PLANCHEREL FORMULAE FOR NON-SYMMETRIC POLAR HOMOGENEOUS SPACES

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Let G be a semisimple Lie group and let H be a closed reductive subgroup of G. The homogeneous space $X = G/H$ is called a semisimple homogeneous space. A fundamental goal of harmonic analysis is to understand the group action of G on the various function spaces of X. In particular, the L^2-harmonic analysis on X is to decompose $L^2(X)$ as a direct integral of irreducibles, i.e., to find a family of irreducible unitary representations $\{V_\omega \mid \omega \in \Omega\}$ of G, and a measure ν on the set Ω, so that

$$(1.1) \qquad L^2(X) \cong \int_\Omega V_\Omega d\nu(\omega) \quad \text{(unitary G-isomophism)}.$$

The above decomposition is called the Plancherel formula for the homogeneous space X. In this paper we prove the Plancherel formulae for some non-symmetric semisimple homogenous spaces.

1. Introduction.

The equivalence classes of the representations of G on the minimal closed invariant subspace of $L^2(X)$ is called the discrete series for X. The discrete series may not exist for some homogeneous spaces. In the case $X = G/\{e\}$ is a semisimple Lie group, Harish-Chandra proved that G has discrete series if and only if the real rank of G is equal to the rank of K ([**HC**]). In other words, G has discrete series if and only if G has a compact Cartan subgroup. Harish-Chandra also obtained a classification of the discrete series for semisimple Lie groups. In the case $X = G/H$ is a semisimple symmetric space, Flensted-Jensen ([**FJ**]) proved that G/H has discrete series if

$$(1.2) \qquad \text{rank of } G/H = \text{ rank of } K/K \cap H.$$

Later on Oshima and Matsuki ([**OM**]) proved if G/H has discrete series then it must satisfy the rank condition (1.2). That is to say, the semisimple symmetric space G/H has discrete series if and only if it has a compact

Cartan subspace. Oshima and Matsuki also gave a description of all discrete series for semisimple symmetric spaces. Since the group G can be regarded as symmetric space $G \times G / \mathrm{d}(G)$, the results of Flensted-Jensen, Othima and Matsuki are the generalizations of that of Harish-Chandra.

It is then a natural question to ask when a semisimple homogeneous space G/H has discrete series? Based on the observation of the results mentioned above, we would guess G/H has discrete series if and and if it has a compact Cartan subspace. Then we face the problem how to define a Cartan subspace for a semisimple homogeneous space. The author succeeded in defining Cartan subspaces for a class of semisimple homogeneous spaces in [**H**]. These spaces are called polar spaces and they are natural generalizations of semisimple symmetric spaces. In this paper we prove that a polar G/H has discrete series if and only if it has a compact Cartan subspace. The main purpose of the paper is to describe all discrete series and prove the Plancherel formulae for non-symmetric polar spaces.

The method we use is as follows. As we will recall in Section 2 the simple non-symmetric noncompact polar spaces are essentially $\mathrm{Spin}_e(3,4) / \mathrm{G}_{2(2)}$, $\mathrm{G}_{2(2)} / \mathrm{SU}(2,1)$ and $\mathrm{G}_{2(2)} / \mathrm{SL}(3,\mathbb{R})$. These non-symmetric polar homogeneous space are all isomorphic to some real hyperbolic symmetric spaces $G'/H' = \mathrm{SO}_e(p,q) / \mathrm{SO}_e(p,q-1)$. We have

$$\mathrm{Spin}_e(3,4) / \mathrm{G}_{2(2)} \cong \mathrm{SO}_e(4,4) / \mathrm{SO}_e(4,3),$$
$$\mathrm{G}_{2(2)} / \mathrm{SU}(2,1) \cong \mathrm{SO}_e(4,3) / \mathrm{SO}_e(4,2),$$
$$\mathrm{G}_{2(2)} / \mathrm{SL}(3,\mathbb{R}) \cong \mathrm{SO}_e(3,4) / \mathrm{SO}_e(3,3).$$

Moreover, these isomorphisms are induced by the embedding of G into G', if we denote by G'/H' the corresponding real hyperbolic spaces. It is our strategy to make use of the powerful results ([**R**]) on real hyperbolic spaces to obtain the decomposition (1.1) for polar spaces. The magic is that any irreducible unitary representation of G' occurring in the decomposition for real hyperbolic space remains irreducible when it is restricted to G, no matter if it is in the continuous spectrum or in the set of discrete series. Therefore the decomposition (1.1) for some real hyperbolic spaces induces the decomposition for polar spaces. The strategy used in this paper may be generalized to study various harmonic analysis problems, such as discrete series, on a larger class of semisimple homogeneous spaces by using known results on semisimple symmetric spaces.

