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GLOBAL ANALYTIC HYPOELLIPTICITY OF $\Box_b$ ON CIRCULAR DOMAINS

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Let D be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, with real analytic boundary. In this paper we show that $\Box_b$ is globally analytic hypoelliptic if D is either circular satisfying $\sum_{j=1}^{n} z_j \frac{\partial r}{\partial z_j}(z) \neq 0$ near the boundary bD, where $r(z)$ is a defining function for D, or Reinhardt.

I. Introduction.

Let D be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, with real analytic boundary, and let $\mathbb{C}^n$ be equipped with the standard Euclidean metric. We consider the real analytic regularity problem of the $\Box_b$- equation on the boundary. Namely, given any $f \in C^{\omega}_{p,q}(bD)$, $0 \leq p \leq n-1$ and $1 \leq q \leq n-1$, let $u = N_b f \in L^2_{p,q}(bD)$ be the solution to the following equation,

$$\Box_b u = \left(\overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b\right) N_b f = f. \tag{1.1}$$

Then we ask: is $u = N_b f \in C^{\omega}_{p,q}(bD)$? For the definitions of these notations the reader is referred to Section II.

The existence of the solution $u = N_b f$ is an immediate consequence of the closedness of the range of $\Box_b$ which was proved by M.C.Shaw [**17**] and Boas and M.C.Shaw [**1**], and independently by Kohn [**15**]. Since $u = N_b f$ is the canonical solution to the equation (1.1), it is unique. It also follows from Proposition 2.7. Next the real analyticity of the boundary bD implies that $u = N_b f$ is smooth, i.e., $u \in C^{\infty}_{p,q}(bD)$. For instance see Kohn [**14**][**16**]. Therefore, the main concern here is about the real analytic regularity of the solution u. The only result we know so far is that the answer is affirmative when D is of strict pseudoconvexity which is due to Tartakoff [**18**][**19**][**20**] and Treves [**21**] for $n \geq 3$ and to Geller [**13**] for $n = 2$.

The purpose of this article is to prove the following main results which presumably yield the first positive result to this problem on weakly pseudoconvex domains.

Theorem 1.2. *Let D be a smoothly bounded pseudoconvex domain with real analytic boundary bD in $\mathbb{C}^n$, $n \geq 2$. Suppose that D is circular and that $\sum_{j=1}^{n} z_j \frac{\partial r}{\partial z_j}(z) \neq 0$ near bD, where $r(z)$ is the defining function for D. Then for any $f \in C^{\omega}_{p,q}(bD)$, $0 \leq p \leq n-1$ and $1 \leq q \leq n-1$, the solution $u = N_b f$ to the $\Box_b$-equation is also in $C^{\omega}_{p,q}(bD)$.*

Here a domain D is called circular if $z \in D$ implies

$$e^{i\theta} \cdot z = \left(e^{i\theta} z_1, \ldots, e^{i\theta} z_n\right) \in D$$

for any $\theta \in \mathbb{R}$. D is called Reinhardt if $z \in D$ implies $\left(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n\right) \in D$ for any $\theta_1, \ldots, \theta_n \in \mathbb{R}$, and D is called complete Reinhardt if $z \in D$ implies $(\lambda_1 z_1, \ldots, \lambda_n z_n) \in D$ for any $\lambda_i \in \mathbb{C}$ with $|\lambda_i| \leq 1$, $i = 1, \ldots, n$. Then we also prove

Theorem 1.3. *Let D be a smoothly bounded Reinhardt pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, with real analytic boundary. Then the same assertion as in the Theorem 1.2 holds.*

Hence, in particular, $\Box_b$ is globally analytically hypoelliptic on any complete Reinhardt domains with real analytic boundary whuch provides a large class of examples. Next we have the following immediate corollary.

