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GENERALIZED MODULAR SYMBOLS AND RELATIVE LIE ALGEBRA COHOMOLOGY

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In this paper we explore the limitations forced on the infinity type of a cohomological automorphic representation given the non-vanishing of an associated period over a generalized modular symbol. After some general remarks, we discuss the example of $GL(2n)$ over a totally real field.

Let G be a reductive group defined over the number field F and $\pi \approx \bigotimes_v \pi_v$ a cuspidal irreducible automorphic representation of $G(\mathbb{A})$, where v runs over all the places of F and $\mathbb{A}$ denotes the adeles of F. Write ω for the central character of π. Let $G_\infty = \Pi G_v$ where v runs over the archimedean places of F and choose K_∞ to be a compact subgroup of G_∞ which contains the connected component of the identity of a maximal compact subgroup of G_∞. Denote by X the symmetric space $G_\infty/K_\infty Z^0_\infty$ where Z is the center of G. We assume X is non-compact.

Set $G_f = \Pi G_v$ where v runs over the non-archimedean places of F and choose a compact open subgroup L of G_f. We let Γ be the arithmetic subgroup of $G(F)$ defined to be the projection of $G(F) \cap G_\infty L$ into G_∞. We assume $\Gamma\backslash X$ is orientable. Let $\mathfrak{g} = \text{Lie } G^0_\infty/Z^0_\infty$ and $\bar{K}_\infty =$ image of K_∞ in G^0_∞/Z^0_∞.

We recall the well-known isomorphism of cohomology groups $H^*_{\text{cusp}}(\Gamma\backslash X, \mathbb{C}) \approx \bigotimes_\omega H^*(\mathfrak{g}, \bar{K}_\infty; L^2_{\text{cusp}}(G(F)\backslash G(\mathbb{A}), \omega))^L$. The latter contains $H^*(\mathfrak{g}, \bar{K}_\infty; \pi_\infty) \otimes \pi_f^L$ as a summand (identifying π with its image in $L^2_{\text{cusp}}(G(F)\backslash G(\mathbb{A}), \omega)$ but taking care to remember that the isomorphism $\pi \approx \bigotimes_v \pi_v$ is an abstract one and doesn't "take place" inside L^2_{cusp}). We let d be a non-negative integer and choose $[\psi] \in H^d_{\text{cusp}}(\Gamma\backslash X, \mathbb{C})$ where ψ is a closed differential d-form on $\Gamma\backslash X$ representing the cohomology class $[\psi]$ – we may even take ψ harmonic. Under the isomorphism above, we suppose ψ goes over to $\alpha \otimes \beta$, with $\alpha \in H^d(\mathfrak{g}, \bar{K}_\infty; \pi_\infty)$ and $\beta \in \pi_f^L$. Recall that $H^d(\mathfrak{g}, \bar{K}_\infty; \pi_\infty) \approx \text{Hom}_{K_\infty}(\wedge^d \mathfrak{g}/\mathfrak{k}, \pi_\infty)$ and we view α as such a homomorphism. (Here $\mathfrak{k} = \text{Lie } \bar{K}_\infty$.)

Now let H denote a reductive F-subgroup of G. We assume H_∞ is connected and $H(\mathbb{A})$ satisfies strong approximation. Choose $e \in X$ fixed by $\bar{K}_\infty$ and set $X_H = H(F_\infty)e \subset X$. We assume $M = (H_\infty \cap \Gamma)\backslash X_H$ is orientable, and we fix an orientation. Then the two propositions of Section 1 of [**AGR**] imply that for some f in the space of π

$$\int_H \psi = \int_{[Z(\mathbb{A})\cap H(\mathbb{A})]H(F)\backslash H(\mathbb{A})} \omega^{-1}(h)f(h)dh.$$

There is a canonical procedure for finding f given ψ or vice versa. Following the argument in Section 5.2 of [**AG**], we take a basis $Y_1, \ldots Y_d$ of Lie $H^0_\infty/(K_\infty \cap H^0_\infty)Z^0_\infty$ and set $Y_M = Y = Y_1 \wedge \cdots \wedge Y_d$. Then up to a nonzero multiplicative constant we may take $f = \alpha(Y)\beta$. In particular, if the integral doesn't vanish then $\alpha(Y) \neq 0$, and of course $d = \dim X_H = \dim M$.

We call f a cohomological vector for π. We call such an integral a period (of the cuspform f or the cohomology class $[\psi]$) over the (generalized) modular symbol M. In our terminology, a modular symbol is an oriented locally finite cycle such as M arising as the projected orbit of a reductive group.

In [**AGR**] it is shown that these integrals are absolutely convergent. Combining the topological methods of [**RS**] with the deRham theorem, it is easy to construct modular symbols M that support non-vanishing periods. Here the reductive group H underlying M will be the fixed points in G of some finite group action.

The non-vanishing of periods seems to be connected with properties of π and its L-functions, e.g. whether π is a lift from some other group, or whether a certain L-function has a pole. This is being investigated by Jacquet, Rallis and others. See [**AG**] for an example, and the references cited there.

On the local level, a non-vanishing period implies the existence of a nontrivial H_∞- invariant functional on π_∞, which should be related to whether π_∞ is a lift.

In this paper we begin to study the question: Does the non-vanishing of a period put a constraint on the isomorphism type of π_∞? The case of $GL(4)$ was studied already in [**AG**] and there led to a proof of the non-vanishing of a p-adic L-function. This paper arose out of an attempt to extend those results to $GL(2n)$ for $n > 2$. We shall see that although many possibilities for π_∞ are ruled out by the nonvanishing of the period, already for $GL(6)$ and $GL(8)$ there are too many possibilities left to allow the use of the trick in Section 5 of [**AG**] for $n > 2$ to prove the non-vanishing of a certain archimedean integral and hence of the p-adic L-function.

In Section 1 we review the Vogan-Zuckerman classification of π_∞ with nontrivial $(\mathfrak{g}, \bar{K}_\infty)$-cohomology. In Section 2 we show how the nonvanishing period enters the picture and prove some propositions that can be used in practice to rule out certain π_∞'s. In Section 3 we outline the example of $GL(8)$ with remarks applying to $GL(m)$ for various m, notably $m = 2, 4, 6$. In the appendix we give a heuristic connection between the existence of a nontrivial $K_\infty^0 \cap H_\infty$-fixed vector in the cohomological K-type of π_∞ and a nontrivial H_∞-invariant continuous linear functional on π_∞ in the case where $G = GL(2n)$ and $H = GL(n) \times GL(n)$.

We close this introduction by pointing out a comparison among the results in [**A**], [**AGR**], and this paper. In [**A**] the existence of a non-vanishing period for π puts constraints on the local component π_v of π at a non-archimedean place, for local reasons. In this paper, we have similarly locally effected results at archimedean places. In [**AGR**], vanishing of certain periods was derived from global considerations.

