PACIFIC JOURNAL OF MATHEMATICS
Vol. 175, No. 1, 1996

ON THE STRUCTURE OF TENSOR PRODUCTS OF ℓ_p-SPACES

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We examine some structural properties of (injective and projective) tensor products of ℓ_p-spaces (projections, complemented subspaces, reflexivity, isomorphisms, etc.). We combine these results with combinatorial arguments to address the question of primarity for these spaces and their duals.

Introduction.

A Banach space X is *prime* if every infinite-dimensional complemented subspace contains a further subspace which is isomorphic to X. A Banach space X is said to be *primary* if whenever $X = Y \oplus Z$, X is isomorphic to either Y or Z. The classical examples of prime spaces are the spaces ℓ_p, $1 \leq p \leq \infty$. Many spaces derived from the ℓ_p-spaces in various ways are primary (see for example [**AEO**] and [**CL**]).

The primarity of $B(H)$ was shown by Blower [**B**] in 1990, and Arias [**A**] has recently developed further techniques which are used to prove the primarity of c_1, the space of trace class operators (this was first shown by Arazy [**Ar1**, **Ar2**]). It has become clear that these techniques are not naturally confined to a Hilbert space context; in the present paper we wish to extend the results to a variety of tensor products and operator spaces of ℓ_p-spaces (and in some cases $\mathcal{L}_p$-spaces). We also include some related results.

Some of the intermediate propositions (on factoring operators through the identity) may actually be true for a wider class of Banach spaces (those with unconditional bases which have nontrivial lower and upper estimates). In fact, the combinatorial aspects of the factorization can be applied quite generally, and may have other applications. The proofs of primarity, however, rely on Pełczyński's decomposition method which is not so readily extended. We have thus kept mainly to the case of injective and projective tensor products of ℓ_p spaces throughout. The results we obtain apply to the growing study of polynomials on Banach spaces since polynomials may be considered as symmetric multilinear operators with an equivalent norm (see [**FJ**], [**M**], or [**R**]).

Our main results are:

(1) If $1 < p < \infty$, then $B(\ell_p) \approx B(L_p)$.

(2) If $\frac{1}{p_i} + \frac{1}{p_j} \leq 1$ for every $i \neq j$, or if all of the p_i's are equal, then $\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$ is primary.

(3) ℓ_p embeds into $\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$ if and only if there exists $A \subset \{1, 2, \cdots, n\}$ such that $\frac{1}{p} = \min\{\sum_{i\in A}\frac{1}{p_i}, 1\}$.

(4) If $1 \leq p < \infty$ and $m \geq 1$, then the space of homogeneous analytic polynomials $\mathcal{P}_m(\ell_p)$ and the symmetric tensor product of m copies of ℓ_p are primary.

The paper is organized as follows. In Section 1 we set notation, definitions and some necessary but more or less known facts. In Section 2 we show that $B(\ell_p)$, the Banach space of bounded linear operators on ℓ_p, is isomorphic to $B(L_p)$, and in fact to $B(X)$ whenever X is a separable $\mathcal{L}_p$-space, along with some more general results we require later. In Section 3 we will construct a multiplier through which a given operator on tensor products may be factored; we then use this to show that some projective tensor products are primary. In Section 4 we will prove that the ℓ_p subspaces of $\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$ are the "obvious" ones and use this to prove that some projective tensor products are not primary (for example, $\ell_2\hat{\otimes}\ell_{1.5}$ is not primary). Section 5 covers the question of primarity in the injective tensor products and operator spaces, a situation not always dual to the projective case and calling for somewhat different techniques. Section 6 is an appendix in which we prove the technical lemmas we use in Section 3.

We would like to thank W.B. Johnson for organizing the summer workshops in Linear Analysis and Probability at Texas A&M University in 1991-1993, and the NSF for funding them.

1. Preliminaries.

Unless explicitly stated, all references to ℓ_p spaces will assume that $1 < p < \infty$, and will adhere the notational convention that $\frac{1}{p_i} + \frac{1}{q_i} = 1$ or sometimes $\frac{1}{r} + \frac{1}{r'} = 1$.

Define
$$X = \ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}.$$
We can identify its predual X_* and dual X^* as follows
$$\begin{aligned} X_* &= \ell_{q_1}\check{\otimes}\cdots\check{\otimes}\ell_{q_N} \\ X^* &= B(\ell_{p_1}, (\ell_{p_2}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N})^*) \\ &\equiv B(\ell_{p_1}, B(\cdots B(\ell_{p_{N-1}}, \ell_{q_N})\cdots)). \end{aligned}$$
The elements of X, X_*, or X^* have representations as an infinite N-dimensional matrix of complex numbers (we must keep in mind, however, that this representation may not be the most efficient for computing the tensor

product norm) where the element in the $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbf{N}^N$ position is the coefficient of the "matrix element" $e_\alpha = e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_N}$ with e_{α_j} being the α_j-th element in the unit vector basis of ℓ_{p_j}. All subspaces we consider are norm-closed, and when we indicate the linear span of elements we always mean the closed span.

The following elementary lemma is very important to the structure of projective tensor products.

Lemma 1.1. *Let X and Y be Banach spaces and $S \in B(X)$, $T \in B(Y)$. Then $S \otimes T \in B(X \hat{\otimes} Y)$ is defined by $S \otimes T(x \otimes y) = S(x) \otimes T(y)$ and satisfies $\|S \otimes T\| \leq \|S\| \|T\|$.*

As a consequence of this we get that projective tensor products of Banach spaces with bases have bases.

Proposition 1.2. *Let X and Y be Banach spaces with bases $(e_n)_n$ and $(f_n)_n$ respectively. Then $X \hat{\otimes} Y$ has a basis. Moreover, we take the elements of the basis from the "shell" $\partial M_n = [e_i \otimes e_j \colon \max\{i,j\} = n]$; i.e., $e_1 \otimes f_1$, $e_2 \otimes f_1, e_2 \otimes f_2, e_1 \otimes f_2,\ \ e_3 \otimes f_1, e_3 \otimes f_2, e_3 \otimes f_3, e_2 \otimes f_3, e_1 \otimes f_3, \cdots$, etc.*

The proof of this is easy. On the one hand it is clear that the span of those vectors is dense and using Lemma 1.1 (with the operators replaced by projections) we see that the initial segments are uniformly complemented, because ∂M_n is clearly complemented.

As a consequence we get that $\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$ has a basis consisting of e_α's. Moreover, we can use Lemma 1.1 to prove that

$$\partial M_n = [e_\alpha \colon \alpha \in \mathbf{N}^N, \max\{\alpha_1, \cdots, \alpha_N\} = n]$$

is 2-complemented and that $(\partial M_n)_n$ forms a Schauder decomposition for $\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$; we also see that $(L_\alpha)_\alpha$ is a Schauder decomposition for $\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$ where $\alpha \in \mathbf{N}^{N-1}$ and $L_\alpha = [e_\alpha \otimes e_j \colon j \in \mathbf{N}]$. (A more complete discussion of this situation appears in [**R**].) We will use these facts in Section 3.

The next theorem gives us the two most basic ingredients of our analysis. We will prove that the main diagonals are 1-complemented and will identify them exactly; we will also state under what conditions the triangular parts of $\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$ are complemented. It is known that the main triangular part of $\ell_p \hat{\otimes} \ell_q$ is complemented if and only if $\frac{1}{p} + \frac{1}{q} > 1$. (See [**KP**], [**MN**] and [**Be**].)

Theorem 1.3. *Let $X = \ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$. Then the main diagonal $\mathcal{D} = [e_n \otimes \cdots \otimes e_n : n \in \mathbf{N}]$ is 1-complemented and satisfies $\mathcal{D} \equiv \ell_r$ where $\frac{1}{r} =$*

$\min\left\{1, \sum_{i=1}^{N} \frac{1}{p_i}\right\}$. *As a consequence we get that* $X \approx (\sum \oplus X)_r$. *Moreover, if* j, k *are fixed, then the canonical projection onto*

$$[e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_N} \ : \ i_k \geq i_j]$$

is bounded if and only if $\frac{1}{p_k} + \frac{1}{p_j} > 1$.

This theorem is known for $n = 2$, and in some respects for larger n as well (see for example [**Z**]). For completeness we show here how the case $n = 2$ may be extended.

Proof. For $1 < k \leq N$, let $P_{1,k} \in B(\ell_{p_1} \hat{\otimes} \ell_{p_k})$ be the main diagonal projection and $I_{1,k}$ be the identity on $\ell_{p_2} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_{k-1}} \hat{\otimes} \ell_{p_{k+1}} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$. Then $P_{1,k} \otimes I_{1,k}$ is the projection on $\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$ defined by $P_{1,k} \otimes I_{1,k} e_\alpha = e_\alpha$ if $\alpha_1 = \alpha_k$ and zero otherwise.

Let $P = (P_{1,2} \otimes I_{1,2}) \cdots (P_{1,N} \otimes I_{1,N})$. It is easy to see that $Pe_\alpha = e_\alpha$ if $\alpha_1 = \cdots = \alpha_N$ and $Pe_\alpha = 0$ otherwise. This tells us that $\mathcal{D}$ is complemented.

When $N = 2$, the main diagonal of $\ell_{p_1} \hat{\otimes} \ell_{p_2}$ is isometric to ℓ_r where $\frac{1}{r} = \min\left\{1, \frac{1}{p_1} + \frac{1}{p_2}\right\}$. We apply an induction step for $N > 2$. The key to the induction step is the following: Let D be the "diagonal-projection" on a projective tensor products of ℓ_p-spaces. Then it is easy to see that $D(\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}) \equiv D(D(\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_{N-1}}) \hat{\otimes} \ell_{p_N})$.

Notice that if the $P_{1,k}$'s above are block projections, then we conclude that the block diagonal projections are bounded. By taking those to be infinite and using the previous paragraph, we see that $X \approx \left(\sum \oplus X\right)_r$.

For the last part let $T_{k,j}$ be the upper triangular projection on $\ell_{p_k} \hat{\otimes} \ell_{p_j}$ and $I_{k,j}$ be the identity on $\hat{\otimes}_{i \neq k,j} \ell_{p_i}$. $T_{k,j}$ is bounded if and only if $\frac{1}{p_k} + \frac{1}{p_j} \leq 1$. Therefore, the same is true for $T_{k,j} \otimes I_{k,j} \in B(\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N})$. □

Remarks. (1) To prove that $X \approx \left(\sum \oplus X\right)_r$ we used Pełczyński's decomposition method. This says that if two Banach spaces X_1 and X_2 embed complementably into each other and if for some $1 \leq p \leq \infty$, $X_1 \approx \left(\sum \oplus X_1\right)_p$, then $X_1 \approx X_2$.

