PACIFIC JOURNAL OF MATHEMATICS
Vol. 176, No. 1, 1996

MODULI SPACE OF ISOMETRIC PLURIHARMONIC IMMERSIONS OF KÄHLER MANIFOLDS INTO INDEFINITE EUCLIDEAN SPACES

HITOSHI FURUHATA

We classify isometric pluriharmonic immersions of a Kähler manifold into an indefinite Euclidean space. The moduli space of such immersions is explicitly constructed in terms of complex matrices. Some examples of these immersions are also given.

1. Introduction.

It has been a fundamental problem in the theory of minimal surfaces to determine the moduli spaces of isometric minimal immersions. An answer to this problem was given by Calabi [**3**], and recently it is generalized to higher dimensional cases by the present author [**8**]. In fact, we prove that the moduli space of isometric minimal immersions of a simply connected Kähler manifold into a *real* Euclidean space can be constructed in an explicit way as a set of certain complex matrices.

The purpose of this paper is to prove the counterpart of this construction in the case that the ambient space is an *indefinite* Euclidean space. Namely, we shall construct a parametrization of the moduli space of isometric pluriharmonic immersions of a simply connected Kähler manifold into an indefinite real Euclidean space in terms of certain complex matrices, which are in fact determined by a full isometric holomorphic immersion of the Kähler manifold into an indefinite complex Euclidean space. The key ingredient of our construction is the pluriharmonicity of these immersions, which eventually enables us to classify them in a similar fashion as in the case of minimal surfaces. However, it should be remarked that isometric maximal immersions of Kähler manifolds into indefinite Euclidean spaces are not pluriharmonic in general, which contrasts with the fact that isometric minimal immersions of Kähler manifolds into Euclidean spaces are always pluriharmonic.

In their paper [**1**, **2**], Abe-Magid proved a rigidity theorem of indefinite complex submanifolds and a representation formula for maximal surfaces in terms of holomorphic curves and complex matrices. Our result can be regarded as a sequel to their work.

This paper is organized as follows. After fixing our notation and terminologies, we recall in Section 2 a rigidity theorem of isometric holomorphic immersions of a Kähler manifold into indefinite complex Euclidean spaces. The precise statement of our classification theorem and its proof are given in Section 3. In Section 4 we illustrate some examples of isometric pluriharmonic immersions of Kähler manifolds into indefinite Euclidean spaces.

The author wishes to express his gratitude to Professors K. Kenmotsu and S. Nishikawa.

2. Preliminaries.

Let $\mathbf{R}_N^{N+P}$ denote a real vector space of dimension $N+P$ endowed with the standard metric

$$-(dx^1)^2 - \cdots - (dx^N)^2 + (dx^{N+1})^2 + \cdots + (dx^{N+P})^2$$

of signature (N,P), and $\mathbf{C}_N^{N+P}$ a complex vector space of dimension $N+P$ endowed with the standard metric

$$-dz^1 dz^{\overline{1}} - \cdots - dz^N dz^{\overline{N}} + dz^{N+1} dz^{\overline{N+1}} \cdots + dz^{N+P} dz^{\overline{N+P}}$$

of signature (N,P), respectively. Let l, t and s be integers such that

$$0 \leq l \leq \min(N,P), \quad 0 \leq t \leq N-l \quad \text{and} \quad 0 \leq s \leq P-l.$$

For each (l,t,s) we denote by $H(l,t,s)$ an $(l+t+s)$-dimensional subspace of $\mathbf{R}_N^{N+P}$ consisting of the elements

$$X = (\underbrace{X^1, \ldots, X^l, X^{l+1}, \ldots, X^{l+t}, 0, \ldots, 0}_{N};$$
$$\underbrace{X^1, \ldots, X^l, X^{l+t+1}, \ldots, X^{l+t+s}, 0, \ldots, 0}_{P}),$$

where $X^j \in \mathbf{R}$ and $1 \leq j \leq l+t+s$. Also, by $H^{\mathbf{C}}(l,t,s)$ we denote an $(l+t+s)$-dimensional subspace of $\mathbf{C}_N^{N+P}$ consisting of the elements

$$Z = (\underbrace{Z^1, \ldots, Z^l, Z^{l+1}, \ldots, Z^{l+t}, 0, \ldots, 0}_{N};$$
$$\underbrace{Z^1, \ldots, Z^l, Z^{l+t+1}, \ldots, Z^{l+t+s}, 0, \ldots, 0}_{P}),$$

where $Z^j \in \mathbf{C}$ and $1 \leq j \leq l+t+s$. For each element $Z \in H^{\mathbf{C}}(l,t,s)$, we set

$$\begin{aligned} Z_0 &:= (Z^1, \ldots, Z^l), \\ Z_- &:= (Z^{l+1}, \ldots, Z^{l+t}), \\ Z_+ &:= (Z^{l+t+1}, \ldots, Z^{l+t+s}), \end{aligned}$$

which are called the 0-component, the $-$-component and the $+$-component of Z, respectively. We often write $Z = (Z_0, Z_-, Z_+)$ for convenience.

