ONE REMARK ON POLYNOMIALS IN TWO VARIABLES

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In this paper, we study some topological properties of rational polynomial maps $f : \mathbb{C}^2 \to \mathbb{C}$. One can extend f to a map $\varphi : X \to \mathbb{P}^1$ where X is a smooth algebraic compactification of $\mathbb{C}^2$. The behaviour of φ on the curve $\mathcal{D} := X \setminus \mathbb{C}^2$ contains all the information on the topology of f at infinity. We study the relationship between the so-called horizontal componentes of $\mathcal{D}$, i.e., irreducible components D of $\mathcal{D}$ such that $\phi_{|}D$ is surjective.

In the article *"Two remarks on polynomials in two variables"* [5] S. Kaliman states two results on polynomial maps $f : \mathbb{C}^2 \to \mathbb{C}$. The first one is a very nice result on the number of irreducible components of reducible fibers of a primitive polynomial (i.e., a polynomial such that its generic fiber is irreducible).

We introduce some notation in order to state the second result. Let $\mathbb{C}^2 \subset X$ a compactification of $\mathbb{C}^2$ where X is a smooth rational compact surface and such that there exists a holomorphic map $\phi : X \to \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ which extends f.

Put $\mathcal{D} := X \setminus \mathbb{C}^2$; $\mathcal{D}$ is a curve whose irreducible components are smooth rational compact curves and all its singularities are ordinary double points. The dual graph of $\mathcal{D}$ is a tree.

Kaliman calls *horizontal* (resp. *sections*) an irreducible component $D \subset \mathcal{D}$ such that $\phi_{|D}$ is surjective (resp. bijective). We recall the second result of [5]:

If f is primitive and the generic fiber of f is a rational curve then there is at most one horizontal component which is not a section.

In this note we are going to give a counter-example to this result and we are going to point out where Kaliman's proof does not apply. In the last section we relate this problem to an article of P. Russell, [8].

§1.-A counter-example.

Let us denote

$$s(x,y) = 1 + xy, \quad p(x,y) = xs(x,y) + 1, \quad u(x,y) = s(x,y)^2 + y.$$

We define:

$$f(x,y) := p(x,y)^4 u(x,y)^2 - (p(x,y)s(x,y) + 1)^2.$$

For any f there exists a canonical way to get X and ϕ. Let

$$f(x,y) = a_0(x,y) + a_1(x,y) + \cdots + a_d(x,y),$$

where $a_j(x,y)$ is a homogeneous form of degree j, $j = 0, 1, \ldots, d$, and $a_d(x,y) \neq 0$; let

$$F(x,y,z) := \sum_{j=0}^{d} z^{d-j} a_j(x,y).$$

Consider the rational map

$$\phi_0 : \quad \mathbb{P}^2 \quad \dashrightarrow \quad \mathbb{P}^1 \quad = \mathbb{C} \cup \{\infty\}$$

$$[x : y : z] \mapsto [F(x,y,z) : z^d] = \frac{F(x,y,z)}{z^d},$$

which is well-defined outside $A(f) := \{[x_0 : y_0 : 0] \mid a_d(x_0, y_0) = 0\}$ (which is a finite set). This set is always contained in the line of infinity L_∞ whose equation is $z = 0$.

The composition $\pi : X \to \mathbb{P}^2$ of a suitable sequence of blowing-up maps (over $A(f)$) gives the required compactification where $\phi := \phi_0 \circ \pi$.

Let's compute X and ϕ in our example; we have $d = 20$ and $a_{20}(x,y) = x^{12}y^8$ so $A(f) = \{[1 : 0 : 0], [0 : 1 : 0]\}$. Let us start with the blowing-up maps over $P := [1 : 0 : 0]$; this is the origin of the chart defined by $x = 1$. We write down ϕ_0 in the coordinates (y, z) of this chart:

$$\phi_0(y,z) = \frac{F(1,y,z)}{z^{20}} = \frac{y^8 + \cdots}{z^{20}},$$

where "$\cdots$" means *terms of higher degree*. We consider $\pi_1 : Y_1 \to \mathbb{P}^2$ the blowing-up of P.

