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DOMAINS OF PARTIAL ATTRACTION IN NONCOMMUTATIVE PROBABILITY

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In this work we focus on infinitely divisible measures relative to free additive convolution. We give the definition of domain of partial attraction of a measure, and we prove that infinitely divisible laws, and only infinitely divisible laws, are characterized by having non-empty domains of partial attraction.

1. Introduction.

The aim of this paper is to prove the noncommutative analogue of a well-known result in classical probability due to Khintchine, namely, a probability measure is infinitely divisible if and only if it has a non-empty domain of partial attraction. Our framework is the noncommutative theory of free products, introduced by Voiculescu in recent years. The key concept in this new theory is the notion of freeness, which leads naturally to the free additive convolution. Many classical results have been proved to have their classical counterpart, such as the central limit theorem [**7**, **10**], the Khintchine characterization of infinitely divisible (with respect to the free additive convolution) laws [**1**, **3**], the weak law of large numbers [**2**, **6**], and, related with this paper, the characterization of stable laws as those laws having a non-empty domain of attraction [**8**]. A background of this noncommutative theory can be found in [**1**, **3**, **9**, **10**]. For reader's convenience, and in order to render this paper self-contained, we begin recalling some basic facts.

2. Definitions and first properties.

A W^*-*probability space* is a pair $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is a noncommutative von Neumann algebra and τ is a normal faithful trace.

A *random variable* is a selfadjoint operator affiliated with $\mathcal{A}$ (via the GNS construction).

An interesting purely noncommutative formal analogue of classical independence is the notion of *freeness*. The analogy is that around freeness, several concepts can be developed similar to those around independence. A family of von Neumann subalgebras $\mathcal{A}_i \subset \mathcal{A}$, $i \in I$ in a W^*-probability space is said to be *free* if $\tau(a_1 a_2 \dots a_n) = 0$ whenever $\tau(a_j) = 0$, $a_j \in \mathcal{A}_{i_j}$, and $i_1 \neq i_2 \neq \dots \neq i_n$.

Given a W^*-probability space $(\mathcal{A}, \tau)$ and a random variable X we define the distribution μ_X of X to be the unique probability measure on $\mathbf{R}$ satisfying the equality $\mu_X(\sigma) = \tau(E_X(\sigma))$, for every $\sigma \in \mathcal{B}(\mathbf{R})$, where E_X is the spectral measure of X.

An important result is that given a family $\{\nu_i\}_{i\in I}$ of probability measures on $\mathbf{R}$, it is possible to find a W^*-probability space $(\mathcal{A}, \tau)$ and a family $\{X_i\}_{i\in I}$ of free random variables such that, for all $i \in I$ $\mu_{X_i} = \nu_i$.

The concept of freeness allows us to define without ambiguity the *free additive convolution* (indicated by $\boxplus$) between two distributions. Indeed it can be shown that if X and Y are two free random variables then μ_{X+Y} depends only on μ_X and μ_Y, therefore it is possible to define the operation $\boxplus$ in the following way: $\mu_X \boxplus \mu_Y = \mu_{X+Y}$. By the above remark, given two probability measures μ and ν, we find a W^*-probability space $(\mathcal{A}, \tau)$ and two free random variables X, Y affiliated with $\mathcal{A}$ such that $\mu_X = \mu$ and $\mu_Y = \nu$. Thus it makes sense to define $\mu \boxplus \nu = \mu_{X+Y}$. Indeed the additive convolution is a binary operation (obviously commutative and associative) defined on the space of probability measures on $\mathbf{R}$.

In the sequel, for α, $\beta > 0$, we denote

$$\Gamma_\alpha = \{z = x + iy : y > 0 \quad \text{and} \quad |x| < \alpha y\} ,$$

and

$$\Gamma_{\alpha,\beta} = \{z \in \Gamma_\alpha : y > \beta\} .$$

It is possible to associate to every probability measure μ a complex function ϕ_μ (the ϕ-function of μ), defined on a domain Ω of the form

$$\Omega = \bigcup_{\alpha>0} \Gamma_{\alpha,\beta(\alpha)} ,$$

with values in $\mathbf{C}^- \cup \mathbf{R}$.

The remarkable property of the ϕ-functions is that, given two probability measures μ_1 and μ_2, setting $\mu = \mu_1 \boxplus \mu_2$, it follows that $\phi_\mu = \phi_{\mu_1} + \phi_{\mu_2}$. Thus the ϕ-function is the noncommutative analogue of the logarithm of the characteristic function in classical probability.