1. Classification of representations with nontrivial $(\mathfrak{g}, K)$ - cohomology.

For simplicity we assume in this section G is a semi-simple, real, connected Lie group with finite center. Let $\mathfrak{g} = \mathrm{Lie}\,(G) \otimes \mathbb{C}$ and $K \subset G$ a maximal compact subgroup. The modifications needed when G is reductive or non-connected are most easily performed on an ad hoc basis. In [**VZ**] a finite list of irreducible admissible $(\mathfrak{g}, K)$ - modules $\{\pi\}$ is given such that $H^*(\mathfrak{g}, K; \pi) \neq 0$ and it is shown that every irreducible unitary G-representation with nontrivial $(\mathfrak{g}, K)$ - cohomology has its Harish-Chandra module isomorphic to some π on the list. Later in [**V**] and [**W**] it was shown that each π on the list is the Harish-Chandra module of a unitary G-representation. Hence the unitary nature of a π_∞ arising from a cohomological cuspform places no restrictions on its isomorphism type. In [**VZ**] twisting π by a finite dimensional representation is also allowed, but we are interested only in untwisted coefficients here. We summarize the properties of the classification that we will use. See [**VZ**] for complete details.

Let $\mathfrak{k} = \mathrm{Lie}(K) \otimes \mathbb{C}$, θ be the corresponding Cartan involution, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition. A finite set $\{\mathfrak{q}\}$ of θ-stable parabolic subalgebras of $\mathfrak{g}$ is defined. Write $\mathfrak{q} = \ell + \mathfrak{u}$, where ℓ is a Levi-factor and $\mathfrak{u}$ the radical of $\mathfrak{q}$. One chooses a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ which is contained in ℓ and let $\mu = \mu(\mathfrak{q})$ = irreducible representation of K with highest weight $2\rho(\mathfrak{u} \cap \mathfrak{p})$. Here $\rho(\mathfrak{u} \cap \mathfrak{p})$ is one-half the sum of the $\mathfrak{t}$ weights on $\mathfrak{u} \cap \mathfrak{p}$.

We shall call the isomorphism class of μ a cohomological K-type. It appears in $\wedge^* \mathfrak{p}$. There is a unique irreducible admissible $(\mathfrak{g}, K)$-module $A_\mathfrak{q}$ such that $H^*(\mathfrak{g}, K; A_\mathfrak{q}) = \mathrm{Hom}_K(\wedge^* \mathfrak{p}, A_\mathfrak{q}) \neq 0$ and the only K-type shared by $\wedge^* \mathfrak{p}$ and $A_\mathfrak{q}$ is $\mu(\mathfrak{q})$. Moreover for different $\mathfrak{q}$'s the $\mu(\mathfrak{q})$'s (and hence the $A_\mathfrak{q}$'s) are distinct. Every irreducible admissible $(\mathfrak{g}, K)$-module π with $H^*(\mathfrak{g}, K; \pi) \neq 0$ is isomorphic to one of the $A_\mathfrak{q}$'s.

2. Enter the nonvanishing period.

We maintain all the preceeding notation.

Now suppose π_∞ is isomorphic to $A_\mathfrak{q}$ for some $\mathfrak{q}$ and that the period of a cohomological vector for π over $H(\mathbb{A})$ doesn't vanish. In this case we shall say that π has a nontrivial H-period. Let d be the dimension of the corresponding modular symbol M, with $Y_M \in \wedge^d\mathfrak{p}$.

Proposition 2.1. *Suppose π has a nontrivial H-period, and $\pi_\infty \approx A_\mathfrak{q}$. Then*

(1) *$\mu(\mathfrak{q})$ appears in $\wedge^d\mathfrak{p}$;*

(2) *$\mu(\mathfrak{q})$ contains a nontrivial vector invariant under $H_\infty \cap K$;*

(3) *The K-submodule of $\wedge^d\mathfrak{p}$ generated by Y_M projected onto the $\mu(\mathfrak{q})$-isotypic component of $\wedge^d\mathfrak{p}$ is non-vanishing.*

Remark. Although (1) and (2) immediately follow from (3) since Y_M is clearly $H_\infty \cap K$-invariant, we stated the three items in order of ease of checking in any given example.

Proof. As stated we need only prove (3). From the hypothesis, there exists $\alpha \in \mathrm{Hom}_K(\wedge^d\mathfrak{p}, A_\mathfrak{q})$ such that $\alpha(Y_M) \neq 0$. Since $\mu(\mathfrak{p})$ is the only K-type shared by $\wedge^d\mathfrak{p}$ and $A_\mathfrak{q}$, (3) follows. □

Proposition 2.2. *Under the hypotheses of the previous proposition, suppose in addition there exists a connected noncompact semi-simple Lie group G_1 with Iwasawa decomposition $G_1 = K_1A_1N_1$ such that* (Lie G_1) $\otimes \mathbb{C}$ *is isomorphic to* (Lie K) $\otimes \mathbb{C}$ *by an isomorphism that takes* (Lie K_1) $\otimes \mathbb{C}$ *onto* Lie$(H_\infty \cap K_1) \otimes \mathbb{C}$. *Extend* Lie A_1 *to a maximal abelian subalgebra $\mathfrak{t}_0$ of $\mathfrak{g}$, so that $\mathfrak{t}_0 = \mathfrak{t}_0 \cap \mathrm{Lie}\ K_1 \oplus \mathrm{Lie}\ A_1$ is a Cartan subalgebra of* Lie G_1. *Let λ be the highest weight of $\mu(\mathfrak{q})$ with respect to $\mathfrak{t}_0$. Then*

(1) $\lambda\left(\sqrt{-1}(\mathfrak{t}_0 \cap \mathrm{Lie}\ K_1)\right) = 0$;

(2) $\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}^+$ *for all* $\alpha \in \sum^+$

where Σ^+ is the set of positive restricted roots on Lie A_1 *with respect to the ordering induced by the choice of N_1.*

Proof. This follows from Proposition 2.1 (2) and Helgason's criterion Theorem 4.1 p. 535 of **[H]** after complexifying the Lie algebras and taking the hypotheses into account. □

In the following, with a view to our examples in the next section, we go back to the notation of Section 1 and allow G to be reductive and not necessarily connected. Thus $\mathfrak{g} = \mathrm{Lie}\ G^0_\infty/Z^0_\infty, \mathfrak{k} = \mathrm{Lie}\ \bar{K}_\infty$, etc.

The group of components $\bar{K}_\infty/\bar{K}^0_\infty$ acts on the set of cohomological K-types $\{\mu(\mathfrak{q})\}$ in the obvious way. If O is an orbit, there is an obvious way to make $\bigoplus_{\mu(\mathfrak{q})\in O} A_\mathfrak{q}$ into an irreducible $(\mathfrak{q}, \bar{K}_\infty)$-module. We will denote it by $B_\mathfrak{q}$ for any $\mathfrak{q}$ such that $\mu(\mathfrak{q}) \in O$. Every irreducible $(\mathfrak{q}, \bar{K}_\infty)$-module with nontrivial cohomology is isomorphic to $B_\mathfrak{q}$ for some $\mathfrak{q}$.

