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LOCAL AND GLOBAL PLURISUBHARMONIC DEFINING FUNCTIONS

ALAN NOELL

Let D be a bounded pseudoconvex domain with real-analytic boundary in $\mathbf{C}^2$ which satisfies a geometric condition on the set of points where the Levi form degenerates. If locally D has smooth defining functions plurisubharmonic on the boundary, then D has a global smooth defining function plurisubharmonic on the boundary.

1. Introduction.

In analysis on pseudoconvex domains one often encounters the problem of passing from local to global information. Here we consider this problem as it relates to defining functions plurisubharmonic on the boundary of certain pseudoconvex domains in $\mathbf{C}^2$, in the sense that at each boundary point the Levi form of the defining function is positive semi-definite on all vectors in $\mathbf{C}^2$, not just complex tangent vectors to the boundary (cf. [**1**]). In [**3**], Fornæss constructed a bounded pseudoconvex domain with real-analytic boundary in $\mathbf{C}^2$ which near each boundary point has a local defining function plurisubharmonic on the boundary, but which nevertheless has no global defining function plurisubharmonic on the boundary. This domain has the geometric property that the set of points where the Levi form degenerates is a curve which always points in the direction of the complex tangent space to the boundary. Our main result is that if a domain is linearly regular—which essentially requires that this geometric property does not hold—then one can pass from local to global defining functions plurisubharmonic on the boundary:

Theorem. *Let D be a linearly regular domain with real-analytic boundary in $\mathbf{C}^2$, and suppose that for each $p \in D$ there is a neighborhood U_p of p on which D has a smooth defining function which is plurisubharmonic on $\partial D \cap U_p$. Then D has a global smooth defining function plurisubharmonic on ∂D.*

The precise definition of linear regularity is given in the next section. For now we note that it has proved to be relevant to similar problems. This condition was introduced in [**5**] for real-analytic domains in $\mathbf{C}^2$ in connection with the study of local and global peak sets. In [**6**] the condition of linear regularity was extended to smooth bounded pseudoconvex domains in $\mathbf{C}^n$ and applied to the study of holomorphic embeddings to convex domains. It was shown that the above example of Fornæss is locally biholomorphic to convex domains even though there can be no global embedding into a convex domain. One motivation for the present paper is the study of this problem of passage from local to global maps. To embed strongly pseudoconvex domains in convex domains, Fornæss in [**2**] patches certain local strongly plurisubharmonic functions.

Based on the background material in §2, a preparatory lemma and the theorem are proved in §3.

2. Background.

If ϕ is a smooth (i.e., infinitely differentiable) function defined near $p \in \mathbf{C}^n(z_1, \dots, z_n)$, for $t = (t_1, \dots, t_n) \in \mathbf{C}^n$ we write $\partial\phi_p(t)$ for $\sum_{j=1}^n \frac{\partial \phi}{\partial z_j}(p)t_j$ and $L_p(\phi, t)$ for $\sum_{j,k=1}^n \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}(p)t_j\bar{t}_k$, the Levi form of ϕ at p applied to t. Let D denote a bounded pseudoconvex domain with smooth boundary in $\mathbf{C}^n$, and let r be a local defining function near $p \in \partial D$. The complex tangent space to ∂D at p is denoted $T_p^{\mathbf{C}}(\partial D)$ and defined to be $\{t \in \mathbf{C}^n : \partial r_p(t) = 0\}$; this is independent of the choice of local defining function. We say D is linearly regular if there does not exist a smooth curve γ in ∂D so that $\gamma'(t)$ lies in $N(\gamma(t))$ for all t; here $N(p) = \{t \in T_p^{\mathbf{C}}(\partial D) : L_p(r, t) = 0\}$, the null space in the complex tangent space of the Levi form at p.

Let Y be a smooth submanifold of ∂D and $p \in Y$. We say Y is complex-tangential at p if $T_p(Y) \subset T_p^{\mathbf{C}}(\partial D)$; here $T_p(Y)$ denotes the (real) tangent space to Y at p. We say Y is totally real if its tangent space at each point contains no nontrivial complex subspace.