Another property of ϕ_μ, which is a quite direct consequence of the definition of ϕ_μ, is the following. If X is a random variable in a W^*-probability space, and c a positive constant, then

$$\phi_{\mu_{cX}}(z) = c\phi_{\mu_X}\left(\frac{z}{c}\right) .$$

We state now two fundamental results from [**1**], along with the definition of a $\boxplus$-infinitely divisible measure.

Proposition 2.1. *Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of probability measures on $\mathbf{R}$. The following assertions are equivalent.*

(i) *The sequence $\{\mu_n\}_{n=1}^{\infty}$ converges weakly to a probability measure μ.*

(ii) *There exist $\alpha, \beta > 0$ such that the sequence $\{\phi_{\mu_n}\}_{n=1}^{\infty}$ converges uniformly on the compact subsets of $\Gamma_{\alpha,\beta}$ to a function ϕ, and $\phi_{\mu_n}(z) = o(z)$ uniformly in n as $|z| \to \infty$, $z \in \Gamma_{\alpha,\beta}$.*

Moreover, if (i) *and* (ii) *are satisfied, we have $\phi = \phi_\mu$ in $\Gamma_{\alpha,\beta}$.*

Definition 2.2. A probability measure μ is said to be $\boxplus$-*infinitely divisible* if for every positive integer n there exists a probability measure μ_n such that

$$\mu = \underbrace{\mu_n \boxplus \ldots \boxplus \mu_n}_{n \text{ times}} .$$

The ϕ-function of a $\boxplus$-infinitely divisible distributions can be written in a canonical form.

Theorem 2.3. *The following hold.*

(i) *A probability measure μ on $\mathbf{R}$ is $\boxplus$-infinitely divisible if and only if ϕ_μ has an analytic extension defined on $\mathbf{C}^+$ with values in $\mathbf{C}^- \cup \mathbf{R}$.*

(ii) *Let $\phi : \mathbf{C}^+ \to \mathbf{C}^-$ be an analytic function. Then ϕ is a continuation of ϕ_μ for some $\boxplus$-infinitely divisible measure μ if and only if*

$$\lim_{|z|\to\infty\ z\in\Gamma_\alpha} \frac{\phi(z)}{z} = 0 ,$$

for some (and hence all) $\alpha > 0$.

(iii) *Let μ be a $\boxplus$-infinitely divisible probability measure on $\mathbf{R}$. Then there exist $a \in \mathbf{R}$ and a positive finite measure σ such that*

$$\phi_\mu(z) = a + \int_{-\infty}^{+\infty} \frac{1 + tz}{z - t} d\sigma(t) .$$

Observe that μ is a Dirac measure if and only if $\sigma = 0$. For $\sigma = 4^{-1} r^2 \delta_0$ we have that $\mu = \gamma_{a,r}$, the *semicircle law*, defined as

$$d\gamma_{a,r}(t) = \begin{cases} \frac{2}{\pi r^2}\sqrt{r^2 - (t-a)^2}\, dt & \text{if } t \in [a - r, a + r], \\ 0 & \text{otherwise,} \end{cases}$$

which is the noncommutative analogue of the normal law (see [**10**]).

A consequence of Theorem 2.3 is the following result.

Proposition 2.4. *Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of $\boxplus$-infinitely divisible probability measures on* $\mathbf{R}$ *converging weakly to a probability measure μ. Then μ is $\boxplus$-infinitely divisible.*

Proof. By Proposition 2.1 there exist $\alpha, \beta > 0$ such that $\phi_{\mu_n}(z) \to \phi_\mu(z)$, uniformly on the compact subsets of the truncated cone $\Gamma_{\alpha,\beta}$. By Theorem 2.3, ϕ_{μ_n} extends to a function $\psi_{\mu_n} : \mathbf{C}^+ \to \mathbf{C}^- \cup \mathbf{R}$. Since $\{\psi_{\mu_n}\}_{n=1}^{\infty}$ is a normal family, by Montel Theorem there exists a subsequence $\psi_{\mu_{n_k}}$ converging to an analytic function $\psi : \mathbf{C}^+ \to \mathbf{C}^- \cup \mathbf{R}$. Being the restriction of ψ on $\Gamma_{\alpha,\beta}$ equal to ϕ_μ, it follows that ϕ_μ has an analytic extension on $\mathbf{C}^+$ with values in $\mathbf{C}^- \cup \mathbf{R}$. Therefore, using again Theorem 2.3, the result is proved. $\square$