Now let $\tilde{K}$ denote the algebraic $\mathbb{R}$-group such that $\tilde{K}(\mathbb{R}) = \bar{K}_\infty$, so Lie $\tilde{K}(\mathbb{C}) = \mathfrak{k}$. Given $\mathfrak{q} = \ell + \mathfrak{u}$, we have the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ contained in ℓ and we choose a Borel subalgebra $\mathfrak{b} = \mathfrak{t} + \mathfrak{n}$ of $\mathfrak{k}$ such that $\mathfrak{u} \subset \mathfrak{n}$. We use capital Roman letters to denote subgroups of $\tilde{K}(\mathbb{C})$ whose Lie-algebra equals the corresponding small Gothic letter. Thus Q is a parabolic subgroup of $K = \tilde{K}(\mathbb{C})$ with Levi decomposition $Q = LU$. Also, B is a Borel subgroup of K with Levi decomposition $B = TN$. We let H stand for $H(\mathbb{C})$.

We now make the following additional hypothesis. For an illustration of it, see Section 3.

Hypothesis 2.3. *There exists a parabolic subgroup P_0 of K with Levi decomposition $P_0 = L_0U_0$ such that*

(i) $P_0 \supset B$ *and hence* $U_0 \subset N$*;*

(ii) $T \subset L_0$*;*

(iii) U_0 *contains a subgroup* W_0 *such that* Lie $N =$ Lie $N \cap H \oplus W_0$*;*

(iv) $L_0 \subset H$ *and* L_0 *stabilizes* W_0 *under conjugation.*

Now choose an order on $\mathfrak{t}^*$ so that B corresponds to the positive roots Φ_+ and for each $\alpha \in \Phi_+$ fix $u_\alpha : \mathbb{C} \tilde{\to} U_\alpha \subset N$. Order the positive roots $\alpha_1, \dots \alpha_r$ and write $u_i = u_{\alpha_i}$. We assume the ordering chosen so that $u_1, \dots u_m$ generate W_0 and $u_{m+1}, \dots u_r$ generate $N \cap H$. Let $x_1, \dots x_r$ be indeterminates and view them as coordinates on N by $x = (x_1, \dots x_r) = u_1(x_1) \dots u_r(x_r) = u(x_1, \dots x_r) = u(x)$. Let $x' = (x_1, \dots x_m) = u_1(x_1) \dots u_m(x_m) = u(x')$. We have an induced action of L_0 on $P \in \mathbb{C}[x'] = \mathbb{C}[x_1, \dots x_m] = \mathbb{C}[W_0]$ by $(g \cdot P)(x') = P(g^{-1} \cdot x') = P(g^{-1}u(x')g)$.

Fix an irreducible K-submodule V of $\wedge^d \mathfrak{g}/\mathfrak{k}$ with highest weight δ (all weights with respect to $\mathfrak{t}$) and let proj denote the K-equivariant projection onto V. Let Y be a generator of the line $\wedge^d$ Lie $H/$ Lie $H \cap \mathfrak{k}$ in $\wedge^d \mathfrak{g}/\mathfrak{k}$. It has weight zero. For any T-module M and weight λ write M_λ for the λ-isotypic component of M. For each weight μ in V choose a $\mathbb{C}$-basis $\{v_{\mu,i} : i = 1, \dots j_\mu\}$ of V_μ. Since V_δ is one-dimensional we write v_δ in place of $v_{\delta,1}$.

Lemma 2.4. *Define* $\{P_{\mu,i}(x) \in \mathbb{C}[x]\}$ *by* proj $u(x) \cdot Y = \Sigma_i P_{\mu,i}(x) v_{\mu,i}$.

(i) $P_{\mu,i}(x)$ *is independent of* $x_{m+1}, \dots x_r$ *for all* μ, i.

(ii) P_δ *is a maximal vector for* $L_0 \cap B^{\mathrm{opp}}$ *of weight* $-\delta$ *and generates an*

L_0^0-module contragredient to a quotient of $\mathrm{Res}^K_{L_0^0} V$.

(iii) *V is contained in the K-span of Y if and only if $P_\delta \neq 0$.*

Proof. Since for $i > m \quad u_i(x_i) \in H \cap N$ and H^0 fixes Y, statement (i) is true. To prove (ii) reindex the $\{v_{\mu,i}\}$ as $\{v_k\}$. Let $g \in L_0^0 \subset H$, so $gY = Y$. Then

$$\text{proj } u(g \cdot x')Y = \text{proj } gu(x')g^{-1}gY = \Sigma P_k(x')gv_k.$$

On the other hand

$$\text{proj } u(g \cdot x')Y = \Sigma P_k(g \cdot x')v_k = \Sigma g^{-1}P_k(x')v_k.$$

Comparing the right hand sides, we see that the matrix representation of g on the span of $\{P_k\}$ is a quotient of the contragredient V^* of V. Thus P_δ generates an L_0^0- module isomorphic to a quotient of $\mathrm{Res}^K_{L_0^0} V^*$.

View δ as a character on T and extend it to B by making it trivial on N. If $g \in B^{\text{opp}}$, since v_δ is a maximal vector in V, we have gv_k has no v_δ-component unless $v_k = v_\delta$ and then $gv_\delta = \delta(g)v_\delta$. Comparing the right hand sides again we get

$$\delta(g)P_\delta = g^{-1}P_\delta.$$

Statement (iii) follows from Lemma 5.5.1 of [**AG**] except we use V in place of the whole isotypic component of type δ. □

Lemma 2.5. *If δ is the cohomological K-type of $B_{\mathfrak{q}}$ and $Q = LU$ is the Levi decomposition, then as $L \cap K$-module, $V = V_\delta \oplus \left(\sum_{\mu\neq\delta} V_\mu\right)$.*

Proof. From the proof of Proposition 3.6 of [**VZ**] we see that $L \cap K$ fixes the line V_δ. Therefore, if α is any root of $L \cap K$, the α-string of weights of V in $\delta + \mathbb{Z}\alpha$ is just $\{\delta\}$. (Incidentally this proves that $< \delta, \alpha >= 0$ in the notation of Section 21.3 of [**Hu**].) So if μ is a weight of $V, \mu \neq \delta$, the α-string through μ can't reach to δ. Hence $\sum_{\mu\neq\delta} V_\mu$ is also $L \cap K$-invariant. □

Lemma 2.6. *Suppose $s \in L \cap L_0$ such that $s \cdot P_\delta = aP_\delta$, $s \cdot Y = bY$, $s \cdot v_\delta = cv_\delta$ with $a, b, c \in \mathbb{C}$. Then if $abc \neq 1$, V is not contained in the K-span of Y in $\wedge^d \mathfrak{g}/\mathfrak{k}$.*

Proof. As in the proof of (ii) of Lemma 2.4 we obtain $b\Sigma P_k s v_k = \Sigma s^{-1} \cdot P_k v_k$. From Lemma 2.5 we can equate the terms involving v_δ to get $bP_\delta s v_\delta = s^{-1}P_\delta v_\delta$ or $abcP_\delta = P_\delta$. If $abc \neq 1, P_\delta = 0$ and the conclusion follows from (iii) of Lemma 2.4. □

3. Examples: $GL(2n)$.

In this section we apply the foregoing to the example whose interest stems from [**AG**]. We refer the reader to the introduction of that paper for motivation.