Now we fix a bounded pseudoconvex domain D with real-analytic boundary in $\mathbf{C}^2$, and we let S denote the set of weakly pseudoconvex boundary points of D, i.e., the set of all $p \in \partial D$ such that $N(p) \neq \{0\}$. The stratification of semi-analytic sets given by Lojasiewicz is basic to the proof of our theorem. In [**4**], Fornæss and Øvrelid applied this stratification to the study of the semi-analytic set S, and their result was modified slightly in [**7**, Prop. 1.1] to give the following.

Proposition. *There exist pairwise disjoint real-analytic manifolds* S_0, S_1, S_2 *so that* $S = S_0 \cup S_1 \cup S_2$. *Further:*

(a) *For each* j, S_j *has finitely many components, and these components are totally real, real-analytic manifolds of dimension* j.

(b) *S_1 is closed in $\partial D \setminus S_0$, and S_2 is closed in $\partial D \setminus (S_0 \cup S_1)$.*

(c) *If E is a component of S_1 then either E is everywhere complex-tangential or E is nowhere complex-tangential.*

Assume now that D is linearly regular. Since $\partial D \subset \mathbf{C}^2$, its complex tangent space has (complex) dimension one at each point, so the condition of linear regularity reduces to the requirement that S contain no complex-tangential curve. Trivially, then, there can be no complex-tangential component in $S_1 \subset S$. Further, S_2 must be empty; otherwise, as each component of S_2 is totally real, it would be possible to find a complex-tangential curve in $S_2 \subset S$.

3. Patching plurisubharmonic functions.

The main difficulty in the proof of the theorem is to patch local defining functions along the set S of weakly pseudoconvex boundary points. The stratification of S described in the previous section essentially reduces this to patching along nowhere complex-tangential curves, and the following patching lemma accomplishes that.

Lemma. *Let D be as in the theorem, and let γ be a nowhere complex-tangential real-analytic curve in ∂D. Suppose there exist subsets E_1, E_2 of γ which are relatively open and connected in γ so that $E = E_1 \cap E_2 \neq \emptyset$, and suppose for each j there exist a neighborhood U_j of E_j and a smooth defining function r_j on U_j which is plurisubharmonic on $\partial D \cap U_j$. Then there exists a smooth defining function r on a neighborhood U of $E_1 \cup E_2$ so that r is plurisubharmonic on $\partial D \cap U$ and $r \equiv r_j$ near $E_j \setminus E$ for each j.*

Proof. Fix a point $p \in E$. We work in a neighborhood V of p which we will shrink without comment. Here is the basic idea of the construction of r. It is not hard to see that there exist holomorphic coordinates on V in which $p = (0,0)$ and $E \cap V$ is a segment of a real coordinate line which is (real) orthogonal to the complex tangent space to ∂D at each point. Assume that the negative axis of this line points in the direction of E_1. Then for r we would like to take the function $(1-\chi)r_1 + \chi r_2$, where χ is a smooth function along this line which is identically 0 near a fixed point on the negative axis and identically 1 near a fixed point on the positive axis. This choice yields a function plurisubharmonic on $E \cap V$; but, because the complex tangent space will be different at points off of $E \cap V$, this function need not be plurisubharmonic at those points. To compensate for this we make a smooth but non-holomorphic change of coordinates in which we can still define χ as a function of a single variable and get the desired result.

Let J denote the almost-complex structure on $T(\mathbf{C}^2)$, corresponding to multiplication by i. We define a smooth vector field X by $X(q) = J$ grad $r_1(q)$ for $q \in V$; so, $X(q)$ is (real) orthogonal to $T_q^{\mathbf{C}}(\partial D)$. It is a standard fact that there exists a smooth change of coordinates Φ on V in which, if τ is the first (real) coordinate of Φ, $X = \frac{\partial}{\partial \tau}$. Then as above we may assume that $E \cap V$ is a segment of the τ-axis in these coordinates, which we may further assume to contain $[-1, 1]$. Choose a smooth function κ of τ so that $0 \leq \kappa \leq 1$, $\kappa \equiv 0$ near -1, and $\kappa \equiv 1$ near 1. Thinking of τ as a function of z, define $\chi = \kappa \circ \tau$ and $r = (1 - \chi)r_1 + \chi r_2$.

