

STOCHASTIC HYPERBOLIC SYSTEMS, SMALL PERTURBATIONS AND PATHWISE APPROXIMATION

ADNAN ABOULALAA

(Received January 5, 2018, revised July 16, 2019)

Abstract

This paper is devoted to the study of hyperbolic systems of linear partial differential equations perturbed by a Brownian motion. The existence and uniqueness of solutions are proved by an energy method. The specific features of this class of stochastic partial differential equations are highlighted and the comparison with standard existence results for SPDEs is discussed. The small perturbations problem is studied and a large deviation principle is stated. A pathwise approximation result, similar to the stochastic differential equations case, is established, with an application to a support theorem.

Contents

1. Introduction	890
2. Hyperbolic stochastic partial differential systems	891
2.1. Notations	891
2.2. Remarks on the infinite dimensional stochastic calculus	892
2.3. Existence and uniqueness of a solution	896
2.4. The case of differential operators and other remarks	904
2.5. Comparison with other existence results for SPDEs	906
3. Small perturbations	908
3.1. Introduction and preliminaries	908
3.2. A large deviation principle	909
4. Pathwise approximation and applications	912
4.1. Wong-Zakai type approximation	913
4.2. Application to a support theorem	914
4.3. Application to the random evolution operator	915
5. On the regularity of laws of the solutions	916
6. Appendix: Proofs of some technical lemmas	919
6.1. Proofs of the lemmas related to small perturbations	919
6.2. Proofs of the lemmas related to pathwise approximation and support theorem	923
References	932

1. Introduction

In this paper we are interested in the following class of stochastic partial differential equations:

$$(E) \begin{cases} du(t) = \sum_{i=1}^n [a_i(t, x, D)u(t) \circ dw_i(t) + f_i(t) \circ dw_i(t)] + b(t, x, D)u(t)dt + g(t)dt, \\ u(0) = u_0 \in (H^s(\mathbb{R}^d))^{d'}, \end{cases}$$

where $H^s(\mathbb{R}^d)$ is a Sobolev space ($s \in \mathbb{R}, d, d' \geq 1$), $a_i(t, x, D), b(t, x, D)$ are smooth families of $d' \times d'$ -matrices of first order pseudodifferential operators (PDO), $w_i(t), t \in I$ are standard Wiener processes, f_i, g are continuous, possibly random, functions from $I = [0, T], T > 0$ to $(H^s(\mathbb{R}^d))^{d'}$, and \circ corresponds to the Fisk-Stratonovich integral or differential. Equation (E) is to be viewed as a random perturbation of a (deterministic) linear symmetric system ($a \equiv 0, f \equiv 0$, cf. Friedrichs[19], Lax[42]). These systems occur often in applications, for example, the wave equation in non-homogeneous media (and more generally any second order linear hyperbolic equation) can be represented by such a system. The Maxwell equations form a symmetric hyperbolic system. The Dirac equation for a relativistic particle of spin 1/2 (whose wave function is a 4-dimensional vector) is also a linear symmetric system.

Hyperbolic partial differential equations (PDE) and systems form an important class in the theory of PDE, and there is already a substantial literature on hyperbolic stochastic partial differential equations (SPDEs), see, e.g., [72], [27], [12], [16], [46], [34], [21], [4], [45], [7]. These works address various models including the stochastic wave and related equations with space-time white noise, non linear models related to conservation laws and hyperbolic systems with additive random perturbation (with assumptions on the diffusion coefficient that exclude the case of the system (E)).

Particular forms of Eq. (E) have been considered by Ogawa [56], and Funaki [20] who used a method of characteristics to construct the solutions. Later, Kunita [40] made use of this method and the theory of stochastic flows to obtain solutions to nonlinear first order partial differential equations in the scalar case. However, it is well known that, in the deterministic case, the method of characteristics no longer applies if u is not scalar (and for more than one space variable). Instead, energy or semigroup methods are used to solve linear symmetric systems, see, e.g., Cordes [13], Hörmander [29], Taylor [66]. In the first part of this paper we use an energy method, based on *a priori* estimates, to solve systems like (E). We also discuss the relation to standard existence results for stochastic partial differential equations (SPDEs).

The second part deals with the small random perturbations of the deterministic system $(E_d) : du(t) = b(t, x, D)u(t)dt, u(0) = u_0$. More precisely, if we consider the solution $u^\epsilon(\cdot)$ to:

$$(S_\epsilon) : du^\epsilon(t) = \sqrt{\epsilon} \sum_{i=1}^n a_i(t, x, D)u^\epsilon(t) \circ dw_i(t) + b(t, x, D)u^\epsilon(t)dt, \quad u^\epsilon(0) = u_0,$$

then it is easily shown that u^ϵ converges in probability (in some Hilbert space) to the solution u of (E_d) as $\epsilon \rightarrow 0$. The objective is to obtain information on how the stochastic solution is close to the deterministic one as the perturbation becomes very small. In the case of stochastic differential equations (SDEs), the deviation probabilities of the stochastic solutions

from the deterministic ones, converge to 0 with an exponential rate (Freidlin-Wentzell estimates for SDEs), i.e., we have a large deviation principle for the small random perturbation of ordinary differential equations. In the case of SPDEs, we face two problems: the dimension is infinite and the coefficients are unbounded. In the literature, these problems have been addressed for various SPDEs, see, e.g., [15], [59], [9], [62] and the references therein. In our case, it turns out that the Freidlin-Wentzell estimates hold, but with the loss of two derivatives: u^ϵ is an H^s -valued process, the large deviation principle is valid in the topology of $C^0(I, H^{s-2})$ associated with the norm $\sup_t |v(t)|_{s-2}$.

The third part of this paper is concerned with the approximation of the stochastic system (E) by deterministic systems depending on a random parameter. In the case of finite dimensional stochastic equations, this is sometimes called the Wong-Zakai or Stroock-Varadhan approximation [73], [67]. Such approximations have received some attention in the case of SDEs for they allow the transfer of some properties of ordinary differential equations to the stochastic case (e.g., the construction of the stochastic flows of diffeomorphisms for SDEs, the initial approach to the stochastic calculus of variations, or the approximation of the solutions), see e.g., Malliavin [47], Ikeda-Watanabe [30] and the references given there. The case of SPDEs has also been considered for several models, see, e.g., Gyöngy [24], Brzezniak-Flandoli [8], Hairer and Pardoux [28], Yastrzhembskiy [74] and Roth [65]; the later reference considered stochastic hyperbolic equations similar to (E) in the scalar case. In this part, the convergence of the deterministic systems associated to (E) to the stochastic one is shown and this is applied to an infinite-dimensional extension of the Stroock-Varadhan support theorem and to the construction of the random evolution semigroup of Eq. (E).

The last part of the paper deals with the regularity of the law of the solutions to Eq. (E) using the Malliavin calculus techniques.

In this paper we restrict ourselves to finite dimensional Wiener processes in order to highlight the specific features of hyperbolic systems, in particular the necessity to use Fisk-Stratonovich integral and the fact that the techniques used in parabolic SPDEs are not suitable in this case.

Finally, as we consider a class of SPDEs that involves pseudodifferential operators (PDO), let us mention some of the few works so far published which use these operators in probabilistic models. Among the references, we quote Kotelenetz [35], Kallianpur and Xiong [33] and Tindel [68] who considered SPDEs with pseudodifferential operators with an assumption on the order ($m > d$) and a space-time white noise; Jacob, Potrykus and Wu studied in [31] the solution of a stochastic Burger equation driven by a space time white noise and using a PDO; and in [44], Liu and Zhang started a study of stochastic pseudodifferential operators with a Calderón-type uniqueness theorem as an application to SPDEs.

2. Hyperbolic stochastic partial differential systems

2.1. Notations. Let $d, d' \geq 1$. We denote by S^m the set of symbols $a(x, \xi)$ of order m on \mathbb{R}^d , i.e. $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and for all $\alpha, \beta \in \mathbb{N}^d$ there is a constant $C(\alpha, \beta)$ such that $|D_\xi^\alpha D_x^\beta a(x, \xi)| \leq C(\alpha, \beta)(1 + |\xi|)^{m - |\alpha|}$, where we have used the notation: for $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, $|\alpha| = \sum_{j=1}^d \alpha_j$ and $D^\alpha = D_1^{\alpha_1} \dots D_d^{\alpha_d}$, where $D_j^{\alpha_j} = (-i)^{\alpha_j} \partial_j^{\alpha_j}$ with $i = \sqrt{-1}$ and $\partial_j = \partial_{x_j}$, for $x = (x_1, \dots, x_d)$.

For $a \in S^m$, $a(x, D)$ will denote the associated pseudodifferential operator defined by

$a(x, D)u(x) = \int a(x, \xi)\hat{u}(\xi)e^{ix\cdot\xi}d\xi$ for $u \in C_0^\infty(\mathbb{R}^d)$. OPS^m will designate the set of such operators and $a^*(x, D)$ is the adjoint of $a(x, D)$. We denote by $\langle \cdot, \cdot \rangle_s$ the scalar product on the Sobolev space $H^s := H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$. We use the same notation for the scalar product on $(H^s)^{d'}$. In the following, we shall consider matrices of pseudodifferential operators $a(x, D)$, which means that $a(x, D) = (a^{ij}(x, D), i, j = 1, \dots, d')$ with $a^{ij}(x, D) \in OPS^m$ for some m . We still denote by OPS^m the set of such matrices of operators.

Throughout this paper, we fix $T > 0$ and we assume that we are given a one-dimensional Brownian motion $w(t), t \in I := [0, T]$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with $\mathcal{F}_t = \sigma(w(\tau), \tau \leq t)$. For $t, t' \in \mathbb{R}$, we set $t \wedge t' = \inf(t, t')$. In the sequel, we deal with $(H^s)^{d'}$ -valued process. For $p > 0$, $M_p^s(I, (H^s)^{d'})$ will designate the set of adapted $(H^s)^{d'}$ -valued processes $u(t), t \in I$ such that $\|u\|_{s,p} := (E \sup_{t \leq T} |u(t)|_s^p)^{1/p} < +\infty$. M_p^s will generally be endowed with the norm $\|\cdot\|_{s,p}$.

The quadratic variation of two processes M, N will be denoted by $\langle M, N \rangle_t$, the time variable subscript will be either t or τ , while the inner product in H^s will be denoted by $\langle u, w \rangle_s$, in which case the subscript will always be the letter s , the order of the Sobolev space; the notation $\langle u, w \rangle$, without subscript, refers to the inner product in $L^2((\mathbb{R}^d)^{d'})$.

2.2. Remarks on the infinite dimensional stochastic calculus. As we are dealing with infinite dimensional processes, the following remarks concern the stochastic calculus concepts that will be used in this paper. The Fisk-Stratonovich integral used in the hyperbolic SPDE (E) is defined by:

$$\int_0^T u(t) \circ dw(t) = \int_0^T u(t)dw(t) + \frac{1}{2} \int_0^T d \langle u, w \rangle_t.$$

where $\langle u, w \rangle_t$ is the quadratic cross variation of the processes $u(\cdot), w(\cdot)$. This integral is usually defined when the integrand $u(t)$ is a semimartingale; we also note that, in our case, $u(\cdot), w(\cdot)$ are in different spaces.

In the following H is a separable Hilbert space and $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a filtered probability space.

• **Martingales and semimartingales in Hilbert spaces** ([49], [48], [15], [22]): An H -valued process $M(t)$ is an \mathcal{F}_t -martingale if $\forall t, t' \geq 0$, with $t' \leq t$: (1) $M(t)$ is an $\mathcal{F}_{t'}$ -measurable random variable, (2) $E(\|M(t)\|) < +\infty$ and (3) $E(M(t)|\mathcal{F}_{t'}) = M(t')$. The last condition is equivalent to $E(\langle M(t), h \rangle_H | \mathcal{F}_{t'}) = \langle M(t'), h \rangle_H, \forall h \in H$. In this case $\|M(t)\|^2$ is a real submartingale and therefore $E\|M(t)\|^2 \leq E\|M(T)\|^2$, for all t in $[0, T]$. We also have the following inequalities:

$$(2.1) \quad E(\sup_{t \in I} \|M(t)\|^p) \leq \left(\frac{p}{p-1}\right)^p \sup_{t \in I} E\|M(t)\|^p, \quad \forall p \geq 1$$

$$(2.2) \quad \Pr(\sup_{t \in I} \|M(t)\| \geq \lambda) \leq \lambda^{-p} \sup_{t \in I} E\|M(t)\|^p, \quad \forall p > 1$$

The space $\mathcal{M}^2(H)$ of square integrable martingales equipped with the norm:

$$\|M(t)\|_{\mathcal{M}^2(H)} = (E(\sup_{t \in I} \|M(t)\|^2))^{1/2}$$

is a Banach space. $M(t)$ is said to be a local martingale if there exists an increasing sequence of stopping times σ_n with $\lim \sigma_n = +\infty$ such that for every n , $M(t \wedge \sigma_n)$ is a martingale. An H -valued process $X(t)$ is a semimartingale if $X(t) = M(t) + V(t)$ where $M(t)$ is a local martingale and $V(t)$ is a process which has a finite variation a.e. in every bounded interval.

• **The quadratic variation and the cross variation tensor of a martingale** ([49], [48], [15]): For a martingale $M(t) \in \mathcal{M}^2(H)$ there exists a unique $H \otimes H$ -valued process, $\langle\langle M(t) \rangle\rangle$, called the quadratic variation of $M(t)$ such that $M(t) \otimes M(t) - \langle\langle M(t) \rangle\rangle$ is a martingale; as usual \otimes denotes the tensor product of Hilbert spaces or elements of Hilbert spaces. In the same way for two square integrable martingales $M_1(t), M_2(t)$ with values in two Hilbert spaces H_1, H_2 , there exists a unique $H_1 \otimes H_2$ -valued process, $\langle\langle M_1(t), M_2(t) \rangle\rangle$, called the quadratic cross variation of $M_1(t), M_2(t)$ such that $M_1(t) \otimes M_2(t) - \langle\langle M_1(t), M_2(t) \rangle\rangle$ is a martingale. If e_i and f_j are orthonormal bases (ONB) of H_1, H_2 , then, with $M_1(t) = \sum_{i=1}^{\infty} M_{1,i}(t)e_i$ and $M_2(t) = \sum_{j=1}^{\infty} M_{2,j}(t)f_j$, we have:

$$(2.3) \quad \langle\langle M_1(t), M_2(t) \rangle\rangle = \sum_{i,j=1}^{\infty} \langle\langle M_{1,i}(t), M_{2,j}(t) \rangle\rangle e_i \otimes f_j,$$

and $\langle\langle M_{1,i}(t), M_{2,j}(t) \rangle\rangle$ is the usual cross variation of the real martingales $M_{1,i}, M_{2,j}$. Let us note that these notions can be defined in another way, for more general processes X, Y with values in the Hilbert spaces H_1, H_2 by:

$$D_t^n(X, Y) = \sum_{k \geq 1} (X(\sigma_{n,k+1} \wedge t) - X(\sigma_{n,k} \wedge t)) \otimes (Y(\sigma_{n,k+1} \wedge t) - Y(\sigma_{n,k} \wedge t))$$

$$\langle\langle X, Y \rangle\rangle_t = \lim_{n \rightarrow \infty} D_t^n(X, Y),$$

if the limit in probability of D_t^n exists. Here, $\sigma_{n,k}$ is family of stopping times such that $\sup_k(\sigma_{n,k+1} - \sigma_{n,k}) \rightarrow 0$ as $n \rightarrow \infty$, so that we have a subdivision of the interval $[0, t]$, see [48], [49].

• **Itô integral and Itô formula in Hilbert spaces** ([15], [22]). The Itô integral $\int Y(t)dX(t)$ can be defined in a general setting, where $X(t)$ is a J -valued process, $Y(t)$ is a $L(J, K)$ -valued process, J, K are separable Banach spaces, $L(J, K)$ is the space of linear operators (not necessarily bounded) from J into K , with the corresponding properties, in particular the Itô formula, see Métivier-Pellaumail [49]. Since we deal with the special case where $Y(t)$ belongs to a Hilbert space and $X(t)$ is a real Brownian motion, we restrict ourselves to the case of stochastic integral with respect to a Q -Wiener process.

Let H, U be two separable Hilbert spaces, $Q \in L(U)$ a linear bounded operator of U which is self-adjoint, non-negative and of finite trace. A Q -Wiener process is an U -valued process $W(t)$ such that: (1) $W(0) = 0$, (2) for all $0 \leq t_1 \leq \dots \leq t_n \leq T$, $W(t_1), W(t_2) - W(t_1), \dots, W(t_n) - W(t_{n-1})$ are independent random variables (rv's), (3) $W(t) - W(t')$ is a Gaussian random variable with the covariance operator $(t - t')Q$ and (4) the paths of W are continuous P -a.e. If e_i is an ONB diagonalizing Q with eigenvalues λ_i , then $W(t)$ can be written as:

$$(2.4) \quad W(t) = \sum_{i \geq 1} \lambda_i^{1/2} W_i(t) e_i$$

where $W_i(t)$ are independent real Brownian motions. Now, let U_0 be the subspace $Q^{1/2}(U)$ of U ; then the stochastic integral $\int_0^T \Phi(\tau)dW(\tau)$ can be defined for all $L(U_0, H)$ -valued predictable processes $\Phi(\tau)$ that are Hilbert-Schmidt for all τ and such that $\int_0^T \|\Phi(\tau)\|_{HS}^2 d\tau < +\infty$.