3. Domains of partial attraction.

In [**8**] we proved that the stable distributions, and only the stable distributions, can be written as limits of weighted sums of the form

$$Z_n = \frac{X_1 + \ldots + X_n}{B_n} - A_n ,$$

where $X_1, X_2, \ldots$ are free, identically distributed (f.i.d.) random variables, $B_n > 0$, $A_n \in \mathbf{R}$. Here we want to investigate the case when Z_n does not necessarily converge, but Z_{n_j} does converge for some sequence n_j. An application of Theorem 2.3 allows us to assert that this limit is necessarily $\boxplus$-infinitely divisible (see Lemma 3.4 below). The much more interesting converse result is also true: every $\boxplus$-infinitely divisible distribution appears as the limit of the sums Z_{n_j}. This result was proved in the classical case by Khintchine [**4, 5**]. We proceed first with a definition.

Definition 3.1. Let $q_1 < q_2 < \ldots < q_n \ldots$ be a sequence of positive integers, let $\{X_n\}_{n=1}^{\infty}$ be a sequence of f.i.d. random variables with distribution ν in a W^*-probability space, and let $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ be sequences of real and positive numbers, respectively. Set

$$Z_n = \frac{X_{q_1} + \ldots + X_{q_n}}{B_n} - A_n .$$

If for a suitable choice of the constants B_n and A_n the distribution of Z_n converges weakly to a measure μ we say that ν is *partially attracted* to μ. The set of all probability measures partially attracted to μ is called the *domain of partial attraction* of μ.

We can now state our result as follows.

Theorem 3.2. *A probability measure has a non-empty domain of partial attraction if and only if it is $\boxplus$-infinitely divisible. Moreover if the domain of partial attraction is non-empty, it contains a $\boxplus$-infinitely divisible measure.*

Remark 3.3. Unlike the stable law case [**8**], a $\boxplus$-infinitely divisible law may not belong to its own domain of partial attraction. Consider the measure defined as

$$d\mu(t) = \begin{cases} \frac{1}{2\pi}\frac{\sqrt{4+4t-t^2}}{t+1}\,dt & \text{if } t \in [2-2\sqrt{2}, 2+2\sqrt{2}], \\ 0 & \text{otherwise.} \end{cases}$$

The ϕ-function of this measure is easy to compute (see [**10**]), and it is given by

$$\phi_\mu(z) = \frac{1+z}{1-z} .$$

By Theorem 2.3 μ is $\boxplus$-infinitely divisible. Suppose now that there exist sequences $\{A_n\}_{n=1}^\infty$ and $\{B_n\}_{n=1}^\infty$ of real and positive numbers, respectively, and a subsequence $\{n_j\}_{j=1}^\infty$ such that

$$\frac{n_j}{B_{n_j}}\phi_\nu(B_{n_j}z) - A_{n_j} \underset{j\to\infty}{\rightrightarrows} \phi_\mu(z) ,$$

uniformly on the compact subsets of $\Gamma_{\alpha,\beta}$, for some $\alpha, \beta > 0$. In particular, for $z = iy$, with $y > \beta$,

$$\Im\left(\frac{n_j}{B_{n_j}}\phi_\nu(iB_{n_j}y)\right) \underset{j\to\infty}{\rightrightarrows} \Im\phi_\mu(iy) ,$$

from where we get

$$\lim_{j\to\infty} \frac{n_j}{1+B_{n_j}^2 y^2} = \frac{1}{1+y^2} .$$

Therefore it follows that

$$\lim_{j\to\infty} \frac{n_j}{B_{n_j}^2} = \frac{y^2}{1+y^2} .$$

Since the above limit must exist for all $y > \beta$, we get that

$$\frac{y^2}{1+y^2} = \text{constant} \quad , \quad y > \beta ,$$

which is a contradiction.

The proof of the theorem is carried out in the following two lemmas.