Let $M_{(N+P)\times(N+P)}(\mathbf{F})$ denote the set of $(N+P)\times(N+P)$-matrices with entries in $\mathbf{F}(=\mathbf{R}$ or $\mathbf{C})$. Let $O(N,P)$ and $U(N,P)$ be the groups of isometries of $\mathbf{R}_N^{N+P}$ and $\mathbf{C}_N^{N+P}$, respectively, that is,

$$O(N,P) := \left\{O \in M_{(N+P)\times(N+P)}(\mathbf{R}) : {}^tO1_{NP}O = 1_{NP}\right\},$$
$$U(N,P) := \left\{U \in M_{(N+P)\times(N+P)}(\mathbf{C}) : {}^*U1_{NP}U = 1_{NP}\right\},$$

where

$$1_{NP} := \begin{bmatrix} -1_N & \\ & 1_P \end{bmatrix} \in M_{(N+P)\times(N+P)}(\mathbf{R}) \quad \text{and} \quad {}^*U = {}^t\overline{U}.$$

Note that each linear subspace of $\mathbf{R}_N^{N+P}$ can be written as $O(H(l,t,s))$ for some (l,t,s) and some $O \in O(N,P)$. As a result, when we discuss $O(N,P)$-congruence classes of maps into $\mathbf{R}_N^{N+P}$, we only have to consider $H(l,t,s)$ as subspaces of $\mathbf{R}_N^{N+P}$. We remark that the induced metric on $H(0,t,s)(\equiv \mathbf{R}_t^{t+s}) \subset \mathbf{R}_N^{N+P}$ is nondegenerate, while for $l>0$ the induced metric on $H(l,t,s) \subset \mathbf{R}_N^{N+P}$ is degenerate.

Throughout this paper, we always denote by M a connected and simply connected Kähler manifold of complex dimension m. An isometric immersion $f : M \to \mathbf{R}_N^{N+P}$ of M into the real indefinite Euclidean space $\mathbf{R}_N^{N+P}$ is said to be *full in* $H(l,t,s)$ if the image $f(M)$ of f is contained in $H(l,t,s)$ and if the coordinate functions $f^1,\dots,f^l,f^{l+1},\dots,f^{l+t},f^{l+t+1},\dots,f^{l+t+s}$ of f are linearly independent over $\mathbf{R}$.

Definition 2.1. We say that an isometric immersion $f : M \to \mathbf{R}_N^{N+P}$ is *pluriharmonic* if the $(1,1)$-component $\alpha^{(1,1)}$ of the complexified second fundamental form α of f vanishes identically :

$$\alpha^{(1,1)} \equiv 0.$$

Note that f is pluriharmonic if and only if the $\mathbf{C}_N^{N+P}$-valued function $\partial f/\partial z^\alpha$ is holomorphic, where $(z^\alpha) := (z^1,\dots,z^m)$ is a local complex coordinate system on M. An isometric holomorphic immersion $f : M \to \mathbf{C}_N^{N+P}$ is pluriharmonic when regarded as a map $f : M \to \mathbf{R}_{2N}^{2N+2P}$ $(\equiv \mathbf{C}_N^{N+P})$. It is also easy to see that the mean curvature of a pluriharmonic immersion vanishes identically. Conversely, we have the following

Proposition 2.2. *Let f be an isometric immersion of M into $\mathbf{R}_N^{N+P}$, and $P = 2m$, the real dimension of the Kähler manifold M. If the mean* *curvature H of f vanishes identically, then f is pluriharmonic.*

Proof. We choose local orthonormal frames $e_1, \dots, e_m, Je_1, \dots, Je_m$ on M and define $\sqrt{2}E_j := e_j + \sqrt{-1}Je_j \in T^{0,1}M$, where J is the complex structure of M. From the Gauss equation of f and the Kähler condition of M, it follows that

$$\begin{aligned} 0 &= g\left(R(E_k, E_r)\overline{E_r}, \overline{E_k}\right) \\ &= \left\langle \alpha\left(E_k, \overline{E_k}\right), \alpha\left(E_r, \overline{E_r}\right)\right\rangle - \left\langle \alpha\left(E_k, \overline{E_r}\right), \alpha\left(E_r, \overline{E_k}\right)\right\rangle, \end{aligned}$$

where g is the Kähler metric on M, R denotes its curvature tensor and $k, r = 1, \dots, m$. Taking sum with respect to k and r then yields

$$0 = m^2\langle H, H\rangle - \left\langle \alpha^{(1,1)}, \overline{\alpha^{(1,1)}}\right\rangle,$$

which implies that $H = 0$ if and only if $\alpha^{(1,1)} = 0$, since H and $\alpha^{(1,1)}$ are both timelike vectors. □

Dajczer - Rodriguez [**6**] and Ferreira - Tribuzy [**7**] proved that for an isometric immersion $f : M \to \mathbf{R}^P (\equiv \mathbf{R}^P_0)$ the same result is true with no restriction on the dimension P of the ambient Euclidean space.

The following proposition is the indefinite version of a result due to Dajczer - Gromoll [**4**], which can be proved in the same way as in the positive definite case.

Proposition 2.3. *For an isometric pluriharmonic immersion $f : M \to \mathbf{R}^{N+P}_N$, there exists an isometric holomorphic immersion $\Phi : M \to \mathbf{C}^{N+P}_N$ such that $f = \sqrt{2}\,\mathrm{Re}\,\Phi$.*

It is well-known by Calabi's rigidity theorem that an isometric holomorphic immersion of a Kähler manifold into a complex Euclidean space is rigid. The counterpart of this result in the indefinte case is given by the following.