In this case, if $\phi(\tau)$ is an adapted H -valued process and $X(t)$ is the H -valued process defined by:

$$X(t) = X(0) + \int_0^t \phi(\tau)ds + \int_0^t \Phi(\tau)dW_\tau,$$

with $X(0)$ an \mathcal{F}_0 -measurable rv, then the Itô formula has the following form ([15], [22]):

$$(2.5) \quad \begin{aligned} F(t, X(t)) = & F(0, X(0)) + \int_0^t DF(\tau, F(X(\tau))).\Phi(\tau)dW(\tau) \\ & + \int_0^t \left[\frac{\partial F}{\partial \tau}(\tau, X(\tau)) + DF(\tau, F(X(\tau))).\phi(\tau) \right. \\ & \left. + \frac{1}{2}\text{tr}(D^2 F(\tau, X(\tau)).(\Phi(\tau)Q^{1/2})(\Phi(s)Q^{1/2})^*)d\tau, \right. \end{aligned}$$

(we use the notation $DF(x).V$ for the image of a vector V by the differential map $DF(x)$). Other forms of the Itô formula may be found in more general settings in [49], [48].

• **Fisk-Stratonovich integral in Hilbert spaces:** In the real case, the Fisk-Stratonovich integral with respect to a semimartingale $X(t)$ is defined by:

$$(2.6) \quad \int_0^T Y(t) \circ dX(t) = \int_0^T Y(t)dX(t) + \frac{1}{2} \int_0^T d\langle X, Y \rangle_t,$$

where $Y(t)$ is also a semimartingale. In the last formula $X(t), Y(t)$ are also assumed to be continuous. See, e.g., [30], [61]. These conditions can be relaxed; for a detailed study we refer to Meyer ([50], Chap. VI, p. 109). In the Hilbert space case, this concept seems not to have received much attention (to the best knowledge of the author); however, the same formula (2.6) is valid with the previous definition of the cross variation (see [2], [71] for some further results).

In our case, for a semimartingale $u(t) = \sum_{i \geq 1} u_i(t)e_i \in H$, where the $u_i(t)$ are real semimartingales, and a real Brownian motion $w(t)$, we note that:

$$\langle u, w \rangle_t = \sum_{i \geq 1} \langle u_i, w \rangle_t e_i,$$

and by expansion in the ONB (e_i) , the calculus rules in this specific Hilbert space setting can also be deduced from their real counterpart. For instance, if $du(t) = \sigma(u(t)) \circ dw(t)$ where $\sigma : H \rightarrow H$ is a differentiable map, then:

$$du(t) = \sigma(u(t))dw(t) + \frac{1}{2}D\sigma(u(t)) \cdot \sigma(u(t))dt$$

• *Application of the Itô formula:* Let us consider a special case of the equation (E): $du(t) = a u(t) \circ dw(t)$, where $u(t)$ is an adapted $(H^s)^d$ -valued process, a is a first order symmetric operator in $(H^s)^d$ (denoted by H) and $w(t)$ is a real Brownian motion. This equation is then written in the Itô form as:

$$du(t) = au(t)dw(t) + \frac{1}{2}a^2u(t)dt$$

Now we shall apply the previous Itô formula to $F(u(t)) = |u(t)|_s^2 = \langle u(t), u(t) \rangle_s$. For the differentials of F , we have:

$$DF(v).h = \langle v, h \rangle_s + \langle h, v \rangle_s \text{ and } D^2F(v).h.k = \langle k, h \rangle_s + \langle h, k \rangle_s$$

In our case, the Q -Wiener process is just a one-dimensional Brownian motion, $U = U_0 = \mathbb{R}$, $Q = Id$ and $\Phi(\tau)$ is the linear operator from \mathbb{R} to H defined by $\Phi(\tau).\alpha = \alpha au(\tau) \in H, \alpha \in \mathbb{R}$ and $\Phi(\tau)^* = \Phi(\tau)$, so that:

(a) The second order term involving D^2F in (2.5) is:

$$D^2F(u(t)).au.au = 2 \langle au(t), au(t) \rangle_s = 2 \langle a^* au(t), u(t) \rangle_s$$

(b) For the first order derivative term $DF(\tau, F(X(\tau))).\phi(\tau)$, we have $\phi(\tau) = 1/2(a^2u(\tau))$ and:

$$\begin{aligned} DF(\tau, F(X(\tau))).\phi(\tau) &= \frac{1}{2}(\langle u(\tau), a^2u(\tau) \rangle_s + \langle a^2u(\tau), (\tau) \rangle_s) \\ &= \frac{1}{2} \langle (a^2 + a^{*2})u(\tau), u(\tau) \rangle_s \end{aligned}$$

(c) The term involving $dw(\tau)$ is:

$$DF(u(\tau)).au(\tau) = \langle u(\tau), au(\tau) \rangle_\tau + \langle au(\tau), u(\tau) \rangle_s = \langle (a + a^*)u(\tau), u(\tau) \rangle_s .$$

This yields:

$$\begin{aligned} |u(t)|^2 &= |u(0)|^2 + \int_0^t \langle (a + a^*)u(\tau), u(\tau) \rangle_s dw(\tau) \\ &\quad + \frac{1}{2} \int_0^t \langle (a^2 + a^{*2})u(\tau), u(\tau) \rangle_s d\tau + \frac{1}{2} \int_0^t 2 \langle a^* au(\tau), u(\tau) \rangle_s d\tau \\ &= |u(0)|^2 + \int_0^t \langle (a + a^*)u(\tau), u(\tau) \rangle_s dw(\tau) \\ &\quad + \frac{1}{2} \int_0^t \langle (a^*(a^* + a) + (a^* + a)a)u(\tau), u(\tau) \rangle_s d\tau \end{aligned}$$

So that, with $A = a + a^*, L = a^*(a^* + a) + (a^* + a)a$, we have:

$$(2.7) \quad |u(t)|^2 = |u(0)|^2 + \int_0^t \langle Au(\tau), u(\tau) \rangle_s dw(\tau) + \frac{1}{2} \int_0^t \langle Lu(\tau), u(\tau) \rangle_s d\tau$$

REMARK. It is convenient to derive formulas like (2.7) by using the symbolic stochastic and differential calculus rules, and since this will be repeatedly used throughout the paper, let us apply these rules in this simple but typical situation: If $X(t) = A_X(t) + M_X(t)$ and $Y(t) = A_Y(t) + M_Y(t)$ are two continuous semimartingales where A and M denote respectively the finite variation and martingale parts, then the cross variation $d \langle M_X, M_Y \rangle_t$ is formally denoted by $dX(t) \cdot dY(t)$ and we have a set of symbolic calculus rules including (we omit here the time parameter):

- $Y \circ dX = YdX + \frac{1}{2}dX \cdot dY,$
- $X \circ (dY \cdot dZ) = (X \circ dY) \cdot dZ = X \circ (dY \cdot dZ),$
- $(XY) \circ dZ = X \circ (Y \circ dZ),$
- $dA_X \cdot dA_Y = dA_X \cdot dY = 0, \quad dw \cdot dw = dt, \quad dX \cdot dY \cdot dZ = 0.$

See [30], pp.99-100. Besides these identities, the usual differential calculus rules can be applied when using the Fisk-Stratonovich differentials (as remarked before, in the present setting with Hilbert space valued processes and finite dimensional Brownian motion, these results are valid under the same conditions of finite dimension processes, via an expansion in an ONB). In our example, we apply this to $|u(t)|_s^2 = \langle u(t), u(t) \rangle_s$:

$$(2.8) \quad \begin{aligned} d|u(t)|_s^2 &= \langle au(t), u(t) \rangle_s \circ dw(t) + \langle u(t), au(t) \rangle_s \circ dw(t) \\ &= \langle (a + a^*)u(t), u(t) \rangle_s \circ dw(t), \end{aligned}$$

and with $A = a + a^*$:

$$(2.9) \quad \langle Au(t), u(t) \rangle_s \circ dw(t) = \langle Au(t), u(t) \rangle_s dw(t) + \frac{1}{2}d \langle Au(t), u(t) \rangle_s \cdot dw(t).$$

So we have to calculate cross variation $d \langle Au(t), u(t) \rangle_s \cdot dw(t)$; we use the above mentioned stochastic rules to get:

$$(2.10) \quad \begin{aligned} d \langle Au(t), u(t) \rangle_s \cdot dw(t) &= (\langle Aau(t), u(t) \rangle_s \circ dw(t)) \cdot dw(t) \\ &\quad + (\langle Au(t), au(t) \rangle_s \circ dw(t)) \cdot dw(t) \\ &= (\langle (Aa + a^*A)u(t), u(t) \rangle_s \circ dw(t)) \cdot dw(t) \\ &= \langle Lu(t), u(t) \rangle_s dw(t) \cdot dw(t) \\ &= \langle Lu(t), u(t) \rangle_s dt \end{aligned}$$

and by replacing (2.9) and (2.10) in (2.8) we get (2.7).

From this calculus we may expect that the existence of solutions of systems like (E) will be related to assumptions about the boundedness of the operators A, L ; this is possible in the case of symmetric hyperbolic systems (as in the deterministic case). We also note that this is possible only when (E) is written in the Stratonovich form, see the remark (c) in § 2.4.

2.3. Existence and uniqueness of a solution. We consider stochastic equations of the type:

$$(\mathcal{E}) : \begin{cases} \frac{\partial u(t, x)}{\partial t} = a_t(x, D)u(t, x) \circ \frac{dw(t)}{dt} + b_t(x, D)u(t, x) + f(t, x) \circ \frac{dw(t)}{dt} + g(t, x) \\ u(0, \cdot) = u_0(\cdot). \end{cases}$$

where $u_0 \in (H^s)^{d'}$, $f, g \in C^0(I, (H^s)^{d'})$ for some s , $w(t), t \in I$ is a one-dimensional Brownian motion and $a_t, b_t, t \in I$ are (matrices of) pseudo-differential operators satisfying the following conditions:

- (i) $a_t(x, D), b_t(x, D)$ form a bounded family in OPS¹ (see below).
- (ii) The $C^\infty(\mathbb{R}^{2d})$ -valued maps: $t \mapsto a_t^{i,j}, b_t^{i,j}$ are continuous ($C^\infty(\mathbb{R}^{2d})$ is equipped with the usual topology of the semi-norms $\max_{x \in \mathbb{R}^d} |\partial^p f|, p = (p_1, \dots, p_k) \in \mathbb{N}^k, k \geq 0$).

Throughout this paper, the space variable x may not be shown and we shall write $v(t, \cdot) = v(t) \in (H^s)^{d'}$ for $v = u, f, g$, etc.

Bounded family of symbols and operators: We shall need some results about the action of pseudodifferential operators on Sobolev spaces and their boundedness:

- A family of operators $p_t(x, D)$ is said to be bounded in H^s if: $\sup_{t \in I} \|p_t(x, D)\|_s \leq C$, where $\|P\| = \sup_{\|u\|=1} \|Pu\|$ is the norm of the operator P acting on some vector space.
- A family of symbols $p_t(x, \xi) \in S^m$ that satisfy $|D_\xi^\alpha D_x^\beta p_t(x, \xi)| \leq C_t(\alpha, \beta)(1 + |\xi|)^{m-|\alpha|}$ is bounded if the constants C_t are bounded w.r.t the parameter t . In other words:

$$(2.11) \quad \forall l > 0, \exists C_l > 0 : |p_t|_l^{(m)} := \max_{|\alpha+\beta \leq l} \sup_{x, \xi} (D_\xi^\alpha D_x^\beta p_t(x, \xi)(1 + |\xi|)^{-(m-|\alpha|)}) \leq C_l$$

Let us recall the important result about the action of pseudodifferential operators on the Sobolev spaces H^s (Hörmander, Kumano-Go, see [37] (Lemma 2.1), [38], p.124): for all $s > 0$ there exist $C_s, l_s > 0$ such that for $p(x, \xi) \in S^m$ we have:

$$(2.12) \quad \forall u \in H^s : \|p(x, D)u\|_s \leq C_{s,m} |p|_{l_s}^{(m)} \|u\|_{s+m}$$

which means that $p(x, D)$ maps H^{s+m} onto H^s . The constants $C_{s,m}, l_s$ are independent of the operators p and therefore, if $p_t(x, \xi) \in S^m$ is a bounded family in S^m then by (2.11) and (2.12) we will have $\|p_t(x, D)u\|_s \leq C'_s \|u\|_{s+m}$ for all $t \in I$.

In particular, for $m = 0$ and a bounded family $p_t(x, \xi) \in S^0$, we have $\|p_t(x, D)u\|_s \leq C'_s \|u\|_s$ for all $t \in I$, which means that the family of operators $p_t(x, D)$ is bounded in OPS⁰:

$$(2.13) \quad \sup_{t \in I} \|p_t(x, D)\|_s \leq C$$

The equation (\mathcal{E}) is to be viewed as

$$(\mathcal{E}_S) : \quad u(t) = u_0 + \int_0^t a_\tau(x, D)u(\tau) \circ dw_\tau + \int_0^t b_\tau(x, D)u(\tau) d\tau + \int_0^t f(\tau) \circ dw_\tau + \int_0^t g(\tau) d\tau,$$

where $\circ dw(t)$ is the Fisk-Stratonovich differential. With

$$g_1(\tau) = g(\tau) + \frac{1}{2} \langle f, w \rangle_\tau,$$

(\mathcal{E}_S) can be written in the Itô form as:

$$(\mathcal{E}_I) : \quad u(t) = u_0 + \int_0^t a_\tau(x, D)u(\tau) dw_\tau + \int_0^t b_\tau(x, D)u(\tau) d\tau + \int_0^t f(\tau) dw_\tau + \frac{1}{2} \int_0^t a_\tau(x, D)(a_\tau(x, D)u(\tau) + f(\tau)) d\tau + \int_0^t g_1(\tau) d\tau.$$

The integrals involve the processes $a_\tau(x, D)u(\tau)$ in the Stratonovich form (\mathcal{E}_S) and $a_\tau(x, D)a_\tau(x, D)u(\tau)$ in the Itô form; the later integrals belong to $(H^{s-2})^{d'}$ when $u(\tau) \in (H^s)^{d'}$, the integrals above and the equations (\mathcal{E}_I) , (\mathcal{E}_S) are to be considered in $(H^{s-2})^{d'}$, but it will be seen that the solution will be actually an $(H^s)^{d'}$ -valued process.

Notation. In the sequel we will set:

$$\begin{aligned} A_t(x, D) &= a_t(x, D) + a_t^*(x, D), \\ B_t(x, D) &= b_t(x, D) + b_t^*(x, D), \\ L_t(x, D) &= A_t(x, D)a_t(x, D) + a_t^*(x, D)A_t(x, D) \end{aligned}$$

and we will often denote the matrices of PDOs $a_t(x, D), b_t(x, D), A_t(x, D)$, etc. by $a(t), b(t), A(t)$, or a_t, b_t, A_t , etc. We have the following existence and uniqueness result:

Theorem 2.1. *Let $a_t(x, D), b_t(x, D)$ be two families of matrices of pseudodifferential operators which satisfy (i)-(ii). We assume that $\sup_t E|f(t)|_{s+1}^4, \sup_t E|g_1(t)|_s^4$ are bounded, with $g_1(t) = g(t) + \langle f, w \rangle_t$, and*

- (iii) $A_t(x, D), B_t(x, D)$ are bounded families in OPS^0
- (iv) $L_t(x, D)$ is a bounded family in OPS^0

Then (\mathcal{E}) has a unique solution $u \in M_2(I, H^s)$ and $(u(t)), t \in I$ is a strong Markov process. Moreover, the solution $u(t)$, viewed as an $H^{s'}$ -valued process ($s' \leq s$), has a modification which is almost-surely γ -Hölder continuous with respect to the norm $|\cdot|_{s-2}$, for all $\gamma \in]0, 1/4[$.

This theorem extends easily to the case of symmetrizable systems: instead of conditions (iii) and (iv), we may suppose that there exist smooth families of $(d' \times d')$ -matrices of pseudodifferential operators $R_t^1, R_t^2 \in OPS^0$ such that the principal symbols $R_0^i(t, x, \xi), i = 1, 2$ are positive definite matrices for $|\xi| \geq 1$, and (iii), (iv) are to be replaced by:

- (iii) $R_t^1 a_t + a_t^* R_t^1, R_t^2 b_t + b_t^* R_t^2$ form bounded families in OPS^0 .
- (iv) $(R_t^1 a_t + a_t^* R_t^1) a_t + a_t^* (R_t^1 a_t + a_t^* R_t^1)$ is a bounded family in OPS^0 .

The rest of this section is devoted to the proof of Theorem 2.1, which is inspired from the corresponding proof for deterministic symmetric systems. We mention that similar *a priori* estimates were used by Mohammed and Sango [52] to obtain existence results for an hyperbolic PDE with additive random term defined by a finite dimensional Brownian motion, and in [25], Grecksch and Tudor used a parabolic regularization method to solve first order stochastic equations similar to (E) in the scalar case.

2.3.1 Energy estimates. Let $u \in M_2(I, (H^s)^{d'}), f, g \in C^0(I, (H^s)^{d'})$ be such that

$$\begin{aligned} u(t) = & u_0 + \int_0^t a_\tau(x, D)u(\tau) \circ dw_\tau + \int_0^t b_\tau(x, D)u(\tau)d\tau \\ & + \int_0^t (f(\tau) \circ dw_\tau + g(\tau)d\tau). \end{aligned}$$

Then by the Itô formula (§2.2) we get:

$$\begin{aligned} (2.14) \quad |u(t)|_s^2 = & |u_0|_s^2 + \int_0^t (\langle A(\tau)u(\tau), u(\tau) \rangle_s + 2Re \langle f(\tau), u(\tau) \rangle_s)dw(\tau) \\ & + \int_0^t \langle B(\tau)u(\tau), u(\tau) \rangle_s d\tau + \frac{1}{2} \int_0^t \langle L(\tau)u(\tau), u(\tau) \rangle_s d\tau \\ & + Re \int_0^t \{ \langle A(\tau)u(\tau), f(\tau) \rangle_s + 2 \langle u(\tau), g_1(\tau) \rangle_s \\ & + \langle a(\tau)u(\tau), f(\tau) \rangle_s + \langle f(\tau), f(\tau) \rangle_s \} d\tau \end{aligned}$$

This formula can be obtained in the same way as for the case treated in §2.2 formula (2.7) where b, f, g are $\equiv 0$, which explains the main terms A, L .

