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PROOF OF LONGUERRE'S THEOREM AND ITS EXTENSIONS BY THE METHOD OF POLAR COORDINATES

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There are several methods to prove the well-known Longuerre's theorem and its extensions in plane geometry. We now prove them by the method of polar coordinates. Our proof is characterized by its directness, simplicity, regularity, originality, and no need for any auxiliary lines.

Longuerre's Theorem. *Let $A_1A_2A_3A_4$ be a quadrilateral inscribed in a circle, on which p is an arbitrary point. Let S_i denote the Simson line of point p with respect to the triangle $A_jA_kA_l$ $(i, j, k, l$ distinct) and let D_i denote the projection of p on S_i.*

The four points D_1, D_2, D_3, D_4 are collinear.

Proof. We establish a polar coordinates system (Fig. 1) with p being the pole and the extension line of po being the polar axis. Let d be the diameter of the circle. Hence the equation of the circle is $\rho = d\cos\theta$. Let $(d\cos\theta_i, \theta_i)((i = 1, 2, 3, 4), \theta_i \in [0, 2\pi])$ be the coordinates of A_1, A_2, A_3, A_4. Hence the two-point form equation of A_1A_2 is

$$\frac{\sin(\theta_2-\theta_1)}{\rho} = \frac{\sin(\theta_2-\theta)}{d\cos\theta_1} + \frac{\sin(\theta-\theta_1)}{d\cos\theta_2},$$

$$\therefore \rho[\sin(\theta_2-\theta)\cos\theta_2 + \sin(\theta-\theta_1)\cos\theta_1]$$
$$= d\sin(\theta_2-\theta_1)\cos\theta_1\cos\theta_2.$$
$$\therefore \frac{1}{2}\rho[\sin(2\theta_2-\theta) + \sin(\theta-2\theta_1)]$$
$$= d\sin(\theta_2-\theta_1)\cos\theta_1\cos\theta_2,$$
$$\therefore\ \rho\sin(\theta_2-\theta_1)\cos(\theta-\theta_1-\theta_2) = d\sin(\theta_2-\theta_1)\cos\theta_1\cos\theta_2,$$
$$\because\ \sin(\theta_2-\theta_1) \neq 0,$$
$$\therefore\ \rho\cos(\theta-\theta_1-\theta_2) = d\cos\theta_1\cos\theta_2.$$

This is exactly the normal form equation of A_1A_2. Hence we have the coordinates of the foot B_1 at which pB_1 is normal to A_1A_2 : $B_1(d\cos\theta_1\cos\theta_2, \theta_1+\theta_2)$. By means of cyclic permutation of indices we get $B_2(d\cos\theta_2\cos\theta_3,$

$\theta_2 + \theta_3)$, $B_3(d\cos\theta_3\cos\theta_1,\ \theta_3+\theta_1)$. Obviously the coordinates of the three feet B_i satisfy the normal form equation

$$\rho\cos(\theta-\theta_1-\theta_2-\theta_3) = d\cos\theta_1\cos\theta_2\cos\theta_3.$$

Hence we get the normal form equation of the Simson line S_1 of point p with respect to $\Delta A_1A_2A_3$:

$$S_1\colon \rho\cos(\theta-\theta_1-\theta_2-\theta_3) = d\cos\theta_1\cos\theta_2\cos\theta_3.$$