Lemma 3.4. *Let $q_1 < q_2 < \ldots < q_n \ldots$ be a sequence of positive integers, let $\{X_n\}_{n=1}^{\infty}$ be a sequence of f.i.d. random variables with distribution ν in a W^*-probability space, and let $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ be sequences of real and positive numbers, respectively. Set*

$$Z_n = \frac{X_{q_1} + \ldots + X_{q_n}}{B_n} - A_n .$$

If for a suitable choice of the constants B_n and A_n the distribution of Z_n converges weakly to a measure μ, then μ is $\boxplus$-infinitely divisible.

Proof. Denote the ϕ-function of Z_n by ψ_n. First we prove that

$$\lim_{n\to\infty} B_n = \infty .$$

Indeed, by Proposition 2.1, there exist $\alpha, \beta > 0$ such that

$$\psi_n(z) = \frac{q_n}{B_n}\phi_\nu(B_n z) - A_n \underset{n\to\infty}{\longrightarrow} \phi_\mu(z) ,$$

uniformly on the compact subsets of the truncated cone $\Gamma_{\alpha,\beta}$. If there exists a subsequence $B_{n_j} \to B < \infty$, then, for every fixed $z \in \Gamma_{\alpha,\beta}$,

$$\left|\Im\psi_{n_j}(z)\right| = \left|\Im\left(\frac{q_{n_j}}{B_{n_j}}\phi_\nu(B_{n_j}z)\right)\right| \underset{j\to\infty}{\longrightarrow} \infty .$$

So the assertion is proved.

Notice further that ϕ_ν is defined in

$$\Omega = \bigcup_{\omega>0} \Gamma_{\omega,\theta(\omega)} ,$$

so if we set

$$\theta(\omega, n) = \frac{\theta(\omega)}{B_n} ,$$

we get that ψ_n is defined in

$$\Omega_n = \bigcup_{\omega>0} \Gamma_{\omega,\theta(\omega,n)} .$$

Note that $\Omega_n \subset \Omega_{n+1}$, and

$$\bigcup_{n=1}^{\infty} \Omega_n = \mathbf{C}^+ .$$

Let $\Phi_k = \{\psi_n\}_{n\geq k}$. For every k the family Φ_k is normal, thus by Montel Theorem there exists a subsequence ψ_{n_i} (with $n_i = n_i(k) \geq k$) converging to an analytic function $\psi^{(k)}$ uniformly on the compact subsets of Ω_k. Since $\psi_{n_i} \to \phi_\mu$ uniformly on the compact subsets of $\Gamma_{\alpha,\beta}$, by the Identity Theorem it follows that ϕ_μ extends on Ω_k. Since this holds for every k, ϕ_μ has an analytic extension on $\mathbf{C}^+$ with values in $\mathbf{C}^- \cup \mathbf{R}$. Hence, by Theorem 2.3, μ is $\boxplus$-infinitely divisible. $\square$

Lemma 3.5. *Let μ be a $\boxplus$-infinitely divisible probability measure. Then there exist a sequence $\{X_n\}_{n=1}^\infty$ of f.i.d. random variables with common distribution ν, a sequence of positive integers $q_1 < q_2 < \ldots < q_n \ldots$, and sequences $\{A_n\}_{n=1}^\infty$ and $\{B_n\}_{n=1}^\infty$ of real and positive numbers, respectively, such that $B_n^{-1}(X_{q_1}+\ldots+X_{q_n})-A_n$ converges in distribution to μ. Moreover, the distribution ν can be chosen to be $\boxplus$-infinitely divisible.*

Proof. By Theorem 2.3 there exist $a \in \mathbf{R}$ and a positive finite measure σ such that

$$\phi_\mu(z) = a + \int_{-\infty}^{+\infty} \frac{1+tz}{z-t}\, d\sigma(t) \, .$$

We may assume that $\mu \neq \delta_a$ (the theorem being trivially true when $\mu = \delta_a$), and also that μ is not the semicircle law (indeed as a consequence of the Central Limit Theorem **[10]**, the semicircle law has a non-empty domain of attraction). This amounts to requiring the measure σ not to be concentrated at zero. Select $s > 1$ such that $\sigma(\{t : s^{-1} < |t| < s\}) > 0$. Consider the domains Δ_k defined as

$$\Delta_k = \{t : s^{-k} < |t| < s^k\} \, ,$$

for $k = 1, 2, \ldots$, and put

$$g_k = \sigma(\Delta_k) \, .$$

Moreover define

$$a_k = \int_{\Delta_k} t\, d\sigma(t) \, ,$$

$$b_k = \int_{\Delta_k} (1+t^2)\, d\sigma(t) \, ,$$

$$c_k = \int_{\Delta_k} |t|(1+t^2)\, d\sigma(t) \, .$$

Let now $\lambda_1 = 1$ and

$$\lambda_k = \frac{1}{\varrho + \frac{1}{k}} \sum_{j=1}^{k-1} \lambda_j b_j \, ,$$

where $\varrho = \sigma(\{0\})$. Choose positive integers $1 = q_1 < q_2 < \dots < q_n \dots$ increasing so fast that the following hold.