Proposition 2.4. *Let $H^{\mathbf{C}}(l,t,s)$ and $H^{\mathbf{C}}(l',t',s')$ be linear subspaces of $\mathbf{C}^{N+P}_N$ as above. Let $\Phi = (\Phi_0, \Phi_-, \Phi_+) : M \to H^{\mathbf{C}}(l,t,s)$ and $\Psi = (\Psi_0, \Psi_-, \Psi_+) : M \to H^{\mathbf{C}}(l',t',s')$ be isometric holomorphic immersions, respectively. If Φ is full in $H^{\mathbf{C}}(l,t,s)$, then*

1) *$s \le s'$ and $t \le t'$, and*
2) *there exists a unitary transformation $U \in U(t', s')$ such that*

$$\begin{bmatrix} \Psi_- \\ \Psi_+ \end{bmatrix} = U \begin{bmatrix} \Phi_- \\ 0_{t'-t} \\ \Phi_+ \\ 0_{s'-s} \end{bmatrix}.$$

To prove this proposition, we only have to apply Calabi's rigidity theorem to the new isometric holomorphic immersions $(\Psi_-;\Phi_+,0_{s'-s})$ and $(\Phi_-,0_{t'-t};\Psi_+) : M \to \mathbf{C}^{t'+s'}$ constructed from Φ and Ψ. We refer the reader to Abe-Magid [**1**] for details. It should be remarked that we have no relation between Φ_0 and Ψ_0 in this case.

Definition 2.5. A *full* isometric holomorphic immersion of M into $H^{\mathbf{C}}(l,t,s) \subset \mathbf{C}_N^{N+P}$ is called the *shape* of M if $l=0$.

Note that by Proposition 2.4, the *shape* of M is unique up to unitary transformations.

3. A parametrization of the moduli space.

We denote by $\mathcal{M}^f\,(M;\mathbf{R}_N^{N+P})$ the moduli space of *full* isometric pluriharmonic immersions, that is, the set of $O(N,P)$-congruence classes of full isometric pluriharmonic immersions of a Kähler manifold M into $\mathbf{R}_N^{N+P}$.

Our aim of this section is to parametrize $\mathcal{M}^f(M;\mathbf{R}_N^{N+P})$ by the set $\mathcal{P}(\Phi;N,P)$ defined in the following manner.

We assume, throughout this section, that $\mathcal{M}^f\,(M;\mathbf{R}_N^{N+P})$ is not empty. Then it follows from Propositions 2.3 and 2.4 that there exists the *shape* $\Phi : M \to \mathbf{C}_n^{n+p}$ of M. For Φ and integers N and P, we define $\mathcal{P}(\Phi;N,P)$ to be the set of $(n+p)\times(n+p)$-complex matrices satisfying the following conditions (P1) – (P4):

(P1) $$\frac{{}^t\partial\Phi}{\partial z^\alpha}P\frac{\partial\Phi}{\partial z^\beta}=0 \quad (\alpha,\beta=1,\dots,m),$$

(P2) $${}^tP=P,$$

(P3a) $${}^*x_-\left(1_{np}-{}^tP1_{np}\overline{P}\right)x_- \leq 0 \quad \text{for } x_- \in H^{\mathbf{C}}(0,n,0),$$

(P3b) $${}^*x_+\left(1_{np}-{}^tP1_{np}\overline{P}\right)x_+ \geq 0 \quad \text{for } x_+ \in H^{\mathbf{C}}(0,0,p),$$

(P4) $$\operatorname{sign}\left(1_{np}-{}^tP1_{np}\overline{P}\right)=(N-n,P-p),$$

where (P4) means that the Hermitian matrix $1_{np}-{}^tP1_{np}\overline{P}$ has $(N-n)$ negative eigenvalues and $(P-p)$ positive eigenvalues.

First, we give another description of $\mathcal{P}(\Phi;N,P)$ for later use.

Lemma 3.1. *An $(n+p)\times(n+p)$-complex matrix P belongs to $\mathcal{P}(\Phi;N,P)$ if and only if P satisfies* (P1) *and there exist a complex matrix*

$$U \in U(n)\times U(p) := \left\{\begin{bmatrix}A & O\\ O & B\end{bmatrix} \in U(n,p) : A\in U(n),\quad B\in U(p)\right\},$$

and real numbers $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_p$ *satisfying*

(P2′) $$P = {}^tU\mathrm{diag}(-\lambda_1, \dots, -\lambda_n; \mu_1, \dots, \mu_p)U,$$

(P3′) $$-1 \le -\lambda_1 \le \cdots \le -\lambda_n \le 0 \le \mu_p \le \cdots \le \mu_1 \le 1,$$

(P4′a) $$-1 = -\lambda_1 = \cdots = -\lambda_{2n-N} < -\lambda_{2n-N+1},$$

(P4′b) $$\mu_{2p-P+1} < \mu_{2p-P} = \cdots = \mu_1 = 1.$$

Proof. In order to see that $P \in \mathcal{P}(\Phi; N, P)$ is diagonalized as in (P2′), we inductively define subsets $\mathcal{S}^{2n-(2j-1)}$ $(j = 1, \dots, n)$ of $H^{\mathbf{C}}(0, n, 0)$ and vectors $x_j \in \mathcal{S}^{2n-(2j-1)}$ as follows.
Step 1. We set

$$\begin{aligned} \mathcal{S}^{2n-1} &:= \{x = (x_-; 0) \in \mathbf{C}^{n+p}_n : {}^*x1_{np}x = -1\}, \\ -\lambda_1 &:= \inf_{x \in \mathcal{S}^{2n-1}} \mathrm{Re}\,({}^txPx). \end{aligned}$$