(2) We will work in Section 3 with $\mathcal{T} = [e_\alpha \colon \alpha_1 < \alpha_2 < \cdots < \alpha_N]$. Some of the results from Theorem 1.3 hold for this space. For instance, the block projections are bounded. This implies that $\mathcal{T} \approx \left(\sum \mathcal{T}\right)_r$ where r is as in Theorem 1.3.

(3) It is clear that when $r = 1$ then $\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$ is not reflexive. It is not very difficult to prove that if $r > 1$ then $\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$ is reflexive.

2. Isomorphisms of Spaces of Operators on ℓ_p.

In this section we will show that $B(\ell_p)$ is isomorphic to $B(X)$ when X is any separable $\mathcal{L}_p$-space. In particular, $B(\ell_p)$ is isomorphic to $B(L_p[0,1])$.

A consequence of this is that $B(\ell_2)$ embeds complementably in $B(\ell_p)$ for $1 < p < \infty$.

Theorem 2.1. *Let X and Y be separable $\mathcal{L}_p$- and $\mathcal{L}_q$-spaces respectively with $1 < p \leq q$. Then* $B(X,Y) \approx B(\ell_p, \ell_q) \approx \left(\sum_{n=1}^{\infty} \oplus B(\ell_p^n, \ell_q^n)\right)_{\infty}$.

We also obtain an isomorphic representation for $(\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N})^*$ when $\sum_{i \leq N} \frac{1}{p_i} \geq 1$.

Theorem 2.2. *Let $X = \ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$ be such that $\frac{1}{r} = \min\left\{1, \sum_{i=1}^{N} \frac{1}{p_i}\right\} = 1$. Then* $X^* \approx \left(\sum_{n=1}^{\infty} \ell_{q_1}^n \check{\otimes} \cdots \check{\otimes} \ell_{q_N}^n\right)_{\infty}$.

The proof of these two theorems is very similar; they use Pełczyński's decomposition method.

For Theorem 2.1 notice that $B(\ell_p, \ell_q) \equiv (\ell_p \hat{\otimes} \ell_{q'})^*$ where $\frac{1}{q} + \frac{1}{q'} = 1$. Hence, if $p \leq q$ (i.e., $\frac{1}{p} + \frac{1}{q'} \geq 1$), Theorem 1.3 tells us that $\ell_p \hat{\otimes} \ell_{q'} \approx \left(\sum \oplus \ell_p \hat{\otimes} \ell_{q'}\right)_1$, and then $B(\ell_p, \ell_q) \approx \left(\sum \oplus B(\ell_p, \ell_q)\right)_{\infty}$. For Theorem 2.2, notice that Theorem 1.3 implies that $X \approx \left(\sum \oplus X\right)_1$; therefore, $X^* \approx \left(\sum \oplus X^*\right)_{\infty}$.

Then it is enough to prove that each space embeds complementably into the other. We prove these facts for Theorem 2.1 in the next two lemmas and indicate how to do it for Theorem 2.2 at the end of the section.

A Banach space X is $\mathcal{L}_p$ if its finite dimensional subspaces are like those of ℓ_p. If $1 < p < \infty$, the separable $\mathcal{L}_p$-spaces are the complemented subspaces of $L_p[0,1]$ not isomorphic to ℓ_2.

We use the following properties of a separable $\mathcal{L}_p$-space X: (1) X contains a complemented copy of ℓ_p, and (2) There is an increasing (by inclusion) sequence of finite dimensional subspaces which are uniformly isomorphic to finite dimensional ℓ_p-spaces. Moreover, they are uniformly complemented and their union is dense in X. For more information on $\mathcal{L}_p$-spaces see [**LP**] or [**JRZ**].

Lemma 2.3. *Suppose that $1 < p \leq q$ and let X and Y be separable $\mathcal{L}_p$ or $\mathcal{L}_q$ spaces. Then $B(X,Y)$ embeds complementably in* $W = \left(\sum_{n=1}^{\infty} \oplus B(\ell_p^n, \ell_q^n)\right)_{\infty}$.

Proof. By the assumptions on X and Y, we can find $\phi_n \colon B(\ell_p^n, \ell_q^n) \to B(X,Y)$ and $\psi_n \colon B(X,Y) \to B(\ell_p^n, \ell_q^n)$ satisfying: (1) $\psi_n \phi_n = I_n$, the identity on $B(\ell_p^n, \ell_q^n)$, and (2) for every $T \in B(X,Y)$, $\phi_n \psi_n(T) \to T$ in the w^*-topology.

Then define $\Psi : B(X,Y) \to W$ by $\Psi(T) = (\psi_n(T))_n$. Let $\mathcal{U}$ be a free ultrafilter in $\mathbf{N}$ and define $\Phi : W \to B(X,Y)$ by $\Phi((T_n)) = \lim_{n \in \mathcal{U}} \phi_n(T_n)$ where the limit is taken in the w^*-topology. We can easily verify that $\Phi\Psi = I$, the identity on $B(X,Y)$, and the conclusion follows. $\square$

Lemma 2.4. *Let X and Y be $\mathcal{L}_p$ and $\mathcal{L}_q$-spaces respectively, with $1 < p \leq q$ and let W be as above. Then W embeds complementably into $B(X,Y)$*

Proof. It is clear that W embeds complementably into $B(\ell_p, \ell_q)$, because $B(\ell_p, \ell_q)$ has ℓ_∞-blocks down the diagonal. Moreover, if X is a separable $\mathcal{L}_p$-space, $1 < p < \infty$, then X contains a complemented copy of ℓ_p. Since the same is true for Y we see that $B(\ell_p, \ell_q)$ embeds complementably into $B(X,Y)$. $\square$

Remark. For Theorem 2.2 notice that

$$Z_n = B(\ell_{p_1}^n, B(\cdots B(\ell_{p_{N-1}}^n, \ell_{q_N}^n)\cdots))$$

is isometric to $\ell_{q_1}^n \check{\otimes} \cdots \check{\otimes} \ell_{q_N}^n$ and is 1-complemented in X^*. (We use this to show that $(\sum_n \oplus Z_n)_\infty$ embeds complementably into X^*.) Moreover, $\bigcup_n Z_n$ is w^*-dense in X^*. (We may use this and an ultrafilter argument to show the reverse complemented inclusion.)

3. Primarity of Projective Tensor Products.

We devote most of this section to the proof of the following theorem.

Theorem 3.1. *Let $X = \ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$ be such that $\frac{1}{p_i} + \frac{1}{p_j} \leq 1$ for every $i \neq j$. Then X is primary.*

The proof of this theorem will follow easily from the next proposition that was inspired by results of Blower [**B**] and was used in [**A**] in a similar context. The ideas involved in this "factorization" approach are well-known (see for example Bourgain [**Bo**]).

We have to introduce some notation.

Let $X = \ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$, $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbf{N}^N$ and denote by $e_\alpha = e_{\alpha_1} \otimes e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_N}$. Then $X = [e_\alpha : \alpha \in \mathbf{N}^N]$. We also define $|\alpha| = \max\{\alpha_1, \cdots \alpha_2\}$; and introduce an order between different multiindices. Let $\alpha \in \mathbf{N}^k$ and $\beta \in \mathbf{N}^m$; we say that

$$\alpha < \beta \quad \text{if} \quad \max\{\alpha_1, \cdots, \alpha_k\} < \min\{\beta_1, \cdots, \beta_m\}.$$

Let $\sigma_1, \sigma_2, \cdots \sigma_N : \mathbf{N} \to \mathbf{N}$ be increasing functions (it will also be useful to think of the σ_i's as infinite subsets of $\mathbf{N}$); and let $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N)$ be a function on $\mathbf{N}^N$ defined by $\sigma(\alpha) = (\sigma_1(\alpha_1), \cdots, \sigma_N(\alpha_N))$. Then define

$$J_\sigma : \ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N} \to \ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$$
$$K_\sigma : \ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N} \to \ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$$

by

$$J_\sigma e_\alpha = e_{\sigma(\alpha)} \quad \text{and} \quad K_\sigma e_\alpha = \begin{cases} e_\beta & \text{if there exists } \beta \text{ such that } \sigma(\beta) = \alpha \\ 0 & \text{otherwise.} \end{cases}$$

J_σ and K_σ have many important algebraic properties. J_σ is one-to one, K_σ is onto and $K_\sigma J_\sigma = I$. Moreover, they compose nicely; that is, if $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N)$ and $\psi = (\psi_1, \cdots, \psi_N)$, then

$$J_\sigma J_\psi = J_{\sigma\psi} \quad \text{and} \quad K_\sigma K_\psi = K_{\sigma\psi}.$$

We are now ready to state the proposition.

Proposition 3.2. *Let $\frac{1}{p_i} + \frac{1}{p_j} \le 1$ for every $i \ne j$. Then if $\Phi \in B(\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N})$ and $\epsilon > 0$ there exist $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N)$ and $\lambda \in \mathbf{C}$ such that*

$$\|K_\sigma \Phi J_\sigma - \lambda I\| < \epsilon.$$

Thus one of $K_\sigma \Phi J_\sigma$ or $K_\sigma(\Phi - I)J_\sigma$ is invertible.

It is immediate from this proposition that I, the identity on $\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$, factors through Φ or through $I - \Phi$ which implies trivially that if $\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N} \approx X \oplus Y$ then $\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$ embeds complementably into X or Y. Since $\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$ is isomorphic to its infinite r-sum, the Pełczyński decomposition method implies that $\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$ is primary.

We will present a sketch of the proof. For $\Phi \in \ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$ and $\alpha \in \mathbf{N}^N$ we have,

$$\Phi e_\alpha = \sum_{\beta \in \mathbf{N}^N} \lambda_{\alpha,\beta} e_\beta,$$

for some $\lambda_{\alpha,\beta} \in \mathbf{C}$. Our goal will be to come with a series of the aforementioned J-maps and K-maps which will allow us to get $K\Phi J \approx \lambda I$. We will do this is several steps, fixing progressively more restrictive portions of the range of β. We can do this since this maps compose nicely; however we must be careful not to destroy previous work (see the assumption below). More precisely, Step 1 asserts that we can find K_1, J_1 such that $K_1 \Phi J_1 \approx \Phi_1$ and for every $n \in \mathbf{N}$,

$$\alpha \in \mathbf{N}^N, \quad |\alpha| = n \quad \Longrightarrow \quad \Phi_1 e_\alpha = \sum_{|\beta| = n} \lambda^{(1)}_{\alpha,\beta} e_\beta,$$

for some $\lambda^{(1)}_{\alpha,\beta} \in \mathbf{C}$. This is clearly an improvement in the range of β, but we still have that $\{\beta \in \mathbf{N}^N : |\beta| = n\}$ is a big set. After Steps 2, 3 and 4 we have K_4, J_4 such that $K_4 \Phi J_4 \approx \Phi_4$ and for every $n \in \mathbf{N}$, $j \in \mathbf{N}$,

$$\alpha \in \mathbf{N}^{N-1}, \quad |\alpha| = n, \quad |\alpha| < j \quad \Longrightarrow \quad \Phi_4(e_\alpha \otimes e_j) = \sum_{\substack{|\beta|=n \\ \beta \in \mathbf{N}^{N-1}}} \lambda^{(4)}_{\alpha,\beta} e_\beta \otimes e_j.$$

Step 5 gives us K_5, J_5 such that $K_5 \Phi J_5 \approx \Phi_5$ and for every $n \in \mathbf{N}$, $\gamma \in \mathbf{N}^2$,

$$\alpha \in \mathbf{N}^{N-2}, \quad |\alpha| = n, \quad \alpha < \gamma \quad \Longrightarrow \quad \Phi_5(e_\alpha \otimes e_\gamma) = \sum_{\substack{|\beta|=n \\ \beta \in \mathbf{N}^{N-2}}} \lambda^{(5)}_{\alpha,\beta} e_\beta \otimes e_\gamma.$$

Finally Step 6 provides the general induction argument.