Using the boundedness of the family a_t, b_t and Schwarz's and martingale inequalities, it follows from (2.14) that

$$\begin{aligned}
 E \sup_{\theta \leq t} |u(\theta)|_s^4 &\leq C_2(E|u_0|_s^4 + E \int_0^t (A^2|u(\tau)|_s^4 + |f(\tau)|_s^2|u(\tau)|_s^2)d\tau \\
 &\quad + \int_0^t ((B^2 + L^2)|u(\tau)|_s^4 + |g_1(\tau)|_s^2|u(\tau)|_s^2)d\tau \\
 &\quad + \int_0^t (A^2|f(\tau)|_s^2|u(\tau)|_s^2 + a^2|f(\tau)|_{s+1}^2|u(\tau)|_s^2 + |f(\tau)|_s^4)d\tau)
 \end{aligned}$$

If we set $\phi^2(t) = E \sup_{\theta \leq t} |u(\theta)|^4$, the last inequality yields

$$\begin{aligned}
 \phi^2(t) &\leq C_3\{E|u_0|^4 + \int_0^t [\phi^2(\tau) + \phi(\tau)((E|f(\tau)|_{s+1}^4)^{1/2} \\
 &\quad + (E|g(\tau)|_s^4)^{1/2}) + E|f(\tau)|_s^4]d\tau\} \\
 &\leq C\{E|u_0|^4 + \int_0^T (E|f(\tau)|_{s+1}^4 + E|g_1(\tau)|_s^4)d\tau + \int_0^t \phi^2(\tau)d\tau\}
 \end{aligned}$$

which implies by the Gronwall lemma the following estimate:

$$(2.15) \quad E \sup_{\theta \leq T} |u(\theta)|_s^4 \leq C(E|u_0|_s^4 + \int_0^T (E|f(\tau)|_{s+1}^4 + E|g_1(\tau)|_s^4)d\tau).$$

REMARK 2.2. We can show that $E \sup_{\theta \leq T} |u(\theta)|_s^p$ is bounded for $p \geq 1$ in a similar way: using the moment inequality for martingales:

$$(2.16) \quad E \sup_{\theta \leq t} |M_\theta|^{2p} \leq K_p E \langle \langle M \rangle \rangle_t^p,$$

we have for all $1 \leq p < \infty$ and for the following term, obtained when estimating $E \sup_{\theta \leq t} |u(\tau)|_s^{2p}$ via the formula (2.14) which expands $|u(\tau)|_s^2$:

$$\begin{aligned}
 E \sup_{\theta \leq t} \left(\int_0^\theta \langle A(\tau)u(\tau), u(\tau) \rangle_s dw(\tau) \right)^{2p} &\leq K_p E \left(\int_0^t \langle A(\tau)u(\tau), u(\tau) \rangle_s^2 d\tau \right)^p \\
 &\leq K_p T^{p-1} E \int_0^t C_A^2 \langle u(\tau), u(\tau) \rangle_s^{2p} d\tau \\
 &\leq K_p T^{p-1} C_A^2 \int_0^t E \sup_{\theta \leq \tau} \langle u(\theta), u(\theta) \rangle_s^{2p} d\tau,
 \end{aligned}$$

where we have used Hölder's inequality. The other terms found by expanding $E \sup_{\theta \leq T} |u(\theta)|_s^p$ are treated in the same way as for $p = 2$, so that by setting $\phi^2(t) = E \sup_{\theta \leq t} |u(\theta)|^{2p}$ we get an estimate of the type:

$$(2.17) \quad \phi^2(t) \leq E|u_0|^{2p} + C_1 + C_2 \int_0^t \phi^2(\tau)d\tau + C_3 \int_0^t \phi(\tau)d\tau$$

which implies the boundedness of $E \sup_{\theta \leq T} |u(\theta)|^{2p}$ by the Gronwall lemma (In the following, by the Gronwall lemma we mean also its extension, cf. Bihari [6]; however, the use of this extension will not be necessary in general. In the previous case for instance, we can just note that $\int_0^t \phi(\tau)d\tau \leq 1 + T \int_0^t \phi^2(\tau)d\tau$ and use the standard Gronwall lemma.)

2.3.2 Construction of the solution.

(a) *Preliminaries.* Let $\chi \in C_0^\infty(\mathbb{R}^d)$ be a test function with $\chi \geq 0, \chi(-x) = \chi(x)$ and $\int \chi(x)dx = 1$. Given $\epsilon > 0$, let J_ϵ be the Friedrichs mollifier defined by

$$J_\epsilon(v)(x) = \int \chi_\epsilon(x - y)v(y)dy \text{ for } v \in L^2 \text{ with } \chi_\epsilon(x) = \frac{1}{\epsilon^n}\chi(x/\epsilon).$$

We recall the following properties:

- J_ϵ maps continuously H^s into $H^\infty := \bigcap_{s \geq 1} H^s$, equipped with the projective topology, in particular we have (see [69], Proposition 4.1, p.114):

$$(2.18) \quad \forall \epsilon > 0, \exists C_{\epsilon,k} > 0 : |J_\epsilon v|_{s+k} \leq C_{\epsilon,k}|v|_s$$

- *Friedrichs lemma on commutators:* The commutators $[J_\epsilon, p_t(x, D)]$ remain in a bounded set of pseudodifferential operators of order $m - 1$ if the $p_t(x, D), t \in I$ belong to a family of bounded pseudodifferential operators of order m ; see [69] Theorem 4.1. p. 116 and Remark 4.1. p. 118, see also [13] p. 79 and p.204 for similar results.

The family of operators $a_t(x, D)J_\epsilon, b_t(x, D)J_\epsilon$ is bounded as a family of $L((H^s)^{d'})$: indeed as by the assumption (i), $a_t(x, D)$ is a bounded family in OPS¹, we have by (2.12) and (2.18):

$$|a_t(x, D)J_\epsilon u|_s \leq C_a|J_\epsilon u|_{s+1} \leq C_a C_\epsilon |u|_s, \forall t \in I$$

$$|J_\epsilon a_t(x, D)u|_s \leq C_\epsilon |a_t(x, D)u|_{s-1} \leq C_a C_\epsilon |u|_s, \forall t \in I$$

We have the same bounds for $b_t(x, D)J_\epsilon, J_\epsilon b_t(x, D)$. Hence $a_t(x, D)J_\epsilon$ and $b_t(x, D)J_\epsilon$ are continuous (bounded) in $L((H^s)^{d'})$ and for ϵ fixed their norm is uniformly bounded in t . Now let us consider the equation:

$$(\mathcal{E}_\epsilon) : u(t) = u_0 + \int_0^t a_\tau(x, D)J_\epsilon u(\tau) \circ dw_\tau + \int_0^t b_\tau(x, D)J_\epsilon u(\tau)d\tau + \int_0^t (f(\tau) \circ dw_\tau + g(\tau)d\tau).$$

The equation (\mathcal{E}_ϵ) can be viewed as an SDE in a Hilbert space: $du(t) = a_\epsilon(t)u(t)dw(t) + b_\epsilon(t)u(t)dt + f(t)dw(t) + g(t)dt$ with $w(t)$ a Brownian motion and as $a_\epsilon(t), b_\epsilon(t)$ are bounded operators; the local Lipschitz property holds for the coefficients, and (\mathcal{E}_ϵ) has a unique solution (see e.g. Métivier-Pellaumail [49] § 6.10 p.74, Métivier [48]).

REMARK 2.3. The solution to (\mathcal{E}) will be constructed as the limit of the solutions u_ϵ of the SDEs (\mathcal{E}_ϵ) in the Hilbert space H^s . The solution $u_\epsilon(t)$ verifies (\mathcal{E}_ϵ) in the Itô and Stratonovich forms and it has a modification which is a.e. Hölder continuous w.r.t the norm of H^s ; this property is proved by the same argument that will be used for u (see the §(c.4) in the proof of Theorem 2.1).

Notation. We set:

$$A_\epsilon(t) = a_t(x, D)J_\epsilon + J_\epsilon a_t^*(x, D) \text{ and } L_\epsilon(t) = A(t)a_t(x, D)J_\epsilon + J_\epsilon a_t^*(x, D)A(t).$$

Lemma 2.1. *Under the conditions of Theorem 2.1, the operators $A_\epsilon(t), L_\epsilon(t)$ form a bounded family of operators in OPS⁰, that is: for all $s > 0$ these operators are continuous in $(H^s)^{d'}$ and their norms are uniformly bounded in t and $\epsilon : |A_\epsilon(t)|_s \leq C_{s,A}$ and $|L_\epsilon(t)|_s \leq C_{s,L}$*

for some constants $C_{s,A}, C_{s,L}$.

Proof. For A_ϵ we write:

$$A_\epsilon(t) = J_\epsilon(a_t + a_t^*) + [a_t, J_\epsilon] = A(t)J_\epsilon + [a_t, J_\epsilon]$$

By assumption (conditions (iii), (iv) of Theorem 2.1), $A(t), L(t)$ are bounded families of operators: $\|A(t)\|_s \leq C_{s,A}, \|L(t)\|_s \leq C_{s,L}$ and the fact that a_t^* is a bounded family in OPS^1 implies by the Friedrichs lemma that $[a_t, J_\epsilon]$ is a bounded family in OPS^0 ; this implies that it is bounded in $L((H^s)^{d'})$ for all $s > 0$ by the above mentioned results about the action of pseudodifferential operators on Sobolev spaces, see the bounds (2.12) and (2.13). As for the family $L_\epsilon(t)$ we write:

$$L_\epsilon(t) = J_\epsilon(A(t)a_t + a_t^*A(t)) + [A(t)a_t, J_\epsilon] = L(t)J_\epsilon + [A(t)a_t, J_\epsilon]$$

and we use the same argument: $L(t)$ is a bounded family in OPS^0 by the assumption (iv) and $A(t)a_t$ is a bounded family in OPS^1 ($A(t), a(t)$ being bounded families in OPS^0 and OPS^1 respectively). \square

Using this lemma we can prove as in §2.3.1 (estimate (2.15)) the following estimates:

$$(2.19) \quad E \sup_{\theta \leq t} |u_\epsilon|_s^4 \leq C(E|u_0|_s^4 + \int_0^t [E|f(\tau)|_{s+1}^4 + E|g(\tau)|_s^4]d\tau),$$

the constant being independent of ϵ .

(b) *The construction.* Let $u_0 \in (H^{s+2})^{d'}$, $f, g \in C^0(I, (H^{s+2})^{d'})$ and u_ϵ be the solution to (\mathcal{E}_ϵ) with the above data. Let $v_{\epsilon, \epsilon'} = u_\epsilon - u_{\epsilon'}$. Then

$$\begin{aligned} dv_{\epsilon, \epsilon'} = & a(t, x, D)J_\epsilon v_{\epsilon, \epsilon'}(t) \circ dw(t) + b(t, x, D)J_\epsilon v_{\epsilon, \epsilon'}(t)dt \\ & + f_{\epsilon, \epsilon'}(t) \circ dw(t) + g_{\epsilon, \epsilon'}(t)dt, \end{aligned}$$

with

$$f_{\epsilon, \epsilon'}(t) = a(t, x, D)(J_\epsilon - J_{\epsilon'})u_\epsilon(t), \quad g_{\epsilon, \epsilon'}(t) = b(t, x, D)(J_\epsilon - J_{\epsilon'})u_\epsilon(t).$$

In order to simplify the proof, we assume that $b = 0$ and then $g_{\epsilon, \epsilon'} = 0$; in the proof below, these purely deterministic terms will give rise to terms that are similar to those which appear in the deterministic hyperbolic systems and have no interaction with the stochastic terms.

Lemma 2.2. *There exists $k(\epsilon, \epsilon') > 0$ with $k(\epsilon, \epsilon') \rightarrow 0$ as $\epsilon, \epsilon' \rightarrow 0$ such that for all $v \in H^{s+1}$.*

$$(2.20) \quad |(J_\epsilon - J_{\epsilon'})v|_s \leq k(\epsilon, \epsilon')|v|_{s+1}$$

Proof. Observe that $((J_\epsilon - J_{\epsilon'})v)(\xi) = (\hat{\chi}(\epsilon\xi) - \hat{\chi}(\epsilon'\xi))\hat{v}(\xi)$, which implies that

$$|(J_\epsilon - J_{\epsilon'})v|_s \leq \sup_{\xi \in \mathbb{R}^d} \frac{|\hat{\chi}(\epsilon\xi) - \hat{\chi}(\epsilon'\xi)|}{(1 + |\xi|^2)^{1/2}} |v|_{s+1}.$$

Then the lemma follows from the fact that $k(\epsilon, \epsilon') := \sup_{\xi \in \mathbb{R}^d} |\hat{\chi}(\epsilon\xi) - \hat{\chi}(\epsilon'\xi)| / (1 + |\xi|^2)^{1/2} \rightarrow 0$ as $\epsilon, \epsilon' \rightarrow 0$. \square

Lemma 2.3. (u_ϵ) is a Cauchy family in $M_4(I, (H^s)^{d'})$, namely:

$$E \sup_{t \in I} |u_\epsilon(t) - u_{\epsilon'}(t)|_s^4 \rightarrow 0 \text{ as } \epsilon, \epsilon' \rightarrow 0.$$

Proof. We have

$$d|v_{\epsilon, \epsilon'}(t)|_s^2 = \langle A_\epsilon(t)v_{\epsilon, \epsilon'}(t), v_{\epsilon, \epsilon'}(t) \rangle_s \circ dw(t) + 2Re \langle f_{\epsilon, \epsilon'}(t), v_{\epsilon, \epsilon'}(t) \rangle_s \circ dw(t)$$

We write $|v_{\epsilon, \epsilon'}(t)|_s^2$ in the Itô form, and using martingale and Schwarz's inequalities we get:

$$\begin{aligned} E \sup_{\theta \leq t} |v_{\epsilon, \epsilon'}(\theta)|_s^4 &\leq CE \int_0^t \{A|v_{\epsilon, \epsilon'}(\tau)|^4 + |f_{\epsilon, \epsilon'}(\tau)|^2 |v_{\epsilon, \epsilon'}(\tau)|^2 \\ &\quad + L|v_{\epsilon, \epsilon'}(\tau)|^4 + |v_{\epsilon, \epsilon'}(\tau)|^2 (|f_{\epsilon, \epsilon'}(\tau)|_{s+1}^2 + |f_{\epsilon, \epsilon'}(\tau)|_{s+2}^2) \\ &\quad + |v_{\epsilon, \epsilon'}(\tau)|^2 (|(J_\epsilon - J_{\epsilon'})(a_\tau(x, D))J_\epsilon u_\epsilon(\tau) + f(\tau)|_{s+1}^2 \\ &\quad + |f_{\epsilon, \epsilon'}(\tau)|_s^4\} d\tau. \end{aligned}$$

We have $|f_{\epsilon, \epsilon'}(t)|_s \leq C|(J_\epsilon - J_{\epsilon'})u_\epsilon(t)|_{s+1} \leq k(\epsilon, \epsilon')|u_\epsilon(t)|_{s+2}$ by Lemma 2.2, therefore

$$\begin{aligned} E \sup_{\theta \leq t} |v_{\epsilon, \epsilon'}(\theta)|_s^4 &\leq CE \int_0^t \{|v_{\epsilon, \epsilon'}(\tau)|_s^4 + k(\epsilon, \epsilon')^2 |v(\tau)|^2 |u_\epsilon(\tau)|_{s+4}^2 \\ &\quad + k(\epsilon, \epsilon')^2 |v_{\epsilon, \epsilon'}(\tau)|^2 |f(\tau)|_{s+2}^2 + k(\epsilon, \epsilon')^4 |u_\epsilon(\tau)|_{s+1}^4\} d\tau. \end{aligned}$$

Hence, by setting $\phi_{\epsilon, \epsilon'}^2(t) = E \sup_{\theta \leq t} |v_{\epsilon, \epsilon'}(\theta)|_s^4$ and using the boundedness of $E \sup_{t \leq T} |u_\epsilon(t)|_s^2$ we get

$$\phi_{\epsilon, \epsilon'}^2(t) \leq C \int_0^t (\phi_{\epsilon, \epsilon'}^2(\tau) + k(\epsilon, \epsilon')\phi_{\epsilon, \epsilon'}(\tau) + k(\epsilon, \epsilon')) d\tau.$$

from which we deduce (by the Gronwall lemma) that $E \sup_{t \in I} |u_\epsilon(t) - u_{\epsilon'}(t)|_s^4 \rightarrow 0$ as $\epsilon, \epsilon' \rightarrow 0$. □

Lemma 2.4. Let u be the limit of (u_ϵ) as $\epsilon \rightarrow 0$. Then u satisfies the equation (\mathcal{E}) .