Corollary 1.4. *Let D be a smoothly bounded pseudoconvex domain with real analytic boundary in $\mathbb{C}^n$, $n \geq 2$. Suppose that either D is Reinhardt or D is circular with $\sum_{j=1}^{n} z_j \frac{\partial r}{\partial z_j}(z) \neq 0$ near bD, where $r(z)$ is the defining function for D. Then we have*

(i) *The Szegö projection S defined on bD preserves the real analyticity globally, and*

(ii) *The canonical solution w to the $\bar{\partial}_b$-equation, i.e., $\bar{\partial}_b w = \alpha$, is in $C^{\omega}_{p,q-1}(bD)$ if the given α is in $C^{\omega}_{p,q}(bD)$ and satisfies $\bar{\partial}_b \alpha = 0$.*

Here the Szegö projection S is defined to be the orthogonal projection from $L^2(bD)$ onto the closed subspace, denoted by $H^2(bD)$, of square-integrable CR- functions defined on the boundary, and by canonical solution w we mean the solution with minimum L^2- norm. We remark that statement (i) has been proved by the author before in **[5]** via a more direct argument, and a special case of (ii), i.e., $n = 2$, is verified by Derridj and Tartakoff in **[11]**.

Now if we combine the above theorems and the main result, i.e., the Theorem B, obtained by the author in Chen **[6]**, then we can conclude the following theorem.

Theorem 1.5. *Let $D \subseteq \mathbb{C}^n$, $n \geq 3$, be a smoothly bounded pseudoconvex domain with real analytic boundary. Then the Szegö projection S associated with D preserves the real analyticity globally whenever D is defined by*

(i) $D = \left\{(z_1, \dots, z_n) \in \mathbb{C}^n \middle| |f(z_1)|^2 + H\left(|z_2|^2, \dots, |z_n|^2\right) < 1\right\}$, *where $f(z_1)$ is holomorphic in z_1 and $H(x_2, \dots, x_n)$ is a polynomial with positive coefficient and $H(0, \dots, 0) = 0$, or*

(ii) $D = \left\{(z_1, \dots, z_n) \in \mathbb{C}^n \middle| |f(z_1)|^2 + |g(z)|^2 + \sum_{j=3}^{n} h_j\left(|z_j|^2\right) < 1\right\}$, *where $f(z_1)$ and $g(z_2)$ are holomorphic in one variable z_1 or z_2 respectively, and $h_j(x)$ is a polynomial with positive coefficients satisfying $h_j(0) = 0$, $h'_j(0) > 0$ for $3 \leq j \leq n$.*

The real analytic regularity of the Bergman projection P, which is defined to be the orthogonal projection from $L^2(D)$ onto the closed subspace $H^2(D)$ of square- integrable holomorphic functions defined on D, on the domains (i) and (ii) defined in Theorem 1.5 has been established in Chen [**6**].

We should point out that in general the analytic pseudolocality of the Szegö projection S is false. Counterexamples have been discovered by Christ and Geller [**7**]. However, so far there is no counterexample to the globally real analytic regularity of S. Meanwhile, a number of positive results of the local analytic hypoellipticity for $\Box_b$ have been established on some model pseudoconvex hypersurface by Derridj and Tartakoff. For instance, see [**8**][**9**][**10**].

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II. Proofs of the Theorems 1.2 and 1.3.

Let D be a smoothly bounded pseudoconvex domain with real analytic boundary in $\mathbb{C}^n$, $n \geq 2$, and let $\mathbb{C}^n$ be equipped with the standard Euclidean metric. Since we assume that the domain D is circular, we can choose a real analytic defining function $r(z)$ for D such that $r(z) = r\left(e^{i\theta} \cdot z\right)$ and that $|\nabla r(z)| = 1$ for $z \in bD$. Let $z_0 \in bD$ be a boundary point. We may assume that $\frac{\partial r}{\partial z_n}\left(z_o\right) \neq 0$. Hence a local basis for $T^{1,0}\left(bD\right)$ near z_0 can be chosen to be

$$L_j = \frac{\partial r}{\partial z_n}\frac{\partial}{\partial z_j} - \frac{\partial r}{\partial z_j}\frac{\partial}{\partial z_n} \cdot \text{for } 1 \leq j \leq n-1.$$