We will apply our arguments on $\mathcal{T} = [e_\alpha : \alpha_1 < \alpha_2 < \cdots < \alpha_N]$ without loss of generality in order to simplify notation, keeping in mind that they will be repeated many times when the order of the α_i's is different. We will choose σ so that J_σ and K_σ "respect" that order. More precisely, consider the permutation group Π_n and a multiindex $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbf{N}^N$. We choose σ so that the (not necessarily complemented) subspaces $\mathcal{T}(\pi) = [e_\alpha : \alpha_{\pi(1)} < \alpha_{\pi(2)} < \cdots < \alpha_{\pi(N)}]$ are invariant for J_σ and K_σ; i.e., $J_\sigma \mathcal{T}(\pi) \subset \mathcal{T}(\pi)$ and $K_\sigma \mathcal{T}(\pi) \subset \mathcal{T}(\pi)$. Notice that the $\mathcal{T}(\pi)$'s "exhaust" the N-dimensional matrix array on which we represent $\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$ (modulo diagonal elements, which we always ignore; see Step 2).

Assumption. Assume from now on that whenever we choose

$$\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N),$$

it always "preserves the order", that is, if $i < j$, then $\sigma_k(i) < \sigma_l(j)$, for every $k, l \leq N$.

We can always satisfy this assumption by passing to subsequences whenever we are choosing the sets σ, which our technical lemmas allow us to do.

Example. It might be instructive to consider the following example "far" from a multiplier. Let $\Phi : \ell_2 \hat{\otimes} \ell_2 \to \ell_2 \hat{\otimes} \ell_2$ be the transpose operator, i.e., $\Phi e_i \otimes e_j = e_j \otimes e_i$. Then choose σ_1 the set of even integers, σ_2 the set of odd integers and $\sigma = (\sigma_1, \sigma_2)$. We verify easily that $K_\sigma \Phi J_\sigma = 0$ thus satisfying the conclusion of Proposition 3.2.

Our steps require the repeated use of two technical lemmas whose proof we delay until Section 6.

Step 1. Let $\partial M_n = [e_\alpha : \alpha \in \mathbf{N}^N, |\alpha| = n]$ with projection Q_n. Then for every $\Phi \in B(\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N})$ and $\epsilon > 0$, there exist $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N)$ and $\Phi_1 \in B(\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N})$ such that $\|\Phi_1 - K_\sigma \Phi J_\sigma\| < \epsilon$ and for every n, $\Phi_1 \partial M_n \subset \partial M_n$.

The proof of this step is an immediate consequence of the following lemma. Remember that $(\partial M_n)_n$ forms a Schauder decomposition for $\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$.

Basic Lemma 1. *Let $\Phi \in B(\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N})$. Then for every $\epsilon_{n,m} > 0$ we can find $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N)$ such that if $x \in \partial M_n$, and $n \neq m$, then $\|Q_m K_\sigma \Phi J_\sigma x\| \leq \epsilon_{n,m}\|x\|$.*

We prove Basic Lemma 1 in the appendix (if $X = \ell_p$ the proof is very easy).

Choose $\epsilon_{n,m}$ in Basic Lemma 1 so that $\epsilon_n = \sum_{m=1}^{\infty} \epsilon_{n,m}$ and $\sum_{n=1}^{\infty} \epsilon_n < \frac{\epsilon}{2}$. Then define Φ_1 on $\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$ as follows: For $x \in \partial M_n$, let

$$\Phi_1(x) = Q_n K_\sigma \Phi J_\sigma x.$$

If $x \in \partial M_n$, then $\|(\Phi_1 - K_\sigma \Phi J_\sigma)x\| \leq \epsilon_n \|x\|$. If $x \in \ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$, we have that $x = \sum_{n=1}^{\infty} x_n$ where $x_n \in \partial M_n$ and $\|x_n\| \leq 2\|x\|$. Therefore,

$$\|(\Phi_1 - K_\sigma \Phi J_\sigma)x\| \leq \sum_{n=1}^{\infty} \|(\Phi_1 - K_\sigma \Phi J_\sigma)x_n\| < \epsilon\|x\|.$$

□

Step 2. Let $\Phi \in B(\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N})$ be such that $\Phi\partial M_n \subset \partial M_n$ for every n, then we can find $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N)$ such that $\Phi_2 = K_\sigma \Phi J_\sigma$ "respects" the place where $\alpha \in \mathbf{N}^N$ takes its maximum; for example, if the maximum takes place in the last coordinate, i.e., $\alpha \in \mathbf{N}^{N-1}$ and $|\alpha| < j$, then

$$\Phi_2(e_\alpha \otimes e_j) = \sum_{|\beta|<j} \lambda_{\alpha,\beta,j} e_\beta \otimes e_j,$$

and we also have similar results for the other coordinates.

We attain this by "disjointifying" the different faces. For $i \leq N$ let $\sigma_i(j) = N(j-1) + i$, and $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N)$. It is easy to see that $\Phi_2 = K_\sigma \Phi J_\sigma$ satisfies the required property. Indeed, if $\alpha \in \mathbf{N}^{N-1}$ and $|\alpha| < n$, then $e_\alpha \otimes e_n \in \partial M_n$, and $J_\sigma(e_\alpha \otimes e_n) \in \partial M_{\sigma_N(n)}$. Hence,

$$\Phi J_\sigma(e_\alpha \otimes e_n) = \sum_{|\gamma|=\sigma_N(n)} \lambda_{\sigma(\alpha,n),\gamma} e_\gamma.$$

Recall that $K_\sigma e_\gamma = e_\eta$ if $\sigma(\eta) = \gamma$ for some η and $K_\sigma e_\gamma = 0$ otherwise. Since the ranges of the σ_i's are disjoint, $\sigma_N(n)$ is nonzero only for the last coordinate. Therefore, if $|\gamma| = \sigma_N(n)$ and $\sigma(\eta) = \gamma$, the last coordinate of η must be n; i.e., $e_\eta = e_\beta \otimes e_n$ for some $\beta \in \mathbf{N}^{N-1}$, and since σ preserves the order, $|\beta| < n$. That is,

$$K_\sigma \Phi J_\sigma(e_\alpha \otimes e_n) = \sum_{|\beta|<n} \lambda_{\sigma(\alpha,n),\sigma(\beta,n)} e_\beta \otimes e_n.$$

We denote $\lambda_{\sigma(\alpha,n),\sigma(\beta,n)}$ by $\lambda_{\alpha,\beta,n}$. $\square$

To make the notation a bit clearer we will state the hypothesis and the conclusion of the steps when the maximum takes place in the Nth coordinate. However the other cases are identical and we will assume that (after repeating the step for the other coordinates) the same result holds for these cases.

Step 3. Let $\Phi \in B(\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N})$ be such that whenever $\alpha \in \mathbf{N}^{N-1}$, $j \in \mathbf{N}$ satisfy $|\alpha| < j$, then $\Phi(e_\alpha \otimes e_j) = \sum_{|\beta|<j} \lambda_{\alpha,\beta,j} e_\beta \otimes e_j$. Then for every $\epsilon > 0$ there exist $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N)$ and $\Phi_3 \in B(\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N})$ such that $\|\Phi_3 - K_\sigma \Phi J_\sigma\| < \epsilon$ and whenever $\alpha \in \mathbf{N}^{N-1}$, $j \in \mathbf{N}$ satisfy $|\alpha| < j$, then

$$\Phi_3(e_\alpha \otimes j) = \sum_{|\beta|<j} \mu_{\alpha,\beta} e_\beta \otimes j.$$

The proof of this step follows from the next lemma.

Basic Lemma 2. *Let $\Phi \in B(\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N})$ be such that whenever $\alpha \in \mathbf{N}^{N-1}$, $j \in \mathbf{N}$ satisfy $|\alpha| < j$, then $\Phi(e_\alpha \otimes e_j) = \sum_{|\beta|<j} \lambda_{\alpha,\beta,j} e_\beta \otimes e_j$. Then for every $\epsilon_{\alpha,\beta,j} > 0$ with $j > \max\{|\alpha|, |\beta|\}$, we can find $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N)$ (respecting the order) such that if we set $\tilde{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_{N-1})$ then*

$$\lim_{j\to\infty} \lambda_{\tilde{\sigma}(\alpha),\tilde{\sigma}(\beta),\sigma_N(j)} = \lambda_{\tilde{\sigma}(\alpha),\tilde{\sigma}(\beta)};$$
$$|\lambda_{\tilde{\sigma}(\alpha),\tilde{\sigma}(\beta),\tilde{\sigma}_N(j)} - \lambda_{\tilde{\sigma}(\alpha),\tilde{\sigma}(\beta)}| \leq \epsilon_{\alpha,\beta,j}.$$

We also give the proof of Basic Lemma 2 in the appendix. Then set $\tilde{\Phi} = K_\sigma \Phi J_\sigma$, and let $L_\alpha = [e_\alpha \otimes e_j : j \in \mathbf{N}]$ with projection P_α. Since $\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_{N-1}}$ has a basis consisting of e_α's, we have that $(L_\alpha)_\alpha$ forms a Schauder decomposition for $\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$.