E. Heintze, R. Palais, C.-L. Terng and G. Thorbergsson have done an extensive study of isometric actions on polar Riemannian manifold and k-flat homogeneous spaces [**HPPT**]. T. Kobayashi and T. Uzawa [**K**] obtained a description of discrete series of $\mathrm{G}_{2(2)} / \mathrm{SU}(2,1)$ and $\mathrm{G}_{2(2)} / \mathrm{SL}(3,\mathbb{R})$ in a

different parameters by their beautiful method of discrete decomposition of the restriction of $A_q(\lambda)$-modules to reductive subgroups. Kobayashi [**K**] also proved the existence of discrete series for a lot of non-symmetric homogeneous spaces.

This paper is organized as following. In Section 2 we recall the definition of polar spaces. In Section 3 we define the principal series for X. In Section 4 we obtain Plancherel formula for X. In Section 5 we discuss the representations of eigenspace spaces of G-invariant differential operators on X and obtain a Poisson integral formula for K-finite eigenfunctions. The author would like to thank Nolan Wallach for helpful discussions.

2. Polar spaces and real hyperbolic spaces.

Suppose G is a connected semisimple compact Lie group and H a closed subgroup. In order for the homogeneous space G/H to have the same kind of properties of a compact symmetric space, the first thing we expect is that G/H has a Cartan subspace. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H respectively and $\mathfrak{q}$ the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$. The tangent space of G/H at the identity eH can be identified with the vector space $\mathfrak{q}$. We want that all maximal abelian subalgebras contained in $\mathfrak{q}$ are conjugated by the action of H. This action of H is the adjoint action of G restricted to H and will be denoted by $\mathrm{Ad}_G(H)$. Note that the representation space of $\mathrm{Ad}_G(H)$ is $\mathfrak{q}$, which is a vector space over $\mathbb{R}$. It is very natural to require $\mathrm{Ad}_G(H)$ on $\mathfrak{q}$ to be a polar representation as defined below (Definition 2.2).

Let H be a compact Lie group and $\mathfrak{h}$ be its Lie algebra. Consider a representation of H on a real vector space V. Let $\langle\,,\,\rangle$ be a H-invariant inner product on V. For a vector $v \in V$ let a_v be the subspace of V defined by
$$a_v = \{w \in V | \langle w, \mathfrak{h} \cdot v \rangle = 0\}.$$
In other words, a_v is the normal space to the H-orbit $H \cdot v$ at v. A easy fact about a_v is that it meets every H-orbit. A vector $v \in V$ is called regular if $\mathfrak{h} \cdot v$ is of maximal possible dimension.

Proposition 2.1 [D]. *Fix a regular vector $v_0 \in V$. The following conditions are equivalent:*
  (i)   *For any regular vector $v \in V$, $\mathfrak{h} \cdot v = h \cdot (\mathfrak{h} \cdot v_0)$ for some $h \in H$.*
  (ii) *For any regular vector $v \in V$, $a_v = h \cdot a_{v_0}$ for some $h \in H$.*
  (iii) *For any $w \in a_{v_0}$, $\langle \mathfrak{h} \cdot w, a_{v_0} \rangle = 0$.*

Definition 2.2. A representation of H on V is called polar if it satisfies one of the three equivalent conditions in Proposition 2.1. If $v \in V$ is regular then

the subspace a_v is called a Cartan subspace. Let a be a Cartan subspace. The Weyl group W of the polar representation (π, V) is defined as $N_H(a)/Z_H(a)$.

Here are a few examples of polar representations:

(a) The natural representations of the classical compact Lie groups.

(b) The nontrivial representations of the exceptional compact Lie groups of minimal possible dimension.

(c) The adjoint representations.

(d) The action of $\mathrm{Ad}_G(H)$ on $\mathfrak{g}/\mathfrak{h}$ provided G/H is a symmetric space.