Put $X(z) = \sum_{j=1}^{n} \frac{\partial r}{\partial \bar{z}_j}\frac{\partial}{\partial z_j} - \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}\frac{\partial}{\partial \bar{z}_j}$. We see that

$$L_1, \dots, L_{n-1}, \overline{L}_1, \dots, \overline{L}_{n-1}$$

and $X(z)$ form a local basis for the complexified tangent space $\mathbb{C}T(bD)$, and $X(z)$ is perpendicular to $T^{1,0}(bD) \oplus T^{0,1}(bD)$. Let $w_1, \dots, w_{n-1}$ be $(1,0)$-form dual to $L_1, \dots, L_{n-1}$ respectively. Put $\eta = 2\left(\partial r - \overline{\partial} r\right)$. Then it is not hard to see that $w_1, \dots, w_{n-1}, \overline{w}_1, \dots, \overline{w}_{n-1}$ and η form a local basis for the complexified cotangent space $\mathbb{C}T^*(bD)$, and η is dual to $X(z)$ and perpendicular to $T^{*^{1,0}}(bD) \oplus T^{*^{0,1}}(bD)$.

Now for any $\theta \in \mathbb{R}$, define

$$\begin{aligned} \Lambda_\theta : &\overline{D} \to \overline{D} \\ &z \mapsto e^{i\theta} \cdot z = \left(e^{i\theta} z_1, \dots, e^{i\theta} z_n\right). \end{aligned}$$

Put $\zeta = e^{i\theta} \cdot z$, then we obtain by direct computation $\frac{\partial r}{\partial z_k}(z) = e^{i\theta} \frac{\partial r}{\partial \zeta_k}(\zeta)$, $\Lambda_{\theta^*}\left(\frac{\partial}{\partial z_k}\right) = e^{i\theta} \frac{\partial}{\partial \zeta_k}$ and $\Lambda_\theta^*(d\zeta_k) = e^{i\theta} dz_k$ for $1 \leq k \leq n$. It follows that we have

(2.1) $\Lambda_{\theta^*}(X(z)) = X(\zeta)$,

(2.2) $\Lambda_{\theta^*}(L_j(z)) = e^{i2\theta} L_j(\zeta)$, $\Lambda_{\theta^*}\left(\overline{L}_j(z)\right) = e^{-i2\theta} \overline{L}_j(\zeta)$, for $1 \leq j \leq n-1$,

(2.3) $\Lambda_\theta^*\left(\overline{\partial} r(\zeta)\right) = \overline{\partial} r(z)$, $\Lambda_\theta^*(\partial r(\zeta)) = \partial r(z)$.

This implies that $\Lambda_\theta^* w_i$ is again a $(1,0)$-form in $\mathbb{C}T^*(bD)$.

Next we recall the definition of $\overline{\partial}_b$ briefly here. let $f \in C_{p,q}^\infty(bD)$, where $C_{p,q}^\infty(bD)$ denotes the space of tangential (p,q)-forms defined on the boundary with smooth coefficients. Namely, any f in $C_{p,q}^\infty(bD)$ can be expressed in the form

$$f = \sum_{\substack{|I|=p \\ |J|=q}}{}' f_{IJ} w_I \wedge \overline{w}_J,$$

where $I = (i_1, \dots, i_p)$ and $J = (j_1 \dots, j_q)$ are strictly increasing multiindices of length p and q respectively, and $w_I = w_{i_1} \wedge \cdots \wedge w_{i_p}$ and $\overline{w}_J = \overline{w}_{j_1} \wedge \cdots \wedge \overline{w}_{j_q}$, and the prime indicates that the summation is carried over only the strictly increasing multiindices. Then consider f as a (p,q)-form in some open neighbourhood U of the boundary, and apply $\overline{\partial}$ to f. We get

$$\overline{\partial} f = F + r(z)G + \overline{\partial} r \wedge H,$$

where F is a $(p, q+1)$-form involving only the w_i's and $\overline{w}_j$'s, and G is a $(p, q+1)$-form, and H is a (p,q)-form. Then the tangential Cauchy-Riemann operator $\overline{\partial}_b$ is defined to be

$$\overline{\partial}_b f = \pi_{p,q+1}\left(\overline{\partial} f\right) = F\Big|_{bD},$$

where $\pi_{p,q+1}$ maps $\overline{\partial} f$ to the restriction of F on the boundary. For the details the reader is referred to Folland and Kohn [**12**].