Proof. Let

$$\begin{aligned} F(t) = &u(t) - u(0) - \int_0^t a_\tau(x, D)u(\tau)dw(\tau) - \int_0^t f(\tau)dw(\tau) \\ &- \frac{1}{2} a_\tau(x, D)a_\tau(x, D)u(\tau)d\tau - \int_0^t g_1(\tau)d\tau. \end{aligned}$$

We want to show that $F(t) = 0$ a.e. Since u_ϵ is a solution to (\mathcal{E}_ϵ) , we have:

$$\begin{aligned} F(t) = &u(t) - u_\epsilon(t) - \int_0^t a_\tau(x, D)(u(\tau) - J_\epsilon u(\tau))dw(\tau) \\ &- \frac{1}{2} \int_0^t a_\tau(x, D)(a_\tau(x, D)u(\tau) - J_\epsilon a_\tau(x, D)J_\epsilon u_\epsilon(\tau))d\tau. \end{aligned}$$

it follows from the boundedness of the family $(a_t(x, D))$ in OPS¹ and martingale inequalities that

$$E|F(t)|_{s-2}^2 \leq C(E|u(t) - u_\epsilon(t)|_{s-2}^2 + \int_0^t E|u(\tau) - J_\epsilon u(\tau)|_{s-1}^2 d\tau + \int_0^t E|u(\tau) - u_\epsilon(\tau)|_s^2 d\tau).$$

Then we get $E|F(t)|_{s-2}^2 = 0$ by letting $\epsilon \rightarrow 0$ in the last inequality. □

(c) *End of the proof of theorem 2.1:*

(c.1) Let $u_0 \in (H^s)^{d'}$, $f \in C^0(I, (H^{s+1})^{d'})$ and $u_0^\epsilon \in (H^{s+2})^{d'}$, $f^\epsilon \in C^0(I, (H^{s+2})^{d'})$ be such that

$$|u_0^\epsilon - u_0|_s \text{ and } E \sup_{t \in I} |f^\epsilon(t) - f(t)|_{s+1}^4 \rightarrow 0,$$

as $\epsilon \rightarrow 0$. Let u^ϵ be the solution of (\mathcal{E}_ϵ) with the data u_0^ϵ, f^ϵ . Then $u^\epsilon - u^{\epsilon'}$ satisfies:

$$(2.21) \quad d(u^\epsilon(t) - u^{\epsilon'}(t)) = a_t(x, D)(u^\epsilon(t) - u^{\epsilon'}(t)) \circ dw(t) + (f^\epsilon(t) - f^{\epsilon'}(t)) \circ dw(t).$$

Let $(\phi_{\epsilon, \epsilon'}(t))^2 = E \sup_{\tau \leq t} |u^\epsilon(\tau) - u^{\epsilon'}(\tau)|_s^4$. The same calculations that give (2.17) applied this time to (2.21) yields

$$(2.22) \quad (\phi_{\epsilon, \epsilon'}(t))^2 \leq C \int_0^t \{(\phi_{\epsilon, \epsilon'}(\tau))^2 + \phi_{\epsilon, \epsilon'}(\tau)(E|f^\epsilon(\tau) - f^{\epsilon'}(\tau)|_{s+1}^4)^{1/2}\} d\tau + \int_0^t E|f^\epsilon(\tau) - f^{\epsilon'}(\tau)|_s^4 d\tau + |u_0^\epsilon - u_0^{\epsilon'}|_s^4.$$

But the same energy inequality (2.15) applied to the equation satisfied by u^ϵ implies that $E \sup_{t \in I} |u^\epsilon(t)|_s^4$ is bounded by a constant independent of ϵ . Therefore $\phi_{\epsilon, \epsilon'}(T)$ is bounded. From this and (2.22), we deduce that

$$(\phi_{\epsilon, \epsilon'}(t))^2 \leq C \int_0^t \{(\phi_{\epsilon, \epsilon'}(\tau))^2 + \int_0^T E|f^\epsilon(\tau) - f^{\epsilon'}(\tau)|_{s+1}^4\}^{1/2} d\tau + \int_0^T E|f^\epsilon(\tau) - f^{\epsilon'}(\tau)|_s^4 d\tau + |u_0^\epsilon - u_0^{\epsilon'}|_s^4,$$

which implies that $\phi_{\epsilon, \epsilon'}(T) \rightarrow 0$ as $\epsilon, \epsilon' \rightarrow 0$. Hence u^ϵ is a Cauchy family in $M_2(I, H^s)$. Finally, the fact that its limit u satisfies the equation $du(t) = a(t, x, D) \circ dw(t)$ can be proved exactly as in Lemma 2.4.

This proves the existence of a solution to equation (\mathcal{E}) .

(c.2) The uniqueness follows from the energy estimate (2.15).

(c.3) The Markov property of the process u can be proved as usual (using the fact that w is of independent increments) see, e.g., Da Prato-Zabczyk [15].

(c.4) To prove the continuity of the solution, we recall the Kolmogorov-Centsov theorem for metric-space valued processes, see Kallenberg [32]: let (E, d) be a complete metric space, and $X(t), t \in I \subset \mathbb{R}^d$ an E -valued process such that there exist $C, \alpha, \beta > 0$:

$$(2.23) \quad E(d(X(t), X(t'))^\alpha \leq C|t - t'|^{d+\beta} \quad \forall t, t' \in I,$$

then $X(t)$ has a modification which is almost-surely γ -Hölder continuous for all $\gamma \in]0, \beta/\alpha[$ and this modification verifies (2.23). To simplify the proof we suppose that $b = 0$ and $f = g = 0$; in this case, we have, for the solution u to Eq. (\mathcal{E}) :

$$u(t) - u(t') = \int_{t'}^t a(\tau)u(\tau)dw(\tau) + \frac{1}{2} \int_{t'}^t a(\tau)^2u(\tau)d\tau$$

and we apply the previous criterion for $\alpha = 4$: the quantities $E|u(t) - u(t')|_{s-2}^4$ are controlled by the sum of terms like:

$$T_{ij} = E(| \int_{t'}^t a(\tau)u(\tau)dw(\tau) |_{s-2}^j | \int_{t'}^t a(\tau)^2u(\tau)d\tau |_{s-2}^j), \quad i, j = 0, \dots, 4.$$

We use the Schwarz and moment martingale inequalities (2.16) to estimate these terms, for instance:

$$\begin{aligned} E| \int_{t'}^t a(\tau)u(\tau)dw(\tau) |_{s-2}^4 &\leq K_2 E(\int_{t'}^t |a(\tau)u(\tau)|_{s-2}^2 d\tau)^2 \\ &\leq K_2 E \sup_{\theta \in I} |a(\theta)u(\theta)|_{s-2}^4 (\int_{t'}^t d\tau)^2 \\ &\leq K_2 C_2 (t - t')^2, \end{aligned}$$

where $C_2 = E \sup_{\theta \in I} |a(\theta)u(\theta)|_{s-2}^4 \leq AE \sup_{\theta \in I} |u(\theta)|_{s-1}^4$, which is bounded (Remark 2.2). For the other terms we can show in the same way that $T_{ij} \leq K_{ij}|t - t'|^k$, with $k \geq 2$. The estimates of the terms T_{ij} which include the quantities $|a(\tau)^2u(\tau)|_{s-2}$ will be done by $E \sup_{\theta \in I} |u(\theta)|_s^k$, this explains the continuity w.r.t the norm $|\cdot|_{s-2}$. (2.23) is then verified with $\alpha = 4, \beta = 1, d = 1$ and then $\gamma \in]0, 1/4[$. \square

2.4. The case of differential operators and other remarks. (a) The case of hyperbolic differential systems: In this section we consider the equation (\mathcal{E}) where a and b are first order differential operators:

$$a_t(x, D) := \sum_{i=1}^d \alpha^i(t, x) \frac{\partial}{\partial x^i}, \quad b_t(x, D) := \sum_{i=1}^d \beta^i(t, x) \frac{\partial}{\partial x^i},$$

where $\alpha^i(t, x), \beta^i(t, x)$ are symmetric $d \times d'$ -matrices; $A(t)$ and the adjoint of a_t are given by:

$$a_t^*u = - \sum_{i=1}^d \alpha^i(t, x) \frac{\partial u}{\partial x^i} - \sum_{i=1}^d \frac{\partial \alpha^i}{\partial x^i} u, \quad A(t)u = - \sum_{i=1}^d \frac{\partial \alpha^i}{\partial x^i} u.$$

The same formulas hold for b , so that the condition (iii) of Theorem 2.1 is satisfied if the first order partial derivatives of $\alpha^i(t, x)$ and $\beta^i(t, x)$ are bounded. As for $L(t) = A_t a_t + a_t^* A_t$, we have:

$$L(t)u = \sum_{i,k=1}^d [-\alpha^i \frac{\partial^2 \alpha^k}{\partial x^i \partial x^k} + \frac{\partial \alpha^k}{\partial x^k} \frac{\partial \alpha^i}{\partial x^i}]u.$$

The condition (iv) is then satisfied if the α^i and their 2d order derivatives are bounded.

(b) Scalar equations: In the rest of this paragraph we focus on the scalar case ($d' = 1, d = 1$) for simplicity, and we consider the equation (\mathcal{E}_S):

$$(\mathcal{E}_S) \begin{cases} du(t) = a(t, x, D)u(t, x) \circ dw(t) + b(t, x, D)u(t, x)dt + f(t, x) \circ dw(t) + g(t, x)dt, \\ u(0, x) = u_0(x), \end{cases}$$

with

$$a(t, x, D)u = a^1(t, x)\partial u/\partial x + a^0(t, x)u, \quad b(t, x, D)u = b^1(t, x)\partial u/\partial x + b^0(t, x)u.$$

Thus, according to Theorem 2.1, for each $u_0 \in H^s, f \in C^0(I, H^{s+1}), g \in C^0(I, H^s)$, Eq. (\mathcal{E}_S) has a unique solution. In the case of regular data u_0, f, g , Ogawa [56] and Funaki [20] gave an expression of the solutions to (\mathcal{E}) in particular cases using a stochastic version of the classical characteristic method. Kunita [39] made a systematic use of this method — by exploiting the theory of stochastic flows— to study the solution of nonlinear first order partial differential equations.

We suppose first that $a^0 = b^0 = 0$ and $f = g = 0$ and denote by $\phi_t(x)$ the flow associated to the stochastic differential equation

$$dx(t) = a(t, x(t)) \circ dw(t) + b(x(t))dt.$$

Then if the initial condition is C^1 , Kunita [39] showed that the equation (\mathcal{E}_S) has a unique global solution $u(t, x)$ in a strong sense, namely:

$$u(t, x) = u_0(x) + \int_0^t a^1(\tau, x)(\partial u/\partial x) \circ dw(\tau) + \int_0^t b^1(\tau, x)(\tau, x)(\partial u/\partial x)d\tau,$$

and furthermore, $u(t, x) = u_0(\phi_t^{-1}(x))$. Funaki showed that the last expression gives a solution in a weak sense (for a similar equation that (\mathcal{E}_S) with boundary conditions).

Proposition 2.4. *Let $u_0 \in H^s$. Then $u(t, \cdot) = u_0(\phi_t^{-1}(x))$ is the unique solution to (\mathcal{E}) , in the case where $f = g = 0$ and $a^0 = b^0 = 0$.*

Proof. It suffices to approximate u_0 by a sequence $u_0^n \in C^1(\mathbb{R}^d)$ with $|u_0^n - u_0|_s \rightarrow 0$ as $n \rightarrow \infty$. Then the solution to the equation (\mathcal{E}_S) with the initial data u_0^n is given by $u^n(t, x) = u_0^n(\phi_t^{-1}(x))$. Now using the energy estimates of (2.15), we get $E \sup_{t \in I} |u^n(t) - u(t)|_s^2 \rightarrow 0$, and the proposition follows from the fact that $E \sup_{t \in I} |u^n(t) - u_0(\phi_t^{-1}(\cdot))|_s^2 \rightarrow 0$ by Lebesgue's theorem. □

In the case where $a^0 \neq 0, f \neq 0$ (and still $b^0 = 0, g = 0$ for simplicity) the solution to (\mathcal{E}_S) has the following expression:

$$(2.24) \quad u(t, x) = \{u_0(\phi_t^{-1}(x)) + \int_0^t f(\tau, \phi_{\tau,t}^{-1}(\cdot)) \exp(\int_{\tau}^t a^1(r, \phi_{r,t}^{-1}(\cdot)) \circ \hat{d}w(r)) \circ \hat{d}w(\tau)\} \times \exp(\int_0^t a^0(\tau, \phi_{\tau,t}^{-1}(\cdot)) \circ \hat{d}w(\tau),$$

where $\int X(t) \circ \hat{d}w(t)$ denotes the backward Stratonovich integral, taken in H^{s-2} . This expression follows from the same argument as in the above proof and the results of Kunita [39].

(c) Remark on the use of the Fisk-Stratonovich differential. We want to show that the use of the Fisk-Stratonovich differential in Eq. (E) is essential for obtaining the existence result. Let us consider a similar equation in which we use the Itô differential: $du(t) = a(x, D)u(t)dw(t)$. Then we will have:

$$d \langle u(t), u(t) \rangle_s = \langle u(t), (a(x, D) + a^*(x, D))u(t) \rangle_s dw(t) + \frac{1}{2} \langle (a^2(x, D) + a^{*2}(x, D))u(t), u(t) \rangle_s dt,$$

and we can not obtain an energy estimate as in § 2.3 because the operator $a^2(x, D) + a^{*2}(x, D)$ is unbounded.

In the case of scalar equations solved with the method of stochastic characteristics, the Fisk-Stratonovich notation is also essential for it allows the use of the same arguments as in the deterministic case.

(d) Remark on the propagation speed. It is well known that the solution to deterministic symmetric systems of the form $du(t) = a_t(x, D)u(t)dt$, $u(0) = u_0$ has a finite propagation speed, i.e. there is a constant $C > 0$ such that if u_0 vanishes on $\{x : |x| > R\}$ then $u(t)$ will vanish on $\{x : |x| > R + Ct\}$. In the stochastic case, things are different: some coefficients are white noises and thus 'unbounded'. Let us consider the simple equation:

$$du(t) = \frac{\partial u}{\partial x}(x, t) \circ dw(t).$$

Its solution is $u(t, x) = u_0(x + w(t))$. But since $\sup_w |w(t)| = +\infty$ a.e. when $t > 0$, we see that we can not have a finite propagation speed or a finite domain of dependence ([66]) in this case.

2.5. Comparison with other existence results for SPDEs. Hyperbolic SPDEs have been studied through several models. To cite only few examples, the stochastic wave equation in a space-time white noise setting is one of the basic models, see Walsh [72]; see also the interesting cases considered by Gaveau [21], Hajek [27]. Hyperbolic equations or systems subject to additive noises are studied, e.g. in Chow [12], Dalang et al. [16], Lototsky-Rosovsky [46]. Kim [34] considered a system of the form $\partial_t u + \sum_i A_i(t, x, u) \partial_{x_i} u = \sum_i f_i(u) dw_i$ where the A_i are symmetric matrices and the f_i are mappings that satisfy a Lipschitz condition on Sobolev spaces; this is a non linear model close to the one we study in this paper, but the last mentioned condition excludes the cases where the $f_i(u)$ are first order operators.

Ascanelli and Süß [5] studied a model of linear *scalar* hyperbolic SPDEs of the type: $Lu(t, x) = \gamma(t, x) + \sigma(t, x) \dot{F}(t, x)$, to which one has to give a sense in the framework of mild solution and stochastic integration with respect to martingale measures; and in [4], Ascanelli, Coriasco and Süß considered a scalar stochastic hyperbolic equation of the type $L(t, x, \partial_t, \partial x)u(t, x) = \gamma(t, x, u(t, x)) + \sigma(t, x, u(t, x)) \dot{\Xi}(t, x)$ with a space-time noise $\dot{\Xi}$.

Finally, a class of hyperbolic-parabolic equations driven by standard Brownian motions received interest by many authors, e.g., in Lions, Perthame and Souganidis [43], Bauzet, Vallet and Wittbold [7] and Gess and Souganidis [23] where the deterministic entropy solution concepts are adapted to the stochastic case; in these cases, the equations are either scalar or the factor of the Brownian motion is a Lipschitz function of u .

These models can be included in one of the two main approaches to SPDEs: the first one considers SPDEs as stochastic evolution equations driven by Brownian motion in a Hilbert space [15], [41], [57], [64]. The second one considers partial differential equations perturbed by a space-time white noise (cf., e.g., Walsh [72]). We refer to [16] for an account and comparison of these approaches and to [36], [45], [58], [22] for further informations and references. As we are in the Hilbert space framework, we briefly explain why the standard existence results are not comparable to those of section 2.3. These results are essentially obtained by two methods: the variational method (see Pardoux [57], Krylov-Rozovskii [41],

Roszovskii [64]) and the semi-group method (see Da Prato-Zabczyk [15] and the references given there), and they are mainly concerned with parabolic type SPDEs.