Definition 2.3. Let G be a semisimple compact Lie group with Lie algebra $\mathfrak{g}$ and H a closed subgroup of G with Lie algebra $\mathfrak{h}$. We say $(\mathfrak{g}, \mathfrak{h})$ is a simple polar pair if the action of $\mathrm{Ad}_G(H)$ on $\mathfrak{g}/\mathfrak{h}$ is an irreducible polar representation. In general, a Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$ is call polar, if $\mathfrak{g} = \oplus_i \mathfrak{g}_i, \mathfrak{h} = \oplus_i \mathfrak{h}_i$ such that $(\mathfrak{g}_i, \mathfrak{h}_i)$ is a polar pair for all i's. The corresponding homogeneous space G/H is called a polar space.

From representation-theoretic point of view the polar spaces are more natural subjects to study than the symmetric spaces. The Cartan subspace $\mathfrak{a}$ and Weyl group W of a polar space can be defined by the corresponding notions of polar representations. All the important results for compact symmetric spaces can be extended to compact polar spaces. For instance we can show that there exists an algebra isomorphism from the set of invariant differential operators on a polar space onto the W-invariant elements $S(\mathfrak{a}_\mathbb{C})^W$ of symmetric algebra $S(\mathfrak{a}_\mathbb{C})$ over $\mathfrak{a}_\mathbb{C}$. This result is even true for noncompact polar spaces since it does not depends on the various real forms. If (G, H) is a compact polar pair with G simply connected and H connected, we can describe the irreducible representations of G with a H-fixed vectors in the same way as for symmetric pairs.

It is clear that all compact simple symmetric pairs are compact simple polar pairs. Therefore all compact symmetric spaces are compact polar spaces. Besides symmetric pairs there are only two simple polar pairs $(\mathfrak{g}, \mathfrak{h})$ of Lie algebras up to automorphisms of $\mathfrak{g}$. They are $(B_3, \mathfrak{g}_2)$ and $(\mathfrak{g}_2, A_2)$ (cf. [**H**] Theorem 3.5.). Let G be a linear semisimple group and H be a closed reductive group. The homogeneous space G/H is called a (non-compact) polar space if its compact real form G^c/H^c is a compact polar space. Here G^c and H^c are compact Lie groups such that their complexifications are isomorphic to the complexifications of G and H respectively.

Theorem 2.4 ([**H**, Theorem 7.3]). *Let U/K be a simple non-symmetric compact polar space, with U simply connected and both G and K connected. In other words, we assume that U/K is either $G_{2(-14)}/SU(3)$ or*

$\mathrm{Spin}(7)/G_{2(-14)}$. *Here $G_{2(-14)}$ is the simply connected compact Lie group with Lie algebra $\mathfrak{g}_2$. Then there are essentially three noncompact real forms G/H of U/K (which means $G_{\mathbb{C}} = U_{\mathbb{C}}$ and $K_{\mathbb{C}} = H_{\mathbb{C}}$). They are $G_{2(2)}/SU(2,1)$, $G_{2(2)}/SL(3,\mathbb{R})$ and $\mathrm{Spin}(3,4)/G_{2(2)}$. Here $G_{2(2)}$ is the noncompact real form of the complex group G_2.*

In this paper we will prove the following theorem.

Theorem 2.5. *Let U/K be a compact polar space with U connected and simple connected and K connected. Let G/H be a noncompact real form of U/K. Then G/H has discrete series if and only if it has a compact Cartan subspace.*

By Theorem 2.4 in order to prove the theorem it suffices to verify it for the three non-symmetric simple polar spaces $G_{2(2)}/SU(2,1)$, $G_{2(2)}/SL(3,\mathbb{R})$ and $\mathrm{Spin}_e(3,4)/G_{2(2)}$. Since all of the three polar spaces have compact Cartan subspaces, we will show that they all have discrete series. We actually obtain the Plancherel formulae for these homogeneous spaces in Section 4. In particular, we obtain all discrete series for these homogeneous spaces.

The three non-symmetric polar spaces $X = G/H$, $G_{2(2)}/SU(2,1)$, $G_{2(2)}/SL(3,\mathbb{R})$ and $\mathrm{Spin}_e(3,4)/G_{2(2)}$, are isomorphic to real hyperbolic spaces in a special way. Now we describe the isomorphisms. Consider the following three quadruplets:

1. Let $G=\mathrm{Spin}_e(3,4)$ be the subgroup of $G'=\mathrm{SO}_e(4,4)$. Let $H'=\mathrm{SO}_e(4,3)$ be the standard subgroup of G'. Then $H = G \cap H'$ is isomorphic to $G_{2(2)}$.