Now the above argument shows Λ_θ^* maps the tangential component to the tangential component and maps the normal component to the normal component. Therefore, if $f \in C_{p,q}^\infty(bD)$ with $1 \le q \le n-2$ we obtain

$$\begin{aligned}
\overline{\partial}(\Lambda_\theta^* f(\zeta)) &= \pi_{p,q+1} \circ \overline{\partial} \circ \Lambda_\theta^* f \\
&= \pi_{p,q+1} \circ \Lambda_\theta^* \circ \overline{\partial} f \\
&= \pi_{p,q+1} \circ \Lambda_\theta^* \left(F + r(\zeta) G + \overline{\partial} r \wedge H\right) \\
&= \pi_{p,q+1} \circ \left(\Lambda_\theta^* F + r(z)\Lambda_\theta^* G + \overline{\partial} r \wedge \Lambda_\theta^* H\right) \\
&= \Lambda_\theta^* F\Big|_{bD} \\
&= \Lambda_\theta^* \circ \pi_{p,q+1} \left(F + r(\zeta) G + \overline{\partial} r \wedge H\right) \\
&= \Lambda_\theta^* \circ \pi_{p,q+1} \circ \overline{\partial} f \\
&= \Lambda_\theta^* \left(\overline{\partial}_b f\right).
\end{aligned}$$

Hence we have proved the following lemma.

Lemma 2.4. $\overline{\partial}_b \Lambda_\theta^* f = \Lambda_\theta^* \overline{\partial}_b f$ *for any* $f \in C_{p,q}^\infty(bD)$ *with* $1 \le q \le n-1$.

In general, $\overline{\partial}_b \circ h^* \ne h^* \circ \overline{\partial}_b$ if h is just smooth CR- mapping. Denote by $L_{p,q}^2(bD)$ the space of tangential (p,q)-forms with square-integrable coefficients. Then we have

Lemma 2.5. *For any* u *in* $L_{p,q}^2(bD)$, *we have* $(\Lambda_\theta^* u, v) = (u, \Lambda_{-\theta}^* v)$ *for any* $\theta \in \mathbb{R}$.

Proof. Put $\zeta = e^{i\theta} \cdot z$, and express u and v in terms of the Euclidean coordinates, we get

$$u(\zeta) = {\sum_{\substack{|I|=p \\ |J|=q}}}' u_{IJ}(\zeta)\, d\zeta_I \wedge d\overline{\zeta}_J \text{ and } v(z) = {\sum_{\substack{|I|=p \\ |J|=q}}}' v_{IJ}(z) dz_I \wedge d\overline{z}_J .$$

Let $d\sigma$ be the surface element defined on bD. We see that $d\sigma$ is invariant under rotation, i.e., $\Lambda_\theta^* d\sigma_\zeta = d\sigma_z$. For instance, see Chen [**5**]. Hence if we set $z = e^{-\theta} \cdot \zeta$, we obtain

$$\begin{aligned}(\Lambda_\theta^* u, v) &= \left(\sum{}' u_{IJ}\left(e^{i\theta} \cdot z\right) e^{i(p-q)\theta} dz_I \wedge d\overline{z}_J, \sum{}' v_{IJ}(z) dz_I \wedge d\overline{z}_J\right) \\ &= \sum{}' \int_{bD} u_{IJ}\left(e^{i\theta} \cdot z\right) e^{i(p-q)\theta} \overline{v_{IJ}(z)} d\sigma_z \\ &= \sum{}' \int_{bD} u_{IJ}(\zeta) \cdot \overline{e^{-i(p-q)\theta} v_{IJ}\left(e^{-i\theta} \cdot \zeta\right)} d\sigma_\zeta \\ &= \left(u, \Lambda_{-\theta}^* v\right) .\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.6. $\overline{\partial}_b^* \Lambda_\theta^* \alpha = \Lambda_\theta^* \overline{\partial}_b^* \alpha$ *for any* $\alpha \in C_{p,q}^\infty(bD)$ *with* $1 \le q \le n-1$, *where* $\overline{\partial}_b^*$ *is the* L^2*-adjoint of* $\overline{\partial}_b$.