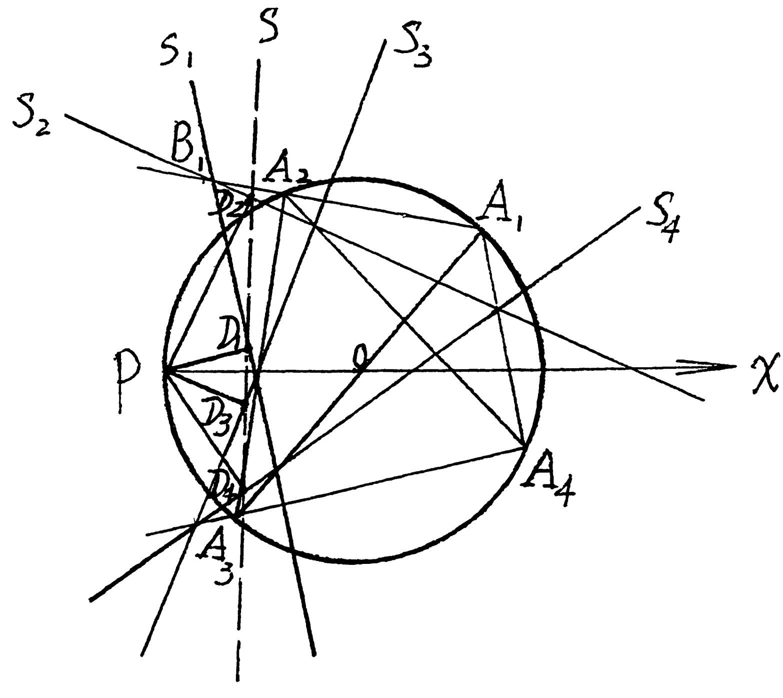

Figure 1.

Similarly, by the same means we can obtain the normal form equations of the other three Simson lines with respect to $\Delta A_jA_kA_l$. They are

$$S_2\colon \rho\cos(\theta-\theta_1-\theta_2-\theta_4) = d\cos\theta_1\cos\theta_2\cos\theta_4,$$
$$S_3\colon \rho\cos(\theta-\theta_2-\theta_3-\theta_4) = d\cos\theta_2\cos\theta_3\cos\theta_4,$$
$$S_4\colon \rho\cos(\theta-\theta_3-\theta_4-\theta_1) = d\cos\theta_3\cos\theta_4\cos\theta_1.$$

Hence the coordinates of the four projections D_i are:

$$D_1(d\cos\theta_1\cos\theta_2\cos\theta_3,\ \theta_1+\theta_2+\theta_3),$$

$$D_2(d\cos\theta_1\cos\theta_2\cos\theta_4,\ \theta_1+\theta_2+\theta_4),$$
$$D_3(d\cos\theta_2\cos\theta_3\cos\theta_4,\ \theta_2+\theta_3+\theta_4),$$
$$D_4(d\cos\theta_3\cos\theta_4\cos\theta_1,\ \theta_3+\theta_4+\theta_1).$$

It is obvious that the above-mentioned coordinates satisfy the normal form equation of the line

$$S\colon \rho\cos(\theta-\theta_1-\theta_2-\theta_3-\theta_4)=d\cos\theta_1\cos\theta_2\cos\theta_3\cos\theta_4.$$

Thus the four points D_i are collinear. □

The above equation of S represents a straight line containing the points D_i and this straight line is named the Simson line of a point p with respect to four concyclic points.

Extension I. *Let A_1, A_2, A_3, A_4, A_5 be five points on a circle and let p be an arbitrary point on this circle. Let S_i denote the Simson line of p with respect to the 4-tuple $A_jA_kA_lA_m$ (i,j,k,l,m distinct) and let D_i denote the projection of p on the line S_i ($i=1,2,3,4,5$). Then the five points D_i are collinear.*

Proof. We establish a polar coordinates system (Fig. 2) with p being the pole and the extension line of po being the polar axis. Let d be the diameter of the circle. The equation of the circle is $\rho=d\cos\theta$. Let $(d\cos\theta_i,\ \theta_i)(\theta_i\in[0,2\pi])$ be the coordinates of A_i. According to the above Longuerre's theorem and its proof we can get the normal form equations of S_i. They are:

$$S_1\colon \rho\cos(\theta-\theta_1-\theta_2-\theta_3-\theta_4)=d\cos\theta_1\cos\theta_2\cos\theta_3\cos\theta_4,$$
$$S_2\colon \rho\cos(\theta-\theta_1-\theta_2-\theta_3-\theta_5)=d\cos\theta_1\cos\theta_2\cos\theta_3\cos\theta_5,$$
$$S_3\colon \rho\cos(\theta-\theta_2-\theta_3-\theta_4-\theta_5)=d\cos\theta_2\cos\theta_3\cos\theta_4\cos\theta_5,$$
$$S_4\colon \rho\cos(\theta-\theta_3-\theta_4-\theta_5-\theta_1)=d\cos\theta_3\cos\theta_4\cos\theta_5\cos\theta_1,$$
$$S_5\colon \rho\cos(\theta-\theta_4-\theta_5-\theta_1-\theta_2)=d\cos\theta_4\cos\theta_5\cos\theta_1\cos\theta_2.$$