2. Let $G=G_{2(2)}$ be the subgroup of $G'=\mathrm{SO}_e(4,3)$. Let $H'=\mathrm{SO}_e(4,2)$ be the standard subgroup of G'. Then $H = G \cap H'$ is isomorphic to $SU(2,1)$.

3. Let $G=G_{2(2)}$ be the subgroup of $G'=\mathrm{SO}_e(3,4)$. Let $H'=\mathrm{SO}_e(3,3)$ be the standard subgroup of G'. Then $H = G \cap H'$ is isomorphic to $SL(3,\mathbb{R})$.

List of the Three Quadruplets.

	G	G$'$	H$'$	H
1.	$\mathrm{Spin}_e(3,4)$	$\mathrm{SO}_e(4,4)$	$\mathrm{SO}_e(4,3)$	$G_{2(2)}$
2.	$G_{2(2)}$	$\mathrm{SO}_e(4,3)$	$\mathrm{SO}_e(4,2)$	$SU(2,1)$
3.	$G_{2(2)}$	$\mathrm{SO}_e(3,4)$	$\mathrm{SO}_e(3,3)$	$SL(3,\mathbb{R})$

In all of the three cases the inclusion of $G \hookrightarrow G'$ induces a diffeomorphism of G/H onto G'/H'. In other words, the natural embedding $G/H \hookrightarrow G'/H'$ is surjective. The very significance of these diffeomorphisms is that we can make use of the known results on real hyperbolic spaces to obtain the desired results on polar spaces.

For q larger than 1 the homogeneous space $G'/H' = \mathrm{SO}_e(p,q)/\mathrm{SO}_e(p,q-1)$ may be realized as the hypersurfaces

$$X = \left\{ x \in \mathbb{R}^{p+q} \mid -x_1^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_{p+q}^2 = 1 \right\}$$

under the left action of the group $G' = \mathrm{SO}_e(p,q)$ of connected linear transformations of $\mathbb{R}^{p+q}$ which leave the bilinear form $(x,y) = -x_1 y_1 - \cdots - x_p y_p + x_{p+1} y_{p+1} + \cdots + x_{p+q} y_{p+q}$ invariant. Therefore the three polar spaces can be identified with X for some proper integers p and q by restricting the action of G' to G. Follow the idea in [**R**] we introduce the asymptotic cone $\Xi = \{\xi \in \mathbb{R}^{p+q} \mid (\xi,\xi) = 0, \xi \neq 0\}$ of X. Denote by K and K' the maximal compact subgroup of G and G' respectively. The manifold $Y = \left\{ y \in \mathbb{R}^{p+q} \mid y_1^2 + \cdots + y_p^2 = y_{p+1}^2 + \cdots + y_{p+q}^2 = 1 \right\} \cong S^{p-1} \times S^{q-1} \subset \Xi$ may be regarded as a compact subspaces of the polar space, since $Y \cong K/K \cap H \cong K'/K' \cap H'$. Note that Y is contained in the asymptotic cone Ξ, but not in X. The three cases we are interested have p and q equal either 3 or 4. So in the rest of the paper, we always have both p and q larger than 1.

3. Principal series for X.

We retain the notations in the previous section. Note that both G' and G act transitively on Ξ, and the actions commute with the multiplicative action of nonzero real scalars $\mathbb{R}^\times$. Hence the space of functions on Ξ which transform according to a fixed character of $\mathbb{R}^\times$ is invariant under the regular representation of G or G'. So, for $\varepsilon = 0, 1$ and $\nu \in \mathbb{C}$, define

$$L^2_{\varepsilon,\nu}(\Xi) = \left\{ f : \Xi \to \mathbb{C} \mid f(\alpha\xi) = |\alpha|^{-\rho+\nu} \operatorname{sign}^\varepsilon(\alpha) f(\xi), \forall \alpha \in \mathbb{R}^\times; \right.$$

$$\left. \text{and } \int_Y |f(y)|^2 dy < \infty \right\},$$

where $\rho = \frac{1}{2}(p+q-2)$. The representation $(\pi_{\varepsilon,\nu}, L^2_{\varepsilon,\nu}(\Xi))$ is called a principal series for X.