Convention. We will identify a curve with its proper transform after a blowing-up, even in the notation, whenever it does not cause any confusion.

Let's call A_1 the exceptional locus of this blowing-up. It is easy to see from last formula that the only point of indeterminacy over P of $\phi_{y1} := \phi_0 \circ \pi_1$ is the infinitely near point P_1 of the line $\{y = 0\}$. We choose a chart of Y_1 with coordinates y_1, z_1 such that P_1 is the origin of this chart and we can write down π_1 as follows:

$$\pi_1(y_1, z_1) = (y_1 z_1, z_1).$$

Note that L_∞ does not intersect this chart and A_1 is defined by $z_1 = 0$ (P_1 is in A_1 but it is not in L_∞). We write down ϕ_{y1} in these coordinates:

$$\phi_{y1}(y_1, z_1) = \frac{F(1, y_1 z_1, z_1)}{z_1^{20}} = \frac{f_{y1}(y_1, z_1)}{z_1^{12}} = \frac{(y_1 + z_1)^8 + \cdots}{z_1^{12}}.$$

We consider $\pi_2 : Y_2 \to Y_1$ the blowing-up of P_1. Let's call A_2 the exceptional locus of this blowing-up. It is easy to see that A_2 does not intersect L_∞. From last formula the only point of indeterminacy over P of $\phi_{y2} := \phi_{y1} \circ \pi_2$ is the infinitely near point P_2 of the line $\{y_1 + z_1 = 0\}$. We choose a chart of Y_2 with coordinates y_2, z_2 such that P_2 is the origin of this chart and we can write down π_2 as follows:

$$\pi_2(y_2, z_2) = ((y_2 - 1)z_2, z_2).$$

Note that A_2 is defined by $z_2 = 0$. We write down ϕ_{y2} in these coordinates:

$$\phi_{y2}(y_2, z_2) = \frac{f_{y1}((y_2 - 1)z_2, z_2)}{z_2^{12}} = \frac{f_{y2}(y_2, z_2)}{z_2^4} = \frac{-z_2^4 + \cdots}{z_2^4}.$$

We consider $\pi_3 : Y_3 \to Y_2$ the blowing-up of P_2. Let A_3 be the exceptional locus of this blowing-up. It is easy to see that A_2 intersects neither L_∞ nor A_1. From last formula the only point of indeterminacy over P of $\phi_{y3} := \phi_{y2} \circ \pi_3$ is $P_3 := A_2 \cap A_3$. We choose a chart of Y_3 with coordinates y_3, z_3 such that P_3 is the origin of this chart and we can write down π_3 as follows:

$$\pi_3(y_3, z_3) = (y_3, y_3 z_3).$$

Note that A_3 is defined by $y_3 = 0$ while A_2 is $z_3 = 0$. We write down ϕ_{y3} in these coordinates:

$$\phi_{y3}(y_3, z_3) = \frac{f_{y2}(y_3, y_3 z_3)}{y_3^4 z_3^4} = \frac{f_{y3}(y_3, z_3)}{z_3^4} = \frac{-z_3^4 + \cdots}{z_3^4}.$$

Note that $\phi_{y3}(L_\infty) = \phi_{y3}(A_1) = \phi_{y3}(A_2 \setminus \{P_3\}) = \infty = [1 : 0]$, but we have $\phi_{y3}(A_3 \setminus \{P_3\}) = -1 = [-1 : 1]$.

If $\pi_4 : Y_4 \to Y_3$ is the blowing-up of P_3 then $\phi_{y4} := \phi_{y3} \circ \pi_4$ has no point of indeterminacy in the exceptional locus A_4 of π_4; this curve separates A_2 and A_3; the restriction of ϕ_{y4} is a surjective map of degree 4 (an horizontal component in the notation of S. Kaliman in [5]).