We let $G = GL(2n)/\mathbb{Q}$ for $n \geq 1$. Choose $K_\infty = O(2n, \mathbb{R})$ and $H = GL(n) \times GL(n)$. Although H doesn't satisfy all the hypotheses made in Section 1, in this particular example all the conclusions there and in Section 2 remain true, as comparison with Section 5 of [**AG**] will show.

We found in [**AG**] that for $n = 2$, the nonvanishing of the H-period determined π_∞ uniquely up to isomorphism. The same is easily seen to be the case for $n = 1$. Here we will investigate $n = 3$ and $n = 4$.

Of particular interest in the following calculations is the invariant theory that comes in.

We will present the $GL(8)$ case in detail and summarize our results for the $GL(6)$ case. The methods in both cases are basically the same, but since $GL(6)$ is smaller that $GL(8)$, less variety appears.

3.1. Case of $GL(N)$. First we present the list of irreducible $(\mathfrak{g}, K)$-modules π with non-trivial cohomology. We thank J.S. Li for providing us with this, which may be derived either from Speh's original article [**S**] or from the general theory of Vogan and Zuckermann [**VZ**].

In this subsection, let $G = GL(N, \mathbb{R}), K = O(N), \mathfrak{g} = \text{Lie } G, \mathfrak{k} = \text{Lie } K$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition. Let $\epsilon_j, 1 \leq j \leq \lfloor N/2 \rfloor$ be the usual basis for the dual of a Cartan subalgebra of $\mathfrak{k}$.

Let $r_1, \ldots r_k$ be positive integers with $m = r_1 + \cdots + r_k \leq N/2$. We allow the case $k = 0$. There corresponds a θ-stable parabolic subalgebra $\mathfrak{q} = \ell + \mathfrak{u}$ whose corresponding Levi subgroup is

$$L = GL(r_1, \mathbb{C}) \times \cdots \times GL(r_k, \mathbb{C}) \times GL(N - 2m, \mathbb{R}).$$

In the notation of Section 2 the $(\mathfrak{g}, K)$- module $B_\mathfrak{q}$ is irreducible, unitarizable and $H^*(\mathfrak{g}, K; B_\mathfrak{q}) \neq 0$. Any such π is isomorphic to $B_\mathfrak{q}$ for some $\mathfrak{q} = \mathfrak{q}(r_1, \ldots r_k)$ arising this way.

Set $m_s = r_1 + \cdots + r_s$, $1 \leq s \leq k$. Then the cohomological K-type of $B_\mathfrak{q}$ has highest weight

$$2_\rho(\mathfrak{u} \cap \mathfrak{p}) = \sum_{1 \leq s \leq k} \quad \sum_{m_{s-1} \leq i \leq m_s} (N + 1 - m_{s-1} - m_s)\epsilon_i.$$

This is the unique K-type of $A(\mathfrak{q})$ that occurs in $\wedge^*(\mathfrak{g}/\mathfrak{k})$.

Let P be the standard parabolic subgroup of G with Levi component

$$M = GL(2r_1, \mathbb{R}) \times \cdots \times GL(2r_k, \mathbb{R}) \times GL(N - 2m, \mathbb{R}).$$

Let π_s be the Speh representation of $GL(2r_s, \mathbb{R})$ which is the Langlands quotient of

$$\mathrm{Ind}\left(\sigma_s|det|^{\frac{r_s-1}{2}} \otimes \cdots \otimes \sigma_s|det|^{\frac{-r_s+1}{2}}\right)$$

where σ_s is the discrete series representation of $GL(2, \mathbb{R})$ given by $\sigma_s = \pi(\mu_s, -\mu_s)$ with $\mu_s = \frac{1}{2}(N - m_{s-1} - m_s)$. We also let 1 denote the trivial representation of $GL(N - 2m, \mathbb{R})$. Then $B_{\mathfrak{q}} \approx \mathrm{Ind}_P^G(\pi_1 \otimes \cdots \otimes \pi_k \otimes 1)$.

For $N = 6$ and 8 we record this information in tabular form. The case $N = 4$ is already treated in [**AG**]. We give each representation an identifying number for later reference.

Table for $GL(6)$

#	k	$r_1, \ldots r_k$	$m_1, \ldots, m_k$	$\delta(\epsilon$- basis)	$\delta(f$- basis)
1	0	–	–	0, 0, 0	0, 0, 0
2	1	1	1	6, 0, 0	0, 6, 0
3		2	2	5, 5, 0	5, 0, 5
4		3	3	4, 4, 4	8, 0, 0
5	2	1, 1	1, 2	6, 4, 0	4, 2, 4
6		1, 2	1, 3	6, 3, 3	6, 3, 0
7		2, 1	2, 3	5, 5, 2	7, 0, 3
8	3	1, 1, 1	1, 2, 3	6, 4, 2	6, 2, 2

In these tables, δ refers to the cohomological K-type. The ϵ-basis was defined above; $(a, \ldots, b, c)$ stands for $a\epsilon_1 \cdots + b\epsilon_{n-1} + c\epsilon_n, N = 2n$. The f-basis refers to the parametrization of K-types in terms of fundamental weights; draw the Dynkin diagram so that the all but two of the nodes lie along a horizontal line, and the outer automorphism switches the two nodes on the far right; then $(a, \ldots, b, c)$ in that basis stands for a times the leftmost weight plus ... plus b times the upper rightmost weight plus c times the lower rightmost weight.

The last entry in each table is the unique representation on the list which could occur as the infinity type of a global cuspidal representation on $GL(N)/\mathbb{Q}$.

If π is isomorphic to $B_{\mathfrak{q}}$ for the $\mathfrak{q}$ from the i-th line on the list, write $\pi = \pi_i = \pi_\delta$ where δ is the corresponding cohomological K-type.

3.2. Case of $GL(8)$. Resumé of notations: $G_\infty = GL(8, \mathbb{R}), K_\infty = O(8)$, $H_\infty = GL(4, \mathbb{R}) \times GL(4, \mathbb{R}), \mathfrak{g}_\infty = \mathfrak{k}_\infty \oplus \mathfrak{p}_\infty$ where $\mathfrak{p}_\infty$ can be viewed as 8×8 symmetric matrices. Let ${}^0\mathfrak{p}_\infty$ denote the traceless matrices in $\mathfrak{p}_\infty$. Identify $\mathfrak{g}_\infty/\mathfrak{k}_\infty$ with $\mathfrak{p}_\infty$ and let $Y \in \wedge^{19}\mathfrak{p}_\infty$ be the wedge of a fixed basis of Lie $H_\infty \cap {}^0\mathfrak{p}_\infty$.