To complete the proof we need to show that r is plurisubharmonic on $\partial D \cap V$. Fix $q \in \partial D \cap V$. We may assume that, in holomorphic coordinates $z = (z_1, z_2)$ with $v = \text{Im } z_2$, we have $X(q) = \frac{\partial}{\partial v}$, and so $T_q^{\mathbf{C}}(\partial D) = \{(t_1, t_2) : t_2 = 0\}$. One computes that, for $t \in \mathbf{C}^2$,

$$\begin{aligned} L(r, t) = (1 - \chi(q))L(r_1, t) - 2\,\text{Re}\left[\partial\chi(t)\overline{\partial r_1(t)}\right] - r_1(q)L(\chi, t) \\ + \chi(q)L(r_2, t) + 2\,\text{Re}\left[\partial\chi(t)\overline{\partial r_2(t)}\right] + r_2(q)L(\chi, t). \end{aligned}$$

(Here, for ease of notation, we have suppressed the subscripts involving q.) The third and sixth terms on the right are zero since $q \in \partial D$. To conclude the proof we need only show that the second and fifth terms are zero also. But direct computation gives that, with $t = (t_1, t_2)$, $\partial\chi(t)$ is a real multiple of it_2, while both $\partial r_1(t)$ and $\partial r_2(t)$ are real multiples of t_2. Hence each term in brackets is pure imaginary, as desired. □

Proof of the theorem. Put $V = \cup\{U_p : p \in S_0\}$ and $K = S \setminus V$. Since K is compact, we may choose finitely many of the neighborhoods U_p (with $p \in S_1$) which cover K. Refining this cover if necessary, we may then apply the lemma finitely many times to get a smooth local defining function plurisubharmonic on $\partial D \cap W$ for a neighborhood W of K. Then we apply the lemma again to get a smooth local defining function r plurisubharmonic on $\partial D \cap U$, where $U = V \cup W$ is a neighborhood of S.

Now we apply a procedure due to Kohn to get defining functions strictly plurisubharmonic away from S. For $M > 0$ put $s = \exp(Mr) - 1$. One computes that, for $p \in U$,

$$(*) \qquad L_p(s, t) = M \exp(Mr(p)) \left[M|\partial r_p(t)|^2 + L_p(r, t)\right].$$

So s is a smooth local defining function plurisubharmonic on $\partial D \cap U$. Further, from $(*)$ and the fact that $L_p(r, t) > 0$ for $p \in (\partial D \cap U) \setminus S$ and nonzero $t \in T_p^{\mathbf{C}}(\partial D)$, it follows that there exists a neighborhood $U' \subset\subset U$ of S so that s is strictly plurisubharmonic on $(\partial D \cap U) \setminus U'$, where we have the estimate

$$(**) \qquad L_p(s,t) \geq c\left[M^2|\partial r_p(t)|^2 + M||t||^2\right];$$

here $c > 0$ is a constant independent of M. Also, let σ be a global defining function for D which, after using Kohn's construction again, we may assume is strictly plurisubharmonic on a neighborhood of $\partial D \setminus U'$, satisfying the appropriate version of the estimate $(**)$ there.

Pick a smooth function χ so that $0 \leq \chi \leq 1$, $\chi \equiv 0$ near $\overline{U'}$, and $\chi \equiv 1$ near $\partial D \setminus U$. We claim that, for large M, $(1-\chi)s + \chi\sigma$ is plurisubharmonic on $\partial D \cap U$. Clearly this holds away from $U \setminus U'$. As in the proof of the lemma, one computes that, for $p \in \partial D \cap U$ and $t \in \mathbf{C}^2$,

$$L_p((1-\chi)s,t) = (1-\chi(p))L_p(s,t) - 2\,\mathrm{Re}\left[\partial\chi_p(t)\overline{\partial s_p(t)}\right].$$

Note that the second term on the right is at least $-aM||t||\,|\partial r_p(t)|$ for some constant $a > 0$ independent of M. Now recall the elementary inequality

$$\alpha\beta \leq A\alpha^2 + \frac{1}{4A}\beta^2,$$

which holds if $A > 0$ and $\alpha, \beta \in \mathbf{R}$. We apply this with $\alpha = ||t||$, $\beta = |\partial r_p(t)|$, and A small. Together with $(**)$, this gives a lower estimate for $L_p((1-\chi)s,t)$. A similar estimate holds for $L_p(\chi\sigma,t)$. Combining these gives the claim. □

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Oklahoma State University
Stillwater, OK 74078-1058
E-mail address: noell@math.okstate.edu