Let Q be the transform of $[0 : 1 : 0]$ by the sequence $\pi_4 \circ \pi_3 \circ \pi_2 \circ \pi_1$. We consider also the transform of the chart $y = 1$. In its coordinates x, z we have:

$$\phi_{y4}(x, z) = \frac{F(x, 1, z)}{z^{20}} = \frac{x^{12} + \cdots}{z^{20}}.$$

We consider $\sigma_1 : X_1 \to Y_4$ the blowing-up of Q. Let's call B_1 the exceptional locus of this blowing-up. It is easy to see that B_1 intersects L_∞. From last formula the only point of indeterminacy of $\phi_{x1} := \phi_{y4} \circ \sigma_1$ is the infinitely near point Q_1 of the line $x = 0$ which does not lie in L_∞. We choose a chart of X_1 with coordinates x_1, z_1 such that Q_1 is the origin of this chart and we can write down σ_1 as follows:

$$\sigma_1(x_1, z_1) = (x_1 z_1, z_1).$$

Note that B_1 is defined by $z_1 = 0$. We write down ϕ_{x1} in these coordinates:

$$\phi_{x1}(x_1, z_1) = \frac{F(x_1 z_1, 1, z_1)}{z_1^{20}} = \frac{f_{x1}(x_1, z_1)}{z_1^8} = \frac{z_1^6 + \cdots}{z_1^8}.$$

We consider $\sigma_2 : X_2 \to X_1$ the blowing-up of Q_1. Let's call B_2 the exceptional locus of this blowing-up. It is easy to see that B_2 intersects B_1 but it is disjoint with L_∞. From last formula the only point of indeterminacy of $\phi_{x2} := \phi_{x1} \circ \sigma_2$ is $Q_2 := B_1 \cap B_2$. We choose a chart of X_2 with coordinates x_2, z_2 such that Q_2 is the origin of this chart and we can write down σ_2 as follows:

$$\sigma_2(x_2, z_2) = (x_2, x_2 z_2).$$

Note that B_2 is defined by $x_2 = 0$ and B_1 is $z_2 = 0$. We write down ϕ_{x2} in these coordinates:

$$\phi_{x2}(x_2, z_2) = \frac{f_{x1}(x_2, x_2 z_2)}{x_2^8 z_2^8} = \frac{f_{x2}(x_2, z_2)}{x_2^2 z_2^8} = \frac{(x_2 + z_2)^6 + \cdots}{x_2^2 z_2^8}.$$

We consider $\sigma_3 : X_3 \to X_2$ the blowing-up of Q_2. Let's call B_3 the exceptional locus of this blowing-up which separates B_1 and B_2. From last formula the only point of indeterminacy of $\phi_{x3} := \phi_{x2} \circ \sigma_3$ is the infinitely near point Q_3 of the affine line $x_2 = -z_2$. We choose a chart of X_3 with coordinates x_3, z_3 such that Q_3 is the origin of this chart and we can write down σ_3 as follows:

$$\sigma_3(x_3, z_3) = ((x_3 - 1)z_3, z_3).$$

Note that B_3 is defined by $z_3 = 0$. We write down ϕ_{x3} in these coordinates:

$$\phi_{x3}(x_3, z_3) = \frac{f_{x2}((x_3 - 1)x_3, z_3)}{(x_3 - 1)^2 z_3^{10}} = \frac{f_{x3}(x_3, z_3)}{(x_3 - 1)^2 z_3^4} = \frac{-(x_3 z_3)^2 + \cdots}{(x_3 - 1)^2 z_3^4}.$$

We consider $\sigma_4 : X_4 \to X_3$ the blowing-up of Q_3. Let's call B_4 the exceptional locus of this blowing-up which intersects only B_3. It is easy to see that B_4 is an horizontal component such that the restriction of $\phi_{x4} := \phi_{x3} \circ \sigma_4$ to B_4 is a map of degree 2; we can also check that $Q_5 := B_4 \cap B_3$ is a point

of indeterminacy of ϕ_{x4}. We choose a chart of X_4 with coordinates x_4, z_4 such that Q_4 is the origin of this chart and we can write down σ_4 as follows:

$$\sigma_4(x_4, z_4) = (x_4, x_4 z_4).$$

Note that B_4 is defined by $x_4 = 0$ and B_3 is $z_4 = 0$. We write down ϕ_{x4} in these coordinates:

$$\phi_{x4}(x_4, z_4) = \frac{f_{x3}(x_4, x_4 z_4)}{(x_4 - 1)^2 x_4^4 z_4^4} = \frac{f_{x4}(x_4, z_4)}{(x_4 - 1)^2 z_4^4} = \frac{(x_4 - z_4)(x_4 + z_4) + \cdots}{(x_4 - 1)^2 z_4^4}.$$