Proof. Let β be any tangential $(p, q-1)$-form, i.e., $\beta \in C_{p,q-1}^\infty(bD)$. We have

$$\begin{aligned}\left(\overline{\partial}_b^* \Lambda_\alpha^*, \beta\right) &= \left(\Lambda_\theta^* \alpha, \overline{\partial}_b \beta\right) \\ &= \left(\alpha, \Lambda_{-\theta}^* \overline{\partial}_b \beta\right) \\ &= \left(\alpha, \overline{\partial}_b \Lambda_{-\theta}^* \beta\right) \\ &= \left(\overline{\partial}_b^* \alpha, \Lambda_{-\theta}^* \beta\right) \\ &= \left(\Lambda_\theta^* \overline{\partial}_b^* \alpha, \beta\right) .\end{aligned}$$

This proves the lemma. □

Now denote by $H_{p,q} = \left\{u \in L_{p,q}^2(bD) \,\middle|\, \Box_b u = 0\right\}$. We have the following fact.

Proposition 2.7. (i) $H_{p,q} = 0$ *for* $1 \le q \le n-2$, *and*

(ii) $H_{p,n-1} = \left\{u \in L_{p,n-1}^2(bD) \,\middle|\, u \in \operatorname{Dom}\left(\overline{\partial}_b^*\right) \text{ and } \overline{\partial}_b^* u = 0\right\}$.

In general, $H_{p,n-1} \ne 0$. Now let $f \in C_{p,q}^\infty(bD)$, $f \perp H_{p,q}$, for $1 \le q \le n-1$ be given, and let $u = N_b f \in C_{p,q}^\infty(bD)$ be the canonical solution to the $\Box_b$-equation,

$$\Box_b u = \Box_b N_b f = f,$$

where N_b is the so-called boundary Neumann operator. Let T be the vector field generated by the rotation, namely, T is defined by

$$\begin{aligned}T(z) &= \frac{1}{2}\pi_{z^*}\left(\left.\frac{\partial}{\partial\theta}\right|_{\theta=0}\right) \\ &= \frac{i}{2}\left(\sum_{j=1}^n z_j \frac{\partial}{\partial z_j} - \sum_{j=1}^n \overline{z}_j \frac{\partial}{\partial \overline{z}_j}\right),\end{aligned}$$

where π_z is the mapping defined for any $z \in bD$ by

$$\begin{aligned}\pi_z : & S^1 \to \overline{D} \\ & e^{i\theta} \mapsto e^{i\theta} \cdot z = \left(e^{i\theta} z_1, \ldots, e^{i\theta} z_n\right).\end{aligned}$$

By our hypotheses stated in the Theorem 1.2, $T(z)$ is tangential and pointing in the bad direction for any $z \in bD$.

From now on, we will assume that f has real analytic coefficients, namely, $f \in C^{\omega}_{p,q}(bD)$ with $1 \leq q \leq n-1$, and that $f \perp H_{p,n-1}$ if $q = n-1$. Write f as

$$f = \sum\nolimits'_{I,J} f_{IJ}(z)\omega_I \wedge \bar{\omega}_J.$$

Define Tf by

$$Tf = \sum\nolimits'_{I,J} Tf_{IJ}(z)\omega_I \wedge \bar{\omega}_J.$$

It is not hard to see that Tf is still a tangential (p,q)-form, i.e., $Tf \in C^{\omega}_{p,q}(bD)$. Then we have the following key lemma.