Proposition 3.1.
  (i) *The pairing of $L^2_{\varepsilon,\nu}(\Xi)$ and $L^2_{\varepsilon,-\nu}(\Xi)$ defined by*

$$\langle \phi_+, \phi_- \rangle = \int_Y \phi_+(y)\phi_-(y)dy$$

is G'-invariant (and hence G-invariant).

  (ii) *The representation $(\pi_{\varepsilon,\nu}, L^2_{\varepsilon,\nu}(\Xi))$ and $(\pi_{\varepsilon,\nu}, L^2_{\varepsilon,\nu}(\Xi))$ are contragredient (as either representation of G or G').*

(iii) *The representation $(\pi_{\varepsilon,\nu}, L^2_{\varepsilon,\nu}(\Xi))$ is unitary for imaginary ν.*

Proof. The statement (i) is proved in Lemma 1 in [**R**]. The statement (ii) and (iii) are consequences of (i).

To study the irreducibility of $L^2_{\varepsilon,\nu}(\Xi)$ as a representation of G, we pass to its K-finite part which is stable under the action of $\mathfrak{g}$ obtained from $\pi_{\varepsilon,\nu}$ by derivation. First of all, the K'-finite functions in $L^2_{\varepsilon,\nu}(\Xi)$ is isomorphic to $L^2_\varepsilon(Y) = \{\phi \in L^2(Y) \mid \phi(-y) = (-1)^\varepsilon \phi(y)\}$. Hence, as a representation of K',

$$L^2_{\varepsilon,\nu}(\Xi) \cong \sum_{\substack{m+n \equiv \varepsilon \\ (\mathrm{mod}\ 2)}} H^{m,n} \text{ (Hilbert space sum)},$$

where $H^{m,n}$ is the space spanned by the products of spherical harmonics of degree m on S^{p-1} with those of degree n on S^{p-1}. As is well known, the $H^{m,n}$'s are irreducible, pairwise inequivalent representations of K'. These spherical harmonics are also irreducible representations of K when the action of K' is restricted to K. We can prove this by using the case by case argument.

Theorem 3.2. *A subspace of $\sum_{m+n \equiv \varepsilon \,\mathrm{mod}\ 2} H^{m,n}$ is an irreducible $(\mathfrak{g}, K)$-module if and only if it is an irreducible $(\mathfrak{g}', K')$-module.*

Proof. Since $\mathfrak{k}$ acts irreducibly on $\mathfrak{g}/\mathfrak{k}$, a subspace of $\sum_{m+n \equiv \varepsilon \,\mathrm{mod}\ 2} H^{m,n}$ is $\mathfrak{g}$-stable if and only if it is stable under $\mathfrak{k}$ and a single element X in $\mathfrak{g}$ not in $\mathfrak{k}$. The element $X \in \mathfrak{g}$ is automatically contained in $\mathfrak{g}'$. The theorem follows from Theorem 2 of [**R**] regarding $L^2_{\varepsilon,\nu}(\Xi)$ as a $(\mathfrak{g}', K')$-module. $\qquad\square$

Corollary 3.3.

(i) *The representation $(\pi_{\varepsilon,\nu}, L^2_{\varepsilon,\nu}(\Xi))$ is an irreducible representation of G or G' unless $-\rho + \nu$ is an integer.*

(ii) *If $-\rho + \nu$ is an integer, then for any pair $(u,v), u, v = \pm 1$, satisfying*

$$-\rho + \nu \equiv \varepsilon + \frac{1}{2}(u-1)p + \frac{1}{2}(v-1)q \ (\mathrm{mod}\ 2)$$

the closure of

$$\Sigma\left\{ H^{m,n} \mid u\left(m + \frac{1}{2}(p-2)\right) + v\left(n + \frac{1}{2}(q+2)\right) < \nu, m+n \equiv \varepsilon \ (\mathrm{mod}\ \overline{2)} \right\}$$

is a G-stable subspace of $L^2_{\varepsilon,\nu}(\Xi)$, and any G-stable subspace is an intersection of some of these.