We consider $\sigma_5 : X_5 \to X_4$ the blowing-up of Q_4. Let's call B_5 the exceptional locus of this blowing-up which separates B_3 and B_4. It is easy to see that there are two points of indeterminacy of $\phi_{x5} := \phi_{x4} \circ \sigma_5$; let's denote Q_5 the infinitely near point of the affine line $x_4 = z_4$ and Q_5' the one of $x_4 = -z_4$. These points are the two points of indeterminacy. They behave in the same way so we study only Q_5. We choose a chart of X_5 with coordinates x_5, z_5 such that Q_5 is the origin of this chart and we can write down σ_5 as follows:

$$\sigma_5(x_5, z_5) = (x_5, x_5(z_5 + 1)).$$

Note that B_5 is defined by $x_5 = 0$. We write down ϕ_{x5} in these coordinates:

$$\begin{aligned}
\phi_{x5}(x_5, z_5) &= \frac{f_{x4}(x_5, x_5(z_5 + 1))}{(x_5 - 1)^2 x_5^4 (z_5 + 1)^4} \\
&= \frac{f_{x5}(x_5, z_5)}{(x_5 - 1)^2 (z_5 + 1)^4 x_5^2} \\
&= \frac{-2(3x_5 + z_5) + \cdots}{(x_5 - 1)^2 (z_5 + 1)^4 x_5^2}.
\end{aligned}$$

It is easy to see that if $\sigma_6 : X_6 \to X_5$ is the blowing-up of Q_5 then $\phi_{x6} := \phi_{x5} \circ \sigma_6$ has only one point of indeterminacy Q_6 in the exceptional locus B_6 of σ_6. It is not difficult to show that if $\sigma_7 : X_7 \to X_6$ is the blowing-up of Q_6 then its exceptional locus B_7 is a section of $\phi_{x7} := \phi_{x6} \circ \sigma_7$.

Let's denote $\sigma_6' : X_6' \to X_7$ and $\sigma_7' : X_7' \to X_6'$ the corresponding blowing-up maps over Q_5' where $\phi_{x6}' := \phi_{x7} \circ \sigma_6'$ $\phi_{x7}' := \phi_{x6}' \circ \sigma_7'$. Then if we denote

$$\pi := \sigma_7' \circ \sigma_6' \circ \sigma_7 \circ \cdots \circ \sigma_1 \pi_4 \circ \cdots \circ \pi_1, \quad X := X_7', \quad \phi := \phi_{x7}'.$$

Then we have found the required compactification of $\mathbb{C}^2$.

Let us summarize the results above. We draw in Figure 1 the dual graph of $\mathcal{D} = X \setminus \mathbb{C}^2$; note that $\mathcal{D}$ is the preimage by π of $L_\infty \subset \mathbb{P}^2$. Weights are self-intersections in X; we recall that the exceptional locus of a blowing-up

has self-intersection -1 and if a smooth curve passing through the center of the blowing-up has self-intersection e, then its proper transform has self-intersection $e - 1$.
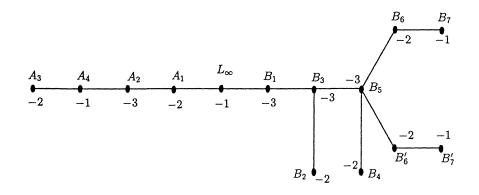

Figure 1.

We are following the next conventions:

- $\pi(A_j) = [1 : 0 : 0]$;
- $\pi(B_j) = \pi(B'_k) = [0 : 1 : 0]$, $j = 1, \ldots, 7$, $k = 6, 7$;
- $\phi(L_\infty) = \phi(A_1) = \phi(A_2) = \phi(B_j) = \phi(B'_6) = \infty = [1 : 0]$, $j = 1, \ldots, 6$;
- $\phi(A_3) = -1 = [-1 : 1]$;
- B_7 and B'_7 are sections;
- A_4 is an horizontal component of degree 4 and
- B_4 is an horizontal component of degree 2.