Lemma 2.8. *$T^k u = T^k N_b f = N_b T^k f$ for any $k \in \mathbb{N}$.*

Proof. Since, in general, $H_{p,n-1} \neq 0$, we need to check that if $u \perp H_{p,n-1}$, then $\Lambda^*_\theta u \perp H_{p,n-1}$. So, let $w \in H_{p,n-1}$. By Lemma 2.6 we have $\Lambda^*_\theta w \in H_{p,n-1}$. It follows that

$$(\Lambda^*_\theta u, w) = (u, \Lambda^*_{-\theta} w) = 0.$$

Hence $\Lambda^*_\theta u \perp H_{p,n-1}$. This proves our assertion.

Now by combining Lemma 2.4 and 2.6, we obtain

$$\begin{aligned}\Box_b \Lambda^*_\theta N_b f &= \Lambda^*_\theta \Box_b N_b f \\ &= \Lambda^*_\theta f \\ &= \Box_b N_b \Lambda^*_\theta f.\end{aligned}$$

Therefore, by Proposition 2.7 and our assertion we conclude that

$$\Lambda^*_\theta N_b f = N_b \Lambda^*_\theta f \text{ for any } \theta \in \mathbb{R}. \tag{2.9}$$

So now one can argue as we did in Chen [**2**] to get $TN_b f = N_b T f$. Inductively we have $T^k N_b f = N_b T^k f$. This completes the proof of the lemma. $\Box$

Lemma 2.8 enables us to estimate the derivatives of the solution $u = N_b f$ in the bad direction as follows,

$$\|T^k u\| = \|T^k N_b f\| = \|N_b T^k f\| \leq C_0 \|f\|_k \leq C C^k k!\ ,$$

for some constant $C > 0$ and any $k \in \mathbb{N}$, where $\| \ \|_k$ is the Sobolev k-norm.

Therefore, what we need to estimate is the mixed derivatives of u, namely, the differentiations involving L_i's, $\overline{L}_i$'s and T. For dealing with the $\overline{\partial}$-Neumann problem we can avail ourselves of the so-called basic estimate to achieve this goal. However, for the $\overline{\partial}_b$- Neumann problem, in general, the energy norm Q_b does not control the barred terms. But if we add the differentiation in T-direction to the right hand side, then we do have the following estimate,

$$\|u\| + \sum_{j=1}^{n-1} \|L_j u\| + \sum_{j=1}^{n-1} \left\|\overline{L}_j u\right\| \le C\left(\left\|\overline{\partial}_b u\right\| + \left\|\overline{\partial}_b^* u\right\| + \|Tu\|\right), \tag{2.10}$$

for any $u \in C^{\omega}_{p,q}(bD)$ with support in some open neighbourhood of z_0. The estimate (2.10) is essentially proved in [**12**]. Since we know how to control the T-derivatives of the solution $u = N_b f$, then a standard argument can be used to obtain the estimates of all the other mixed derivatives. For the details the reader is referred to Chen [**2**][**3**][**4**]. This completes the proof of Theorem 1.2.

A similar argument can be applied to prove the Theorem 1.3. Let D be a smoothly bounded Reinhardt pseudoconvex domain with real analytic boundary in $\mathbb{C}$, $n \ge 2$. Let $z_0 \in bD$ be a boundary point. First one can choose a direction, say z_n, such that $\left(z_n \frac{\partial r}{\partial z_n}\right)(z_0) \ne 0$, where $r(z)$ is the defining function for D. Next we simply consider the rotation in z_n-direction, namely, for each $\theta \in \mathbb{R}$, define

$$\begin{aligned} \Lambda_\theta : &\overline{D} \to \overline{D} \\ &z \mapsto e^{i\theta} \cdot z = \left(z_1, \ldots, z_{n-1}, e^{i\theta} z_n\right). \end{aligned}$$

Then by following the proof we present here for circular domains we can show without difficulty that $\Box_b$ is globally analytically hypoelliptic on any smoothly bounded Reinhardt pseudoconvex domain with real analytic boundary . Details can be found in Chen [**3**]. This also completes the proof of the Theorem 1.3.