Define $\Phi_3 \in B(\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N})$ by

$$P_\beta \Phi_3(e_\alpha \otimes e_j) = \begin{cases} \lambda_{\tilde{\sigma}(\alpha),\tilde{\sigma}(\beta)} e_\beta \otimes e_j & \text{if } \max\{|\alpha|, |\beta|\} < j; \\ P_\beta \tilde{\Phi}(e_\alpha \otimes e_j) & \text{otherwise.} \end{cases}$$

Let $\alpha, \beta \in \mathbf{N}^N$; $\epsilon_{\alpha,\beta} = \sum_{j>\max\{|\alpha|,|\beta|\}} \epsilon_{\alpha,\beta,j}$; and $x \in L_\alpha$; i.e., $x = \sum_{j=1}^\infty e_\alpha \otimes c_j e_j$. Then,

$$P_\beta \Phi_3 x - P_\beta \tilde{\Phi} x = \sum_{j>\max\{|\alpha|,|\beta|\}} e_\beta \otimes \left(\lambda_{\tilde{\sigma}(\alpha),\tilde{\sigma}(\beta)} - \lambda_{\tilde{\sigma}(\alpha),\tilde{\sigma}(\beta),\sigma_N(j)}\right) c_j e_j.$$

Hence,

$$\|P_\beta \Phi_3 x - P_\beta \tilde{\Phi} x\| \leq \sum_{j > \max\{|\alpha|,|\beta|\}} \epsilon_{\alpha,\beta,j} \max |c_j| \leq \epsilon_{\alpha,\beta} \|x\|.$$

If we choose $\sum_\alpha \sum_\beta \epsilon_{\alpha,\beta} < \epsilon$ small enough, Φ_3 is well defined and satisfies the required properties. $\square$

Step 4. Let $\Phi \in B(\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N})$ be such that whenever $\alpha \in \mathbf{N}^{N-1}$, $j \in \mathbf{N}$ satisfy $|\alpha| < j$, then $\Phi(e_\alpha \otimes e_j) = \sum_{|\beta|<j} \lambda_{\alpha,\beta} e_\beta \otimes e_j$. Then for every $\epsilon > 0$, there exist $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N)$ and $\Phi_4 \in B(\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N})$ such that $\|\Phi_4 - K_\sigma \Phi J_\sigma\| < \epsilon$ and whenever $\alpha \in \mathbf{N}^{N-1}$, $j \in \mathbf{N}$ satisfy $|\alpha| < j$, we have

$$\Phi_4(e_\alpha \otimes e_j) = \sum_{|\beta|=|\alpha|} \mu_{\alpha,\beta} e_\beta \otimes e_j.$$

Define $\Psi \in B(\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_{N-1}})$ by

$$(\Psi e_\alpha, e_\beta) = \lambda_{\alpha,\beta}.$$

Since $(\Psi e_\alpha, e_\beta) = \lim_{j\to\infty}(\Phi e_\alpha \otimes e_j, e_\beta \otimes e_j)$, Ψ is a bounded map.

Remark. Ideally we would like to apply an induction step and replace Ψ, after a factorization of the form $K\Psi J$, by a multiple of the identity and then combine this with Φ. Controlling the norm of the perturbation requires a more delicate argument, however.

Apply Basic Lemma 1 to Ψ with its respective ∂M_n and projections Q_n; then find $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_{N-1})$ such that whenever $x \in \partial M_n$, and $m \neq n$,

$$\|Q_m K_\sigma \Psi J_\sigma x\| \leq \epsilon_{n,m} \|x\|.$$

Let $\tilde{\Psi} = K_\sigma \Psi J_\sigma$, $\tilde{\sigma} = (\sigma_1, \cdots, \sigma_{N-1}, \sigma_{N-1})$ and $\tilde{\Phi} = K_{\tilde{\sigma}} \Phi J_{\tilde{\sigma}}$. Denote by $P_n = Q_1 + \cdots + Q_n$ the projection onto $[e_\alpha : \alpha \in \mathbf{N}^{N-1}, |\alpha| \leq n]$. Notice now that if $|\alpha| < j$ then $\tilde{\Phi}(e_\alpha \otimes e_j) = (P_j \tilde{\Psi} e_\alpha) \otimes e_j$.

Let $L_\alpha = [e_\alpha \otimes e_j : j \in \mathbf{N}]$. Then as we explained after the Basic Lemma 2, $(L_\alpha)_\alpha$ forms a Schauder decomposition for $\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$.

Define Φ_4 on $\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$ as follows

$$\Phi_4(e_\alpha \otimes e_j) = \begin{cases} \tilde{\Phi}(e_\alpha \otimes e_j) & \text{if } |\alpha| \geq j; \\ (Q_{|\alpha|} \tilde{\Psi} e_\alpha) \otimes e_j & \text{if } |\alpha| < j. \end{cases}$$

Let $n \in \mathbf{N}$; $\alpha \in \mathbf{N}^{N-1}$ with $|\alpha| = n$; $\epsilon_n = \sum_{m=1, m\neq n}^{\infty} \epsilon_{n,m}$ and $x \in L_\alpha$; i.e., $x = \sum_{j=1}^{\infty} e_\alpha \otimes c_j e_j$. Then,

$$\tilde{\Phi}(x) - \Phi_4(x) = \sum_{j>n} (P_j \tilde{\Psi} e_\alpha - Q_n \tilde{\Psi} e_\alpha) \otimes c_j e_j$$

$$= \sum_{j>n} \sum_{\substack{k=1 \\ k\neq n}}^{j} (Q_k \tilde{\Psi} e_\alpha) \otimes c_j e_j$$

$$= \sum_{\substack{k=1 \\ k\neq n}}^{\infty} (Q_k \tilde{\Psi} e_\alpha) \otimes \sum_{j>\max\{k,n\}} c_j e_j .$$

Since $\| \sum_{j>\max\{k,n\}} c_j e_j \| \leq \|x\|$, we have that

$$\|\Phi_4(x) - \tilde{\Phi}(x)\| \leq \epsilon_n \|x\|.$$

Since $\mathrm{card}\{\alpha : |\alpha| = n\}$ is finite, it is enough to choose ϵ_n so that

$$\sum_{n=1}^{\infty} \mathrm{card}\ \{\alpha : |\alpha| = n\}\, \epsilon_n < \epsilon$$

to insure that Φ_4 is well defined and satisfies the required properties. $\square$

Step 5. Let $\Phi \in B(\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N})$ be such that whenever $\alpha \in \mathbf{N}^{N-1}$ and $j \in \mathbf{N}$ satisfy $|\alpha| < j$, then $\Phi(e_\alpha \otimes e_j) = \sum_{|\beta|=|\alpha|} \lambda_{\alpha,\beta} e_\beta \otimes e_j$. Then for every $\epsilon > 0$, there exist $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N)$ and $\Phi_5 \in B(\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N})$ such that $\|\Phi_5 - K_\sigma \Phi J_\sigma\| < \epsilon$ and whenever $i, j \in \mathbf{N}$ and $\alpha \in \mathbf{N}^{N-2}$ satisfy $|\alpha| < i, |\alpha| < j$, then

$$\Phi_5(e_\alpha \otimes e_i \otimes e_j) = \sum_{|\beta|=|\alpha|} \mu_{\alpha,\beta} e_\beta \otimes e_i \otimes e_j .$$

Proof. Disjointifying for $\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_{N-1}}$ as in Step 2, we can assume without loss of generality that whenever $i, j \in \mathbf{N}$, $\alpha \in \mathbf{N}^{N-2}$ satisfy $|\alpha| < i < j$, then

$$\Phi(e_\alpha \otimes e_i \otimes e_j) = \sum_{|\beta|<i} \lambda_{\alpha,\beta,i} e_\beta \otimes e_i \otimes e_j .$$

Apply Basic Lemma 2 to the sequence $\{\lambda_{\alpha,\beta,i}\}$ and assume that (after factoring Φ through $K_\sigma \Phi J_\sigma$ and renaming it Φ again) this sequence satisfies the conclusions of that lemma.

Let $L_\alpha = [e_\alpha \otimes e_i \otimes e_j : i, j \in \mathbf{N}]$ with projection P_α. Since $\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_{N-2}}$ has a basis consisting of e_α's, then (L_α) forms a Schauder decomposition for $\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$.

Define $\tilde{\Phi} \in B(\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N})$ as follows:

$$P_\beta \tilde{\Phi}(e_\alpha \otimes e_i \otimes e_j) = \begin{cases} \lambda_{\alpha,\beta} e_\beta \otimes e_i \otimes e_j & \text{if } |\alpha| \vee |\beta| < i < j; \\ P_\beta \Phi(e_\alpha \otimes e_i \otimes e_j) & \text{otherwise.} \end{cases}$$

Let $\alpha, \beta \in \mathbf{N}^{N-2}$; $\epsilon_{\alpha,\beta} = \sum_{i>|\alpha|\vee|\beta|} \epsilon_{\alpha,\beta,i}$ and $x \in L_\alpha$; i.e.,

$$x = \sum_i \sum_j c_{i,j} e_\alpha \otimes e_i \otimes e_j.$$

Then,

$$\begin{aligned} &P_\beta \Phi x - P_\beta \tilde{\Phi} x \\ &\quad = \sum_{i>|\alpha|\vee|\beta|} \left(\sum_{j=i+1}^{\infty} c_{i,j} \left[P_\beta \Phi(e_\alpha \otimes e_i \otimes e_j) - P_\beta \tilde{\Phi}(e_\alpha \otimes e_i \otimes e_j) \right] \right) \\ &\quad = \sum_{i>|\alpha|\vee|\beta|} (\lambda_{\alpha,\beta,i} - \lambda_{\alpha,\beta}) e_\beta \otimes e_i \otimes \left(\sum_{j=i+1}^{\infty} c_{i,j} e_j \right). \end{aligned}$$

Since $\| \sum_{j=i+1}^{\infty} c_{i,j} e_j \| \leq \|x\|$,

$$\|P_\beta \Phi x - P_\beta \tilde{\Phi} x\| \leq \epsilon_{\alpha,\beta} \|x\|.$$

If we choose $\epsilon_{\alpha,\beta}$ small enough so that $\sum_{\alpha,\beta} \epsilon_{\alpha,\beta} < \frac{\epsilon}{2}$, then $\tilde{\Phi}$ is well defined and satisfies $\|\Phi - \tilde{\Phi}\| < \frac{\epsilon}{2}$; moreover, whenever $|\alpha| < i < j$, we have

$$\tilde{\Phi}(e_\alpha \otimes e_i \otimes e_j) = \sum_{|\beta|<i} \lambda_{\alpha,\beta} e_\beta \otimes e_i \otimes e_j.$$

Let $T : L_\alpha \to L_\beta$ be defined by $T(x) = P_\beta \tilde{\Phi} x$. T is clearly a bounded map and $L_\alpha \equiv L_\beta \equiv \ell_{p_{N-1}} \hat{\otimes} \ell_{p_N}$. It follows that if $|\alpha| \vee |\beta| < i < j$ then $T(e_\alpha \otimes e_i \otimes e_j) = \lambda_{\alpha,\beta} e_\beta \otimes e_i \otimes e_j$. Since all the arguments work if the maximum is attained at the $(N-1)$-st coordinate and the next maximum is attained in the last coordinate, we can also assume that if $|\alpha| \vee |\beta| < j < i$ then $T(e_\alpha \otimes e_i \otimes e_j) = \mu_{\alpha,\beta} e_\beta \otimes e_i \otimes e_j$. Thus T takes value $\lambda_{\alpha,\beta}$ in the upper triangular part of a copy of $\ell_{p_{N-1}} \hat{\otimes} \ell_{p_N}$ and the value $\mu_{\alpha,\beta}$ in the lower part. Since we assumed that

$$\frac{1}{p_{N-1}} + \frac{1}{p_N} \leq 1,$$

we have that $\lambda_{\alpha,\beta} = \mu_{\alpha,\beta}$. (If $\lambda_{\alpha,\beta} \neq \mu_{\alpha,\beta}$, then $(T - \mu_{\alpha,\beta} I)/(\lambda_{\alpha,\beta} - \mu_{\alpha,\beta})$ would be a projection onto the upper triangular part of $\ell_{p_{N-1}} \hat{\otimes} \ell_{p_N}$, contradicting Theorem 1.3.)