4. Plancherel formula for X.

Now we introduce the polar coordinate on the polar space $X = G/H$:

$$(1, \infty) \times Y \to X, (\alpha, y) \mapsto x(\alpha, y),$$

where

$$x(\alpha, y) = \frac{1}{2} \left(\alpha - \alpha^{-1} y' \right) + \frac{1}{2} \left(\alpha - \alpha^{-1} y'' \right),$$

if we decompose $y = y' + y''$, with $y' \in S^{p-1} \subset Y$ and $y'' \in S^{q-1} \subset Y$ in the obvious way. This map is analytically invertible on its image $X' = \left\{ x \in X \mid x_1^2 + \cdots x_p^2 \neq 0 \right\}$.

The G-invariant differential operators on X were described in [**H**]. They are polynomials in Δ_X, the Laplace operator on X. In polar coordinates, one has

$$\Delta_X = \left(\alpha \frac{d}{d\alpha} \right)^2 + \left((p-1) \frac{\alpha + \alpha^{-1}}{\alpha - \alpha^{-1}} + (q-1) \frac{\alpha - \alpha^{-1}}{\alpha + \alpha^{-1}} \right) \alpha \frac{d}{d\alpha}$$
$$+ \left(\frac{1}{2} \left(\alpha - \alpha^{-1} \right) \right)^{-2} \Delta_{S^{p-1}} - \left(\frac{1}{2} \left(\alpha + \alpha^{-1} \right) \right)^{-2} \Delta_{S^{q-1}},$$

where $\Delta_{S^{p-1}}$ and $\Delta_{S^{q-1}}$ are the Laplacians on S^{p-1} and on S^{q-1} (cf. (4) in [**R**]).

Denote by $D(X)$ the space of compactly supported smooth functions on X. The dual space $D'(X)$ of $D(X)$ is the space of distributions on X. The Laplacian Δ_X, considered as an operator on $D'(X)$, can be restricted to a self-adjoint operators on $L^2(X)$ in a natural way.

Lemma 4.1 ([**R**] Lemma 9).

The Laplacian Δ_X is a self-adjoint operator on $L^2(X)$ with domain

$$\text{Dom}(\Delta_X) = \left\{ f \in D'(X) \mid f \in L^2(X) \text{ and } \Delta_X f \in L^2(X) \right\}.$$

As a self-adjoint operator on $L^2(X)$, Δ_X has a spectral decomposition. This decomposition can be identified with the decomposition in the sense of (1.1). It is in this way that we obtain the Plancherel formula for X.

Theorem 4.2. *The direct integral decomposition of $L^2(X)$ is as follows*

$$(4.1) \qquad L^2(X) \cong \sum_{\varepsilon} \frac{1}{2\pi i} \int_{i\mathbb{R}_+} L^2_{\varepsilon, \nu}(\Xi) |c(\varepsilon, \nu)|^{-2} d\nu + \sum_{\varepsilon, \nu} L^2_{\varepsilon, \nu}(\Xi)^0,$$

where the first sum is over $\varepsilon = 0, 1$; $c(\varepsilon, \nu) = 2^{2\rho+1} \pi^\rho |\Gamma(\nu)/\Gamma(\nu + \rho)|$ if $G = G_{2(2)}$ and $c(\varepsilon, \nu) = 2^{2\rho+1} \pi^\rho |\Gamma(\nu)/\Gamma(\nu + \rho)| \times |\tan(\frac{1}{2}\pi(\rho + \nu + \varepsilon))|$ if

$G = \mathrm{Spin}_e(3,4)$; *the second sum is over $\varepsilon = 0, 1$ and $\nu > 0$ for which $\nu + \rho$ is a positive integer and $\varepsilon \cong \rho + \nu - q(\mathrm{mod}\ 2)$; $L^2_{\varepsilon,\nu}(\Xi)^0$ is the unitary irreducible quotient of $L^2_{\varepsilon,\nu}(\Xi)$.*

The theorem follows from Theorem 3.3 and the main results of [**R**] (Theorem 10, 12 and 13). The irreducible representations of G occurring in the first sum in (4.1) are called the continuous series for X and those occurring in the second sum are called the discrete series for X. As a consequence of the theorem, we have following two corollaries.