Hence the coordinates of the five projections D_i are:

$$D_1(d\cos\theta_1\cos\theta_2\cos\theta_3\cos\theta_4,\ \theta_1+\theta_2+\theta_3+\theta_4),$$
$$D_2(d\cos\theta_1\cos\theta_2\cos\theta_3\cos\theta_5,\ \theta_1+\theta_2+\theta_3+\theta_5),$$
$$D_3(d\cos\theta_2\cos\theta_3\cos\theta_4\cos\theta_5,\ \theta_2+\theta_3+\theta_4+\theta_5),$$
$$D_4(d\cos\theta_3\cos\theta_4\cos\theta_5\cos\theta_1,\ \theta_3+\theta_4+\theta_5+\theta_1),$$
$$D_5(d\cos\theta_4\cos\theta_5\cos\theta_1\cos\theta_2,\ \theta_4+\theta_5+\theta_1+\theta_2).$$

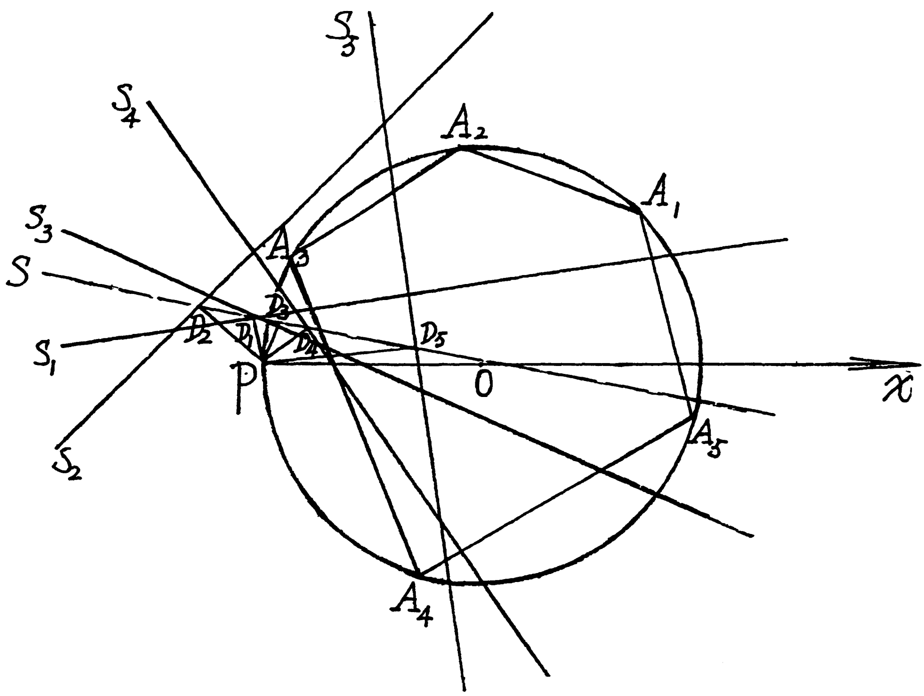

Figure 2.

Clearly the coordinates of D_i satisfy the normal form equation of the line

$$S\colon \rho\cos\left(\theta - \sum_{i=1}^{5}\theta_i\right) = d\prod_{i=1}^{5}\cos\theta_i.$$

Thus the five points D_i are collinear. □

Extension II. *Let $A_1, A_2, \ldots, A_n$ be n points on a circle and let p be an arbitrary point on this circle. Let S_i denote the Simson line of p with respect to the $(n-1)$-tuple $(n-1)$-gonal polygon $A_jA_k\cdots A_x$ $(i, j, k, \ldots, x$ distinct) and let D_i denote the projection of p on the line S_i $(i = 1, 2, \ldots, n)$.*

Then the n points D_i are collinear.

Proof. We again establish a polar coordinates system with p being the pole and the extension line of po being the polar axis. One can immediately verify that

$$\rho\cos\left(\theta - \sum_{i=1}^{n}\theta_i\right) = d\prod_{i=1}^{n}\cos\theta_i$$

represents a straight line containing the n points D_i. Hence the n points D_i are collinear. $\square$

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