Let $\alpha = (\alpha_1, \cdots, \alpha_{N-2})$ and $\gamma = (\alpha_{N-1}, \alpha_N)$. We now have that if $\alpha < \gamma$, then

$$\tilde{\Phi}(e_\alpha \otimes e_\gamma) = \sum_{\beta<\gamma} \lambda_{\alpha,\beta} e_\beta \otimes e_\gamma.$$

Define a map $\Psi \in B(\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_{N-2}})$, as in Step 4, by

$$(\Psi e_\alpha, e_\beta) = \lambda_{\alpha,\beta}.$$

Since Ψ is bounded, we apply Basic Lemma 1 to it and assume without loss of generality (after factoring $K_\sigma\tilde{\Phi}J_\sigma$ and then renaming it $\tilde{\Phi}$ again) that if $x \in \partial M_n$ (here ∂M_n is a subset of $\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_{N-2}}$) and $m \neq n$,

$$\|Q_m \Psi x\| \leq \epsilon_{n,m}\|x\|.$$

Let $L_\alpha = [e_\alpha \otimes e_i \otimes e_j : i,j \in \mathbf{N}]$ with projection P_α, and define $\Phi_1 \in B(\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N})$ as follows

$$\Phi_5(e_\alpha \otimes e_\gamma) = \begin{cases} (Q_{|\alpha|}\Psi e_\alpha) \otimes e_\gamma & \text{if } \alpha < \gamma \\ \tilde{\Phi}(e_\alpha \otimes e_\gamma) & \text{otherwise.} \end{cases}$$

Let $x \in L_\alpha$, $|\alpha| = n$; i.e., $x = \sum_\gamma c_\gamma e_\alpha \otimes e_\gamma$. Hence,

$$\begin{aligned} \tilde{\Phi}x - \Phi_5 x &= \sum_{\gamma>\alpha} \left[c_\gamma \tilde{\Phi} e_\alpha \otimes e_\gamma - c_\gamma (Q_n \Psi e_\alpha) \otimes e_\gamma\right] \\ &= \sum_{\substack{k=1 \\ k\neq n}}^{\infty} (Q_k \Psi e_\alpha) \otimes \left(\sum_{\gamma>n,k} c_\gamma e_\alpha \otimes e_\gamma\right). \end{aligned}$$

Since $\|\sum_{\gamma>n,k} c_\gamma e_\alpha \otimes e_\gamma\| \leq \|x\|$, the result follows. □

The induction step is an extension of Step 5.

Step 6. Let $\Phi \in B(\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N})$ be such that whenever $\alpha \in \mathbf{N}^k$, $\gamma \in \mathbf{N}^{N-k}$ satisfy $\alpha < \gamma$, then $\Phi(e_\alpha \otimes e_\gamma) = \sum_{|\beta|=|\alpha|} \lambda_{\alpha,\beta} e_\beta \otimes e_\gamma$. Then for every $\epsilon > 0$, there exist $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N)$ and $\Phi_k \in B(\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N})$ such that $\|\Phi_k - K_\sigma \Phi J_\sigma\| < \epsilon$ and whenever $\alpha \in \mathbf{N}^{k-1}$ and $\gamma \in \mathbf{N}^{N-k+1}$ satisfy $\alpha < \gamma$, we have

$$\Phi_k(e_\alpha \otimes e_\gamma) = \sum_{|\beta|=|\alpha|} \mu_{\alpha,\beta} e_\beta \otimes e_\gamma.$$

Sketch of proof. Disjointifying as in Step 2 we assume that whenever $\alpha \in \mathbf{N}^{k-1}$, $i \in \mathbf{N}$ and $\gamma \in \mathbf{N}^{N-k}$ satisfy $\alpha < i < \gamma$, then

$$\Phi(e_\alpha \otimes e_i \otimes e_\gamma) = \sum_{\beta<i} \lambda_{\alpha,\beta,i} e_\beta \otimes e_i \otimes e_\gamma.$$

Assume also that the sequence $\{\lambda_{\alpha,\beta,i}\}$ satisfies the conclusion of Basic Lemma 2.

Let $\alpha \in \mathbf{N}^{k-1}$; $L_\alpha = [e_\alpha \otimes e_i \otimes e_\gamma : i \in \mathbf{N}, \gamma \in \mathbf{N}^{N-k}]$ with projection P_α and define $\tilde{\Phi}$ as in Step 5. Since for every i_0, $\|\sum_{\gamma > i_0} c_{i_0,\gamma} e_{i_0} \otimes e_\gamma\| \leq \|\sum_{i,\gamma} c_{i,\gamma} e_i \otimes e_\gamma\|$, then $\|\Phi - \tilde{\Phi}\| < \epsilon$ and whenever $\alpha \in \mathbf{N}^{k-1}$, $\gamma \in \mathbf{N}^{N-k}$ satisfy $\alpha < i < \gamma$, then

$$\tilde{\Phi}(e_\alpha \otimes e_i \otimes e_\gamma) = \sum_{\beta < i} \lambda_{\alpha,\beta} e_\alpha \otimes e_i \otimes e_\gamma.$$

Fix $\alpha, \beta \in \mathbf{N}^{k-1}$ and define $T : L_\alpha \to L_\beta$ by $Tx = P_\beta \tilde{\Phi} x$. Since T is bounded and $L_\alpha \equiv L_\beta \equiv \ell_{p_k} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$ we assume that T is defined on $Z = \ell_{p_k} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$.

Decompose Z into $(E_j)_{j=k}^N$, where $E_j = [e_\theta : \theta_j < \theta_i$ for every $i \neq j]$; (i.e., E_j is the span of those e_θ where the minimum occurs at the jth coordinate). For instance, if $e_\theta \in E_k$, then $e_\theta = e_i \otimes e_\gamma$ for some $i < \gamma$ and hence $Te_\theta = \lambda_{\alpha,\beta} e_\theta = \lambda^{(k)} e_\theta$. Since all the arguments work for the other permutations of the coordinates we can assume that there exist $\lambda^{(j)}$ such that if $x \in E_j$, then $Tx = \lambda^{(j)} x$, $j = k, k+1, \cdots, N$.

We will use that $\frac{1}{p_i} + \frac{1}{p_j} \leq 1$ for every $i \neq j$ to conclude that the $\lambda^{(j)}$'s have to be equal. Indeed, let $\bar{m} = (m, m, \cdots, m) \in \mathbf{N}^{N-k-3}$ and consider $K_m = [e_i \otimes e_j \otimes e_{\bar{m}} : i, j \leq m]$. It is clear that $K_m \equiv \ell_{p_k}^m \hat{\otimes} \ell_{p_{k+1}}^m$ and that T restricted to it gives us $\lambda^{(k)}$ in the upper triangular part and $\lambda^{(k+1)}$ in the lower one. If $\lambda^{(k)} \neq \lambda^{(k+1)}$, we would have that the m-triangular parts are uniformly complemented and this is not true. A similar argument proves that the $\lambda^{(j)}$'s are all equal.

In conclusion, if $\alpha \in \mathbf{N}^{k-1}$, $\gamma \in \mathbf{N}^{N-k+1}$, and $\alpha < \gamma$, then

$$\tilde{\Phi}(e_\alpha \otimes e_\gamma) = \sum_{\beta < \gamma} \lambda_{\alpha,\beta} e_\beta \otimes e_\gamma.$$

Define $\Psi \in B(\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_{k-1}})$ by $(\Psi e_\alpha, e_\beta) = \lambda_{\alpha,\beta}$; apply Basic Lemma 1 to it and finish the proof as in Step 5. □

Iterating Step 6 we finish the proof of the proposition.

We will see in the next section that, for most cases, if $\frac{1}{p_i} + \frac{1}{p_j} > 1$, X is not primary. This is not always true, however.

Theorem 3.3. *Let $1 \leq p < \infty$ and $n \in \mathbf{N}$. Then $X = \ell_p \hat{\otimes} \cdots \hat{\otimes} \ell_p$ (n times) is primary.*

Proof. We divide the proof into two cases. If $\frac{2}{p} \leq 1$, this is a particular case of Theorem 3.1. If $\frac{2}{p} > 1$ then the triangular projections are bounded. This implies that the "tetrahedrals" are complemented. (An example of this is $\mathcal{T} = [e_\alpha : \alpha_1 < \alpha_2 < \cdots < \alpha_n]$.) Since all of them are isometrically isomorphic and there are finitely many of them we conclude that

$X \approx [e_\alpha : \alpha_1 < \alpha_2 < \cdots < \alpha_n]$ by Pełczyński's decomposition method. Then the proofs of Theorem 3.1 and Proposition 3.2 apply to this space. $\square$

The proof of Theorem 3.1 dualizes (formally) to $X_* = \ell_{q_1}\check{\otimes}\cdots\check{\otimes}\ell_{q_N}$. Define J_σ and K_σ on $B(\ell_{q_1}\check{\otimes}\cdots\check{\otimes}\ell_{q_N})$ as in $B(\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N})$. The key to the dualization argument is that $(J_\sigma)^* = K_\sigma$ and $(K_\sigma)^* = J_\sigma$.

Theorem 3.4. *Let $X_* = \ell_{q_1}\check{\otimes}\cdots\check{\otimes}\ell_{q_N}$ be such that $\frac{1}{q_i} + \frac{1}{q_j} \geq 1$ for every $i \neq j$. Then X_* is primary.*

Proof. Let $\Phi \in B(\ell_{q_1}\check{\otimes}\cdots\check{\otimes}\ell_{q_N})$, and $0 < \epsilon < \frac{1}{2}$. Then $\Phi^* \in B(\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N})$ and whenever $i \neq j$ we have $\frac{1}{p_i} + \frac{1}{p_j} \leq 1$. Therefore, Theorem 3.1 tells us that there exist σ and $\lambda \in \mathbf{C}$ such that $\|K_\sigma \Phi^* J_\sigma - \lambda I_X\| < \epsilon$.