Proposition. *f is a primitive polynomial.*

Proof. If f would be not primitive then there exists $p \in \mathbb{C}[t]$ and $g \in \mathbb{C}[x, y]$ such that $f(x, y) = p(g(x, y))$ and $\deg p > 1$. This fact uses Stein factorisation ([**1**, pp. 25-26]). An incomplete proof may be found in [**4**]. It is easy to see that if we construct $\psi_0 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ for g in the same way as we construct ϕ for f then the map $\psi := \psi_0 \circ \pi : X \to \mathbb{P}^1$ is holomorphic and it is an extension of g.

We deduce that the horizontal components are the same in the two cases: in fact if $D \subset \mathcal{D}$ is an irreducible component we have $\phi_{|D} = p \circ \psi_{|D}$, where p is considered as a map $\mathbb{P}^1 \to \mathbb{P}^1$ such that $p(\infty) = \infty$.

Then for an horizontal component D we have that $\deg \phi_{|D} = \deg p \deg \psi_{|D}$. The existence of sections implies that $\deg p = 1$ so f is primitive. $\square$

Proposition. *The generic fiber of f is rational.*

Proof. Note that the general fiber is smooth; if we consider its compactification in $\mathbb{P}^2$ its singularities are on $A(f)$. It is known (see [**2**, pp. 569 and 574] and [**6**, p. 85]) that the genus of an irreducible plane projective C curve is $(d-1)(d-2)/2 - \sum_P \delta_P$, where d is the degree, the sum is over the singular points and δ_P may be calculated as follows.

We recall that the sequence of multiplicities of a singularity is the list of the multiplicities of the proper transform of the singularity at all infinitely near points (see [**2**, p. 504]). Then

$$\delta_P = \sum_{P_i} \frac{m(P_i)(m(P_i)-1)}{2},$$

where the sum is over the infinitely near points of C at P and $m(P_i)$ is their multiplicity.

Note that we have constructed a resolution of the singular points of the compactification of the general fiber as its proper transform in X is smooth. We can get the sequence of multiplicities from this construction. In $[1:0:0]$ the generic fiber has four branches and the sequence of multiplicities is $8, 8, 4, 4, \{1,1,1,1\}$; in $[0:1:0]$ the generic fiber has also four branches and the sequence of multiplicities is $12, 6, 6, 4, \{1,1,2\}, \{1,1,1,1\}$.

Putting these multiplicities in the formula we get zero. $\square$

Corollary. *f is a counterexample to Kaliman's second result in* [**5**].

Finally we shoud note that the number of reducible fibers of f is equal to three (each of them with exactly two components); it agrees with the first result of Kaliman in [**5**]. Using that $ps = xu + 1$ we have

$$f = (pu - ps - 1)(pu + ps + 1),$$
$$f + 1 = p\left(p^3 u^2 - s(ps + 2)\right),$$
$$f + 4 = u\left(p^3 u - x(xu + 4)\right).$$

§2.- Eisenbud-Neumann diagrams and horizontal sections.

These computations may be done using Eisenbud-Neumann diagrams of the link at infinity of the generic fiber of f. In order to find the diagram in Figure 2 we may use the computations for the compactification of $\mathbb{C}^2$ to find the Puiseux expansions of the singularities of the generic fiber of f at

infinity. Using [**3**, pp. 50] and [**7**, p. 450] we get the diagram in Figure 2, called D_{EN}.
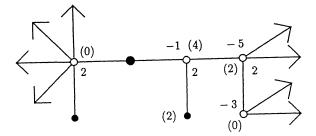

Figure 2.

We can see from D_{EN} that f is primitive. We must construct the fibered multilink associated to f, see [**7**], and verify that its fiber is connected using [**3**, p. 37]. The generic fiber of f is an n-punctured Riemann surface since arrows in the EN-diagram correspond to punctures. The EN-diagram (Theorem 4.3 from [**7**]) also enables us to compute the Euler characteristic of the generic fiber which is -6. Hence the genus of the generic fiber is 0.