Finaly we make a concluding remark that the method we present here can be used to obtain the Sobolev H^s-regularity for $\Box_b$ on any smoothly bounded pseudoconvex domain which is either Reinhard or circular with $\sum_{j=1}^{n} z_j \frac{\partial r}{\partial z_j} \ne 0$ near bD, where $r(z)$ is a smooth defining function for D. For instance, see Chen [**4**].

References

[1] H. Boas and M.C. Shaw, *Sobolev estimates for the Lewy operator on weakly pseudoconvex boundaries*, Math. Ann., **274** (1986), 221-231.

[2] S.C. Chen, *Global analytic hypoellipticity of the $\bar{\partial}$-Neumann problem on circular domains*, Invent. Math., **92** (1988), 173-185.

[3] ———, *Global real analyticity of solutions to the $\bar{\partial}$-Neumann problem on Reinhardt domains*, Indiana Univ. Math. J., **37(2)** (1988), 421-430.

[4] ———, *Global regularity of the $\bar{\partial}$-Neumann problem on circular domains*, Math. Ann., **285** (1989), 1-12.

[5] ———, *Real analytic regularity of the Szegö projection on circular domains*, Pacific J. Math., **148(2)** (1991), 225-235.

[6] ———, *Real analytic regularity of the Bergman and Szegö projections on decoupled domains*, Math. Zeit., **213** (1993), 491-508.

[7] M. Christ and D. Geller, *Counterexamples to analytic hypoellipticity for domains of finite type*, Ann. of Math., **135** (1992), 551-566.

[8] M. Derridj and D.S. Tartakoff, *Local analyticity for $\Box_b$ and the $\bar{\partial}$-Neumann problem at certain weakly pseudoconvex points*, Commun. in P.D.E., **13(12)** (1988), 1521-1600.

[9] ———, *Local analyticity for the $\bar{\partial}$-Neumann problem and $\Box_b$ - some model domains without maximal estimates*, Duke Math. J., **64(2)** (1991), 377-402.

[10] ———, *Microlocal analyticity for $\Box_b$ in block-decoupled pseudoconvex domains*, Preprint.

[11] ——— *Global analyticity for $\Box_b$ on three dimensional pseudoconvex domains*, Preprint.

[12] G.B. Folland and J.J. Kohn, *The Neumann problem for the Cauchy-Riemann complex*, Ann. Math. Studies, **75** Princeton, Princeton University Press 1972.

[13] D. Geller, *Analytic pseudodifferential operators for the Heisenberg group and local solvability*, Math. Notes, **37** Princeton, Princeton University Press 1990.

[14] J.J. Kohn, *Estimates for $\bar{\partial}_b$ on pseudoconvex CR manifolds*, Proc. Sympos. Pure Math., **43** Amer. Math. Soc., Providence, R.I. (1985).

[15] ———, *The range of the tangential Cauchy-Riemann operator*, Duke Math. J., **53(2)** (1986), 525-545.

[16] ———, *Subellipticity of the $\bar{\partial}$-Neumann problem on pseudo-convex domains: sufficient conditions*, Acta Math., **142** (1979), 79-122.

[17] M.C. Shaw, *L^2 estimates and existence theorems for the tangential Cauchy-Riemann complex*, Invent. Math., **82** (1985), 133-150.

[18] D.S. Tartakoff, *On the global analyticity of solutions to $\Box_b$ on compact manifolds*, Commun. in P.D.E., **1** (1976), 283-311.

[19] ———, *Local analytic hypoellipticity for $\Box_b$ on non-degenerate Cauchy-Riemann manifolds*, Proc. Natl. Acad. Sci. USA, **75(7)** (1978), 3027-3028.

[20] ———, *On the local real analyticity of solutions to $\Box_b$ and the $\bar{\partial}$-Neumann problem*, Acta Math., **145** (1980), 177-204.

[21] F. Treves, *Analytic hypo-ellipticity of a class of pseudodifferential operators with double characteristics and applications to the $\bar{\partial}$-Neumann problem*, Commun. in

P.D.E., **3** (1978), 475-642.

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Added in proof: M.Christ has recently proved the following, M.Christ, The Szegö projection need not preserve global analyticity, Annals of Math., **143** (1996), 301-330.