Then there exists $x_1 \in \mathcal{S}^{2n-1}$ such that $-\lambda_1 = {}^tx_1Px_1 \le 0$. In fact, since $\mathcal{S}^{2n-1}$ is compact, we have a vector $x_1 \in \mathcal{S}^{2n-1}$ such that $-\lambda_1 = \mathrm{Re}\,({}^tx_1Px_1) \le 0$. Note that if $x \in \mathcal{S}^{2n-1}$ and $\theta := 1/2(\pi - \arg {}^txPx)$, then the vector $e^{\sqrt{-1}\theta}x$ belongs to $\mathcal{S}^{2n-1}$ and $e^{2\sqrt{-1}\theta}({}^txPx) \le \mathrm{Re}\,({}^txPx)$. Hence, $\mathrm{Re}\,({}^tx_1Px_1) = {}^tx_1Px_1$.

Step j. We set

$$\begin{aligned} \mathcal{S}^{2n-(2j-1)} &:= \left\{x = (x_-; 0) \in \mathcal{S}^{2n-(2j-3)} : {}^*x1_{np}x_{j-1} = {}^*x\overline{Px_{j-1}} = 0\right\}, \\ -\lambda_j &:= \inf_{x \in \mathcal{S}^{2n-(2j-1)}} \mathrm{Re}\,({}^txPx). \end{aligned}$$

Then the same argument as in Step 1 assures that there exists $x_j \in \mathcal{S}^{2n-(2j-1)}$ such that $-\lambda_j = {}^tx_jPx_j \le 0$.

Consequently, we obtain vectors $x_1, \dots, x_n \in H^{\mathbf{C}}(0, n, 0)$ such that

$$\begin{aligned} {}^*x_j1_{np}x_k &= -\delta_{jk}, \\ {}^tx_jPx_k &= -\lambda_j\delta_{jk}, \qquad -\lambda_1 \le \cdots \le -\lambda_n \le 0. \end{aligned}$$

In a similar fashion we also obtain vectors $x_{n+1}, \dots, x_{n+p} \in H^{\mathbf{C}}(0, 0, p)$ such that

$$\begin{aligned} {}^*x_{n+j}1_{np}x_{n+k} &= \delta_{jk}, \\ {}^tx_{n+j}Px_{n+k} &= \mu_j\delta_{jk}, \qquad \mu_1 \ge \cdots \ge \mu_p \ge 0. \end{aligned}$$

It is immediate from these that P is diagonalized as in (P2′):

$$\begin{aligned}
&U^{-1} := (x_1, \dots , x_n; x_{n+1}, \dots , x_{n+p}) \in U(n) \times U(p),\\
&{}^tU^{-1}PU^{-1} = \operatorname{diag}(-\lambda_1, \dots , -\lambda_n; \mu_1, \dots , \mu_p),\\
&-\lambda_1 \leq \cdots \leq -\lambda_n \leq 0 \leq \mu_p \leq \cdots \leq \mu_1.
\end{aligned}$$

Now we note that

$$1_{np} - {}^tP1_{np}\overline{P} = {}^*\overline{U}\operatorname{diag}(-(1-\lambda_1^2), \dots , -(1-\lambda_n^2); 1-\mu_1^2, \dots , 1-\mu_p^2)\overline{U}.$$

Then (P3) means that $-(1-\lambda_j^2) \leq 0$ and $1-\mu_j^2 \geq 0$, which implies (P3′). (P4) means that $\operatorname{sign}\left(1_{np} - {}^tP1_{np}\overline{P}\right) = (n-(2n-N), p-(2p-P))$, which is equivalent to (P4′).

Conversely, it is easy to see that matrices satisfying (P1) and (P2′) – (P4′) belong to $\mathcal{P}(\Phi; N, P)$. □

In order to construct a bijection from $\mathcal{M}^f\left(M; \mathbf{R}_N^{N+P}\right)$ to $\mathcal{P}(\Phi; N, P)$, we prepare the following lemmas.

Lemma 3.2. *For each full isometric pluriharmonic immersion $f : M \to \mathbf{R}_N^{N+P}$, there exists an $(N+P) \times (n+p)$-complex matrix S such that*

(S0) $$f = \sqrt{2}\operatorname{Re} S\Phi,$$

(S1) $$\frac{{}^t\partial\Phi}{\partial z^\alpha}{}^tS1_{NP}S\frac{\partial\Phi}{\partial z^\beta} = 0 \qquad (\alpha, \beta = 1, \dots , m),$$

(S2) $${}^*S1_{NP}S = 1_{np},$$

(S3) $$\operatorname{rank}\left(S, \overline{S}\right) = N+P,$$

where $\left(S, \overline{S}\right)$ denotes the $(N+P) \times 2(n+p)$-matrix consisting of S and its complex conjugate $\overline{S}$.