Remark. In our example horizontal components which are not sections are detected at vertices v in D_{EN} such that $l_v = 0$ (weight in brackets). In the general case we detect in this way all horizontal components where f is not equisingular; an horizontal component D is equisingular if for all $t \in \mathbb{C}$ the closure of $f^{-1}(t)$ in X intersects D at exactly one point and transversally. We are going to explain how.

Let us consider the diagram in Figure 1. If we attach four arrows to the vertex A_4, two arrows to B_4 and one arrow to B_7 and B'_7 then we obtain a plumbing diagram of the link at infinity of f. The dictionnary between these two kinds of diagrams may be found in [**3**, Chap. 5].

This diagram can be interpretted as a resolution diagram of the singularities of the closure of the generic fiber in $\mathbb{P}^2$; this diagram includes also the proper transform of the line at infinity. In this way is not minimal: we may eliminate vertices with weight -1 and belonging to at most two edges (this operation increases by one the weight of their neighbours). We call D_{pb} the diagram obtained in this way. This construction correspond to a sequence of contractions of (-1)-rational curves which we denote $\tilde{\pi} : X \to \tilde{X}$; D_{pb} is the dual graph of the curve $\tilde{\mathcal{D}} = \tilde{\pi}(\mathcal{D})$. Let's denote $\tilde{\phi} : \tilde{X} \dashrightarrow \mathbb{P}^1$ the meromorphic function such that $\phi = \tilde{\phi} \circ \tilde{\pi}$. Let us emphasize that $\hat{\pi} : \tilde{X} \to \mathbb{P}^2$ is the minimal embedded resolution of the singularities of the closure of the

generic fiber in $\mathbb{P}^2$ where $\hat{\pi}$ is the compositon of the blowing-up maps such that $\pi = \hat{\pi} \circ \tilde{\pi}$. Let us note that in general $\tilde{\phi}$ is not everywhere defined.

Lemma. *Let D_{EN} be the Eisenbud diagram of the link at infinity of the generic fiber of f. Let D_{pb} be the dual tree of the total transform of the line at infinity in the minimal embedded resolution of the singularities of the closure of the generic fiber in $\mathbb{P}^2$.*

Then there exists a natural bijection between vertices with attached arrows in D_{EN} and D_{pb} such that if v is such a vertex in D_{EN} and D_v is the corresponding vertex in D_{pb} then l_v is the multiplicity of D_v as pole of the meromorphic function $\tilde{\phi}$.

Proof. We recall the definition of l_v. Each vertex of D_{EN} corresponds to a submanifold of S^3 which is an oriented fibration in circles. Let's denote f_v a fiber corresponding to v and disjoint from K_f which is the link at infinity of a generic fiber of f. Then l_v is the linking number of f_v and K_f. Let us take an oriented Seifert surface F_f of K_f which is transversal to f_v; by definition of linking number l_v is the algebraic intersection number of f_v and F_f.

We are going to take a model of S^3 in $\mathbb{C}^2$ adapted to our situation. Note that $\tilde{X} \setminus \tilde{\mathcal{D}} = \mathbb{C}^2$ and the boundary of a well-chosen tubular neighbourhood of $\tilde{\mathcal{D}}$ is S^3. Let's call $\mathcal{D}_\infty$ the union of irreducible components of $\tilde{\mathcal{D}}$ whose image by $\tilde{\phi}$ is ∞. We take $\tilde{\phi}^{-1}(D_R)$ as tubular neighbourhood of $\mathcal{D}_\infty$ where

$$D_R := \{t \in \mathbb{C} \mid |t| \geq R\} \cup \{\infty\} \subset \mathbb{P}^1,$$

and R is big enough.

We may choose a Hermitian metric in $\tilde{X}$ such that:

(a)   For each double point of $\tilde{\mathcal{D}}$ there exists local coordinates (u, v) in a neignbourhood U of this point such that $\tilde{\mathcal{D}} \cup U$ has $uv = 0$ as equation and the restriction of the metric to U is the standard one with respect to (u, v).