Table for $GL(8)$

#	k	$r_1, \dots r_k$	$m_1, \dots, m_k$	$\delta(\epsilon$- basis)	$\delta(f$- basis)
1	0	–	–	0 0 0 0	0 0 0 0
2	1	1	1	8 0 0 0	8 0 0 0
3		2	2	7 7 0 0	0 7 0 0
4		3	3	6 6 6 0	0 0 6 6
5		4	4	5 5 5 0	0 0 0 10
6	2	1, 1	1, 2	8 6 0 0	2 6 0 0
7		1, 2	1, 3	8 5 5 0	3 0 5 5
8		1, 3	1, 4	8 4 4 4	4 0 0 8
9		2, 1	2, 3	7 7 4 0	0 3 4 4
10		2, 2	2, 4	7 7 3 3	0 4 0 6
11		3, 1	3, 4	6 6 6 2	0 0 4 8
12	3	1, 1, 1	1, 2, 3	8 6 4 0	2 2 4 4
13		1, 1, 2	1, 2, 4	8 6 3 3	2 3 0 6
14		1, 2, 1	1, 3, 4	8 5 5 2	3 0 3 7
15		2, 1, 1	2, 3, 4	7 7 4 2	0 3 2 6
16	4	1, 1, 1, 1	1, 2, 3, 4	8 6 4 2	2 2 2 6

Theorem.

(i) *If* $\alpha \in \mathrm{Hom}_{K_\infty}(\wedge^{19}\mathfrak{p}_\infty, \pi)$ *and* $\alpha(Y) \neq 0$ *then* π *is type* 8, 11 *or* 16.

(ii) *Conversely, if* π *is one of those three types, there exists* α *such that* $\alpha(Y) \neq 0$.

Proof. We do part (i) by eliminating possibilities.

Because Y is invariant under $SO(4) \times SO(4)$ we can apply Proposition 2.2: π_δ contains a nontrivial $K_\infty^0 \cap H_\infty$-fixed vector if and only if

$$\frac{(\delta|\beta)}{(\beta|\beta)} \in \mathbb{Z} \quad \text{for all roots } \beta \text{ of } \mathfrak{k}_\infty.$$

In the ϵ-basis we have $(\epsilon_i|\epsilon_j) = \delta_{ij}$. Since $(\beta|\beta) = 2$ for all β, the criterion becomes $(\delta|\beta) \in 2\mathbb{Z}$. Write $\delta = \Sigma c_i \epsilon_i$. Each β has the form $\epsilon_i \pm \epsilon_j$ for $i \neq j$. Thus $(\delta|\beta) \in 2\mathbb{Z} \iff$ all c_i's have same parity $\iff$ either all r_s's have same parity or $m_k < n$ and all r_s's are odd. This eliminates types 3, 7, 9, 13, 14, 15.

The other cases require a more detailed analysis. It will be convenient to complexify and work with a split version of K_∞. We let $K = O(2n, \mathbb{C})$, $\mathfrak{p} = \mathfrak{p}_\infty \otimes \mathbb{C}$, ${}^0\mathfrak{p} = {}^0\mathfrak{p}_\infty \otimes \mathbb{C}$. However we have to keep track of H_∞ when we do this. Let θ be the standard Cartan involution $g \to {}^tg^{-1}$ and σ be the involution $\begin{pmatrix} I_n & \\ & -I_n \end{pmatrix}$ so that H is the fixed-points of σ. Then we conjugate θ

and σ by the same complex $2n \times 2n$ matrix to get a split form of K and the new H. Let

$$J_m = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} \in GL(m),$$

$$J = \begin{pmatrix} 0 & J_n \\ J_n & 0 \end{pmatrix} \in GL(2n),$$

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} J_n & I_n \\ -iI_n & iJ_n \end{pmatrix}.$$

Then $AJ^tA = I$ and $A^{-1}\sigma A = AdJ$, so conjugation by A takes $O(2n)$ to $O(J)$ and σ to AdJ. We can then conjugate further by $g \in O(J)$ such that $gJg^{-1} = \begin{pmatrix} I_{n/2} & & \\ & -I_n & \\ & & I_{n/2} \end{pmatrix} = \xi$ assuming n is even. (The odd n case is a little more complicated – see the section on $GL(6)$.)

From now on, assume n even and set $K = O(J)(\mathbb{C}), H = Ad(\xi)$-fixed points in $GL(2n, \mathbb{C})$:

$$H = \left\{ \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix} \begin{matrix} n/2 \\ n \\ n/2 \end{matrix} \right\}.$$

If $X \in M_{2n}$, let X_T denote the transpose of X about the non-main diagonal. Then

$$\begin{aligned} K &= \{g \in GL(2n, \mathbb{C}) | g_T^{-1} = g\}, \\ \mathfrak{p} &= \{X \in M_{2n}(\mathbb{C}) | X_T = X\}, \\ Y &= \text{ generator of } \wedge^{\text{top}} ({}^0\mathfrak{p} \cap \text{Lie } H). \end{aligned}$$

Now define some groups that will satisfy Hypothesis 2.3. First let

$$\begin{aligned} B &= \{\text{upper triangular matrices in } K\}, \\ N &= \{\text{unipotent matrices in } B\}, \\ T &= \{\text{diagonal matrices in } B\}. \end{aligned}$$

Then set

$$P_0 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \begin{matrix} n/2 \\ n \\ n/2 \end{matrix} \right\} \cap K,$$

$$P_0 = L_0 U_0 \text{ with } U_0 = R_u(P_0) \text{ and}$$

$$L_0 = \left\{ \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \right\} \cap K.$$

Finally set

$$W_0 = \exp\left(\left\{ \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\} \cap K \right).$$

We make the choices prescribed after Hypothesis 2.3 so that we are in a position to apply the rest of Section 2.

Now set $n = 4$. Fix a cohomological K-type $\delta = (\delta_1, \delta_2, \delta_3, \delta_4)$ in the ϵ-coordinates. Put coordinates on T and W_0 as follows:

$$t = \begin{pmatrix} d_1 & & & & & & & \\ & d_2 & & & & & & \\ & & d_3 & & & & & \\ & & & d_4 & & & & \\ & & & & d_4^{-1} & & & \\ & & & & & d_3^{-1} & & \\ & & & & & & d_2^{-1} & \\ & & & & & & & d_1^{-1} \end{pmatrix} \in T$$

$$w(X,Y) = w = \begin{pmatrix} 1 & 0 & X_1 & X_2 & X_3 & X_4 & * & * \\ 0 & 1 & Y_1 & Y_2 & Y_3 & Y_4 & * & * \\ & & 1 & 0 & 0 & 0 & -Y_4 & -X_4 \\ & & & 1 & 0 & 0 & -Y_3 & -X_3 \\ & & & & 1 & 0 & -Y_2 & -X_2 \\ & & & & & 1 & -Y_1 & -X_1 \\ & & & & & & 1 & 0 \\ & & & & & & 0 & 1 \end{pmatrix} \in W_0$$

so $\epsilon_i(t) = d_i$. If

$$\ell = \begin{pmatrix} A & & \\ & B & \\ & & A_T^{-1} \end{pmatrix} \in L_0$$

then ℓ acts by conjugation on W_0 via $M \to AMB^{-1}$ where

$$M = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 \\ Y_1 & Y_2 & Y_3 & Y_4 \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}.$$