Proof. Recall that by Proposition 2.3, there exists an isometric holomorphic immersion $\Psi : M \to \mathbf{C}_N^{N+P}$ such that $f = \sqrt{2}\operatorname{Re}\Psi$. It also follows from Proposition 2.4 that for Φ and Ψ there exists $U = (u_{IJ}) \in U(N, P)$ $(I, J = 1, \dots , N+P)$ such that

$$\begin{bmatrix}\Psi_-\\ \Psi_+\end{bmatrix} = U\begin{bmatrix}\Phi_-\\ 0_{N-n}\\ \Phi_+\\ 0_{P-p}\end{bmatrix}.$$

Let S be the $(N+P)\times(n+p)$-matrix defined by

$$S := \begin{array}{c} \begin{array}{cc} \overbrace{}^{n} & \overbrace{}^{p} \end{array} \\ \left[\begin{array}{cc} S_1 & S_2 \end{array}\right] \end{array} \}_{N+P},$$

where

$$\begin{aligned} S_1 &:= \left[u_{Ij}\right] \quad (j=1,\dots,n),\\ S_2 &:= \left[u_{I(N+a)}\right] \quad (a=1,\dots,p). \end{aligned}$$

Then we have

$$f = \sqrt{2}\,\mathrm{Re}\ \Psi = \sqrt{2}\,\mathrm{Re}\ S\Phi = \frac{1}{\sqrt{2}}\left(S, \overline{S}\right)\begin{bmatrix}\Phi\\ \overline{\Phi}\end{bmatrix}, \tag{3.1}$$

$$\partial f = \frac{1}{\sqrt{2}}\partial\Psi = \frac{1}{\sqrt{2}}S\partial\Phi. \tag{3.2}$$

Since f and Φ are isometric,

$$0 = 2\frac{{}^t\partial f}{\partial z^\alpha}1_{NP}\frac{\partial f}{\partial z^\beta}, \qquad 2\frac{{}^*\partial f}{\partial z^\alpha}1_{NP}\frac{\partial f}{\partial z^\beta} = \frac{{}^*\partial\Phi}{\partial z^\alpha}1_{np}\frac{\partial\Phi}{\partial z^\beta},$$

which together with (3.2) implies (S1) and (S2). By (3.1), the fullness of f in $\mathbf{R}^{N+P}_N$ is equivalent to (S3). $\square$

Conversely, by reversing the above process it is easy to see the following:

Lemma 3.3.

(1) *Let S be an $(N+P)\times(n+p)$-complex matrix satisfying* (S1), (S2) *and* (S3)*. If we define f as in* (S0)*, then the congruence class $[f]$ of f belongs to $\mathcal{M}^f(M;\mathbf{R}^{N+P}_N)$.*

(2) *Let $f_1 = \sqrt{2}\,\mathrm{Re}\ S_1\Phi$ and $f_2 = \sqrt{2}\,\mathrm{Re}\ S_2\Phi : M^{2n}\to\mathbf{R}^{N+P}_N$ be isometric pluriharmonic immersions. Then $[f_1]=[f_2]$ if and only if ${}^tS_1 1_{NP}S_1 = {}^tS_2 1_{NP}S_2$.*

We also have the following lemma.

Lemma 3.4. *If an $(N+P)\times(n+p)$-matrix S satisfies* (S1),(S2) *and* (S3)*, then ${}^tS1_{NP}S$ belongs to $\mathcal{P}(\Phi;N,P)$.*

Proof. Step 1. By (S1), ${}^tS1_{NP}S$ satisfies (P1).

Step 2. By the same argument as in the proof of Lemma 3.1, we obtain $U\in U(n)\times U(p)$ such that

$$\begin{aligned} &{}^tS1_{NP}S = {}^tU\mathrm{diag}(-\lambda_1,\dots,-\lambda_n;\mu_1,\dots,\mu_p)U,\\ &-\lambda_1\le\cdots\le-\lambda_n\le 0\le\mu_p\le\cdots\le\mu_1. \end{aligned}$$

It follows from (S2) that $-1 \leq -\lambda_1 \leq 0 \leq \mu_1 \leq 1$. In fact, let $V \in U(N,P)$ be a matrix such that

$$VS(\mathcal{S}^{2n-1}) \subset \{y \in H^{\mathbf{C}}(0,N,0) \subset \mathbf{C}^{N+P}_N : {}^*y 1_{NP} y = -1\}\,.$$

Then we have

$$\begin{aligned}
-\lambda_1 &= \inf_{x \in \mathcal{S}^{2n-1}} \operatorname{Re}({}^t x\, {}^t S 1_{NP} S x) = \inf_{y \in S(\mathcal{S}^{2n-1})} \operatorname{Re}({}^t y 1_{NP} y) \\
&= \inf_{y \in VS(\mathcal{S}^{2n-1})} \operatorname{Re}({}^t(V^{-1}y) 1_{NP} (V^{-1}y)) \\
&\geq \inf_{y \in VS(\mathcal{S}^{2n-1})} {}^*(V^{-1}y) 1_{NP} (V^{-1}y) = -1.
\end{aligned}$$

Also, a similar argument applied to μ_1 implies $\mu_1 \leq 1$. Consequently, ${}^t S 1_{NP} S$ satisfies (P2$'$) and (P3$'$).

We proceed to prove that (S3) is equivalent to (P4$'$).

Step 3A. Since $-1 \leq -\lambda_i \leq 0 \leq \mu_j \leq 1$, we can choose complex numbers a_i, b_i, c_j and d_j so that

$$\begin{aligned}
\lambda_i &= a_i^2 + b_i^2, \qquad & 1 &= |a_i|^2 + |b_i|^2 \\
\mu_j &= c_j^2 + d_j^2, \qquad & 1 &= |c_j|^2 + |d_j|^2.
\end{aligned} \tag{3.3}$$

In particular, if $\lambda_i = 1$ (resp. $\mu_j = 1$), we take $a_i = 1$, $b_i = 0$ (resp. $c_j = 1$, $d_j = 0$).