For $\nu \in i\mathbb{R}$ and $f \in D(X)$ we define $\hat{f}_{\varepsilon,\nu}(\xi) \in L^2_{\varepsilon,\nu}(\Xi)$ by

$$\hat{f}_{\varepsilon,\nu}(\xi) = \int_X |(x,\xi)|^{-\rho+\nu} \mathrm{sign}^\varepsilon(x,\xi) f(x) dx.$$

Corollary 4.3. *Every K-finite functions $f \in D(X)$ can be written as*

$$f(x) = \sum_\varepsilon \frac{1}{2\pi i} \int_{i\mathbb{R}_+} \int_Y |(x,y)|^{-\rho-\nu} \mathrm{sign}^\varepsilon(y) \hat{f}_{\varepsilon,\nu}(y) dy |2^{-\rho} c(\varepsilon,\nu)|^{-2} d\nu$$

$$+ \sum_{\substack{\varepsilon \equiv \rho + \nu_0 - q \\ \rho + \nu_0 \in \mathbb{N}, \nu > 0}} \mathrm{res}_{\nu=\nu_0} \left\{ c(\varepsilon,\nu)^{-1} \int_Y |(x,y)|^{-\rho-\nu} \mathrm{sign}^\varepsilon(x,y) \hat{f}_{\varepsilon,\nu}(y) dy \right\}.$$

Corollary 4.5. *A polar semisimple homogeneous space has discrete series if and only if it has a compact Cartan subspace.*

This corollary gives a proof of Theorem 2.5.

Remark 4.6.   For the case $G/H = \mathrm{Spin}_e(3,4)/G_{2(2)}$ if we only sum over $\varepsilon = 0$ in (4.1), we get the Plancherel formula for $G/H = \mathrm{SO}_e(3,4)/G_{2(2)}$. Note that the corresponding representations of $\mathrm{Spin}_e(3,4)$ factor through $\mathrm{SO}_e(3,4)$.

5. Eigenspace representations on X.

In this section we will discuss the two spaces of eigenfunctions of the Laplacian Δ_X under the regular representation of G. One is a subspace of smooth functions on X,

$$C^\infty_{\varepsilon,\lambda}(X) = \{f \in C^\infty(X) \mid \Delta_X f = \lambda f, \text{ and } f(-x) = (-1)^\varepsilon f(x)\}.$$

The other is a subspace of L^2-functions on X,

$$L^2_{\varepsilon,\lambda}(X) = \{f \in L^2(X) \mid \Delta_X f = \lambda f \text{ and } f(-x) = (-1)^\varepsilon f(x)\}.$$

We will also give a Poisson integral-type for K-finite eigenfunctions of Δ_X.

We set

$$C_{\varepsilon,\nu}^{\infty}(\Xi) = \left\{ \phi \in C^{\infty}(\Xi) \mid \phi(\alpha\xi) = |\alpha|^{-\rho+\nu} \operatorname{sign}^{\varepsilon}(\alpha)\phi(\xi) \right\}.$$

Restrict this space of functions to Y is a linear isomorphism $C_{\varepsilon,\nu}^{\infty}(\Xi) \to C_{\varepsilon}^{\infty}(Y) = \{\phi \in C^{\infty}(Y) \mid \phi(-y) = (-1)^{\varepsilon}\phi(y)\}$.

We define a map

$$F_{\varepsilon,\nu} : C_{\varepsilon,\nu}^{\infty}(\Xi) \to C^{\infty}(X)$$

by

$$F_{\varepsilon,\nu}\phi(x) = \int_{Y} |(x,y)|^{-\rho-\nu} \operatorname{sign}^{\varepsilon}(x,y)\phi(y)dy.$$

The integral converges for $\operatorname{Re}(\rho + \nu) < 1$ and admits an interpretation by analytic continuation in ν, as long as ν is not a positive integer congruence to $\varepsilon + 1 \pmod 2$.

Lemma 5.1 ([**R**] Lemma 5).

(i) *The map $F_{\varepsilon,\nu}$ depends meromorphically on ν with simple poles for $\rho + \nu \in \mathbb{N}$ and $\equiv \varepsilon + 1 \mod 2$.*

(ii) *$F_{\varepsilon,\nu}$ is a G-map: $F_{\varepsilon,\nu}\pi_{\varepsilon,\nu}(g)\phi = R(g)F_{\varepsilon,\nu}\phi$, for any $\phi \in C_{\varepsilon,\nu}^{\infty}(\Xi)$.*

(iii) *For any $\phi \in C_{\varepsilon,\nu}^{\infty}(\Xi)$, $\Delta_x F_{\varepsilon,\nu}\phi = (-\rho^2 + \nu^2)F_{\varepsilon,\nu}\phi$.*

Here R denotes the regular representation.