$$\sqrt{\frac{q_k\lambda_k}{q_j\lambda_j}} > s^k \ , \tag{1}$$

for every $j = 1, 2, \dots, k-1$,

$$\sum_{k=1}^{\infty}\sqrt{\frac{\lambda_k}{q_k}}\, s^{4k} g_k = C < \infty \ , \tag{2}$$

$$\lim_{n\to\infty}\sqrt{\frac{q_n}{\lambda_n}}\sum_{k=n+1}^{\infty}\sqrt{\frac{\lambda_k}{q_k}}\, s^k g_k = 0 \ , \tag{3}$$

$$\lim_{n\to\infty}\frac{1}{\sqrt{q_n\lambda_n^3}}\sum_{k=1}^{n-1}\sqrt{q_k\lambda_k^3}\, c_k = 0 \ . \tag{4}$$

Finally define

$$B_k = \sqrt{\lambda_k\, q_k} \ .$$

Denote

$$\varphi_k(z) = \int_{\Delta_k}\frac{1+tz}{z-t}\, d\sigma(t) \ ,$$

$$\psi_n(z) = \sum_{k=1}^{n}\frac{B_k}{q_k}\,\varphi_k\left(\frac{z}{B_k}\right) \ ,$$

and

$$\psi(z) = \sum_{k=1}^{\infty}\frac{B_k}{q_k}\,\varphi_k\left(\frac{z}{B_k}\right) \ .$$

We first show that $\psi(z)$ is the ϕ-function of a positive $\boxplus$-infinitely divisible measure ν. Notice that $\psi_n(z)$ is the ϕ-function of a $\boxplus$-infinitely divisible measure ν_n. Indeed this is an immediate consequence of Theorem 2.3, since $\psi_n(z) : \mathbf{C}^+ \to \mathbf{C}^-$ and

$$\lim_{|z|\to\infty\ z\in\Gamma_1}\frac{\psi_n(z)}{z} = \sum_{k=1}^{n}\frac{1}{q_k}\left(\lim_{|z|\to\infty\ z\in\Gamma_1}\frac{B_k}{z}\varphi_k\left(\frac{z}{B_k}\right)\right) = 0 \ ,$$

the above limit being zero since $\varphi_k(z)$ is the ϕ-function of a $\boxplus$-infinitely divisible measure. Let now $K \subset \Gamma_{1,1}$ be a compact set and let $z \in K$ (hence in particular $|z| \le M < \infty$). Observe that if $w \in \Gamma_1$ and $t \in \Delta_k$, then

$$\frac{1}{|w-t|} < \sqrt{2}\, s^k \ ,$$

therefore we get

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{B_k}{q_k}\left|\varphi_k\left(\frac{z}{B_k}\right)\right| &= \sum_{k=1}^{\infty}\frac{B_k}{q_k}\left|\int_{\Delta_k}\frac{1+t\frac{z}{B_k}}{\frac{z}{B_k}-t}\,d\sigma(t)\right| \\
&\le \sum_{k=1}^{\infty}\frac{B_k}{q_k}\int_{\Delta_k}\left|\frac{1+t\frac{z}{B_k}}{\frac{z}{B_k}-t}\right|d\sigma(t) \\
&< \sum_{k=1}^{\infty}\frac{B_k}{q_k}\int_{\Delta_k}\sqrt{2}\,s^k\left(1+s^k\frac{M}{B_k}\right)d\sigma(t) \\
&< \sum_{k=1}^{\infty}\frac{B_k}{q_k}\sqrt{2}\,s^k(1+M)g_k \\
&= \sqrt{2}\,(1+M)\sum_{k=1}^{\infty}\sqrt{\frac{\lambda_k}{q_k}}\,s^k g_k \\
&< \sqrt{2}\,(1+M)\sum_{k=1}^{\infty}\sqrt{\frac{\lambda_k}{q_k}}\,s^{4k} g_k \\
&= \sqrt{2}\,(1+M)C < \infty\,,
\end{aligned}$$