□

Lemma 3.4. *The space of polynomials $P(X,Y)$ fixed under the induced action by $L_0 \cap R_u(B^{\mathrm{opp}})$ is the $\mathbb{C}$-span of the 6 polynomials $P_1, \dots P_6$ in the table. Each of these is an eigenpolynomial for the action of T. The right hand column of the table gives the character χ_i such that $P_i(t \cdot w) = \chi_i(t) P_i(w)$.*

Table of semi-invariants for $L_0 \cap B^{\mathrm{opp}}$ in $\mathrm{Sym}^*(W_0)$:

$P_1 = X_4$	$\chi_1 = d_1 d_3$
$P_2 = X_2 Y_4 - Y_2 X_4$	$\chi_2 = d_1 d_2 d_3 d_4^{-1}$
$P_3 = X_3 Y_4 - Y_3 X_4$	$\chi_3 = d_1 d_2 d_3 d_4$
$P_4 = X_1 X_4 + X_2 X_3$	$\chi_4 = d_1^2$
$P_5 = Y_1 X_4^2 - X_1 X_4 Y_4 + X_2 Y_3 X_4 + Y_2 X_3 X_4 - 2 X_2 X_3 Y_4$	$\chi_5 = d_1^3 d_3$
$P_6 = \det(M J_4 {}^t M)$	$\chi_6 = d_1^2 d_2^2$

Proof. It is easily checked each P_i is semi-invariant with the designated character. To show these span the space of semi-invariants one can use a result from [**P-SR**]. The local unramified computation in that paper induces a decomposition of the symmetric algebra of $GL(2,\mathbb{C})^3 \approx GL(2,\mathbb{C}) \times GO(4,\mathbb{C})$. Using this decomposition one gets the desired assertion. □

Now consider types 2, 4, 6, 12. They all have $\delta_4 = 0$. Writing $P_\delta = \Pi P_i^{e_i}$, as we may by Lemma 2.4 (iii), we see that necessarily $e_2 = e_3$, since $\delta = \Pi \chi_i^{e_i}$. Set

$$s = \begin{pmatrix} I_3 & & \\ & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & \\ & & I_3 \end{pmatrix} \in K.$$

Then s induces the permutation (23) on the indices of X and Y. Since $s(P_i) = P_i$ for $i \neq 2, 3$ and $s(P_2) = P_3, s(P_3) = P_2$, we have $sP_\delta = P_\delta$ in the case where $\delta_4 = 0$.

Also, s acts on ${}^0\mathfrak{p}$ by conjugation and preserves H, hence Y. It's easy to see $sY = -Y$.

Now let V be an irreducible K-submodule of $\wedge^{19}\mathfrak{p}$ of type δ, with highest weight vector v_δ. Since $\delta_4 = 0$, s preserves δ and hence $sv_\delta = \pm v_\delta$.

Lemma 3.5. *If δ is type* 2, 4, 6, *or* 12 *then* $sv_\delta = v_\delta$.

Proof. By the proof of Theorem 3.3 p. 64 of [**VZ**], if $\mathfrak{q} = \ell + \mathfrak{u}$ corresponds to type δ, then $v_\delta = \alpha \wedge \beta$ for some $\beta \in \wedge^R(\mathfrak{u} \cap \mathfrak{p})$ and some $\alpha \in (\wedge^{19-R}\ell \cap \mathfrak{p})^{\ell \cap \mathfrak{k}}$, where $R = \dim \mathfrak{u} \cap \mathfrak{p}$. Now $(\wedge^* \ell \cap \mathfrak{p})^{\ell \cap \mathfrak{k}}$ is isomorphic to the space of L^0-invariant differential forms on the symmetric space for L^0, which is in turn isomorphic to the cohomology of the compact dual. The latter is explicitly computed in [**B**].

We need only consider V contained in the K-span of Y, hence contained in ${}^0\mathfrak{p}$. So we may assume $\alpha \in (\wedge^{d-R}\ell \cap {}^0\mathfrak{p})^{\ell \cap \mathfrak{k}}$. Of course $\beta \in \wedge^R(\mathfrak{u} \cap {}^0\mathfrak{p}) = \wedge^R(\mathfrak{u} \cap \mathfrak{p})$.

A case by case calculation based on [**B**] now shows that in the cases under consideration $s\alpha = (-1)^{m_k}\alpha$ and $s\beta = (-1)^{m_k}\beta$. Hence $sv_\delta = v_\delta$.

We omit the details, but sketch out one case as an example. Consider type 6. Then $k = 2, (r_1, r_2) = (1,1), m_k = 2, R = 12$. In this case, $L \approx \prod_{i=1}^{k} GL(r_i, \mathbb{C}) \times GL(8 - 2m_k, \mathbb{R}) \approx \mathbb{C}^\times \times \mathbb{C}^\times \times GL(4, \mathbb{R})$ and

$$L(\mathbb{C}) \approx \left\{ \begin{pmatrix} t_1 & & & & \\ & t_2 & & & \\ & & g & & \\ & & & t_3 & \\ & & & & t_4 \end{pmatrix} \middle| \begin{matrix} t_1 \dots t_4 \in \mathbb{C}^\times \\ g \in GL(4, \mathbb{R}) \end{matrix} \right\}$$

and s acts on L as conjugation by

$$\begin{pmatrix} I_3 & & \\ & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & \\ & & I_3 \end{pmatrix}.$$

The compact dual symmetric space for L is $\prod_{i=1}^{k} U(r_i) \times U(8 - 2m_k)/SO(8 - 2m_k)$. Since we consider only the traceless matrices in $\ell \cap {}^0\mathfrak{p}$ we have that $(\wedge^{d-R}\ell \cap {}^0\mathfrak{p})^{\ell \cap \mathfrak{k}}$ is isomorphic to the cohomology of

$$Y = \prod_{i=1}^{k} U(r_i) \times SU(8 - 2m_k)/SO(8 - 2m_k).$$

In our case $Y = U(1) \times U(1) \times SU(4)/SO(4)$ and s acts nontrivially only on the last factor, and there as conjugation by an element of determinant -1 in $O(4)$.

By [**B**] we know that $H^*(SU(4)/SO(4)) \approx E[x_4, x_5]$ where E stands for the exterior algebra generated by generators x_i in deg i. Also s acts on x_i as multiplication by $(-1)^{i+1}$. We also know that $H^*(U(1)) = E[y_1]$.