Proposition 5.2.

(i) *Every K-finite distribution f on X satisfying*

$$\Delta_X f = \lambda f$$

for some $\lambda \in \mathbb{C}$ is an analytic function.

(ii) *Fix a K-type τ and a complex number λ. The space of all K-finite analytic functions f of type τ satisfying*

$$\Delta_X f = \lambda f$$

has dimension equal to $\dim V_{\tau}$, which is independent of eigenvalue λ.

Proof. The statement (i) of the proposition follows from (a) of Lemma 3 in [**R**]. The same lemma guarantee the dimension in (ii) is less than or equal to $\dim V_{\tau}$. Applying an argument of analytic continuation such as in Proposition 2.21 in [**OS**] one can get the other half inequality. □

The following theorem and Theorem 5.5 below can be regarded as a Poisson integral formula for X.

Theorem 5.3 (cf. [**R**, Theorem 6]).   *If $\rho + \nu$ is not an integer, then every K-finite distribution f on X satisfying*

$$\Delta_X f = \left(-\rho^2 + \nu^2\right) f$$

is an analytic function and can be represented in the form

$$f(x) = \sum_\varepsilon \int_Y |(x,y)|^{-\rho-\nu} \operatorname{sign}^\varepsilon(x,y) \phi_\varepsilon(y) dy$$

for unique K-finite function $\phi_\varepsilon \in C_\varepsilon^\infty(Y)$.

Corollary 5.4.

(i) *If $\lambda = -\rho^2 + \nu^2$ (or $\rho + \nu$) is not an integer (note that λ is an integer if and only if $\rho + \nu$ is an integer), then $C_{\varepsilon,\lambda}^\infty(X)$ is irreducible under the regular representation of G. The map $F_{\varepsilon,\lambda}$ is an intertwining operator between $C_{\varepsilon,\pm\nu}^\infty(\Xi)$ and $C_{\varepsilon,\lambda}^\infty(X)$, and the K-finite part of $C_{\varepsilon,\pm\nu}^\infty(\Xi)$ and $C_{\varepsilon,\lambda}^\infty(X)$ are equivalent $(\mathfrak{g}, K)$-modules.*

(ii) *If $\lambda = \lambda_0 = -\rho^2 + \nu_0^2$ and $\rho + \nu_0 \in \mathbb{N}, \equiv \varepsilon + q \mod 2$, $\nu_0 > 0$, then the kernel of $\operatorname{res}_{\nu=\nu_0} \{c(\varepsilon, \nu)^{-1} F_{\varepsilon,\nu}\}$ is the G-stable subspace of $C_{\varepsilon,\nu_0}^\infty(\Xi)$ described by (cf. Theorem 3.2)*

$$-\left(m + \frac{1}{2}(p-2)\right) + \left(n + \frac{1}{2}(q+2)\right) < \nu_0.$$

As representations of G, $C_{\varepsilon,\nu_0}^\infty(\Xi)$ and $C_{\varepsilon,\lambda_0}^\infty(X)$ have the same distribution character.

Now we turn our attention to the square-integrable eigenfunctions of Δ_X. The following theorem is a consequence of Theorem 12 of [**R**] and Theorem 3.2.

Theorem 5.5.   *For $\lambda = -\rho^2 + \nu^2$, the eigenspaces $L_{\varepsilon,\lambda}^2(X)$ of Δ_X are irreducible under the regular representation of G. For $\rho + \nu = \rho + \nu_0 \in \mathbb{N}, \equiv \varepsilon + q \pmod 2$, $\nu_0 > 0, \operatorname{res}_{\nu=\nu_0} \{c(\varepsilon, \nu)^{-1} F_{\varepsilon,\nu}\}$ induces an intertwining operator between an irreducible quotient $L_{\varepsilon,\nu}^2(\Xi)^0$ and $L_{\varepsilon,\lambda}^2(X)$. The corresponding K-finite functions in $L_{\varepsilon,\lambda}^2(X)$ are precisely the functions f of the form*

$$f(x) = \operatorname{res}_{\nu=\nu_0} \left\{ c(\varepsilon, \nu)^{-1} \int_Y (x,y)^{-\rho-\nu} \operatorname{sign}^\varepsilon(x,y) \phi(y) dy \right\},$$

where ϕ is a K-finite function on Y.

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