,b]$. If $p \in \mathrm{Int}_C\, J$, then $f^{N(i)}(J) = J$. Then $J = C$. If $p \notin \mathrm{Int}_C\, J$, there is $1 \leq k$ such that $p \in f^{k \cdot N(i)}(J)$. Put $J' = f^{k \cdot N(i)}(J)$. Then we see that $f^{N(i)}(J \cup J') = J \cup J'$, which implies that $J \cup J' = C$. Hence we see that $k = 1$. Then $C = J \cup f^{N(i)}(J)$, and $J \cap f^{N(i)}(J) = \{p\}$. Hence $f^{2 \cdot N(i)}(J) = J$. $\square$

6. Examples.

In this section, we give some examples which are related to the results obtained in the previous sections.

Example 6.1. In (3.1), we can not conclude that f (or f^{-1}) is chaotic on some closed subset of X in the sense of Devaney. Let $\sigma : M \to M$ be the shift map as in [**7**, 12.39 Theorem]. Then $\dim M = 0$, σ is an expansive homeomorphism and there are no periodic points of σ in M. Let $g : Y \to Y$ be any expansive homeomorphism of a compactum Y with $\dim Y > 0$. Consider the product $f = \sigma \times g : M \times Y \to M \times Y$. Then f is an expansive homeomorphism of a compactum $X = M \times Y$ with $\dim X > 0$, but there are no periodic points of f in X. By (3.1), f is chaotic in the sense of Ruelle-Takens, but f is chaotic on no closed subset of X in the sense of Devaney. Also, in (4.5), we can not omit the condition that G is a graph.

Example 6.2. In the statement of (3.1), we can not replace the condition that f is continuum-wise expansive by the condition that f is sensitive. Let S be the unit circle and I the unit interval. Let r_α denote the rotation of length $2\pi\alpha$ on S. Put $X = S \times I$. Define a homeomorphism $f : X \to X$ by $f(x,t) = (r_t(x), t)$ for $x \in S$ and $t \in I$. Then f is sensitive, but f is not strongly sensitive on any closed subset. Of course, f is not continuum-wise expansive. If $t \in I$ is an irrational number, then $f|S_t : S_t \to S_t$ is two-sided topologically transitive, but $f|S_t : S_t \to S_t$ is not sensitive, where $S_t = \{(x,t) \mid x \in S\}$. We see that there is no closed set Y of X such that f is chaotic on Y in the sense of Ruelle-Takens. Also, this example implies that in the statement of (4.4), we can not replace the condition that G is a graph by the condition that G is a n-dimensional polyhedron ($n \geq 2$). Note that the set of periodic points of f is dense in X.

Example 6.3. Let $D = \{0,1\}$ and $C = \Pi_{-\infty<n<+\infty} D_n$, where $D_n = D$ for each n. Let $\sigma : C \to C$ be the shift of C, i.e., $\sigma((a_n)_n) = (a_{n-1})_n$. Consider the cone X of C, i.e., $X = (C \times I)/(C \times \{0\})$ is obtained from $C \times I$ by shrinking $C \times \{0\}$ to a point. Then X is called a *Cantor fan.* Define a homeomorphism $f : X \to X$ by $f([x,t]) = \left[\sigma(x), \sqrt{t}\right]$ for each $x \in C$ and $t \in I$. Then f is strongly sensitive, but it is not continuum-wise expansive. Note that there is no point x of X such that $\dim(Cl\{f^n(x) \mid n = 0, \pm 1, \pm 2, \dots\}) > 0$. Hence f is not one-sided topologically transitive on any closed subset Y of X with $\dim Y > 0$. This implies that in the statement of (4.4), we can not replace the condition that G is a graph by the condition that G is a one-dimensional continuum.

Example 6.4. Let $G = [0,2]$ and let $f : G \to G$ be the map defined by

$$f(x) = \begin{cases} 2 \cdot x, & \text{if } 0 \leq x \leq 1/2, \\ -2 \cdot x + 2, & \text{if } 1/2 \leq x \leq 1, \\ 4 \cdot x - 4, & \text{if } 1 \leq x \leq 3/2, \\ -4 \cdot x + 8, & \text{if } 3/2 \leq x \leq 2. \end{cases}$$

Then f is sensitive and (G, f) is an indecomposable continuum, but f is not one-sided topologically transitive. Hence f is not topologically mixing (see (5.5)).