Since $K_\sigma \Phi^* J_\sigma - \lambda I_X = (K_\sigma \Phi J_\sigma - \lambda I_{X_*})^*$ we have that $\|K_\sigma \Phi J_\sigma - \lambda I_{X_*}\| < \epsilon$. Therefore, Φ or $I_{X_*} - \Phi$ factors through X_*, and since X_* is isomorphic to its r'-sum, we conclude that X_* is primary. $\square$

4. ℓ_p subspaces of $\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$.

Theorem 1.3 tells us that ℓ_p embeds into $\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$ if there exists a non-empty $A \subset \{1, \cdots, N\}$ for which $p = r_A$, where $\frac{1}{r_A} = \min\left\{1, \sum_{i\in A} \frac{1}{p_i}\right\}$. We will see in the next theorem that the converse holds.

Theorem 4.1. *ℓ_p embeds into $\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$ if and only if there exists a non-empty $A \subset \{1, \cdots, N\}$ such that $p = r_A$.*

We will use this theorem to prove the following:

Theorem 4.2. *Let $X = \ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$ and assume that for some $i \neq j$, $\frac{1}{p_i} + \frac{1}{p_j} > 1$ and that $p_k \notin \{r_A : k \notin A\}$ for $k = i, j$. Then X is not primary.*

Remark. Theorem 4.1 could probably be generalized to characterize when m-fold tensor products embed into n-fold tensor products for $m \leq n$ and this would slightly improve Theorem 4.2.

We will use Theorem 1.3 to decompose $X \approx [e_\alpha : \alpha_i > \alpha_j] \oplus [e_\alpha : \alpha_i \leq \alpha_j]$. The condition $p_i \notin \{r_A : i \notin A\}$ insures that ℓ_{p_i} does not embed into $[e_\alpha : \alpha_i \leq \alpha_j]$. (This is easily seen for example when $N = 2$. In this case $X = \ell_{p_1}\hat{\otimes}\ell_{p_2}$ and $p_k \notin \{r_A : k \notin A\}$ means $p_1 \neq p_2$; we can then observe that ℓ_{p_1} does not embed into $[e_i \otimes e_j : i \leq j]$.)

We will prove Theorem 4.1 by induction. Assume for the remainder of this section that $X = \ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$; $\frac{1}{r} = \min\left\{1, \sum_{i=1}^{N} \frac{1}{p_i}\right\}$ and that $\Phi : \ell_p \to$

$\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$ is an isomorphism. We can assume without loss of generality that there is a sequence of increasing natural numbers n_i such that

$$(*) \qquad \Phi e_i \in [e_\alpha : n_i < |\alpha| < n_{i+1}] \quad \text{for every } i.$$

If $p > 1$ this is true because $\Phi e_j \to 0$ weakly. If $p = 1$ and P_{M_n} is the projection onto $[e_\alpha : \alpha \leq n]$, we can find infinitely many pairs of e_i's (say e_k and e_l) such that $P_{M_n}\Phi(e_k - e_l) \approx 0$. Then we replace the e_i's by differences of unit vectors and get $(*)$.

We say that $\Psi : \ell_p \to \ell_p$ is an ℓ_p-average isometry if there exist a sequence of subsets of $\mathbf{N}$, $\sigma_1 < \sigma_2 < \cdots$ and scalars a_k such that

$$\Psi e_i = \sum_{k\in\sigma_i} a_k e_k \quad \text{and} \quad \sum_{k\in\sigma_i} |a_k|^p = 1 \quad \text{for every } i.$$

Finally we will let $E_n = [e_\alpha : \min\{\alpha\} \leq n]$ for every $n \in \mathbf{N}$. The key to the induction step is that

$$E_n \approx (\ell_{p_2}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}) \oplus (\ell_{p_1}\hat{\otimes}\ell_{p_3}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}) \oplus \cdots \oplus (\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_{N-1}}) .$$

(The isomorphism constant goes to infinity with n.) Notice that each one of those summands is an $(N-1)$-projective tensor product.

We need two lemmas.

Lemma 4.3. *Let $\Phi : \ell_p \to \ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$ be as in $(*)$ with $p > r$. Then for every $\epsilon > 0$ we can find $n \in \mathbf{N}$ such that $\|(I - P_{E_n})\Phi\| \leq \epsilon$.*

Lemma 4.4. *Let $\Phi : \ell_p \to \ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$ be as in $(*)$ with $p < r$, then for every $\epsilon > 0$ there exists $\Psi : \ell_p \to \ell_p$ an ℓ_p-average isometry such that $\|\Phi\Psi\| < \epsilon$.*

Proof of Theorem 4.1. The theorem is clearly true for $N = 1$. Assume that the result is true for $(N-1)$-projective tensor products and let $\Phi : \ell_p \to \ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$ be an isomorphism satisfying $(*)$.

It follows from Lemma 4.4 that $p \geq r$. If $p = r$ there is nothing to prove since ℓ_p clearly embeds in the main diagonal. If $p > r$, Lemma 4.3 tells us that $\Phi\ell_p$ is essentially inside E_n and therefore it is inside one of the $(N-1)$-tensor products. Hence it has to be of the form r_A for some nonempty A by induction. $\square$

We used in the proof the well-known fact that if ℓ_p embeds into $X \oplus Y$ then ℓ_p embeds into X or into Y.

For the proof of Theorem 4.2 we need one more lemma.

Lemma 4.5. *Let $X = \ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$, $i, j \leq N$, $i \neq j$ and assume that $p_i \notin \{r_A : i \notin A\}$. Then ℓ_{p_i} does not embed into $[e_\alpha : \alpha_i \leq \alpha_j]$.*

Proof of Theorem 4.2. Use Theorem 1.3 to decompose $X \approx [e_\alpha : \alpha_i > \alpha_j] \oplus [e_\alpha : \alpha_i \leq \alpha_j]$. Lemma 4.5 tells us that ℓ_{p_j} does not embed into $[e_\alpha : \alpha_i > \alpha_j]$, and that ℓ_{p_i} does not embed into $[e_\alpha : \alpha_i \leq \alpha_j]$. Therefore neither of them is isomorphic to X, and so X is not primary. □

Proof of Lemma 4.3. If the lemma were false, we could find some $\epsilon_0 > 0$; a sequence of normalized vectors $\{x_i\}_{i\in\mathbf{N}}$ in ℓ_p satisfying $\operatorname{supp}\{x_i\} < \operatorname{supp}\{x_{i+1}\}$ for every i; and an increasing sequence $n_i \in \mathbf{N}$ satisfying $\|P_i\Phi x_i\| \geq \epsilon_0$ where P_i is the projection onto the diagonal block

$$[e_\alpha : n_i \leq \alpha < n_{i+1}].$$

Theorem 3.1 implies that $[P_i\Phi x_i : i \in \mathbf{N}] \approx \ell_r$. Let P be the diagonal projection onto $[P_i\Phi x_i : i \in \mathbf{N}]$ and consider $P\Phi : \ell_p \to \ell_r$. Since $\|P\Phi e_i\| \geq \epsilon_0$ for every $i \in \mathbf{N}$ we have that $P\Phi$ is not compact. This is a contradiction. □

Sketch of the proof of Lemma 4.4. For $N = 1$ the result is easy. The condition $(*)$ says that $\Phi : \ell_p \to \ell_r$ is diagonal; i.e., $\Phi e_i = \lambda_i e_i$. Moreover since Φ is bounded, there exists $M > 0$ such that $|\lambda_i| \leq M$ for every i. We get the blocks by taking the a_k's constant in every σ. Let $\sigma \subset \mathbf{N}$ be of cardinality n (say). Then $\|\sum_{k\in\sigma}\left(\frac{1}{n}\right)^{1/p} e_k\|_p = 1$ but $\|\sum_{k\in\sigma}\left(\frac{1}{n}\right)^{1/p}\Phi e_k\|_r \leq Mn^{1/r-1/p}$ goes to zero as n goes to infinity.

Assume the result for $N-1$ and let $\Phi : \ell_p \to \ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$ be as in $(*)$. The idea is to find an ℓ_p-average isometry $\Psi \in B(\ell_p)$ such that $\Phi\Psi$ is essentially supported in a diagonal block; then since the diagonal block is like ℓ_r, the case $N = 1$ takes care of it.

To find Ψ we have to find an increasing sequence $n_i \in \mathbf{N}$ and a normalized sequence $\{x_i\}_{i\in\mathbf{N}}$ in ℓ_p satisfying $\operatorname{supp}\{x_i\} \leq \operatorname{supp}\{x_{i+1}\}$ for every $i \in \mathbf{N}$ and $\Phi x_i \in [e_\alpha : n_i \leq \alpha < n_{i+1}]$. (The last inclusion is an "almost" inclusion; that is, for a given $\epsilon_i > 0$ there exists $n_i \in \mathbf{N}$ such that the distance from Φx_i to $[e_\alpha : n_i \leq \alpha < n_{i+1}]$ is less that ϵ_i.)

It is clear that it is enough to do this for x_1 and x_2 because we can iterate it to conclude the lemma. Clearly $\Phi x_1 \in [e_\alpha : \alpha < n]$ for some n. We want to find x_2 such that Φx_2 is supported outside E_n. Since E_n is isomorphic to the sum of $(N-1)$-projective tensor products, we can apply the induction step to insure the existence of x_2. □

Sketch of the proof of Lemma 4.5. The proof of this goes by induction too. The result is clear for $N = 2$. Suppose it is true for $N-1$ and false for N.

Then let $Z = [e_\alpha : \alpha_i \leq \alpha_j] \subset \ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$ and, by the assumption, find $\Phi : \ell_{p_i} \to Z$, an isomorphism satisfying $(*)$.

The main diagonal of Z is isomorphic to ℓ_r and $p_i > r$. Hence, by Lemma 4.3, there exists $n \in \mathbf{N}$ such that $\Phi\ell_{p_i}$ is essentially inside E_n. We will look at the N-summands of $Z \bigcap E_n$ to get a contradiction.

One of those summands does not contain the ith component and hence is isomorphic to

$$\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_{i-1}}\hat{\otimes}\ell_{p_{i+1}}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}.$$

The condition $p_i \notin \{r_A : i \notin A\}$ and Theorem 4.1 imply that ℓ_{p_i} does not embed there.