(a) The variational method. It is a generalization of the variational approach to PDEs; The framework is the following: let V be a separable Banach space which is (continuously and densely) imbedded in a Hilbert space H : $V \subset H \equiv H' \subset V'$ and we denote by $\|\cdot\|$ and $|\cdot|$ the norms in V and H respectively. Now consider some operators $A \in \mathcal{L}(V, V')$, $B_i \in \mathcal{L}(V, H)$, $i = 1, \dots, n$ and the equation:

$$(2.25) \quad du(t) = Au(t)dt + B_iu(t)dw^i(t)$$

with $u(0) = u_0 \in H$ and $w^i(t), t \in I, i = 1, \dots, n$ are standards independent Brownian motions defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. The case of non linear equations is also treated by this approach. The main assumption is the following coercivity condition: there exist $\lambda, \gamma > 0$ such that for all $v \in V$:

$$-2 \langle Av, v \rangle_{V',V} + \lambda|v|^2 \leq \gamma\|v\|^2 + \sum_{i=1}^n |B_iv|^2.$$

Under this condition, the equation (2.25) has a unique solution in $M^2(I, V)$, the set of adapted V -valued processes $u(t), t \in I$ with $E \int_I \|u(t)\|^2 < \infty$. In order to apply this result to our situation, we choose $V = H^s(\mathbb{R}^d), H = H^{s-1}(\mathbb{R}^d)$. For simplicity we choose $s = 1$, so that $V = H^1, H = L^2$. Now, let us consider the equation $du(t) = a(x, D)u(t) \circ dw(t) + b(x, D)u(t)dt, u(0) = u_0 \in H$ where $a, b \in OPS^1$. This equation can be written in the Itô form:

$$(2.26) \quad du(t) = (\frac{1}{2}a(x, D)a(x, D) + b(x, D))u(t)dt + a(x, D)u(t)dw(t).$$

The coercivity condition for this equation would be: there exist $\lambda, \gamma > 0$ such that:

$$-\langle (a^2 + 2b)v, v \rangle_{H^{-1},H^1} + \lambda|v|_{L^2}^2 \geq \gamma\|v\|_{H^1}^2 + |av|_{L^2}^2.$$

If we consider the simplest case where $a(x, D)u = \alpha\partial u/\partial x, b(x, D)u = \beta\partial u/\partial x$, then, this condition implies that $\lambda|u|_{L^2}^2 \geq \gamma\|u\|_{H^1}^2$ for all $u \in H^1(\mathbb{R}^d)$, which is not possible.

(b) The semi-group method. We consider again the equation

$$(2.27) \quad du(t) = Au(t)dt + B_iu(t)dw^i(t).$$

Here A is assumed to be the infinitesimal generator of a C^0 -semigroup $S(t)$ in a Hilbert space H and $u(0) = u_0 \in H$ with $E|u_0|^2 < \infty$. In this approach we look generally for a mild solution to Eq. (2.27) i.e. $u(t)$ satisfies

$$u(t) = S(t)u_0 + \int_0^t S(\tau)B_iu(\tau)dw^i(\tau).$$

Different assumptions are used; the first one is to suppose that the operators B_i are bounded, in which case (2.27) has a unique mild solution. In the second one, the operators B_i are allowed to be unbounded but the semigroup $S(t)$ is assumed to be analytic; a third one is to assume some Lipshitz conditions on A, B_i . These conditions are, however, not fulfilled in the case of (\mathcal{E}) or the example of equation (2.26).

3. Small perturbations

3.1. Introduction and preliminaries. This section is devoted to the study of the small random perturbations of linear hyperbolic systems. More precisely, let $u^\epsilon(\cdot)$ be the solution to

$$(S_\epsilon) : \begin{cases} du^\epsilon(t) = \sqrt{\epsilon}a_t(x, D)u^\epsilon(t) \circ dw(t) + b_t(x, D)u^\epsilon(t)dt, \\ u^\epsilon(0) = u_0 \in H^s, \end{cases}$$

where a_t, b_t are smooth families of (matrices of) pseudodifferential operators which satisfy the conditions of Theorem 2.1. We denote by $u(\cdot)$ the solution of (S_0) , the corresponding deterministic system. We are then interested in the limiting behavior of $u^\epsilon(\cdot)$ as $\epsilon \rightarrow 0$. In the finite-dimensional case, problems of this type have been studied by many authors, see, e.g., Freidlin-Wentzell [18], Deushel-Stroock [14] for references. In the infinite dimensional case, similar problems have been addressed mainly for stochastic parabolic equations under various conditions, see Daprato-Zabczyk[15] for references to earlier works on the subject. In [11], Chow considered a small perturbation problem for the SPDE:

$$du^\epsilon(t) = (Au^\epsilon(t) + F(u^\epsilon(t)))dt + \sqrt{\epsilon}\Sigma(u^\epsilon(t))dw(t),$$

where A satisfies a coercivity condition (see § 2.4) and Σ is assumed to be Lipschitz in some sense. In [59], Peszat considered the same problem in the semi-group framework with a set of technical conditions which are not satisfied in our case. For other techniques that may be used in this context see also [9], [62] and the references therein.

The method we use here is an adaptation of that of Priouret [60] in the finite dimensional case who follows an idea of Azencott [3]. First, we state the following proposition which shows the convergence in probability of u^ϵ to u with respect to the norm $\sup_t |v(t)|_{s-2}$:

Proposition 3.1. *For each $\delta > 0$ we have*

$$(3.28) \quad \lim_{\epsilon \rightarrow 0} \Pr(\sup_{t \leq T} |u^\epsilon(t) - u(t)|_{s-2} > \delta) = 0.$$

Proof. The proof is similar to the finite dimensional case (see [18]); we shall give it in order to explain the loss of two derivatives in (3.28). First, we recall that for $\epsilon > 0$ bounded (≤ 1 say) we have from the previous section

$$(3.29) \quad E \sup_{t \leq T} |u^\epsilon(t)|_s^4 < K < \infty,$$

for some $K > 0$ (in particular $\sup_{t \leq T} |u(t)|_s^4 \leq K$). Now by a simple calculation we have

$$\begin{aligned} |u^\epsilon(t) - u(t)|_{s-2}^2 &= \sqrt{\epsilon} \int_0^t \sigma_t(u^\epsilon(\tau))dw(\tau) + \int_0^t \beta_t(u^\epsilon(\tau))d\tau \\ &\quad + \int_0^t \langle B_\tau(x, D)(u^\epsilon(\tau) - u(\tau)), u^\epsilon(\tau) - u(\tau) \rangle_{s-2} d\tau, \end{aligned}$$

with

$$\begin{aligned} \sigma_t(v) &= \langle A_t(x, D)v, v \rangle_{s-2} - 2Re \langle v, a_t^*(x, D)u(t) \rangle_{s-2}, \\ \beta_t(v) &= \frac{\epsilon}{2} [\langle L_t(x, D)v, v \rangle_{s-2} - 2Re \langle v, a_t^{*2}(x, D)u(t) \rangle_{s-2}]. \end{aligned}$$

From the boundedness of the B_t and the Gronwall lemma, it follows that

$$\begin{aligned} \sup_{t \leq T} |u^\epsilon(t) - u(t)|_{s-2}^2 \leq C(T) & \left[\frac{\epsilon}{2} \int_0^T (|u^\epsilon(t)|_{s-2}^2 + |u^\epsilon(t)|_{s-2} |u(t)|_s) dt \right. \\ & \left. + \sqrt{\epsilon} \sup_{t \leq T} \left| \int_0^t \sigma_t(u^\epsilon(\tau)) dw(\tau) \right| \right]. \end{aligned}$$

From (3.29) and the last inequality we get

$$E \sup_{t \leq T} |u^\epsilon(t) - u(t)|_{s-2}^2 \leq C'(T) \sqrt{\epsilon}$$

($C'(T)$ is another constant). Now Proposition 3.1 follows from the last inequality. □

The objective is to give the exact rate of convergence in (3.28). It turns out that this rate is exponential w.r.t ϵ . More precisely the family of the laws of u^ϵ satisfies a large deviation principle as in the finite dimensional case.

Let E be a topological space endowed with a σ -field \mathcal{B} . We assume here that E is Polish \mathcal{B} is its Borel σ -field (although many results in large deviation theory hold in a more general setting). A function $I : E \rightarrow [0, +\infty]$ is said to be a rate function if it is lower semi-continuous. If in addition the level sets $\{x \in E : I(x) \leq L\}, L \geq 0$ are compact, then I is said to be a good rate function.

DEFINITION 3.2. A family $P^\epsilon, \epsilon > 0$ of probability measures on (E, \mathcal{B}) satisfies a large deviation principle (LDP) with a rate function I if

$$-\inf_{\text{int}A} I(x) \leq \liminf_{\epsilon \rightarrow 0} \epsilon \log P^\epsilon(A) \leq \limsup_{\epsilon \rightarrow 0} \epsilon \log P^\epsilon(A) \leq -\inf_{\text{cl}A} I(x),$$

for all $A \in \mathcal{B}$.

We shall use the following standard result of large deviation theory (contraction principle):

Proposition 3.3. Let (E_1, d_1) and (E_2, d_2) be two metric spaces and $X_\epsilon^1, X_\epsilon^2$ be two families of random variables with values in E_1 and E_2 respectively. Assume that the family of laws $\text{Pr}(X_\epsilon^1 \in \cdot)$ satisfies a large deviation principle with a good rate function I and that there is a map $\Phi : E_1 \cap \{I < +\infty\} \rightarrow E_2$ such that:

- (i) For all $L > 0, \Phi_{\{I \leq L\}}$ is continuous.
- (ii) For each $h \in E_1$ with $I(h) < +\infty$ and $\eta > 0$ we have

$$\lim_{\delta \rightarrow 0} \limsup_{\epsilon \rightarrow 0} \epsilon \log \text{Pr}(d_2(X_\epsilon^2, \Phi(h)) > \eta, d_1(X_\epsilon^1, h) < \delta) = -\infty.$$

Then the family $\text{Pr}(X_\epsilon^2 \in \cdot)$ satisfies a large deviation principle with the good rate function

$$I'(y) = \inf_x \{I(x) : \Phi(x) = y\}.$$

3.2. A large deviation principle. Let $u^\epsilon(t)$ be the solution to (S_ϵ) . For notational simplicity we shall drop the index t in a_t, b_t , etc. In this paragraph $u_0 \in H^s$ is fixed and we denote by \mathcal{H}^s the space $C_{u_0}(I, H^s), I = [0, T]$ of continuous paths in H^s starting from u_0 . It will be equipped with the norm $\|v\|_{s,\infty} = \sup_{t \in I} |v(t)|_s$ and the corresponding Borel σ -field. Finally, let P^ϵ be the law of $u^\epsilon(\cdot)$ which is defined on \mathcal{H}^s . It is also defined on all $\mathcal{H}^{s'}$ with

$s' < s$. We can now state the main result of this section:

Theorem 3.4. *The family (P^ϵ) satisfies a large deviation principle in \mathcal{H}^{s-2} with the following good rate function*

$$I_u(\phi) = \inf\left\{\frac{1}{2} \int_0^T |\dot{h}(t)|^2 dt : \Psi(h) = \phi\right\},$$

where $\Psi : C_0([0, T], \mathbb{R}) \rightarrow \mathcal{H}^s$ is given by

$$(3.30) \quad \Psi(h)(t) = u_0 + \int_0^t a(x, D)\Psi(h)(\tau)\dot{h}(\tau)d\tau + \int_0^t b(x, D)\Psi(h)(\tau)d\tau.$$

First let us observe that the equation satisfied by $\Psi(h)(\cdot)$ is a deterministic hyperbolic system and has a unique solution in H^s . The rest of this paragraph is devoted to the proof of Theorem 3.4.

The theorem will be proved by applying the contraction principle (Proposition 3.3) with $\Phi = \Psi$, and $X_\epsilon^1 = \sqrt{\epsilon}w(\cdot)$. From the Schilder theorem we know that μ^ϵ , the law of X_ϵ^1 , satisfies a LDP with the good rate function $I_w(h) = (1/2) \int_0^T |\dot{h}(t)|^2 dt$. Hence it suffices to verify the conditions (i) and (ii) of Proposition 3.3.

Lemma 3.1. *The map $\Psi : X := (C_0([0, T], \mathbb{R}) \cap \{I(w) < \infty\}, |\cdot|_\infty) \rightarrow \mathcal{H}^{s-1}$ is continuous.*

Proof. Let $n \geq 0$ and $h \in H^1$. Define the polygonal approximation of h by

$$h_n(t) = h([t]_n) + (t - [t]_n) \frac{h([t]_n + T/n) - h([t]_n)}{T/n},$$

where we have used the following notation: if $t \in [iT/n, (i + 1)T/n[$ then we set $[t]_n = iT/n$ (i.e. $[t]_n = [nt/T]T/n$). Now let $\Psi^n(h)(\cdot)$ be the solution to the following equation:

$$\Psi_n(h)(t) = u_0 + \int_0^t (a(x, D)\Psi_n(h)(\tau)\dot{h}_n(\tau) + b(x, D)\Psi_n(h)(\tau))d\tau$$

The map $\Psi_n : X \rightarrow \mathcal{H}^{s-1}$ is continuous since $\Psi_n(h)$ depends only on $h(iT/n), i = 1, \dots, n$.

Next, we shall prove that for each $L > 0$ the sequence $\Psi_n(h)(\cdot)$ converges uniformly (w.r.t. h) on the set $X_L := (C_0([0, T], \mathbb{R}) \cap \{I_w < L\})$, namely:

$$(3.31) \quad \lim_{n \rightarrow +\infty} \sup_{h \in X_L} \|\Psi_n(h)(\cdot) - \Psi(h)(\cdot)\|_{s-1} = 0.$$

Let $h \in X_L$. We have:

$$\begin{aligned} \Psi(h)(t) - \Psi_n(h)(t) &= \int_0^t (a(x, D)\dot{h}_n(\tau) + b(x, D))(\Psi(h)(\tau) - \Psi_n(h)(\tau))d\tau \\ &\quad + \int_0^t a(x, D)\Psi(h)(\tau)(\dot{h}(\tau) - \dot{h}_n(\tau))d\tau \end{aligned}$$

and if we set $q_n(t) = \Psi(h)(t) - \Psi_n(h)(t)$, we get

$$\begin{aligned} d \langle q_n(t), q_n(t) \rangle_{s-1} &= (\langle (A(x, D) + B(x, D))q_n(t), q_n(t) \rangle_{s-1} \dot{h}_n(t) \\ &\quad + 2\text{Re} \langle q_n(t), a(x, D)q_n(t) \rangle_{s-1} (\dot{h}(t) - \dot{h}_n(t)))dt \end{aligned}$$

Since $q_n(t)$ is uniformly bounded in H^s , it follows that

$$\begin{aligned} \phi_n(t) &\leq C \int_0^t \{\phi_n(\tau)|\dot{h}_n(\tau)|d\tau + \sqrt{\phi_n(\tau)}|\dot{h}(\tau) - \dot{h}_n(\tau)|\}d\tau \\ &\leq C \int_0^t (\phi_n(\tau)|\dot{h}_n(\tau)| + |\dot{h}(\tau) - \dot{h}_n(\tau)|)d\tau + C \int_0^T |\dot{h}(\tau) - \dot{h}_n(\tau)|d\tau \end{aligned}$$

But $\int_0^T |\dot{h}_n(t)|^2 dt \leq \int_0^T |\dot{h}(t)|^2 dt \leq L$ (by convexity), hence from the last inequality and the Gronwall lemma it follows that:

$$\phi_n(t) \leq C \left(\int_0^T |\dot{h}(\tau) - \dot{h}_n(\tau)|^2 d\tau \right)^{1/2}$$

On the other hand:

$$\begin{aligned} \int_0^T |\dot{h}(\tau) - \dot{h}_n(\tau)|^2 d\tau &= \sum_{i=1}^n \int_{iT/n}^{(i+1)T/n} (\dot{h}(\tau)^2 - \frac{(h((i+1)T/n) - h(iT/n))^2}{(T/n)^2})d\tau \\ &= \int_0^T \dot{h}(t)^2 dt - \sum_{i=1}^n \frac{(h((i+1)T/n) - h(iT/n))^2}{T/n}. \end{aligned}$$

It is well known that the r.h.s of the above equality tends to 0 as $n \rightarrow \infty$ for h absolutely continuous and with derivative in L^2 . This implies that $\sup_t \phi_n(t) \rightarrow 0$ and the convergence is uniform on $\{h : I_w(h) \leq L\}$. \square

We shall prove that Ψ satisfies the condition (ii) of the proposition. First we consider the condition (ii) in the case $h = 0$ and for a different map in the following lemma which is proved in the appendix:

Lemma 3.2. *Let $h \in X$ be fixed and consider $v^\epsilon(t)$ the solution to the equation*

$$dv^\epsilon(t) = \sqrt{\epsilon}a(x, D)v^\epsilon(t) \circ dw(t) + (b(x, D) + a(x, D)\dot{h}(t))v^\epsilon(t)dt, \quad v^\epsilon(0) = u_0.$$

Then, we have for each $\eta > 0$

$$(3.32) \quad \limsup_{\delta \rightarrow 0} \epsilon \log \Pr(\sup_t |v^\epsilon(t) - \Psi(h)(t)|_{s-2} > \eta, |\sqrt{\epsilon}w|_\infty < \delta) = -\infty.$$

($\Psi(h)$ is defined by (3.30)).

Now, the passage from Lemma 3.2 to the condition (ii) of Proposition 3.3 can be done as in the finite-dimensional case by the following lemma; for completeness we give its proof.

Lemma 3.3. *For each $h \in X$ and $\eta > 0$, we have:*

$$(3.33) \quad \lim_{\delta \rightarrow 0} \limsup_{\epsilon \rightarrow 0} \epsilon \log \Pr(\sup_t |u^\epsilon(t) - \Psi(h)(t)|_{s-2} \geq \eta, \sqrt{\epsilon}|w - h|_\infty < \delta) = -\infty,$$

where u^ϵ is the solution to (\mathcal{E}_ϵ) .