by (1) and (2). Therefore by the Weierstrass Test $\psi_n(z)$ converges to $\psi(z)$ uniformly on K. Let now $w = x + iy \in \Gamma_1$. Since $|x| < y$, and since

$$\frac{1}{|w-t|^2} < \min\left\{\frac{1}{y^2}, 2s^{2k}\right\},$$

we get

$$\begin{aligned}
|\varphi_k(w)| &= \left|\int_{\Delta_k}\frac{1+tw}{w-t}\,d\sigma(t)\right| \\
&= \left|\int_{\Delta_k}\frac{t(x^2+y^2)+x(1-t^2)-t}{|w-t|^2}\,d\sigma(t) - i\int_{\Delta_k}\frac{y(1+t^2)}{|w-t|^2}\,d\sigma(t)\right| \\
&< \int_{\Delta_k}\frac{2y^2|t|+|t|+2y(1+t^2)}{|w-t|^2}\,d\sigma(t) \\
&< \int_{\Delta_k}\left(2|t|+2|t|s^{2k}+2y(1+t^2)\min\left\{\frac{1}{y^2},2s^{2k}\right\}\right)d\sigma(t) \\
&< \int_{\Delta_k}\left(2|t|+2|t|s^{2k}+4s^{2k}(1+t^2)\min\left\{\frac{1}{y},y\right\}\right)d\sigma(t) \\
&< \left(2s^k+2s^{3k}+4s^{2k}(1+s^{2k})\right)g_k \\
&< 12s^{4k}g_k\,.
\end{aligned}$$

Thus for $z \in \Gamma_{1,1}$ we have

$$\begin{aligned}|\psi_n(z)| &= \left|\sum_{k=1}^{n} \frac{B_k}{q_k}\varphi_k\left(\frac{z}{B_k}\right)\right| \\ &< \sum_{k=1}^{n} \frac{B_k}{q_k} 12 s^{4k} g_k = 12 \sum_{k=1}^{n} \sqrt{\frac{\lambda_k}{q_k}}\, s^{4k} g_k \\ &\leq 12 \sum_{k=1}^{\infty} \sqrt{\frac{\lambda_k}{q_k}}\, s^{4k} g_k \\ &= 12C = o(|z|)\end{aligned}$$

as $z \to \infty$ in $\Gamma_{1,1}$. So by Proposition 2.1, $\psi(z)$ is the ϕ-function of a positive measure ν. Moreover by Proposition 2.4 ν is $\boxplus$-infinitely divisible.

Let now $\{X_n\}_{n=1}^{\infty}$ be a sequence of f.i.d. random variables in a W^*-probability space with common distribution ν (hence $\phi_\nu(z) = \psi(z)$). We want to study the behavior of the distribution of the random variable

$$S_n = \frac{X_{q_1} + \ldots + X_{q_n}}{B_n} ,$$

whose associated ϕ-function is $B_n^{-1} q_n\, \psi(B_n z)$. Let K be as above, and $z \in K$. Observe that

$$\frac{q_n}{B_n}\psi(B_n z) = \Psi_n^{(1)}(z) + \varphi_n(z) + \Psi_n^{(2)}(z) ,$$

where

$$\Psi_n^{(1)}(z) = \frac{q_n}{B_n} \sum_{k=1}^{n-1} \frac{B_k}{q_k}\varphi_k\left(z\frac{B_n}{B_k}\right) ,$$

and

$$\Psi_n^{(2)}(z) = \frac{q_n}{B_n} \sum_{k=n+1}^{\infty} \frac{B_k}{q_k}\varphi_k\left(z\frac{B_n}{B_k}\right) .$$

Notice that $\varphi_n(z)$ converges to $\phi_\mu(z) - a - \varrho z^{-1}$ uniformly on K.