Note that a_i, b_i (resp. c_j, d_j) are linearly dependent over $\mathbf{R}$ if and only if $\lambda_i = 1$ (resp. $\mu_j = 1$).

Step 3B. For these complex numbers a_i, b_i, c_j, d_j and the matrix

$$S = \overbrace{\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}}^{n+p} \begin{array}{l} \}_N \\ \}_P \end{array} \in M_{(N+P)\times(n+p)}(\mathbf{C}),$$

we consider $(2n+2p) \times (n+p)$-matrices $\widetilde{T}$ and $\widetilde{S}$ defined by

$$\widetilde{T} := \overbrace{\begin{bmatrix} \widetilde{T_1} \\ \widetilde{T_2} \end{bmatrix}}^{n+p} \begin{array}{l} \}_{2n} \\ \}_{2p} \end{array} := \begin{bmatrix}
a_1 & & & & & \\
b_1 & & & & & \\
& \ddots & & & & \\
& & a_n & & & \\
& & b_n & & & \\
& & & c_1 & & \\
& & & d_1 & & \\
& & & & \ddots & \\
& & & & & c_p \\
& & & & & d_p
\end{bmatrix} \quad \text{and} \quad \widetilde{S} := \begin{bmatrix} S_1 \\ 0_{2n-N} \\ S_2 \\ 0_{2p-P} \end{bmatrix}.$$

By definition, we have

$$
\begin{aligned}
{}^{t}\left(\widetilde{T}U\right)1_{2n2p}\left(\widetilde{T}U\right) &= {}^{t}\widetilde{S}1_{2n2p}\widetilde{S},\\
{}^{*}\left(\widetilde{T}U\right)1_{2n2p}\left(\widetilde{T}U\right) &= {}^{*}\widetilde{S}1_{2n2p}\widetilde{S} = 1_{np},
\end{aligned}
$$

which implies that there exists $O \in O(2n,2p) = U(2n,2p) \cap O(2n,2p;\mathbf{C})$ such that $O\widetilde{S} = \widetilde{T}U$.

Step 3C. (S3) holds if and only if rank $\left(\widetilde{T_1},\overline{\widetilde{T_1}}\right) = N$ and rank $\left(\widetilde{T_2},\overline{\widetilde{T_2}}\right) = P$.

In fact, rank $\left(S,\overline{S}\right) = N+P$ if and only if we can choose N timelike vectors and P spacelike vectors from the image of $\left(S,\overline{S}\right)$. By Step 3B, this is equivalent to being able to choose these vectors from the image of $\left(\widetilde{T},\overline{\widetilde{T}}\right)$, which means that rank $\left(\widetilde{T_1},\overline{\widetilde{T_1}}\right) = N$ and rank $\left(\widetilde{T_2},\overline{\widetilde{T_2}}\right) = P$.

Step 3D. rank $\left(\widetilde{T_1},\overline{\widetilde{T_1}}\right) = N$ if and only if $1 = \lambda_1 = \cdots = \lambda_{2n-N} > \lambda_{2n-N+1}$, and rank $\left(\widetilde{T_2},\overline{\widetilde{T_2}}\right) = P$ if and only if $1 = \mu_1 = \cdots = \mu_{2p-P} > \mu_{2p-P+1}$.

In fact, by the definition of $\widetilde{T_1}$, rank $\left(\widetilde{T_1},\overline{\widetilde{T_1}}\right) = N$ if and only if there exist $2n-N$ pairs of $\mathbf{R}$-linearly dependent vectors $(a_i,\overline{a_i})$ and $\left(b_i,\overline{b_i}\right)$. Step 3A then implies that this is equivalent to $1 = \lambda_1 = \cdots = \lambda_{2n-N} > \lambda_{2n-N+1}$. The proof for $\widetilde{T_2}$ is similar.

Step 3C combined with Step 3D now implies that (S3) and (P4′) are equivalent, which completes the proof of the lemma. □

We are now in a position to define a natural map $\mathcal{F}$ from $\mathcal{M}^f\left(M;\mathbf{R}^{N+P}_N\right)$ to $\mathcal{P}(\Phi;N,P)$.

Let $[f]$ be an element of $\mathcal{M}^f\left(M;\mathbf{R}^{N+P}_N\right)$. By Lemma 3.2, for each full isometric pluriharmonic immersion $f \in [f]$, we can choose an $(N+P)\times(n+p)$-matrix S satisfying (S0) – (S3). By Lemma 3.4, ${}^{t}S1_{NP}S$ belongs to $\mathcal{P}(\Phi;N,P)$. We then define the map $\mathcal{F}$ by

$$\mathcal{F}([f]) := {}^{t}S1_{NP}S,$$

which is well-defined by Lemma 3.3 2).

With these preparations, we obtain a parametrization of the moduli space of full isometric pluriharmonic immersions as follows.

Theorem 3.5. *Let M be a connected and simply connected Kähler manifold. Suppose M has the* shape *$\Phi : M \to \mathbf{C}^{n+p}_n$. Then the map $\mathcal{F} : \mathcal{M}^f\left(M;\mathbf{R}^{N+P}_N\right) \to \mathcal{P}(\Phi;N,P)$ is bijective.*

Proof. It follows from Lemma 3.3 (2) that $\mathcal{F}$ is injective.