Another summand does not contain the jth component. This really means that $\alpha_j \leq n$. Therefore, $\alpha_i \leq n$ as well and the summand is isomorphic to $\hat{\otimes}_{k\neq i,j}\ell_{p_k}$. We conclude as before that ℓ_{p_i} does embed here.

The remaining summands will have the same structure but with $N-1$ terms. Then the induction hypothesis implies that ℓ_{p_i} does not embed into any one of them.

Therefore, ℓ_{p_i} does not embed in Z. This is a contradiction. $\square$

5. Primarity of Polynomials and Operator Spaces.

In this section we discuss the primarity of $(\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N})^*$. There will be really only one case to consider; namely that of $r = 1$ $\left(\text{recall that } \frac{1}{r} = \min\left\{1, \sum_{i=1}^N \frac{1}{p_i}\right\}\right)$, which we demonstrate below using techniques of Bourgain [**Bo**] and Blower [**B**].

It is interesting to note that completely different factors determine the primarity of $(\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N})^*$ when $r = 1$ and $r > 1$. When $r > 1$ it is the unboundedness of the main triangle projection in each pair (taken separately) that is the most important factor, while for $r = 1$ we will see that the main point is that we have ℓ_∞-blocks down the diagonal.

Theorem 5.1. *Let $X = \ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N}$ be such that $\frac{1}{r} = \min\left\{1, \sum_{i=1}^N \frac{1}{p_i}\right\} = 1$. Then $(\ell_{p_1}\hat{\otimes}\cdots\hat{\otimes}\ell_{p_N})^*$ is primary.*

This result will solve the question of primarity for spaces of polynomials. Since the space of analytic polynomials of degree m on ℓ_p is isomorphic (with constant $\frac{m^m}{m!}$) to the dual of the symmetric m-fold tensor product $\hat{\otimes}_s^m \ell_p$. That is $\mathcal{P}_m \approx (\hat{\otimes}_s^m \ell_p)^*$. (Here m is the number of times that one takes the tensor product.)

Lemma 5.2. *For any $1 \leq p < \infty$ and $m \in \mathbf{N}$ we have that $\ell_p\hat{\otimes}\cdots\hat{\otimes}\ell_p \approx$*

$\hat{\otimes}_s^m \ell_p$.

Proof. We use Pełczyński's decomposition method again. Since $\ell_p \hat{\otimes} \cdots \hat{\otimes} \ell_p$ is isomorphic to its infinite s-sum ($s = \max\{1, \frac{p}{m}\}$) we only have to prove that they embed complementably into each other. It is clear that $\hat{\otimes}_s^m \ell_p$ embeds into $\ell_p \hat{\otimes} \cdots \hat{\otimes} \ell_p$. Indeed, $S \in B(\ell_p \hat{\otimes} \cdots \hat{\otimes} \ell_p)$ defined by $S e_\alpha = \frac{1}{m!} \sum_{\pi \in \Pi_m} e_{\pi(\alpha)}$ shows that the embedding is 1-complemented. On the other hand, for $i \leq N$ let $\sigma_i(j) = m(j-1) + i$, $\sigma = (\sigma_1, \cdots, \sigma_m)$ and define $T \in B(\ell_p \hat{\otimes} \cdots \hat{\otimes} \ell_p)$ by $T = K_\sigma S J_\sigma$. It is clear that T factors through $\hat{\otimes}_s^m \ell_p$ and it is easy to see that $T e_\alpha = \frac{1}{m!} e_\alpha$. Hence, $\ell_p \hat{\otimes} \cdots \hat{\otimes} \ell_p$ embeds complementably into $\hat{\otimes}_s^m \ell_p$ and the result follows. □

Corollary 5.3. *Let $1 \leq p < \infty$ and $m \geq 1$. The space of homogeneous analytic polynomials $\mathcal{P}_m(\ell_p)$ and the symmetric tensor product of m copies of ℓ_p are primary.*

We now proceed to the proof of the theorem. Notice that if

$$X = \ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$$

is such that $\frac{1}{r} = \min\left\{1, \sum_{i=1}^N \frac{1}{p_i}\right\} = 1$, then Theorem 2.2 tells us that

$$(\ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N})^* \approx \left(\sum_{n=1}^{\infty} \ell_{q_1}^n \check{\otimes} \cdots \check{\otimes} \ell_{q_N}^n \right)_\infty .$$

This decomposition allows us to use the technique developed by Bourgain [**Bo**] to prove that H^∞ is primary; namely, one obtains the general theorem from the finite dimensional version.

The proof is an exact generalization of the proof of Blower [**B**] that $B(H)$ is primary; it has no surprises, and so we will simply sketch the part that is different for the case $N > 2$, and refer the interested reader to [**B**] for other details. The proof follows from the following 2 lemmas, as indicated in [**Bo**].

Proposition 5.4. *Given $n \in \mathbf{N}$, $\epsilon > 0$ and $K < \infty$, there exists $N_0 = N_0(n, \epsilon, K)$ such that if $M \geq N_0$ and $T \in B(\ell_{q_1}^M \check{\otimes} \cdots \check{\otimes} \ell_{q_N}^M)$ with $\|T\| \leq K$, then there exist subsets $\sigma_1, \sigma_2, \cdots \sigma_N \subset \{1, \cdots, M\}$ of cardinality n, and a constant λ such that if $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N)$ then,*

$$\|K_\sigma T J_\sigma - \lambda I_n\| \leq \epsilon.$$

Thus, one of $K_\sigma T J_\sigma$ and $K_\sigma (I_N - T) J_\sigma$ is invertible.

Remark. Here $J_\sigma \colon \ell_{q_1}^n \check{\otimes} \cdots \check{\otimes} \ell_{q_N}^n \to \ell_{q_1}^M \check{\otimes} \cdots \check{\otimes} \ell_{q_N}^M$ is defined by $J_\sigma e_\alpha = e_{\sigma(\alpha)}$ where $\sigma(\alpha) = (\sigma_1(\alpha_1), \cdots, \sigma_N(\alpha_N))$, and $\sigma_i = \{\sigma_i(1), \sigma_i(2), \cdots, \sigma_i(n)\}$. Moreover, $\sigma_i(k) < \sigma_i(l)$ iff $k < l$. The definition for K_σ is similar.

Proposition 5.5. *Given $n \in \mathbf{N}$ and $\epsilon > 0$ there exists $N_0 = N_0(n,\epsilon)$ such that if $M \geq N_0$ and E is an n-dimensional subspace of $\ell_{q_1}^M \check{\otimes} \cdots \check{\otimes} \ell_{q_N}^M$ then there exists a subspace F of $\ell_{q_1}^M \check{\otimes} \cdots \check{\otimes} \ell_{q_N}^M$, isometrically isomorphic to $\ell_{q_1}^n \check{\otimes} \cdots \check{\otimes} \ell_{q_N}^n$, and a block projection Q from $\ell_{q_1}^M \check{\otimes} \cdots \check{\otimes} \ell_{q_N}^M$ to F such that $\|Qx\| < \epsilon\|x\|$ for every $x \in E$.*

Sketch of the proof of Proposition 5.4. Let $T \in B(\ell_{q_1}^M \check{\otimes} \cdots \check{\otimes} \ell_{q_N}^M)$ such that $\|T\| \leq K$. We will find a copy of $\ell_{q_1}^n \check{\otimes} \cdots \check{\otimes} \ell_{q_N}^n$ inside $\ell_{q_1}^M \check{\otimes} \cdots \check{\otimes} \ell_{q_N}^M$ such that T is essentially a multiple of the identity when restricted to this subspace. We accomplish this in two steps.

Step 1. Find a large subset $\psi \subset \{1, \cdots, M\}$ and $\lambda \in \mathbf{C}$ such that whenever $\alpha = (\alpha_1, \cdots, \alpha_N)$ is such that $\alpha_1 < \cdots < \alpha_N$ and $\alpha_k \in \psi$ for $i \leq N$, then $|(Te_\alpha, e_\alpha) - \lambda| < \epsilon$.

Step 2. Find $\sigma_1 < \sigma_2 < \cdots < \sigma_N \subset \psi$ each of cardinality n, such that whenever $\alpha = (\alpha_1, \cdots, \alpha_N), \alpha' = (\alpha_1', \cdots, \alpha_N')$ are are such that $\alpha_k, \alpha_k' \in \sigma_k$ for every $k \leq N$ and $\alpha \neq \alpha'$, then $|(Te_\alpha, e_{\alpha'})| < \epsilon$. Then define $\mathcal{S} = [e_\alpha \colon \alpha_k \in \sigma_k]$. One can easily verify that if $\epsilon > 0$ is chosen small enough then T restricted and projected into $\mathcal{S}$ is essentially a multiple of the identity and that $\mathcal{S}$ is isometrically isomorphic to $\ell_{q_1}^n \check{\otimes} \cdots \check{\otimes} \ell_{q_N}^n$.

Both steps depend on Ramsey's Theorem and they are very minor modifications of Blower's argument.

For Step 1 divide the disk $\{z \colon |z| \leq K\}$ into finitely many disjoint subsets V_k of diameter less than ϵ, and define the coloring on N-sets of $\{1, \cdots, M\}$ by $\{\alpha_1, \cdots, \alpha_N\} \to \ell$ if $(Te_\alpha, e_\alpha) \in V_\ell$ where $\alpha = (\alpha_1, \cdots, \alpha_N)$ for $\alpha_1 < \cdots < \alpha_N$. Then use Ramsey's Theorem fo find a large monochromatic set ψ.

The proof of Step 2 involves many different cases (but all of them are similar). One has to look at all the different ways that $(\alpha_1, \cdots, \alpha_N) \neq (\alpha_1', \cdots, \alpha_N')$. We will illustrate the case when $\alpha_k < \alpha_k'$ for every $k \leq N$.

Color the $2N$-elements of $\{1, \cdots, M\}$ by: $\{\alpha_1, \alpha_1', \alpha_2, \alpha_2', \cdots, \alpha_N, \alpha_N'\}$ is *bad* if $\alpha_1 < \alpha_1' < \alpha_2 < \alpha_2' < \cdots < \alpha_N < \alpha_N'$ and $|(Te_\alpha, e_{\alpha'})| \geq \epsilon$ where $\alpha = (\alpha_1, \cdots, \alpha_N)$ and $\alpha' = (\alpha_1', \cdots, \alpha_N')$; it is *good* otherwise.

Ramsey's Theorem gives us a large monochromatic subset $\psi_1 \subset \psi$. We will show that ψ_1 has to be *good*. Let $\alpha_1' < \alpha_2 < \alpha_2' < \cdots < \alpha_N < \alpha_N'$ be the $2N-1$ largest elements of ψ_1, and let $\beta = (\alpha_2, \cdots, \alpha_N)$, $\alpha' = (\alpha_1', \cdots, \alpha_N')$, and $F = [e_i \otimes e_\beta \colon i \in \psi_1, i < \alpha_1']$.