So $H^*(Y) \approx E[y_1, y_1', x_4, x_5]$ and $sy_1 = y_1$, $sy_1' = y_1'$, $sx_4 = -x_4$, $sx_5 = x_5$. Now α corresponds to an element in $H^{19-R}(Y) = H^7(Y)$, so the only possibility is $y_1 \wedge y_1' \wedge x_5$. It follows that the K-type δ_6 appears with multiplicity one in $\wedge^{19}\,{}^0\mathfrak{p}$, and that $s\alpha = \alpha$. (The only case among 2, 4, 6, 12 with more than one linearly independent choice of α is case 12, with multiplicity two. One simply checks that for all possible α, $s\alpha = (-1)^{m_k}\alpha$.)

Next β is the wedge of 12 vectors in $\mathfrak{u} \cap \mathfrak{p}$, indicated schematically as

$$\beta = \wedge^{\text{top}} \left\{ \begin{pmatrix} 0 & a & b & c & d & e & f & g \\ & 0 & h & i & j & k & l & f \\ & & 0 & 0 & 0 & 0 & k & e \\ & & & 0 & 0 & 0 & j & d \\ & & & & 0 & 0 & i & c \\ & & & & & 0 & h & b \\ & & & & & & 0 & a \\ & & & & & & & 0 \end{pmatrix} \right\}.$$

Now s switches c and d, and i and j. Hence s acts as $+1$ on β. So $s\alpha = \alpha, s\beta = \beta$ and $sv_\delta = v_\delta$.

So by Lemma 2.6, since $sP_\delta = P_\delta$, $sv_\delta = v_\delta$ and $sY = -Y$ for types 2, 4, 6, 12, they can't occur in the K-span of Y.

Finally, using $v_\delta = \alpha \wedge \beta$ and [**B**] again, one sees that types 1, 5 and 10 can't occur in $\wedge^{19}\,{}^0\mathfrak{p}$.

To prove (ii) we must exhibit v_δ in the K-span of Y for δ of type 8, 11 and 16. First we treat cases 8 and 11. Setting $\delta = \Pi\chi_i^{e_i}$ we find that we must have $P_{\delta_8} = c_8 P_3^4 P_4^2$ and $P_{\delta_{11}} = c_{11} P_3^4 P_2^2$ where c_8 and c_{11} are constants.

Let's treat case 11; case 8 is similar. In the notation of Section 2, we have after specialization

$$\text{proj } w \begin{pmatrix} 0\,0\,0\,1 \\ 0\,1\,1\,0 \end{pmatrix} \cdot Y = c_{11} P_3^4 P_2^2 \begin{pmatrix} 0\,0\,0\,1 \\ 0\,1\,1\,0 \end{pmatrix} v_\delta + \text{lower–weight–terms.}$$

Set

$$w_0 = w \begin{pmatrix} 0\,0\,0\,1 \\ 0\,1\,1\,0 \end{pmatrix}.$$

Thus proj $w_0 \cdot Y = c_{11} v_\delta + \ell.w.t.$, and we must show $c_{11} \neq 0$.

Now Y is a wedge of 19 vectors in the 35-dimensional space ${}^0\mathfrak{p}$. Even with computer aided symbolic algebra it is not feasible just to ask for $w_0 \cdot Y$ and pick out the v_δ-term.

Instead, we write $v_\delta = \alpha \wedge \beta$ as above. Writing Y as the wedge of 19 vectors taken from a basis of ${}^0\mathfrak{p}$ which includes a basis of $\mathfrak{u} \cap \mathfrak{p}$, we then apply w_0 to Y. We see that if, as we remove the parentheses and expand terms, we are to get a term of the form (something)$\wedge\beta$ then certain choices are forced. For a schematic example, if $Y = a \wedge b \wedge c \wedge \dots$ and $w_0 Y = (w_0 a) \wedge (w_0 b) \wedge (w_0 c) \wedge \cdots = (d+e+f) \wedge (g+h) \wedge (e+j+k+\ell) \wedge \dots$ and if d is a basis vector appearing in the pure wedge β, and if d doesn't appear in the other 18 terms, then we must keep d from the first term and discard e and f. Now if e is also in β and appears only in the terms shown, we can't get e from the first term any more, so we must get it from the third term and discard $j+k+\ell$.

In this way, we can actually write down the exact formula $w_0 Y = \psi \wedge \beta +$ other-weight-terms for an explicit $\psi \in \wedge^6\, {}^0\mathfrak{p}$. Moreover ψ is a weight zero wedge of vectors from $\tilde{\ell} \cap {}^0\mathfrak{p}$ where $\tilde{\ell}$ is the Lie-subalgebra of $\mathfrak{q}$

$$\tilde{\ell} = \left\{ \begin{pmatrix} X_1 & & \\ & 0 & \\ & & X_2 \end{pmatrix} \begin{matrix} 3 \\ 2 \\ 3 \end{matrix} \right\}.$$

It follows that proj $w_0 Y = c_{11} v_\delta + \ell.w.t.$ and $c_{11} \neq 0$ only if the projection of ψ to $\left(\wedge^6 (\ell \cap {}^0\mathfrak{p})\right)^{\ell \cap \mathfrak{k}}$ is nonzero.

Computing this projection of ψ is a problem in $GL(3)$. For convenience we apply the Hodge $*$ operator and work in $\wedge^3$. To see if our explicit form has a nonzero projection to the $\tilde{\ell} \cap \mathfrak{k}$-invariants we look instead (by duality) to see if it fails to lie in the linear span C of vectors of the form $\langle gv - v \rangle, g \in GL(3)$. We compute C and find that $*\psi$ is not in C.

The proof of (ii) in case 16 is similar but easier because we don't have to worry about invariant theory in $GL(3)$. We do have to pick judiciously an element $w \in W_0$ such that proj $wY = c_{16} v_\delta + \ell.w.t.$ In fact, we let

$$w = w \begin{pmatrix} x_1 & x_2 & x_3 & 0 \\ y_1 & y_2 & 0 & y_4 \end{pmatrix}$$

and compute:

$$\text{proj } wY = cf(x,y) v_\delta + \ell.w.t.$$

for some $c \neq 0$ and

$$f(x,y) = y_1 x_1 x_3 + y_1 x_2 x_3 - \frac{1}{2}(x_3 y_2 + y_4 x_1) \left(x_1 + \frac{y_2 y_3}{y_4} \right).$$

Clearly $f(x,y) \neq 0$ for some choice of x, y. Again this computation is performed completely by hand. □

3.3. Case of $GL(6)$**.** Here H does not have as simple a form as in the $GL(2n)$ cases with n even. We may take

$$\mathrm{Lie}\, H \cap \mathfrak{p} = \begin{pmatrix} a & 0 & b & b & 0 & c \\ 0 & d & e & -e & f & 0 \\ i & j & g & h & -e & b \\ i & -j & h & g & e & b \\ 0 & k & -j & j & d & 0 \\ m & 0 & i & i & 0 & a \end{pmatrix},$$

$$\mathrm{Lie}\, H \cap \mathfrak{k} = \left\{ \begin{pmatrix} t_1 & 0 & y & y & 0 & 0 \\ 0 & t_2 & x & -x & 0 & 0 \\ y' & x' & t_3 & 0 & x & -y \\ y' & -x' & 0 & -t_3 & -x & -y \\ 0 & 0 & x' & -x' & -t_2 & 0 \\ 0 & 0 & -y' & -y' & 0 & -t_1 \end{pmatrix} \right\},$$

$$s = \begin{pmatrix} I_2 & & \\ & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & \\ & & I_2 \end{pmatrix}.$$