Proof. For $\epsilon > 0$ let $w^\epsilon(t) = w(t) - h(t)/\sqrt{\epsilon}, t \in [0, T]$. By the Girsanov theorem, w^ϵ is a Brownian motion under the probability Q^ϵ with:

$$\frac{dQ^\epsilon}{dP} = \exp\left(\int_0^T \left(\frac{\dot{h}(t)}{\sqrt{\epsilon}}dw(t) - \frac{1}{2} \int_0^T \frac{\dot{h}^2(t)}{\epsilon} dt\right)\right).$$

Under the probability Q^ϵ , u^ϵ satisfies

$$du^\epsilon(t) = \sqrt{\epsilon}a(x, D)u^\epsilon(t) \circ dw(t) + (\dot{h}(t)a(x, D) + b(x, D))u^\epsilon(t)dt$$

By Lemma 3.2 we have proved that (ii) holds for the above equation when $h = 0$ i.e.

$$(3.34) \quad \lim_{\delta \rightarrow 0} \limsup_{\epsilon \rightarrow 0} \epsilon \log Q^\epsilon(F(\epsilon, \eta, \delta)) = -\infty,$$

with $F(\epsilon, \eta, \delta) = \{\sup_t |u^\epsilon(t) - \Psi(h)(t)|_{s-2} \geq \eta, \sqrt{\epsilon}|w^\epsilon|_\infty < \delta\}$. To prove the lemma we have to show that $\lim_{\delta \rightarrow 0} \limsup_{\epsilon \rightarrow 0} \epsilon \log P(F(\epsilon, \eta, \delta)) = -\infty$. But

$$P(F(\epsilon, \eta, \delta)) = E^{Q^\epsilon} 1_{F(\epsilon, \eta, \delta)} \frac{dP}{dQ^\epsilon} \leq (Q^\epsilon(F(\epsilon, \eta, \delta)))^{1/2} (E(\frac{dP}{dQ^\epsilon})^2)^{1/2}.$$

Using the fact that $E^{Q^\epsilon} \exp(-\int_0^T 2\dot{h}(t)/\sqrt{\epsilon}dw(t) - 1/2 \int_0^T 4\dot{h}^2(t)/\epsilon dt) = 1$, it follows that

$$P(F(\epsilon, \eta, \delta)) \leq Q^\epsilon(F(\epsilon, \eta, \delta))^{1/2} \exp \int_0^T \frac{\dot{h}^2(t)}{\epsilon} dt,$$

and

$$\epsilon \log P(F(\epsilon, \eta, \delta)) \leq \frac{\epsilon}{2} \log Q^\epsilon(F(\epsilon, \eta, \delta)) + \frac{1}{2} \int_0^T \dot{h}^2(t) dt.$$

Now, (3.33) follows from (3.34) and the last inequality. □

4. Pathwise approximation and applications

In this part, we consider the problem of pathwise approximation, also called Wong-Zakai [73] or Stroock-Varadhan [67] approximation in the case of SDEs. This aims at approximating the stochastic solutions by the solutions of ordinary differential equations where the Brownian motion is regularized. To be specific, let $(t_i^n), n \geq 1, 0 \leq i \leq n$ be the subdivision of the interval $[0, T]$ with $t_i^n = iT/n$ and consider the equation:

$$(\mathcal{E}_n) : \begin{cases} \frac{\partial u}{\partial t} = a_t(x, D)u(t)\dot{w}^n(t) + b_t(x, D)u + f(t, x)\dot{w}^n(t) + g(t, x) \\ u(0, \cdot) = u_0(\cdot), \end{cases}$$

where $w^n(t)$ is the polygonal approximation of the Brownian motion given by:

$$w^n(t) = w([t]) + (t - [t])\frac{\Delta w(t)}{\Delta t}.$$

We use here the following notation: if $t \in [t_i^n, t_{i+1}^n[$, then we set

$$[t] = t_i^n, \Delta w(t) = w(t_{i+1}^n) - w(t_i^n)$$

and

$$\Delta t = t_{i+1}^n - t_i^n = T/n.$$

For each w , the (deterministic) equation (\mathcal{E}_n) has a unique solution in $C^0(I, H^s)$ which we denote by u^n . Then we are interested in the convergence of (u^n) to u . As we have mentioned in the introduction, this kind of approximation has been extensively studied in the case of SDEs, see, e.g., [17]. In the case of SPDEs, Gyöngy (see e.g., [24]) stud-

ied this problem in an abstract variational framework which concerns parabolic SPDEs while Twardovska [70] obtained other results by using the semi-group method. Brzezniak and Flandoli addressed this problem in the case of *scalar* parabolic (possibly degenerate) SPDEs with an application to scalar first order equation; in fact they use a representation of the solutions to these equation via a Feynman-Kac type formula which reduces the problem to proving the approximation for a stochastic flow of an associated SDE. In [65], Roth considered scalar stochastic hyperbolic equations for which a pathwise approximation is used together with finite difference scheme in order to approximate the solutions. More recently, Hairer and Pardoux [28] studied the case of non linear parabolic SPDEs driven by a space-time white noise by addressing the issue of the Stratonovich integration for space-time Brownian motion, and in [74], Yastrzhembskiy considered the SPDE: $du(t, x) = [a^{ij}(t, x)D_{ij}u(t, x) + f(u, t, x)]dt + \sum_{k=1}^m g_k(u(t, x))dw^k(t)$, for which he proved a pathwise approximation result and a Stroock-Varadhan type support theorem in a suitable path space.

These results do not seem to be applicable for the hyperbolic systems (\mathcal{E}). Instead, we observe that the approximation is valid if the operator are bounded, like in the finite-dimensional case. In the general case, we approximate the operators by a family of bounded operators and we prove a uniform estimate (Lemma 4.1). In section 4.2 we prove a support type theorem for the SPDE (\mathcal{E}) which extends the Stroock-Varadhan support theorem for SDEs (see [67], [30]) to the infinite dimensional case of hyperbolic systems (\mathcal{E}), and in section 4.3, we mention an application of the pathwise approximation to the random semigroup associated to Eq. (\mathcal{E}). The results of this section, especially Proposition 4.5, are used in [1] to extend the Hörmander propagation of singularities theorem for the stochastic hyperbolic equations considered in this paper.

4.1. Wong-Zakai type approximation. The purpose of this section is to prove the following

Theorem 4.1. *Under the assumptions of Theorem 2.1 we have:*

$$\lim_{n \rightarrow \infty} E \sup_{t \in I} |u^n(t) - u(t)|_{s-2}^2 = 0.$$

For the sake of simplicity, the proof will be done in the following case: $b = 0, f = g = 0$ and we shall assume that the operator $a_t(x, D)$ does not depend on t . It will appear that the proof is valid for the setting of section 2.

Let $J_\epsilon, \epsilon \in]0, 1]$ be a Friedrichs mollifier and consider the solutions u^ϵ and $u^{\epsilon,n}$ to the equations

$$(\mathcal{E}_\epsilon) : \quad du^\epsilon(t) = a(x, D)J_\epsilon u^\epsilon(t) \circ dw(t)$$

$$(\mathcal{E}_{\epsilon,n}) : \quad du^{\epsilon,n}(t) = a(x, D)J_\epsilon u^{\epsilon,n}(t)\dot{w}^n(t)dt$$

with the initial conditions $u^\epsilon(0) = u^{\epsilon,n}(0) = u_0$. Let $y^{\epsilon,n} = u^{\epsilon,n} - u^\epsilon$. The proof of Theorem 4.1 will be based on the following theorem and lemma.

Theorem 4.2. *For each fixed $\epsilon > 0$ we have*

$$\lim_{n \rightarrow \infty} E \sup_{t \in I} |u^{\epsilon,n}(t) - u^\epsilon(t)|_s^2 = 0.$$

Proof. As the operators $a(x, D)J_\epsilon$ are bounded for each ϵ fixed, this theorem is proved in the same way as the corresponding result of Nakao-Yamato [53] in the case of stochastic differential equations. \square

Lemma 4.1. *There exist two functions $\alpha(\epsilon), \beta(n)$ with $\alpha(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$ and $\beta(n) \rightarrow 0$ as $n \rightarrow \infty$, such that*

$$E \sup_{t \in I} |u^{\epsilon, n}(t) - u^n(t)|_{s-2}^2 \leq \alpha(\epsilon)(1 + \beta(n)).$$

The proof of this lemma is made in the Appendix.

Proof of theorem 4.1. Let $\delta > 0$. By Lemma 4.1, there exists $\epsilon_1 > 0$ such that $E \sup_{t \in I} |u^{\epsilon_1, n}(t) - u^n(t)|_{s-2}^2 < \delta$ for all $n \geq 1$. On the other hand, by Lemmas 2.3 and 2.4 we can choose ϵ_1 such that $E \sup_{t \in I} |u^{\epsilon_1}(t) - u(t)|_{s-2}^2 < \delta$. Now Theorem 4.2 with $\epsilon = \epsilon_1$ implies that there is $N \geq 1$ such that $E \sup_{t \in I} |u^{\epsilon_1, n}(t) - u^{\epsilon_1}(t)|_s^2 < \delta$ for all $n \geq N$. Summarizing,

$$\begin{aligned} E \sup_{t \in I} |u^n(t) - u(t)|_{s-2}^2 &\leq 3(E \sup_{t \in I} |u^n(t) - u^{\epsilon_1, n}(t)|_{s-2}^2 + E \sup_{t \in I} |u^{\epsilon_1, n}(t) - u^{\epsilon_1}(t)|_s^2 \\ &\quad + E \sup_{t \in I} |u^{\epsilon_1}(t) - u(t)|_{s-2}^2) \\ &\leq 9\delta \end{aligned}$$

for $n \geq N$, which completes the proof of Theorem 4.1. \square

4.2. Application to a support theorem. In this section we apply the previous pathwise approximation to prove a support theorem for the equation (\mathcal{E}) . Several extensions of this theorem to infinite dimensional settings have been carried out. In [2], Aida proved a support theorem for diffusions in a Hilbert space where some of the complications of the infinite dimension are pointed out, and Nakayama [54] proved a support theorem for the mild solution to equations of the type $dX(t) = AX(t)dt + b(X(t))dt + \sigma(X(t))dB(t)$, where W is a cylindrical Brownian motion, A is the infinitesimal generator of a (C^0) -semigroup $(S(t), t \geq 0)$ of bounded linear operators on H , and b, σ are bounded and Lipschitz (which, as mentioned before, is different from the case we consider: for the system (\mathcal{E}) , the factor $\sigma(X(t))$ is a linear unbounded operator).

In the case of parabolic SPDEs similar results have been obtained by Gyöngy [26]; the same problem has been addressed for space-time white noise driven SPDEs of hyperbolic type by A. Millet and M. Sanz-Solé [51].

As in Section 3, let $\mathcal{H}_{u_0}^s$ be the space of continuous maps $I \rightarrow (H^s(\mathbb{R}^d))^d$ endowed with the norm $\sup_{t \in I} |v|_s$ and the corresponding Borel σ -field. We know that the law P_{u_0} of the solution to Equation (\mathcal{E}) defines a probability measure on $\mathcal{H}_{u_0}^{s-2}$. In this section we describe the topological support of P_{u_0} , that is, the smallest closed subset A of $\mathcal{H}_{u_0}^{s-2}$ such that $P_{u_0}(A) = 1$. Let H^∞ and H_p^∞ be the sets of $\phi : I \rightarrow \mathbb{R}$ which are respectively infinitely differentiable and piecewise infinitely differentiable, with $\phi(0) = 0$. To each such function we associate the solution $(v(t, \phi), t \in I)$ to the following first order system

$$(4.35) \quad dv(t) = a_t(x, D)v(t)\dot{\phi}(t)dt + b(x, D)v(t)dt, \quad v(0) = u_0$$

As in the finite-dimensional case, we define

$$\mathcal{S}^{u_0} = \{v(t, \phi) : \phi \in H^\infty\}, \quad \mathcal{S}_p^{u_0} = \{v(t, \phi) : \phi \in H_p^\infty\},$$

In this section we shall assume that the family $a_t(x, D)$ satisfies the additional condition:

(iv') the operators $M(t) := L(t)a(t) + a^*(t)L(t)$ form a bounded family in OPS^0 .

Then we have the following result which extends the Stroock and Varadhan support theorem ([67], [30]) to the infinite dimensional case of Eq. (E):

Theorem 4.3. *The support $S(P_{u_0})$ of P_{u_0} is equal to $\overline{S}^{u_0} = \overline{S}_p^{u_0}$ where the closure is taken in $\mathcal{H}_{u_0}^{s-2}$.*

Proof. First, it is easy to see that $\overline{S}^{u_0} = \overline{S}_p^{u_0}$. Next, for $n \geq 1$, let $P_{u_0}^n$ be the law of the solution u^n to (E_n) . Then $Q^n(\overline{S}^{u_0}) = Q^n(\overline{S}_p^{u_0}) = 1$ for all n . By Theorem 4.1 it follows that $P_{u_0}^n \rightarrow P_{u_0}$ weakly, which implies that $P_{u_0}(\overline{S}^{u_0}) \geq \limsup P_{u_0}^n(\overline{S}^{u_0}) = 1$. Hence $S(P_{u_0}) \subset \overline{S}^{u_0}$. The inclusion $\overline{S}^{u_0} \subset S(P_{u_0})$ is a consequence of the following

Theorem 4.4. *Let $\phi \in H^\infty$ and $\eta > 0$. Then under the additional assumption (iv') we have*

$$(4.36) \quad \lim_{\delta \rightarrow 0} \Pr(\sup_{t \in I} |u(t, w) - u(t, \phi)|_{s-2} > \eta | |w - \phi|_\infty < \delta) = 0.$$

This theorem is proved in the Appendix; the proof consists in reducing the theorem to the case of a stochastic evolution equation with bounded operators which could be treated as in the finite dimensional case.

4.3. Application to the random evolution operator. One of the main applications of Wong-Zakai type approximations in the case of a stochastic differential equation is the construction of its stochastic flow of diffeomorphisms. In this paragraph we give a similar application which will be used later in the study of the singularities of Eq. (E). For notational simplicity, we still consider the case where $b = 0, f = g = 0$. Let $U_f(t', t)\phi$ denotes the solution to the forward equation:

$$(E_F) : \quad u(t) = \phi + \int_{t'}^t a_\tau(x, D)u(\tau) \circ dw(\tau), \quad 0 \leq t' \leq t \leq T,$$

where $\phi \in H^s$ (we could also assume that ϕ is random and $\mathcal{F}_{t'}$ -measurable). Now, let us consider the backward equation

$$(E_B) : \quad u(t') = \phi - \int_{t'}^t a_\tau(x, D)u(\tau) \circ \hat{d}w(\tau), \quad 0 \leq t' \leq t \leq T,$$

where $\phi \in H^s$. For a fixed $t \in]0, T]$ we denote by $\mathcal{F}_{t', t}$ the σ -field $\sigma(w(\tau) - w(\tau'), t' \leq \tau' \leq \tau \leq t)$.

Proposition 4.5. (i) *The equation (E_B) has a unique solution $(u(t'))_{0 \leq t' \leq t}$ which is $\mathcal{F}_{t', t}$ -adapted. We denote it by $U_b(t, t')\phi$.*

(ii) *We have $U_b(t, t')U_f(t', t) = U_b(t, t')U_f(t', t) = Id$, a.e.*

Proof. (i) is proved exactly as in the case of forward equations. Also, the Wong-Zakai approximations holds for backward equations.

(ii) Let us denote by $U_f^n(t', t)\phi$ and $U_b^n(t', t)\phi$ the solutions to the following equations

$$u(t) = \phi + \int_{t'}^t a_\tau(x, D)u(\tau)\dot{w}^n(\tau)d\tau,$$

$$u(t') = \phi - \int_{t'}^t a_\tau(x, D)u(\tau)\dot{w}^n(\tau)d\tau.$$

Then we have for all $\phi \in H^s$:

$$U_b^n(t, t')U_f^n(t', t)\phi = U_f^n(t', t)U_b^n(t, t')\phi = \phi.$$

Now the assertion (ii) follows from the approximation theorem of this section. □

5. On the regularity of laws of the solutions

In this section we consider the equation:

$$(5.37) \quad u(t) = u_0 + \int_0^t a_\tau(x, D)u(\tau) \circ dw(\tau) + \int_0^t b_\tau(x, D)u(\tau)d\tau,$$

in the scalar case, where $u_0 \in H^s(\mathbb{R}^d)$ and $a_t(x, D), b_t(x, D)$ are smooth and bounded families of pseudodifferential operators in OPS¹ such that their principal symbols are imaginary. By Theorem 2.1, there is a unique solution to (5.37) in $M^2(I, H^s)$. We shall assume that $s > d/2$ so that $x \mapsto u(t, x)$ is continuous for each t . Then we are interested in the regularity of the law of the random variable $u(t, x)$ for a given (t, x) . Similar problems of regularity of laws have been studied for other classes of stochastic partial differential equations of parabolic type in the case of one parameter driven white noise such as the Zakai equation of nonlinear filtering. On the other hand the same problems have been addressed for parabolic and hyperbolic equations in the case of space-time noise (and one space dimension). See Nualart [55] for references.

We recall now some definitions and notations of the Malliavin calculus. Let X be a Hilbert space. We denote by $S(X)$ the set of “simple” X -valued random variables F of the form

$$F(w) = f(w(t_1), \dots, w(t_n)), \quad 0 \leq t_1 < \dots < t_n \leq T,$$

where $f : \mathbb{R}^n \rightarrow X$ is a function which, together with all its partial derivatives, has a polynomial growth. We denote by H the Cameron-Martin space i.e. $H := \{h \in H^1(\mathbb{R}) : h(0) = 0\}$. Given $h \in H$ and $F \in X$, the derivative of F in the direction h is defined by

$$D_h F(w) := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} F(w + \epsilon h) = \sum_{i=1}^n \partial_i f(w) h(t_i).$$

The gradient of an X -valued random variables is the operator $D : S(X) \rightarrow L^2(\Omega \times [0, 1] \times X)$ defined by

$$D_\theta F := \sum_{i=1}^n \partial_i f(w) 1_{\theta \leq t_i},$$

so that

$$D_h F = \int_0^1 D_\theta F \cdot \dot{h}(\theta) d\theta, \quad h \in H.$$

The operator D is closable in $L^p(\Omega, X)$, $p \geq 1$ and $\mathbb{D}^{1,2}$ will designate the domain of its closure in $L^2(\Omega \times X)$ (still denoted by D). We recall the following criterion of absolute continuity of laws (see [55]):

Proposition 5.1. *Let F be a real valued random variable in $\mathbb{D}^{1,2}(\mathbb{R})$. Assume that*

$$(5.38) \quad \|DF\|_{L^2([0,1])} > 0 \text{ a.e.}$$

then the law of F is absolutely continuous with respect to the Lebesgue measure.