(b)   If the double point is in one component $D_1 \subset \tilde{\mathcal{D}}$ whose equation is $u = 0$ and another component $D_2 \notin \tilde{\mathcal{D}}$ then $\phi_{|U} = u^{-1}$.

For each component of $\tilde{\mathcal{D}}$ which is not in $\mathcal{D}_\infty$ we choose small tubular neighbourhoods in this metric. The union of these tubular neighbourhoods with $\tilde{\phi}^{-1}(D_R)$ produces a tubular neighbourhood of $\tilde{\mathcal{D}}$. Let's take S^3 as the boundary of this big tubular neighbourhood.

Let's take as generic fiber F_f the intersection of $f^{-1}(R)$ with S^3. Let us take a vertex v such that D_v is not sent to ∞; then we may choose f_v disjoint from F_f and so $l_v = 0$ (f_v is the boundary of a meridian disk of D_v). In this case as the generic point of D_v is not a pole its multiplicity as pole is also zero.

If $D_v \subset \mathcal{D}_\infty$ then choose a point $P \in D_v$ which is smooth in $\widetilde{\mathcal{D}} \cup \overline{f^{-1}(t)}$, for a generic $t \in \mathbb{C}$. We can choose analytic coordinates u, v around P such that the equation of D_v is $v = 0$ and $\tilde{\phi}(u, v) = v^{-a}$ where a is the multiplicity of D_v as pole of $\tilde{\phi}$. Note that we may take $f_v = \{(u, v) \mid u = 0, |v| = R^{-a}\}$. Then the algebraic intersection of F_f and f_v is equal to a. $\square$

Let us use this lemma to prove the remark. We recall that $l_v \geq 0$ for regular links. If $l_v > 0$ we deduce from the lemma that the generic point of D_v is sent to infinity by $\tilde{\phi}$. A generic fiber of $\tilde{\phi}$ is smooth at D_v and transversal to it; let's denote $P_1, \ldots, P_r$ the points of this generic fiber in D_v. These points are points of indeterminacy of $\tilde{\phi}$ as two different fibers pass through them (a generic one and $\tilde{\phi}^{-1}(\infty)$). As $\hat{\pi}$ is an embedded resolution of the closure of the generic fiber in $\mathbb{P}^2$ P_i is a smooth point of $\tilde{\phi}^{-1}(t)$, $t \in \mathbb{C}$, and each fiber is transversal to D_v. The intersection number of two fibers at P_i is equal to l_v. Then blowing up over each P_i l_v times, we obtain r equisingular horizontal components which are sections.

If $l_v = 0$ there are two possibilities for D_v. It may be an horizontal component; minimality of D_{pb} and the fact that this graph is a tree assure that this component is not equisingular and its degree equals the number of attached arrows. It may be also a component on which ϕ is constant and finite. In this case we cannot have arrows of the generic fiber attached to this component because only one fiber can intersect this component.

§3.-About Kaliman's proof.

The scheme of Kaliman's proof of this second result is as follows (see [5]): Lemma 4 $\Rightarrow$ Lemma 5 and then the result follows from Lemma 5 and Corollary 3. Let us recall the statement of Lemma 4:

Lemma 4 [5]. *Let f be a polynomial whose generic fiber is rational and let $\phi : X \to \mathbb{P}^1$ an extension of f. Let H_1, H_2 two horizontal components of $\mathcal{D}$. Then there is a commutative diagram*

$$
\begin{array}{ccc}
X & \overset{\sigma}{\dashrightarrow} & \widetilde{X} \\
{\scriptstyle\phi}\searrow & & \swarrow{\scriptstyle\tilde{\phi}} \\
& \mathbb{P}^1 &
\end{array}
$$

such that
(1) *$\tilde{\phi} : \widetilde{X} \to \mathbb{P}^1$ is an extension of f where $\widetilde{X}$ is a rational smooth compact surface and σ is a birational isomorphism.*
(2) *The restriction of σ to $\phi^{-1}(\mathbb{C})$ is a regular isomorphism from $\phi^{-1}(\mathbb{C})$ to $\tilde{\phi}^{-1}(\mathbb{C})$.*
(3) *$\tilde{\phi}^{-1}(\infty)$ is irreducible.*

(4) If $\widetilde{H}_k$ is the closure of the transform of H_k by σ, $k = 1, 2$, then $\widetilde{H}_1 \cap \widetilde{H}_2 \cap \tilde{\phi}^{-1}(\infty) = \emptyset$.