To show that $\mathcal{F}$ is surjective, we claim that for each $P \in \mathcal{P}(\Phi; N, P)$ there exists an $(N+P) \times (n+p)$-matrix S satisfying (S1), (S2) and (S3). First, by Lemma 3.1, there exist $U \in U(n,p)$ and $\lambda_i, \mu_j \in \mathbf{R}$ such that

$$P = {}^tU \operatorname{diag}(\underbrace{-1, \dots, -1}_{2n-N}, -\lambda_{2n-N+1}, \dots, -\lambda_n; \underbrace{1, \dots, 1}_{2p-P}, \mu_{2p-P+1}, \dots, \mu_p)U.$$

Choose complex numbers a_i, b_i, c_j and d_j such that (3.3) holds for these λ_i and μ_j. Then we define an $(N+P) \times (n+p)$-matrix S by

$$S := \begin{bmatrix} -1_{2n-N} & & & & & & & \\ & a_{2n-N+1} & & & & & & \\ & b_{2n-N+1} & & & & & & \\ & & \ddots & & & & & \\ & & & a_n & & & & \\ & & & b_n & & & & \\ & & & & 1_{2p-P} & & & \\ & & & & & c_{2p-P+1} & & \\ & & & & & d_{2p-P+1} & & \\ & & & & & & \ddots & \\ & & & & & & & c_p \\ & & & & & & & d_p \end{bmatrix} U.$$

It can be verified without difficulty that S satisfies (S1), (S2) and (S3), which together with Lemma 3.3 (1) implies that $\mathcal{F}$ is surjective. $\square$

Before closing this section, we now consider the moduli space without assuming the fullness of immersions. Let $\mathcal{M}\left(M; \mathbf{R}^{N+P}_N\right)$ denote the set of $O(N,P)$-congruence classes of isometric pluriharmonic immersions of a Kähler manifold M into $\mathbf{R}^{N+P}_N$. Then we have

$$\mathcal{M}\left(M; \mathbf{R}^{N+P}_N\right) = \coprod_{\substack{0 \le l \le \min(N,P),\\ 0 \le t \le N-l,\\ 0 \le s \le P-l}} \mathcal{M}^f(M; H(l,t,s)).$$

In particular, we have

$$\mathcal{M}\left(M; \mathbf{R}^{N+P}_N\right) \supsetneqq \coprod_{\substack{0 \le t \le N,\\ 0 \le s \le P}} \mathcal{M}^f\left(M; \mathbf{R}^{t+s}_t\right).$$

It should be remarked that $\mathcal{M}\left(M; \mathbf{R}^{N+P}_N\right)$ is not finite-dimensional in general, which contrasts with the fact that the moduli space is finite-dimensional in the positive definite case. In fact, it is not true that $\mathcal{M}^f(M; H(l,t,s))$ is of finite dimension when $l \ge 1$.

4. Examples.

In this section we illustrate some examples of pluriharmonic immersions.

Example 4.1. *Holomorphic immersions.* As mentioned above, all holomorphic immersions are pluriharmonic. For instance,

$$f(z) := {}^t\left(\mathrm{Re}\, z, \mathrm{Im}\, z; \mathrm{Re}\left(\frac{1}{2}z^2 + 2z\right), \mathrm{Im}\left(\frac{1}{2}z^2 + 2z\right)\right)$$

gives rise to a (pluri)harmonic immersion of a non simply connected and non complete Kähler manifold defined by $(\{z \in \mathbf{C} : |z+2| > 1\}, 2(|z+2|^2 - 1)|dz|^2)$ into $\mathbf{R}^4_2$.

Example 4.2. *Product immersions.* Given two pluriharmonic immersions $f_1 : M_1 \to \mathbf{R}^{N_1+P_1}_{N_1}$ and $f_2 : M_2 \to \mathbf{R}^{N_2+P_2}_{N_2}$, we can define product immersion

$$f_1 \times f_2 : M_1 \times M_2 \to \mathbf{R}^{N_1+P_1}_{N_1} \times \mathbf{R}^{N_2+P_2}_{N_2} \equiv \mathbf{R}^{N_1+P_1+N_2+P_2}_{N_1+N_2},$$

which is also pluriharmonic.

Now, let $f = \sqrt{2}\,\mathrm{Re}\,\Phi$ be an isometric pluriharmonic immersion of a simply connected Kähler manifold M into $\mathbf{R}^{N+P}$ $(= \mathbf{R}^{N+P}_0)$, where $\Phi : M \to \mathbf{C}^{N+P}$ is an isometric holomorphic immersion such that

$$\frac{{}^t\partial\Phi}{\partial z^\alpha}\frac{\partial\Phi}{\partial z^\beta} = 0. \tag{4.1}$$

For $\Phi = {}^t(\Phi^1, \dots, \Phi^N, \Phi^{N+1}, \dots, \Phi^{N+P})$ we consider a new immersion

$$\widetilde{\Phi} := {}^t\left(\sqrt{-1}\Phi^1, \dots, \sqrt{-1}\Phi^N; \Phi^{N+1}, \dots, \Phi^{N+P}\right) : \widetilde{M} \to \mathbf{C}^{N+P}_N,$$

where $\widetilde{M}$ is a Kähler manifold defined by

$$\left(\left\{z \in M : \widetilde{\Phi}^*\langle\cdot, \bar{\cdot}\rangle_{\mathbf{C}^{N+P}_N}(z) > 0\right\}, \quad \tilde{g} := \widetilde{\Phi}^*\langle\cdot, \bar{\cdot}\rangle_{\mathbf{C}^{N+P}_N}\right).$$