Now we return to Equation (5.37). We first state the following

Proposition 5.2. *Let u be the solution to Eq. (5.37). Then for each $t \in I$ we have $u(t) \in \mathbb{D}^{1,2}(H^{s-2})$ with $D_\theta u(t) = 0$ if $\theta > t$ and*

$$(5.39) \quad D_\theta u(t) = a_\theta(x, D)u(\theta) + \int_\theta^t a_\theta(x, D)D_\theta u(\tau) \circ dw(\tau) + \int_\theta^t b_\theta(x, D)D_\theta u(\tau) \circ d\tau$$

In other words,

$$(5.40) \quad D_\theta u(t) = U(\theta, t)a_\theta(x, D)u(\theta),$$

where $U(\theta, t)$ is the stochastic evolution semigroup associated to Eq. (5.37).

Proof. For the sake of simplification, we suppose that $b \equiv 0$. Let $\epsilon > 0$ and u^ϵ be the solution to

$$u^\epsilon(t) = u_0 + \int_0^t a_\tau(x, D)J_\epsilon u^\epsilon(\tau) \circ dw(\tau).$$

Since the operators $a_\tau(x, D)J_\epsilon$ are bounded in H^s , it can be shown that $u^\epsilon(t) \in \mathbb{D}^{1,2}(H^s)$ for $t \in I$ and

$$D_\theta u^\epsilon(t) = a_\theta(x, D)J_\epsilon u^\epsilon(\theta) + \int_\theta^t a_\theta(x, D)D_\theta J_\epsilon u^\epsilon(\tau) \circ dw(\tau).$$

The proof of these is exactly the same as for finite dimensional SDEs, see e.g. [55]. Next we shall show that $Du^\epsilon(t)$ is a Cauchy family in $L^2(\Omega \times I \times H^{s-2})$. Before doing this, we remark that

$$(5.41) \quad \sup_{0 < \epsilon < 1} \sup_{\theta \in I} E|D_\theta u^\epsilon(t)|_{s-1}^4 < +\infty,$$

Indeed, for θ fixed we have

$$d|D_\theta u^\epsilon(t)|_{s-1}^2 = \langle A_\epsilon(t)D_\theta u^\epsilon(t), D_\theta u^\epsilon(t) \rangle_{s-1} dw(t) + \frac{1}{2} \langle L_\epsilon(t)D_\theta u^\epsilon(t), D_\theta u^\epsilon(t) \rangle_{s-1},$$

where we have used the notation of section 2. Hence

$$E|D_\theta u^\epsilon(t)|_{s-1}^4 \leq C(E|u^\epsilon(\theta)|_s^4 + \int_0^t E|D_\theta u^\epsilon(\tau)|_{s-1}^4 d\tau)$$

which implies (5.41). Now to show that $Du^\epsilon(t)$ is a Cauchy family, we have to estimate $v_{\epsilon, \epsilon'}^\theta(t) := D_\theta u^\epsilon(t) - D_\theta u^{\epsilon'}(t)$ which satisfies

$$dv_{\epsilon, \epsilon'}^\theta(t) = a_\tau(x, D)J_\epsilon v_{\epsilon, \epsilon'}(t) \circ dw(t) + a_\tau(x, D)(J_\epsilon - J_{\epsilon'})D_\theta u^{\epsilon'}(t) \circ dw(t).$$

To this end, we use (5.41) and the same calculations as in 2.3.2 (b). We omit the details. Now

we have: $u(t) \in \mathbb{D}^{1,2}(H^{s-2})$ and $Du(t)$ is the limit of $Du^\epsilon(t)$ in $L^2(\Omega \times I \times H^{s-2})$. The fact that $D_\theta u(t)$ satisfies (5.39) for $0 \leq \theta \leq t$ can be proved easily by showing that $E \sup_{t \in I} |D_\theta u^\epsilon(t) - v_\theta(t)|_{s-2}^2 \rightarrow 0$ as $\epsilon \rightarrow 0$; here $v_\theta(t)$ is the solution to $dv_\theta(t) = a_t(x, D)v_\theta(t) \circ dw(t)$ for $t \geq \theta$ and $v_\theta(\theta) = a_\theta(x, D)u(\theta)$. \square

REMARK 5.3. We also see that for $h \in H$, $D_h u(t)$ satisfies

$$D_h u(t) = \int_0^t a_\tau(x, D)D_h u(\tau) \circ dw(\tau) + \int_0^t a_\tau(x, D)u(\tau)\dot{h}(\tau)d\tau$$

From this equation, we deduce an equivalent form of (5.40) (“Duhamel’s principle”)

$$D_h u(t) = \int_0^t U(\tau, t)a_\tau(x, D)u(\tau)\dot{h}(\tau)d\tau.$$

We return now to the absolute continuity of the law of $u(t, x)$ for a given (t, x) . We assume from now on that $s - 1 > d/2$. This implies that $u(t, x) \in \mathbb{D}^{1,2}(\mathbb{R})$. To see this, we note that if $s > d/2$ then $u^\epsilon(t, x) \in D^{1,2}(\mathbb{R})$ and (5.39) can be written ‘pointwise’. This is done as in the finite dimensional case; indeed it suffices to estimate the pointwise norms by Sobolev norms. Now assuming that $s - 1 > d/2$, we get from (5.41) that

$$(5.42) \quad \sup_{\theta} \sup_{\epsilon} E|D_\theta u^\epsilon(t, x)|^4 \leq C \sup_{\theta} \sup_{\epsilon} E|D_\theta u^\epsilon(t)|_s^4 < \infty.$$

Since $u^\epsilon(t, x) \rightarrow u(t, x)$ in $L^2(\Omega)$, we deduce from (5.42) that $u(t) \in \mathbb{D}^{1,2}(\mathbb{R})$, by using e.g. Lemma 1.2.3 in Nualart [55]. Furthermore, from (5.39) we deduce that

$$D_\theta u(t, x) = (U(\theta, t)a_\theta(x, D)u(\theta))(x), \quad 0 \leq \theta \leq t.$$

Using Proposition 5.1 we see that a sufficient condition for the law of $u(t, x)$ to be absolutely continuous w.r.t. the Lebesgue measure is

$$(5.43) \quad \int_0^t |(U(\theta, t)a_\theta(x, D)u(\theta))(x)|^2 d\theta > 0, \quad a.e.$$

Since we know that $U(\theta, t) : H^s \rightarrow H^s$ is continuous for almost all w we deduce that $\theta \mapsto U(\theta, t)a_\theta(x, D)u(\theta)$ is continuous (w.r.t. the norm $|\cdot|_{H^{s-1}}$) and then $\theta \mapsto (U(\theta, t)a_\theta(x, D)u(\theta))(x)$ is continuous a.s. Hence a sufficient condition to have (5.43) is

$$|U(t, t)a_t(x, D)u(t)(x)| := |a_t(x, D)u(t)(x)| > 0 \quad a.s.$$

or

$$|U(0, t)a_0(x, D)u_0(x)| > 0 \quad a.s.$$

To go further, let us see the particular case of differential operators: $a_t(x, D) := a^i(t, x)\partial/\partial x^i$, $b(x, D) := b^i(t, x)\partial/\partial x^i$ for which $U(0, t)v(x)$ is given by Eq. (2.24). This implies that a sufficient condition for (5.43) to hold is that

$$a_0(x, D)u_0(x) \neq 0 \quad \text{for all } x.$$

In fact, to deduce this last condition we have used the strict positivity of the semi-group $U(0, t)$ in the case of differential operator i.e. $U(0, t)\phi(x) > 0$ for all x whenever ϕ is continuous and $\phi(x) > 0$ for all x . Now the remaining question is whether the semigroup $U(0, t)$ is

“strictly positive” when $a(x, D)$ is a pseudodifferential operator.

6. Appendix: Proofs of some technical lemmas

In order to simplify the proofs and notations, we suppose that the operators $a_t(x, D)$ and $b_t(x, D)$ do not depend on the time variable t , and they will be denoted by a, b ; this will be therefore the case for the operators A_t, B_t and L_t which will be denoted by A, B and L .

6.1. Proofs of the lemmas related to small perturbations. Proof of Lemma 3.2. The proof of this lemma will use the following:

Lemma 6.1. *Let $Z_i(t), i = 1, 2, 3, t \in [0, T]$ be three adapted processes such that*

$$\begin{cases} dZ_1(t) = \epsilon Z_2(t) \circ dw(t) + Z_3(t)dt, \\ |Z_2(t)| \leq A_c Z_1(t) \text{ with } A_c > 0, \\ \int_0^T \left[\frac{|Z_3(t)|}{Z_1(t)} + \frac{|Z_2(t)|}{Z_1(t)^2} + \frac{|\langle Z_2, w \rangle_t|}{Z_1(t)} \right] dt \leq K \text{ a.e.} \end{cases}$$

Then, assuming $Z_1(0) = 1$, we have

$$\Pr(\sup_{t \in [0, T]} |Z_1(t)| \geq M) \leq \exp\left[-\frac{(\log M - K)^2}{2T\epsilon^2 A_c^2}\right].$$

Proof. Let $\gamma > 0$. By the stochastic calculus rules and Itô formula we have:

$$\begin{aligned} d \log(Z_1(t) + \gamma) &= \frac{1}{Z_1(t) + \gamma} \circ dZ_1(t) \\ &= \epsilon \frac{Z_2(t)}{Z_1(t) + \gamma} \circ dw(t) + \frac{Z_3(t)}{Z_1(t) + \gamma} dt \\ &= \epsilon \frac{Z_2(t)}{Z_1(t) + \gamma} dw(t) - \epsilon^2 \frac{Z_2^2(t)}{2(Z_1(t) + \gamma)^2} dt + \epsilon \frac{\langle Z_2, w \rangle_t}{2(Z_1(t) + \gamma)} + \frac{Z_3(t)}{Z_1(t) + \gamma} dt \end{aligned}$$

Then, using the assumptions of the lemma we get for $\epsilon \leq 1$:

$$(6.44) \quad |\log(Z_1(t) + \gamma)| \leq |\log(1 + \gamma)| + K + \sup_{t \leq T} \epsilon \left| \int_0^t \frac{Z_2(\tau)}{Z_1(\tau) + \gamma} dw(\tau) \right|.$$

Now the lemma follows from the exponential inequality for martingales; we recall a particular case of this inequality that will be also used in the proofs of some lemmas below: if M_t is a martingale such that $\langle M \rangle_t \leq ct, \forall t \in [0, T]$ for some constant c , then for $a \geq 0$:

$$(6.45) \quad \Pr(\sup_{t \leq T} M_t \geq at) \leq e^{-a^2 T / 2c}$$

See, e.g. Revuz-Yor [63] (Exercise 3.16, p. 145). In our case we take:

$$M_t = \epsilon \left| \int_0^t \frac{Z_2(\tau)}{Z_1(\tau) + \gamma} dw(\tau) \right|$$

and we have:

$$\begin{aligned} \langle M \rangle_t &= \epsilon^2 \int_0^t \frac{Z_2^2(\tau)}{(Z_1(\tau) + \gamma)^2} d\tau \\ &\leq \epsilon^2 A_c^2 t \end{aligned}$$

By letting $\gamma \rightarrow 0$ in (6.44) we get:

$$\begin{aligned} \Pr(\sup_{t \in [0, T]} |Z_1(t)| \geq M) &\leq \Pr(\sup_{t \in [0, T]} M_t \geq \frac{(\log M - K)}{T} T) \\ &\leq \exp\left[-\frac{(\log M - K)^2}{T^2} \times \frac{T}{2\epsilon^2 A_c^2}\right] = \exp\left[-\frac{(\log M - K)^2}{2T\epsilon^2 A_c^2}\right] \end{aligned}$$

where we have applied (6.45) with $c = \epsilon^2 A_c^2$ and $a = (\log M - K)/T$. □

Now we turn to the proof of Lemma 3.2. Let us denote $q_\epsilon(t) = v^\epsilon(t) - \Psi(h)(t)$ and:

$$F(\epsilon, \eta, \delta) = \{\sup_t |v^\epsilon(t) - \Psi(h)(t)|_{s-2} > \eta, |\sqrt{\epsilon}w|_\infty < \delta\}.$$

First, by a standard localization argument, $v^\epsilon(t)$ may be assumed bounded in H^{s-2} . Indeed, let τ^ϵ be the stopping time defined by:

$$\tau^\epsilon = \inf\{t : |v^\epsilon(t) - \Psi(h)(t)|_{s-2} \geq \eta\} \wedge T$$

then we have:

$$\{\sup_{t \leq T} |v^\epsilon(t) - \Psi(h)(t)|_{s-2} \geq \eta\} = \{\sup_{t \leq \tau^\epsilon} |v^\epsilon(t) - \Psi(h)(t)|_{s-2} \geq \eta\}.$$

But if $t \leq \tau^\epsilon$ then $|v^\epsilon(t)|_{s-2} \leq \sup_{t \leq T} |\Psi(h)(t)|_{s-2} + \eta =: M$. Hence we have:

$$(6.46) \quad F(\epsilon, \eta, \delta) = F \cap \{t \leq \tau^\epsilon\} \subset F \cap \{\sup_{t \leq T} |v^\epsilon(t)|_s \leq M\}.$$

Next, we have:

$$\begin{aligned} v^\epsilon(t) - \Psi(h)(t) &= \int_0^t \sqrt{\epsilon} a(x, D) u^\epsilon(\tau) \circ dw(\tau) \\ &\quad + \int_0^t (a(x, D) \dot{h}(\tau) + b(x, D))(u^\epsilon(\tau) - \Psi(h)(\tau)) \dot{h}(\tau) d\tau, \end{aligned}$$

which yields:

$$\begin{aligned} \langle q_\epsilon(t), q_\epsilon(t) \rangle_{s-2} &= \sqrt{\epsilon} \int_0^t \sigma(v^\epsilon(\tau)) dw(\tau) + \int_0^t \beta(v^\epsilon(\tau)) d\tau \\ &\quad + \int_0^t \langle q_\epsilon(\tau), (\dot{h}(\tau)A(x, D) + B(x, D))q_\epsilon(\tau) \rangle_{s-2} d\tau, \end{aligned}$$

where $q_\epsilon(t) = v^\epsilon(t) - \Psi(h)(t)$ and

$$\begin{aligned} \sigma(v_\epsilon(t)) &= \langle v^\epsilon(t), A(x, D)v^\epsilon(t) \rangle_{s-2} - 2\text{Re} \langle v^\epsilon(t), a^*(x, D)\Psi(h)(t) \rangle_{s-2}, \\ \beta(v_\epsilon(t)) &= \langle L(x, D)v^\epsilon(\tau), v^\epsilon(\tau) \rangle_{s-2} + \langle v^\epsilon(\tau), a^{*2}(x, D)\Psi(h)(\tau) \rangle_{s-2} \end{aligned}$$

Using the boundedness of $A(x, D), B(x, D)$ and $\int_0^T \dot{h}^2(t) dt$ it follows by the Gronwall lemma that:

$$\begin{aligned} q_\epsilon(t) &\leq C \sup_{\theta \in [0, T]} |\sqrt{\epsilon} \int_0^\theta \sigma(v_\epsilon(\tau)) dw(\tau)| \\ &\quad + \epsilon \int_0^T |\beta(v_\epsilon(\tau))| d\tau. \end{aligned}$$

Then, according to (6.46), we have $F(\epsilon, \eta, \delta) = F_1 \cup F_2$ with

$$F_1 = \{\sup_{\theta \leq T} |\sqrt{\epsilon}| \int_0^{\theta \wedge \tau^\epsilon} \sigma(v_\epsilon(\tau)) dw(\tau)| \geq \eta^2/C, \sqrt{\epsilon}|w|_\infty < \delta\},$$

$$F_2 = \{\epsilon \int_0^T |\beta(v_\epsilon(\tau))| d\tau \geq \eta^2/C, \sup_{t \leq T} |v^\epsilon(t)|_s \leq M, \sqrt{\epsilon}|w|_\infty < \delta\}.$$

By the boundedness of $L(x, D)$ and $\sup_t |\Psi(h)(t)|_s$, we have $F_2(\epsilon, \eta, \delta) = \emptyset$ for $\epsilon \leq \epsilon_0$ with δ_0 sufficiently small. Now, for $n \geq 1$ we set: $v^{n,\epsilon}(t) = v^\epsilon([t]_n)$ and we have for $\gamma > 0$:

$$F_1(\epsilon, \delta, \eta) \subset A(\epsilon, \gamma, n) \cup B(\epsilon, \eta, \gamma, n) \cup C(\epsilon, \eta, \delta, n)$$

with:

$$\begin{aligned} A(\epsilon, \gamma, n) &= \{\sup_{t \leq \tau^\epsilon} |v^\epsilon(t) - v^{\epsilon,n}(t)|_{s-1} > \gamma\}, \\ B(\epsilon, \eta, \gamma, n) &= \{\sup_{t \leq \tau^\epsilon} |v^\epsilon(t) - v^{\epsilon,n}(t)|_{s-2} \leq \gamma, \\ &\quad \sup_{t \leq T} |\sqrt{\epsilon} \int_0^t (\sigma(v^\epsilon(\tau)) - \sigma(v^{\epsilon,n}(\tau))) dw(\tau)| \geq \eta^2/2C\}, \\ C(\epsilon, \eta, \delta, n) &= \{\sup_{t \leq T} |\sqrt{\epsilon} \int_0^t \sigma(v^{\epsilon,n}(\tau)) dw(\tau)| \geq \eta^2/2C, \sqrt{\epsilon} \sup_t |w(t)| \leq \delta\}. \end{aligned}$$