As Kaliman says it is easy to see that (applying Kodaira-Morrow's theorem) there exists a commutative diagram

$$X \overset{\sigma_1}{\to} X_1$$
$$\phi \searrow \quad \swarrow \phi_1$$
$$\mathbb{P}^1$$

satisfying (1),(2) and (3) (σ_1 is regular). In our case σ_1 is obtained collapsing one after the other the curves $L_\infty, A_1, B_1, A_2, B_3, B_2, B_5, B_6$.

In order to simplify the notation we keep the notation from Figure 1 for the proper transforms of the curves in X. We have:

- $B_6' = p_1^{-1}(\infty)$,
- $P = B_6' \cap A_4 \cap B_4$,
- (A_4, P) is a simple singularity of type $\mathbb{A}_4$,
- (B_4, P) is smooth,
- $(B_4 \cdot A_4)_P = 5$ and
- $(B_4 \cdot B_6')_P = 2$.

Let us take $H_1 = A_4$ and $H_2 = B_4$. Kaliman asserts that if $H_1 \cap H_2 \cap \phi_1^{-1}(\infty) \neq \emptyset$ then we may blow up P and then contract the proper transform of B_6' (a -1-curve). We obtain $\phi_2 : X_2 \to \mathbb{P}^1$ and σ_2 satisfying (1), (2) and (3); he claims that the intersection number of the proper transforms of H_1 and H_2 decrease and we can apply induction.

We see in our case that the intersection number does not decrease (it does for the blowing-up but as the curves are tangents to B_6' it increases with the contraction). In any case it is not possible to separate H_1 and H_2 with the hypothesis of Lemma 4 because if it would be true we may apply the scheme of Kaliman's proof showing that f has at most one horizontal component which is not a section and this statement is not true.

§4.- Remarks on an article of Russell.

In [8] the author constructs an example of *bad field generator*, i.e., a polynomial $f \in \mathbb{C}[x, y]$, such that its generic fiber is rational and such that there is no polynomial $g \in \mathbb{C}[x, y]$ verifying $k(f, g) = k(x, y)$. This property is related with the non-existence of sections: a rational polynomial is a *bad field generator* if and only if no horizontal component is a section. The example given by Russell is a polynomial of degree 21; it has only two horizontal

components, one of degree 3 and the other one of degree 2. It gives also a counter-example to Kaliman's theorem. In fact, from the above characterisation of *bad field generator*, [**8**, 2.2], we deduce that all bad field generators are counter-examples to Kaliman's theorem; note that our counter-example is a *good field generator* because there are two sections. We write down his polynomial because the polynomial which appears in [**8**, p. 324] is not the right one:

$$
\begin{aligned}
f(x,y) := \; & y^3(xy+1)^9 + 4x^7y^9 + 25x^6y^8 + 66x^5y^7 \\
& + 6x^5y^6 + 95x^4y^6 + 23x^4y^5 + 80x^3y^5 + 34x^3y^4 + 4x^3y^3 \\
& + 39x^2y^4 + 24x^2y^{3*} + 7x^2y^2 + 10xy^3 + 8xy^{2*} + 3xy + x + y^2 + y^*.
\end{aligned}
$$

(The * shows the coefficients which do not match with [**8**].) From the first remark of [**5**] we deduce that this polynomial has one factorisation. In fact we have:

$$
\begin{aligned}
f := \; & \left(y^2(xy+1)^4 + y(2xy+1)(xy+1) + 1\right) \\
& \qquad\qquad\qquad \times \left(y(xy+1)^5 + 2xy(xy+1)^2 + x\right).
\end{aligned}
$$

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