Then the map $\tilde{f}$ defined by $\tilde{f} := \sqrt{2}\,\mathrm{Re}\,\widetilde{\Phi} : \widetilde{M} \to \mathbf{R}^{N+P}_N$ gives rise to a pluriharmonic immersion, since

$$\frac{{}^t\partial\widetilde{\Phi}}{\partial z^\alpha} 1_{NP} \frac{\partial\widetilde{\Phi}}{\partial z^\beta} = 0.$$

To sum up, in order to obtain (locally defined) pluriharmonic immersions into $\mathbf{R}^{N+P}_N$, we only have to construct holomorphic immersions into $\mathbf{C}^{N+P}$ satisfying the condition (4.1).

Example 4.3. *Cone immersions.* As an example, we shall construct pluriharmonic immersions of subsets of $\mathbf{C}^2$ into $\mathbf{R}_1^5$.

As remarked above, it suffices to define holomorphic immersions into $\mathbf{C}^{1+4}$ satisfying the condition (4.1). Let C and D be simply connected domains of $\mathbf{C}$. Suppose that $\psi : C \to \mathbf{C}$ is a holomorphic function and $\phi : D \to \mathbf{C}^{1+4}$ is a holomorphic immersion such that

$$ {}^t\phi\phi = {}^t\phi\frac{\partial\phi}{\partial z} = \frac{{}^t\partial\phi}{\partial z}\frac{\partial\phi}{\partial z} = 0, \tag{4.2}$$

where z is a coordinate of D. Then the holomorphic immersion $\Phi(w, z) := \psi(w)\phi(z) : C \times D \to \mathbf{C}^{1+4}$ satisfies (4.1), from which we obtain a pluriharmonic immersion of a subset of $C \times D$ into $\mathbf{R}_1^5$.

We can construct ϕ as follows. For any holomorphic function h on D we set

$$\begin{aligned} g(z) &:= {}^t(g^1(z), g^2(z), g^3(z)) \\ &:= \int^z {}^t(1 - h(\zeta)^2, \sqrt{-1}(1 + h(\zeta)^2), 2h(\zeta))d\zeta. \end{aligned}$$

Then

$$\phi(z) := {}^t(1 - {}^tg(z)g(z), \sqrt{-1}(1 + {}^tg(z)g(z)), 2g^1(z), 2g^2(z), 2g^3(z))$$

gives rise to a holomorphic immersion satisfying the condition (4.2).

If we choose $\psi(w) := w$ and $h(z) := z$, the corresponding pluriharmonic immersion is

$$\begin{aligned} \widetilde{f}(w, z) &= \sqrt{2}\,\mathrm{Re}\;\widetilde{\Phi}(w, z) \\ &= \sqrt{2}\,\mathrm{Re}\left(w \begin{bmatrix} \sqrt{-1}\ (1 + \frac{1}{3}z^4) \\ \sqrt{-1}\ (1 - \frac{1}{3}z^4) \\ 2(z - \frac{1}{3}z^3) \\ \sqrt{-1}\ 2(z + \frac{1}{3}z^3) \\ 2z^2 \end{bmatrix} \right) : \mathbf{C}^2 \supset \widetilde{C \times D} \to \mathbf{R}_1^5. \end{aligned}$$

It should be pointed out that we may use a class of complex ruled immersions obtained by Dajczer - Gromoll **[5]** as the above Φ, which provides us with a larger class containing cone immersions.

References

[1] K. Abe and M.A. Magid, *Indefinite rigidity of complex submanifolds and maximal surfaces*, Math. Proc. Camb. Phil. Soc., **106** (1989), 481-494.

[2] ———, *Complex analytic curves and maximal surfaces*, Monatsh. Math., **108** (1989), 255-276.

[3] E. Calabi, *Quelques applications de l'analyse complexe aux surfaces d'aire minima*, Topics in complex manifolds (by Rossi, H.), Univ. Montreal, (1968), 59-81.

[4] M. Dajczer and D. Gromoll, *Real Kaehler submanifolds and uniqueness of the Gauss maps*, J. Differential Geom., **22** (1985), 13-28.

[5] ———, *The Weierstrass representation for complete minimal real Kaehler submanifolds of codimension two*, Invent. Math., **119** (1995), 235-242.

[6] M. Dajczer and L. Rodriguez, *Rigidity of real Kaehler submanifolds*, Duke Math. J., **53** (1986), 211-220.

[7] M.J. Ferreira and R. Tribuzy, *On the type decomposition of the second fundamental form of a Kähler submanifold*, Rend. Sem. Mat. Univ. Padova, **94** (1995), 17-23.

[8] H. Furuhata, *Construction and classification of isometric minimal immersions of Kähler manifolds into Euclidean spaces*, Bull. London Math. Soc., **26** (1994), 487-496.

Received August 26, 1994 and revised October 5, 1995. The author was partially supported by the JSPS Research Fellowships for Young Scientists, and the Grants-in-Aid for Encouragement of Young Scientists, The Ministry of Education, Science and Culture, Japan.

TÔHOKU UNIVERSITY
KATAHIRA SENDAI 980-77, JAPAN
E-mail address: furuhata@math.is.tohoku.ac.jp