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References

[1] N. Aoki, *Topological dynamics*, Topics in general topology (eds. K. Morita and J. Nagata), Elsevier Science Publishers B.V. (1989), 625-740.

[2] J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey, *On Devaney's definition of chaos*, Amer. Math. Monthly, **99** (1992), 332-334.

[3] M. Barge and J. Martin, *Chaos, periodicity and snakelike continua*, Trans. Amer. Math. Soc., **289** (1985), 355-365.

[4] B.F. Bryant, *Unstable self-homeomorphisms of a compact space*, Vanderbilt University, Thesis, 1954.

[5] R. Devaney, *An Introduction to Chaotic Dynamical Systems*, second edition, Addison-Wesley, 13046.

[6] W. Gottschalk, *Minimal sets; an introduction to topological dynamics*, Bul. Amer. Math. Soc., **64** (1958), 336-351.

[7] W. Gottschalk and G. Hedlund, *Topological dynamics*, Amer. Math. Soc. Colloq., **34** (1955).

[8] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton University Press, Princeton, NJ, 1948.

[9] J.F. Jacobson and W.R. Utz, *The nonexistence of expansive homeomorphisms of a closed 2-cell*, Pacific J. Math., **10** (1960), 1319-1321.

[10] H. Kato, *The nonexistence of expansive homeomorphisms of Peano continua in the plane*, Topology and its appl., **34** (1990), 161-165.

[11] ———, *On expansiveness of shift homeomorphisms of inverse limits of graphs*, Fund. Math., **137** (1991), 201-210.

[12] ———, *Expansive homeomorphisms in continuum theory*, Topology and its appl., **45** (1992), 223-243.

[13] ———, *Expansive homeomorphisms and indecomposability*, Fund. Math., **139** (1991), 49-57.

[14] ———, *Continuum-wise expansive homeomorphisms* , Canad. J. Math., **45** (1993), 576-598.

[15] ———, *Striped sructures of stable and unstable sets of expansive homeomorphisms and a theorem of K. Kuratowski on independent sets*, Fund. Math., **143** (1993), 153-165.

[16] ———, *Chaotic continua of (continuum-wise) expansive homeomorphisms and chaos in the sense of Li and Yorke*, Fund. Math., **145** (1994), 261-279.

[17] ———, *Concerning continuum-wise fully expansive homeomorphisms of continua*, Topology and its appl., **53** (1993), 239-258.

[18] K. Kuratowski, *Applications of Baire-category method to the problem of independent sets*, Fund. Math., **81** (1974), 65-72.

[19] T.Y. Li and J.A. Yorke, *Period three implies chaos*, Amer. Math. Monthly, **82**

(1975), 985-992.

[20] R. Mañé, *Expansive homeomorphisms and topological dimension*, Trans. Amer. Math. Soc., **252** (1979), 313-319.

[21] S.B. Nadler, Jr., *Hyperspaces of sets*, Pure and Appl. Math., **49**, Dekker, New York, 1978.

[22] W. Reddy, *The existence of expansive homeomorphisms of manifolds*, Duke Math. J., **32** (1965), 627-632.

[23] D. Ruelle and F. Takens, *On the nature of turbulence*, Comm. Math. Phys., **20** (1971), 167-192.

[24] R.M. Schori, *Chaos: An introduction to some topological aspects*, Continuum theory and dynamical systems, M.B. Brown, Editor, Contemporary Mathematics, **117**, 149-161.

[25] W. Utz, *Unstable homeomorphisms* , Proc. Amer. Math. Soc., **1** (1950), 769-774.

[26] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Math., **79**, Springer.

[27] R.F. Williams, *A note on unstable homeomorphisms* , Proc. Amer. Math. Soc., **6** (1955), 308-309.

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