We examine $\Psi_n^{(1)}(z)$ and $\Psi_n^{(2)}(z)$ separately. Setting $z = x + iy$, we have

$$\begin{aligned}|\Psi_n^{(2)}(z)| &\leq \frac{q_n}{B_n} \sum_{k=n+1}^{\infty} \frac{B_k}{q_k} \int_{\Delta_k} \left| \frac{1 + tz\frac{B_n}{B_k}}{z\frac{B_n}{B_k} - t} \right| d\sigma(t) \\ &\leq \frac{q_n}{B_n} \sum_{k=n+1}^{\infty} \frac{B_k}{q_k} \int_{\Delta_k} \left(\frac{1}{|z\frac{B_n}{B_k} - t|} + \frac{|t||z|\frac{B_n}{B_k}}{|z\frac{B_n}{B_k} - t|} \right) d\sigma(t)\end{aligned}$$

$$
\begin{aligned}
&< \frac{q_n}{B_n} \sum_{k=n+1}^{\infty} \frac{B_k}{q_k} \int_{\Delta_k} \left(\sqrt{2}\, s^k + \frac{\sqrt{2}\,|t| y \frac{B_n}{B_k}}{y \frac{B_n}{B_k}} \right) d\sigma(t) \\
&\leq 2\sqrt{2}\, \frac{q_n}{B_n} \sum_{k=n+1}^{\infty} \frac{B_k}{q_k} s^k g_k \, .
\end{aligned}
$$

Hence by (3)

$$
\limsup_{n\to\infty} \left| \Psi_n^{(2)}(z) \right| \leq \limsup_{n\to\infty} 2\sqrt{2} \sqrt{\frac{q_n}{\lambda_n}} \sum_{k=n+1}^{\infty} \sqrt{\frac{\lambda_k}{q_k}}\, s^k g_k = 0 \, .
$$

Moreover, since the above estimate does not depend on z, it is clear that

$$
\Psi_n^{(2)}(z) = o(|z|) \, ,
$$

uniformly in n as $z \to \infty$ in $\Gamma_{1,1}$. Let us now turn our attention to $\Psi_n^{(1)}(z)$. Notice that the following equality holds:

$$
\frac{1 + tz\frac{B_n}{B_k}}{z\frac{B_n}{B_k} - t} = \frac{1}{z} \frac{B_k}{B_n} \left(1 + t^2 + tz\frac{B_n}{B_k} + \frac{t(1+t^2)}{z\frac{B_n}{B_k} - t} \right) .
$$

Therefore we have

$$
\begin{aligned}
\Psi_n^{(1)}(z) &= \frac{q_n}{B_n^2} \sum_{k=1}^{n-1} \frac{B_k^2}{q_k} \frac{1}{z} \int_{\Delta_k} \left(1 + t^2 + tz\frac{B_n}{B_k} + \frac{t(1+t^2)}{z\frac{B_n}{B_k} - t} \right) d\sigma(t) \\
&= \frac{q_n}{B_n^2} \sum_{k=1}^{n-1} \frac{B_k^2}{q_k} \frac{1}{z} \left(b_k + z\, a_k \frac{B_n}{B_k} + \int_{\Delta_k} \frac{t\,(1+t^2)}{z\,\frac{B_n}{B_k} - t}\, d\sigma(t) \right) \\
&= \frac{\varrho_n}{z} + C_n + \mathcal{I}_n(z) \, ,
\end{aligned}
$$

where

$$
\begin{aligned}
\varrho_n &= \frac{q_n}{B_n^2} \sum_{k=1}^{n-1} \frac{B_k^2}{q_k}\, b_k = \frac{1}{\lambda_n} \sum_{k=1}^{n-1} \lambda_k\, b_k = \varrho + \frac{1}{n} \, , \\
C_n &= \frac{q_n}{B_n} \sum_{k=1}^{n-1} \frac{B_k}{q_k}\, a_k \, ,
\end{aligned}
$$

and

$$
\mathcal{I}_n(z) = \frac{q_n}{B_n^2} \sum_{k=1}^{n-1} \frac{B_k^2}{q_k} \frac{1}{z} \int_{\Delta_k} \frac{t(1+t^2)}{z\frac{B_k}{B_n} - t}\, d\sigma(t) \, .
$$