Types 3, 6, 7 are ruled out by Proposition 2.2. As in the $GL(8)$ case we use Lemma 2.6 to rule out types 1, 2 and 5. The invariant theory for finding P_δ reduces to finding weights, since L_0 in the $GL(6)$ case is a torus. We get $sP_\delta = P_\delta$ in these cases. A twist occurs for $GL(6)$ because now $sY = Y$. However computation of $\ell \cap k$ invariants in $\wedge^* \ell \cap {}^0\mathfrak{p}$ using [**B**] gives that $s\alpha = (-1)^{m_k+1}\alpha$ in these three cases. We also see that $s\beta = (-1)^{m_k}\beta$ so that $sv_\delta = -v_\delta$.

We rule in types 4 and 8 by explicit computations similar to the $GL(8)$ case. Thus we prove:

Theorem.
(i) *If* $\alpha \in \mathrm{Hom}_{K_\infty}(\wedge^{11}\mathfrak{p}_\infty, \pi)$ *and* $\alpha(Y) \neq 0$ *then* π *is type* 4 *or* 8.
(ii) *Conversely if* π *is one of these two types, there exists* α *such that* $\alpha(Y) \neq 0$.

Appendix. Periods and Liftings.

Several relationships between the existence of a nonzero period for an automorphic representation π and the fact that π is a lift from another group (in the sense of "Langlands' philosophy") are known, and more are conjectured. In particular, if π is a cuspidal irreducible automorphic representation for $GL(2n)/F$ it is conjectured that π has a nonzero period over $GL(n) \times GL(n)$ if and only if π is a lift from $GO(2n+1)$ (cf. the introduction to [**AG**]).

We can rephrase this locally at a place v in terms of L-groups by conjecturing that an irreducible admissible representation π_v of $GL(2n, F_v)$ possesses a $GL(n, F_v) \times GL(n, F_v)$-invariant continuous functional if and only if the L-parameter classifying π_v factors through the symplectic group.

In this appendix we prove the following proposition which is a heuristic analog of this conjecture in the "geometric" setting for v a real place:

Proposition. *Let π be an irreducible admissible representation for $GL(2n,\mathbb{R})$ with nontrivial $(\mathfrak{g},K)$-cohomology, and let V_δ be a representative of its cohomological K-type $(K = O(2n,\mathbb{R}))$. Then V_δ contains a vector invariant under $SO(n)\times SO(n)$ if and only if the L-parameter corresponding to π*

$$\Phi : W_{\mathbb{R}} \to GL(2n, \mathbb{C})$$

factors through $GSp(2n,\mathbb{C})$.

Remark. The connection with a nonvanishing period for $H = GL(n) \times GL(n)$ is given by Proposition 2.1.

Proof. Suppose π is given by the data $(r_1, \ldots r_k)$ as in Section 3.1. As in the proof of the theorem in Section 3.2, we apply Proposition 2.2 to show that V_δ contains an $SO(n) \times SO(n)$-invariant if and only (i) all the r_s have the same parity and (ii) if $m_k < n$ then that parity is odd. So we must show that Φ factors through $GSp(2n,\mathbb{C})$ if and only if (i) and (ii) hold.

From the description of π as a Langlands' quotient in Section 3.1 it is easy to write down Φ (or more precisely a representative for Φ, which is only determined up to choice of a basis in $GL(2n, \mathbb{C})$).

Recall that $W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times$ with $j^2 = -1$ and $jzj^{-1} = \bar{z}$ for any $z \in \mathbb{C}^\times$. Let $a(z) = z/|z|$ and $t(z) = z\bar{z}$. For any integers M, r with $r > 0$ let $A(M, r)$ denote the $2r \times 2r$ matrix:

$$A(M,r) = \operatorname{diag}\left(a^M t^{\frac{r-1}{2}}, a^{-M} t^{\frac{r-1}{2}}, a^M t^{\frac{r-3}{2}}, a^{-M} t^{\frac{r-3}{2}}, \ldots a^M t^{\frac{1-r}{2}}, a^{-M} t^{\frac{1-r}{2}}\right).$$

Also let $I(M, r)$ denote the $2r \times 2r$ matrix

$$I(M,r) = \begin{pmatrix} 0 & I_r \\ (-I_r)^M & 0 \end{pmatrix}.$$

For $s = 1, \ldots, k$, let $m_s = r_1 + \cdots + r_s$ and $M_s = (2n - m_{s-1} - m_s)$. Also $r_0 = 2n - 2m_k$. Recall that $r_s > 0$ for all s and $r_1 + \cdots + r_k \leq n$. Hence $M_1 > M_2 > \cdots > M_k > 0$.

Then we can give Φ in block diagonal form by

$$\Phi(z) = \operatorname{diag}\left(A(M_1, r_1), \ldots A(M_k, r_k), A(0, r_0)\right);$$

$$\Phi(j) = \operatorname{diag}\left(I(M_1, r_1), \dots I(M_k, r_k), I_{2r_0}\right).$$

Now suppose Φ factors through $GSp(2n, \mathbb{C})$ up to conjugacy. That means there exists a skew symmetric $2n \times 2n$ matrix J and a character λ of $W_{\mathbb{R}}$ such that for any $w \in W_{\mathbb{R}}$,

$${}^t\Phi(w) J \Phi(w) = \lambda(w) J.$$

Applying this to $\Phi(z)$, which has determinant 1, we first see that $\lambda(z)^{2n} = 1$ and then (by taking a generic z) that $J_{ij} = 0$ except for the entries of J along the non-main diagonal of each block. In other words $J = \operatorname{diag}(J_1, \dots J_k, J_0)$ with

$$J_i = \begin{pmatrix} 0 & B_i \\ -B_i & 0 \end{pmatrix} \quad \text{where}$$

$$B_i = \begin{pmatrix} & & \star \\ & \cdot^{\cdot^{\cdot}} & \\ \star & & \end{pmatrix} \quad (r \times r).$$

Now apply the same formula to $\Phi(j)$. Since

$${}^tI(M_s, r_s) J_s I(M_s, r_s) = (-1)^{M_s+1} J_s$$

for $s = 1, \dots k$ and $I_{2r_0} J_0 I_{2r_0} = J_0$, we see that $\lambda(j) = (-1)^{M_s+1}$ for all s and further that $\lambda(j) = 1$ if $r_0 \neq 0$, i.e. if $m_k < n$. Since $r_s \equiv M_s \pmod 2$ for all s, we are finished.

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