It is clear that $F \equiv \ell_{p_1}^{|\psi_1|-2N+1}$. Define $\tilde{T} : F \to \mathbf{C}$ by $\tilde{T}(x) = (Tx, e_{\alpha'})$. Then we have that $\tilde{T}$ is a map from $\ell_{p_1}^s$ into $\mathbf{C}$, with norm less than or equal to K and maps the canonical basis into "large" elements. Since we assumed that $p_1 > 1$ this is a contradiction.

Now we have to look at all the other possibilities; e.g., $\alpha_1' > \alpha_1$ and

$\alpha_k < \alpha'_k$ for $2 \leq k \leq N$ etc. We have to look also at the cases when some of the coordinates are equal, but these are not very different. We prove the proposition by choosing M large enough. $\square$

Sketch of the proof of Proposition 5.5. It is enough to prove that if $x \in \ell_{q_1}^M \check{\otimes} \cdots \check{\otimes} \ell_{q_N}^M$ with $\|x\| \leq 1$, then we can find Q, a large block projection, such that $\|Q(x)\| \leq \epsilon$. Then take an ϵ-net of the sphere of E, $\{x_i\}_{i=1}^s$. Find Q_1 a large block projection such that $\|Q_1 x_1\| \leq \epsilon$; then find Q_2 a large block projection contained in the range of Q_1 such that $\|Q_2 Q_1 x_2\| \leq \epsilon$. Proceeding in this way we get that $Q = Q_s \cdots Q_2 Q_1$; this Q does it.

To check the first claim let $x \in \ell_{q_1}^M \check{\otimes} \cdots \check{\otimes} \ell_{q_N}^M$ with $\|x\| \leq 1$ and let $\rho > 0$ (to be fixed later). Then define a coloring on the N sets of $\{1, \cdots, M\}$ by: $\{\alpha_1, \cdots, \alpha_N\}$ is *bad* if $|(x, e_\alpha)| \geq \rho$ where $\alpha = (\alpha_1, \cdots, \alpha_N)$ and $\alpha_1 < \cdots < \alpha_N$. And *good* otherwise. Ramsey's theorem gives us a large monochromatic subset, and this subset has to be *good*. $\square$

6. Appendix

In this section we will prove Basic Lemmas 1 and 2 from Section 3.

Proof of Basic Lemma 1. For this proof let $M_n = [e_\alpha : \alpha \leq n]$ with projection P_n. We will divide the proof into two parts, one for $m > n$ and the other one for $m < n$. In both cases, $\sigma = (\sigma_1, \cdots, \sigma_N)$ satisfies $\sigma_1 = \sigma_2 = \cdots = \sigma_N$.

The case $m > n$ is simpler; we start with it.

If $K \subset \ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$ is a compact set, then K is essentially inside one of the M_n's. The following elementary lemma states this fact quantitatively (we omit its proof as it is an easy exercise). The proof of the case $m > n$ follows easily from it.

Lemma 6.1. *Let $K \subset \ell_{p_1} \hat{\otimes} \cdots \hat{\otimes} \ell_{p_N}$ be a compact set and $\epsilon_k > 0$ be given. Then we can find a sequence $n_k \in \mathbf{N}$ such that* $\sup_{x \in K} \|(I - P_{n_k})x\| < \epsilon_k$.

We start the inductive construction of σ_1. Set $A_1 = \mathbf{N}$ and $\sigma_1(1) = \min A_1$. Let $K = \Phi$ Ball $\partial M_{\sigma_1(1)}$ and $\epsilon_k = \epsilon_{1,k}$. Then find $A_2 \subset A_1 \setminus \{\sigma_1(1)\}$ according to Lemma 6.1; and set $\sigma_1(2) = \min A_2$.

Let $K = \Phi$ Ball $\partial M_{\sigma_1(2)}$, $\epsilon_k = \epsilon_{2,k}$ and find $A_3 \subset A_2 \setminus \{\sigma_1(2)\}$ according to Lemma 6.1. Then set $\sigma_1(3) = \min A_3$.

Continuing in this fashion we get σ_1 and construct $\sigma = (\sigma_1, \cdots, \sigma_1)$. It is easy to see that if $x \in \partial M_n$ and $m > n$, then

$$\|Q_m K_\sigma \Phi J_\sigma x\| \leq \epsilon_{n,m} \|x\|.$$

We will now prove the case $m < n$.

The construction of σ_1 is similar to the previous case. We need the following elementary lemma (which as before do not prove).

Lemma 6.2. *Let $1 < p < \infty$, F a finite dimensional space, and $T : \ell_p \to F$ a bounded linear map. Then for every $\epsilon > 0$, the set $\{i : \|Te_i\| > \epsilon\}$ is finite.*

We will only present the induction step for the construction of σ_1. Assume that $\Lambda \subset \mathbf{N}$ is an infinite set with first n elements $\sigma_1(1), \cdots, \sigma_1(n)$. We want to find an infinite $\Lambda' \subset \Lambda$ with the same first n elements as Λ such that whenever $\alpha \in (\Lambda')^N$ is such that $e_\alpha \notin M_{\sigma_1(n)}$, then $\|P_{\sigma_1(n)}\Phi e_\alpha\| < \epsilon$. Then we will choose $\sigma_1(n+1) = \min \Lambda' \setminus \{\sigma_1(1), \cdots, \sigma_1(n)\}$.

The construction of Λ' uses Ramsey's Theorem as in Section 5. We look at all the different ways that $e_\alpha \notin M_{\sigma_1(n)}$. We will illustrate this for two different cases. The others are very similar.

Case 1. $\sigma_1(n) < \alpha_1 < \alpha_2 < \cdots < \alpha_N$.

Color the N-sets of $\{i \in \Lambda : i > \sigma_1(n)\}$ as follows: $\{\alpha_1, \cdots, \alpha_N\}$ is *good* if $\alpha_1 < \cdots < \alpha_N$ and $\|P_{\sigma_1(n)}\Phi e_\alpha\| < \epsilon$ and *bad* otherwise.

Ramsey's Theorem gives us a monochromatic infinite set $\Lambda_1 \subset \Lambda$. It is easy to see that Lemma 6.2 implies that the set has to be *good.* (Let $\beta_1 < \cdots < \beta_{N-1}$ be the $N-1$ smallest elements of Λ_1 and define $T : \ell_{p_N} \to M_{\sigma_1(n)}$ as follows: if $i > \beta_{N-1}$, then $Te_i = P_{\sigma_1(n)}\Phi e_{(\beta_1,\cdots,\beta_{N-1},i)}$ and if $i \leq \beta_{N-1}$, then $Te_i = 0$. If Λ_1 were *bad* this would contradict Lemma 6.2.)

Case 2. $\alpha_1, \alpha_2 \leq \sigma_1(n) < \alpha_3 < \cdots < \alpha_N$.

Color the $(N-2)$-sets of $\{i \in \Lambda : i > \sigma_1(n)\}$ as follows: $\{\alpha_3, \cdots, \alpha_N\}$ is *good* if $\alpha_3 < \cdots < \alpha_N$ and $\|P_{\sigma_1(n)}\Phi e_\alpha\| < \epsilon$ for every $\alpha_1, \alpha_2 \leq \sigma_1(n)$, (notice that $\alpha = (\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_N)$) and *bad* otherwise.

Once again Ramsey's Theorem gives an infinite monochromatic subset of Λ. And as before it has no choice but to be *good.* This follows because there are only finitely many $\alpha_1, \alpha_2 \leq \sigma_1(n)$.

There are finitely many ways in which $e_\alpha \notin M_{\sigma_1(n)}$. They are very similar to the two cases just considered, and repeating the above argument for all of them we get $\tilde{\Lambda} \subset \Lambda$ that is *good* in all the cases. Then let $\Lambda' = \tilde{\Lambda} \bigcup \{\sigma_1(1), \cdots, \sigma_1(n)\}$. We choose $\epsilon > 0$ small enough so that whenever $x \in \partial M_{n+1}$, then

$$\|P_{\sigma_1(n)} K_\sigma \Phi J_\sigma x\| \leq \min_{k \leq n}\{\epsilon_{n+1,k}\}\|x\|.$$

$\square$

Proof of Basic Lemma 2. Assume that we have a sequence of complex numbers $\{\lambda_{\alpha,\beta,j} : \alpha, \beta \in \mathbf{N}^{N-1}, |\alpha| \vee |\beta| < j\}$ and a sequence of positive numbers,

$\{\epsilon_{\alpha,\beta,j} : j > |\alpha| \vee |\beta|\}$.

For α, β fixed, find a subsequence $\{j_k\}$ of $\{j : j > |\alpha| \vee |\beta|\}$ and some $\lambda_{\alpha,\beta} \in \mathbf{C}$ satisfying:

$$\lim_{j_k \to \infty} \lambda_{\alpha,\beta,j_k} = \lambda_{\alpha,\beta}$$

$$(**) \qquad |\lambda_{\alpha,\beta,j_k} - \lambda_{\alpha,\beta}| < \epsilon_{\alpha,\beta,k}.$$

Moreover, if we have finitely many $\{\alpha_l, \beta_l\}_{l \leq m}$, we can find a subsequence $\{j_k\}$ such that $(**)$ is true for every $l \leq m$.

The condition $j > |\alpha| \vee |\beta|$ is the key to extend the argument to all $\alpha, \beta \in \mathbf{N}^{N-1}$. The basic idea is that once we have fixed $\sigma_1(1), \cdots, \sigma_1(n)$, we take the subsequence j_k from $\{j : j > \sigma_1(n)\}$; hence, we do not affect the initial segment.

We will only present the induction step for σ_1. Assume that $\Lambda \subset \mathbf{N}$ is an infinite set with first elements $\sigma_1(1), \sigma_1(2), \cdots, \sigma_1(n)$. We want to find an infinite $\Lambda' \subset \Lambda$ with the first n elements as in Λ, and such that $(**)$ is satisfied for every $\alpha, \beta \leq \sigma_1(n)$. We can do that because there are only finitely many of them. We take the subsequence j_k from $\{j \in \Lambda : j > \sigma_1(n)\}$ and let $\Lambda' = \{j_k : k \in \mathbf{N}\} \bigcup \{\sigma_1(1), \cdots, \sigma_1(n)\}$. Then set $\sigma_1(n+1) = j_1$, the minimum of the j_k's (remember that $j_1 > \sigma_1(n)$).

Repeating the process we finish the proof. $\square$

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Received August 10, 1993. This research was partially supported by NSF DMS-8921369. Portions of this paper form a part of the second author's Ph.D. dissertation under the supervision of W.B. Johnson.

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