Notice that, by (4), for every $z \in \Gamma_{1,1}$ (so in particular for every $z \in K$),

$$\begin{aligned}
|\mathcal{I}_n(z)| &\le \frac{q_n}{B_n^2}\sum_{k=1}^{n-1}\frac{B_k^2}{q_k}\frac{1}{|z|}\int_{\Delta_k}\frac{|t|(1+t^2)}{|z\frac{B_k}{B_n}-t|}\,d\sigma(t) \\
&\le \frac{q_n}{B_n^2}\sum_{k=1}^{n-1}\frac{B_k^2}{q_k}\int_{\Delta_k}\frac{|t|(1+t^2)}{\frac{B_k}{B_n}}\,d\sigma(t) \\
&= \frac{q_n}{B_n^3}\sum_{k=1}^{n-1}\frac{B_k^3}{q_k}c_k \\
&= \frac{1}{\sqrt{q_n\lambda_n^3}}\sum_{k=1}^{n-1}\sqrt{q_k\lambda_k^3}\,c_k \underset{n\to\infty}{\longrightarrow} 0\,.
\end{aligned}$$

Setting $A_n = C_n - a$, and $\mathcal{J}_n(z) = \Psi_n^{(2)}(z) + \mathcal{I}_n(z)$, we get that

$$\frac{q_n}{B_n}\psi(B_n z) = \varphi_n(z) + \frac{\varrho_n}{z} + a + A_n + \mathcal{J}_n(z)\,.$$

Since $\varrho_n \to \varrho$, we have that

$$\varphi_n(z) + \frac{\varrho_n}{z} + a \underset{n\to\infty}{\longrightarrow} \phi_\mu(z)\,,$$

uniformly in K. Indeed, denoting

$$\Sigma_n = [-s^{-n}, 0) \cup (0, s^{-n}]\,,$$

and

$$\Lambda_n = (-\infty, -s^n] \cup [s^n, +\infty)\,,$$

we get, for every $z \in K$ and $n > \log_s M$,

$$\begin{aligned}
&\left|\varphi_n(z) + \frac{\varrho_n}{z} + a - \phi_\mu(z)\right| \\
&= \left|\int_{\Sigma_n}\frac{1+tz}{z-t}\,d\sigma(t) + \int_{\Lambda_n}\frac{1+tz}{z-t}\,d\sigma(t) + \frac{1}{z}(\varrho-\varrho_n)\right| \\
&\le \int_{\Sigma_n}\frac{|1+tz|}{|z-t|}\,d\sigma(t) + \int_{\Lambda_n}\frac{|1+tz|}{|z-t|}\,d\sigma(t) + \frac{1}{|z|}|\varrho-\varrho_n| \\
&\le \int_{\Sigma_n}(1+s^{-n}M)\,d\sigma(t) + \int_{\Lambda_n}\frac{1+|t|M}{|t|-M}\,d\sigma(t) + |\varrho-\varrho_n|\,.
\end{aligned}$$

Since for $|t|$ big enough $(1+|t|M)(|t|-M)^{-1}$ is increasing in $|t|$, and

$$\lim_{|t|\to\infty}\frac{1+|t|M}{|t|-M} = M\,,$$

and using the fact that σ is a finite measure, we finally get

$$\begin{aligned}&\limsup_{n\to\infty}\left|\varphi_n(z)+\frac{\varrho_n}{z}+a-\phi_\mu(z)\right|\\&\le\limsup_{n\to\infty}\Big((1+s^{-n}M)\,\sigma(\Sigma_n)+M\,\sigma(\Lambda_n)+|\varrho-\varrho_n|\Big)\\&=0\,,\end{aligned}$$

which proves the assertion. By our preceding observations we know that $\mathcal{J}_n(z)\to 0$, uniformly in K, moreover $\mathcal{J}_n(z)=o(|z|)$, uniformly in n as $z\to\infty$ in $\Gamma_{1,1}$. To prove this fact, observe that for $z\in\Gamma_{1,1}$, we have that $|\mathcal{I}_n(z)|\le k_n$, where k_n is a sequence converging to zero, and we already showed that $\Psi_n^{(2)}(z)=o(|z|)$ uniformly in n as $z\to\infty$ in $\Gamma_{1,1}$. Thus it follows that

$$\frac{q_n}{B_n}\psi(B_nz)-A_n \underset{n\to\infty}{\longrightarrow} \phi_\mu(z)\,,$$

uniformly in K, and

$$\frac{q_n}{B_n}\psi(B_nz)-A_n=o(|z|)\,,$$

uniformly in n as $z\to\infty$ in $\Gamma_{1,1}$.

Therefore by Theorem 2.3 the sequence S_n-A_n, whose ϕ-function is given by $B_n^{-1}q_n\psi(B_nz)-A_n$, converges in distribution to μ. □

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