First, observe that σ is uniformly Lipschitz (constant k) on $\{y \in H^{s-2} : |y|_{s-2} \leq M\}$; hence if $|v(t) - v^{\epsilon,n}(t)|_s \leq \gamma$ and $|v(t)|_s \leq M$ then $|\sqrt{\epsilon}(\sigma(v^\epsilon(t)) - \sigma(v^{\epsilon,n}(t)))| \leq k\gamma\sqrt{\epsilon}$. Then, by the exponential inequality of martingales (6.45), we have:

$$(6.47) \quad \Pr(B(\epsilon, \eta, \gamma, n)) \leq 2 \exp\left(-\frac{\eta^4}{8C^2 k^2 \gamma^2 \epsilon}\right).$$

Next, we turn to estimate $\Pr(A(\epsilon, \gamma, n))$. We have

$$v^\epsilon(t) - v^{\epsilon,n}(t) = \sqrt{\epsilon} \int_{[t]_n}^t a(x, D)v^\epsilon(\tau) \circ dw(\tau) + \int_{[t]_n}^t (a(x, D)v^\epsilon(\tau)\dot{h}(t) + b(x, D)v^\epsilon(\tau))d\tau.$$

hence:

$$\langle v^\epsilon(t) - v^{\epsilon,n}(t), v^\epsilon(t) - v^{\epsilon,n}(t) \rangle_{s-2} = \int_{[t]_n}^t Y(v^\epsilon)(\tau) dw(\tau) + \int_{[t]_n}^t Z(v^\epsilon)(\tau) d\tau,$$

with:

$$\begin{aligned} Y(v^\epsilon)(\tau) &= \sqrt{\epsilon} \langle a(x, D)v^\epsilon(\tau), v^\epsilon(\tau) \rangle_{s-2} - 2\text{Re} \langle a(x, D)v^\epsilon(\tau) \rangle_{s-2}, \\ Z(v^\epsilon)(\tau) &= \frac{1}{2} \langle L(x, D)v^\epsilon(\tau), v^\epsilon(\tau) \rangle_{s-2} - 2\text{Re} \langle a^2(x, D)v^\epsilon(\tau), v^\epsilon(\tau) \rangle_{s-2} \\ &\quad + 2\text{Re} \langle v^\epsilon(\tau) - v^{\epsilon,n}(\tau), a(x, D)v^\epsilon(\tau)\dot{h}(\tau) + b(x, D)v^\epsilon(\tau) \rangle_{s-2}. \end{aligned}$$

On the other hand we have:

$$\Pr(A(\epsilon, \gamma, n)) \leq \Pr(A(\epsilon, \gamma, n), \sup_{t \leq T} |v^\epsilon(t)|_s \leq M) + \Pr(\sup_{t \leq T} |v^\epsilon(t)|_s \geq M).$$

Now let $Z_1(t) = |v^\epsilon(t)|_s^2$. By the Itô formula it follows that:

$$(6.48) \quad dZ_1(t) = \sqrt{\epsilon}Z_2(t) \circ dw(t) + Z_3(t)dt$$

where Z_2, Z_3 are given by:

$$Z_2(t) = \langle A(x, D)v^\epsilon(t), v^\epsilon(t) \rangle_s$$

$$Z_3(t) = \frac{1}{2} \langle L(x, D)v^\epsilon(\tau), v^\epsilon(\tau) \rangle_s + \langle (B(x, D) + A(x, D)\dot{h}(t))v^\epsilon(\tau), v^\epsilon(\tau) \rangle_s.$$

By the boundedness of the operators A, B, L , we see that Z_1, Z_2 and Z_3 satisfy the assumptions of Lemma 6.1 (which we use with $\sqrt{\epsilon}$ in the Eq. (6.48) instead of the corresponding ϵ of the above-mentioned lemma; we also use the fact that $\int_0^T |\dot{h}(t)|dt \leq \int_0^T (1 + \dot{h}(t)^2)dt < \infty$). This implies that there is a constant K such that:

$$(6.49) \quad \Pr(\sup_{t \leq T} |v^\epsilon(t)|_s \geq M) \leq \exp\left(-\frac{(\log M - K)^2}{4TA_c\epsilon}\right).$$

Where A_c is a bound of the operator $A = a + a^*$. Next we have

$$\Pr(A(\epsilon, \gamma, n), \sup_{t \leq T} |v^\epsilon(t)|_s \leq M) \leq S_1 + S_2,$$

where:

$$S_1 = \sum_{i=1}^{n-1} \Pr(\sqrt{\epsilon} \sup_{t \in [iT/n, (i+1)T/n]} \left| \int_{iT/n}^{t \wedge \tau^\epsilon} Y(v^\epsilon(\tau))dw(\tau) \right| > \gamma^2/4, \sup_{t \leq T} |v^\epsilon(t)|_s \leq M),$$

$$S_2 = \sum_{i=1}^{n-1} \Pr(\sup_{t \in [iT/n, (i+1)T/n]} \left| \int_{iT/n}^{t \wedge \tau^\epsilon} |Z(v^\epsilon(\tau))|d\tau \right| > \gamma^2/4, \sup_{t \leq T} |v^\epsilon(t)|_s \leq M).$$

Noting that for $t \leq \tau^\epsilon$ we have that $|Z((v^\epsilon)(t))| \leq CM(1 + |\dot{h}(t)|)$ for some constant C , we get:

$$S_2 \leq n\Pr\left(\int_{iT/n}^{(i+1)T/n} CM(1 + |h(t)|)dt > \gamma^2/4\right) = 0,$$

for $n \geq n_0$ sufficiently large. Also, concerning S_1 , we have $|Y(v^\epsilon(t))| \leq CM$ and by the exponential inequality (6.45), we get

$$S_1 \leq \sum_{i=1}^{n-1} 2 \exp\left(-\frac{\gamma^4}{32\epsilon C^2 M^2 n^{-1}}\right),$$

hence, for $n \geq n_0$ we have:

$$(6.50) \quad \Pr(A(\epsilon, \gamma, n) \leq 2n \exp\left(-\frac{n\gamma^4}{32\epsilon C^2 M^2}\right) \leq \exp\left(-\frac{n\gamma^4}{64\epsilon C^2 M^2}\right),$$

provided that n_0 is sufficiently large.

Finally, as regards $C(\epsilon, \eta, \delta, n)$, noting that on $\{\sqrt{\epsilon}|w|_\infty \leq \delta\}$:

$$|\sqrt{\epsilon} \int_0^t \sigma(v^{\epsilon, n}(\tau))dw(\tau)| \leq \sqrt{\epsilon} \left| \sum_{i=1}^n \sigma(v(i\Delta_n))(w((i+1)\Delta_n) - w(i\Delta_n)) \right| \leq 2C'Mn\delta,$$

we get: $C = 0$ if $\delta < \eta^2 C' / MCn$. To summarize, let $R > 0$. By (6.47) there is a real $\gamma_0 > 0$ such that: for all $n \geq 1$ we have

$$\Pr(B(\epsilon, \eta, \gamma_0, n)) \leq \exp\left(\frac{-R}{\epsilon}\right).$$

By (6.49) there exists $M > 0$ such that

$$\Pr(\sup_{t \leq T} |v^\epsilon(t)|_s^2 \geq M) \leq \exp(-\frac{R}{\epsilon}).$$

By (6.50), $(\gamma_0, M$ being fixed) there exists $n_1 \geq n_0$ such that

$$\Pr(A(\epsilon, \gamma, n_1, \sup_{t \leq T} |v^\epsilon(t)|_s^2 \leq M) \leq \exp(-\frac{R}{\epsilon}),$$

and if we choose $\delta \leq \delta_0 := \eta^2 C' / CMn_1$ (so that $\Pr(C(\epsilon, \eta, \delta, n_1)) = 0$ we get

$$\Pr(F(\epsilon, \delta, \eta)) \leq 2 \exp(-\frac{R}{\epsilon}),$$

i.e $\epsilon \log \Pr F(\epsilon, \delta, \eta) \leq -R + 2\epsilon$, which completes the proof of Lemma 3.2. □

6.2. Proofs of the lemmas related to pathwise approximation and support theorem.

6.2.1. **Proof of Lemma 4.1.** First we state the following lemma which will be used in the proof.

Lemma 6.2. *Let $u^n(t), u^{\epsilon,n}(t)$ be the solutions to $\mathcal{E}_n, \mathcal{E}_{\epsilon,n}$ respectively, with the same initial value $u_0 \in H^s$. Then*

$$E \sup_{t \in I} |u^n(t)|_s^8 + E \sup_{t \in I} |u^{\epsilon,n}(t)|_s^8 \leq C,$$

where C is a constant which depends only on $E|u_0|_s^8$ (and not on n, ϵ).

Proof. We have:

$$\begin{aligned} \langle u^n(t), u^n(t) \rangle_s &= |u_0|_s^2 + \int_0^t \langle Au^n(\tau), u^n(\tau) \rangle \dot{w}(\tau) d\tau \\ &= |u_0|_s^2 + \int_0^t \langle Au^n([\tau]), u^n([\tau]) \rangle \dot{w}([\tau]) d\tau \\ &\quad + \int_0^t (\langle Au^n(\tau), u^n([\tau]) \rangle - \langle Au^n([\tau]), u^n([\tau]) \rangle) \dot{w}([\tau]) d\tau \\ &= |u_0|_s^2 + \int_0^t \langle Au^n([\tau]), u^n([\tau]) \rangle dw(\tau) \\ &\quad + \int_0^t \langle Lu^n(c_\tau), u^n(c_\tau) \rangle (\dot{w}(\tau))^2 (\tau - [\tau]) d\tau, \end{aligned}$$

where $c_\tau \in][\tau, \tau[$. Let $p^n(t) = \sup_{\tau \leq t} |u^n(\tau)|_s^8$, then using the boundedness of A, L and the martingale inequality, we get:

$$\begin{aligned} E \sup_{t \leq t} |u^n(\tau)|_s^8 &\leq C \int_0^t E \sup_{\theta \leq t} |u^n(\theta)|_s^8 d\tau \\ &\quad + CE \left(\int_0^t |u_n(c_\tau)|^2 (\Delta w(\tau))^2 \frac{(\tau - [\tau])}{(\Delta \tau)^2} d\tau \right)^4. \end{aligned}$$

The use of Schwarz's inequality for the last term does not permit to conclude (via the Gronwall lemma). However we have:

Claim: $E(\int_0^t |u_n(c_\tau)|^2 (\Delta w(\tau))^2 \frac{(\tau - [\tau])}{(\Delta\tau)^2} d\tau)^4 \leq C \int_0^t \sup_{\theta \leq t} |u^n(\theta)|_s^8 \frac{(\tau - [\tau])^4}{(\Delta\tau)^4} d\tau.$

Using this and denoting $\psi^n(t) = E \sup_{t \leq \tau} |u^n(\tau)|_s^8$ we get

$$\psi^n(t) \leq C \int_0^t \psi^n(\tau) \left(1 + \frac{(\tau - [\tau])^4}{(\Delta\tau)^4}\right) d\tau.$$

Since:

$$\int_0^T \frac{(\tau - [\tau])^4}{(\Delta\tau)^4} d\tau = \frac{T}{\Delta\tau} \sum_{i=1}^n \int_{\tau_i}^{\tau_{i+1}} \frac{(\tau - [\tau])^4}{(\Delta\tau)^4} d\tau = \frac{T}{5},$$

we get by the Gronwall lemma:

$$E \sup_{t \leq T} |u^n(t)|_s^8 = \psi^n(T) \leq E|u_0|_s^2 (1 + e^{T^2/5} (1 + T/5)).$$

□

Proof of Claim. We write for $\theta \in][\tau], [\tau]^+[$:

$$\langle u^n(\theta), u^n(\theta) \rangle = \langle u^n([\tau]), u^n([\tau]) \rangle + \int_{[\tau]}^\theta \langle Au^n(\lambda), u^n(\lambda) \rangle \frac{\Delta w(\tau)}{\Delta\tau} d\tau.$$

and then:

$$|u^n(\theta)|_s^2 \leq |u^n([\tau])|_s^2 + C \int_{[\tau]}^\theta |u^n(\lambda)|_s^2 \frac{|\Delta w|}{\Delta\tau} d\lambda.$$

By the Gronwall lemma this yields:

$$|u^n(\theta)|_s^2 \leq |u^n([\tau])|_s^2 e^{\frac{A|\Delta w|}{\Delta\tau}(\theta - [\tau])}.$$

Now, using this inequality and the fact that the increment $\Delta w([\tau])$ is independent of the $\mathcal{F}_{[\tau]}$ (and then on $u([\tau])$) we get:

$$E(\int_0^t |u_n(c_\tau)|^2 (\Delta w(\tau))^2 \frac{(\tau - [\tau])}{(\Delta\tau)^2} d\tau)^4 \leq T^3 \int_0^t d\tau [E|u^n([\tau])|_s^8] \times \\ \times E(e^{\frac{4A|\Delta w|}{\Delta\tau}(\tau - [\tau])} (\Delta w(\tau))^8 \frac{(\tau - [\tau])^4}{(\Delta\tau)^8}).$$

But:

$$E e^{\frac{4A|\Delta w|}{\Delta\tau}(\tau - [\tau])} (\Delta w(\tau))^8 = \int \frac{x^8}{\sqrt{2\pi\Delta\tau}} e^{\frac{4A|x|}{\Delta\tau}(\tau - [\tau])} e^{-\frac{x^2}{2\Delta\tau}} dx \\ = (\Delta\tau)^4 \int e^{\frac{4A|x|}{\Delta\tau}(\tau - [\tau])} e^{-x^2} 2dx \\ \leq (\Delta\tau)^4 \int e^{4A\sqrt{T}|x| - \frac{x^2}{2}} dx.$$

Consequently,

$$E(\int_0^t |u_n(c_\tau)|^2 (\Delta w(\tau))^2 \frac{(\tau - [\tau])}{(\Delta\tau)^2} d\tau)^4 \leq C \int_0^t E \sup_{\theta \leq t} |u^n(\theta)|_s^8 \frac{(\tau - [\tau])^4}{(\Delta\tau)^4} d\tau.$$

This proves the claim and the uniform boundedness of $E \sup_{t \in I} |u^n(t)|_S^8$. The proof is similar in the case of $u^{\epsilon, n}$. □

Proof of Lemma 4.1. We have, with $y^{\epsilon, n} = u^{\epsilon, n} - u^\epsilon$:

$$dy^{\epsilon, n}(t) = aJ_\epsilon y^{\epsilon, n}(t)\dot{w}(t)dt + a(J_\epsilon u^n - u^n)\dot{w}(t)dt$$

$$\begin{aligned} d \langle y^{\epsilon, n}(t), y^{\epsilon, n}(t) \rangle &= \langle A_\epsilon y^{\epsilon, n}(t), y^{\epsilon, n}(t) \rangle \dot{w}(t)dt \\ &\quad + 2 \langle y^{\epsilon, n}(t), a(J_\epsilon u^n(t) - u^n(t)) \rangle \dot{w}(t)dt, \end{aligned}$$

with $A_\epsilon = aJ_\epsilon + J_\epsilon a^*$. We rewrite the above equation as

$$\begin{aligned} \langle y^{\epsilon, n}(t), y^{\epsilon, n}(t) \rangle &= \int_0^t (\langle A_\epsilon y^{\epsilon, n}([\tau]), y^{\epsilon, n}([\tau]) \rangle \\ &\quad + 2 \langle y^{\epsilon, n}([\tau]), a(J_\epsilon u^n([\tau]) - u^n([\tau])) \rangle) \dot{w}(\tau) d\tau \\ &\quad + \int_0^t (\langle A_\epsilon y^{\epsilon, n}(\tau), y^{\epsilon, n}(\tau) \rangle - \langle A_\epsilon y^{\epsilon, n}([\tau]), y^{\epsilon, n}([\tau]) \rangle) d\tau \\ &\quad + \int_0^t 2 \langle y^{\epsilon, n}(\tau), a(J_\epsilon u^n(\tau) - u^n(\tau)) \rangle \\ &\quad - \langle y^{\epsilon, n}([\tau]), a(J_\epsilon u^n(\tau) - u^n([\tau])) \rangle) \dot{w}(\tau) d\tau. \end{aligned}$$

Noting that for an adapted process G we have $\int_0^t G([\tau])\dot{w}(\tau)d\tau = \int_0^t G([\tau])dw(\tau)$, we get

$$\begin{aligned} \langle y^{\epsilon, n}(t), y^{\epsilon, n}(t) \rangle &= \int_0^t (\langle A_\epsilon y^{\epsilon, n}([\tau]), y^{\epsilon, n}([\tau]) \rangle \\ &\quad + 2 \langle y^{\epsilon, n}([\tau]), a(J_\epsilon u^n([\tau]) - u^n([\tau])) \rangle) dw(\tau) \\ &\quad + \int_0^t \{ \langle L_\epsilon y^{\epsilon, n}(c_\tau), y^{\epsilon, n}(c_\tau) \rangle \\ &\quad + \langle y^{\epsilon, n}(c_\tau), K_\epsilon (J_\epsilon u^n(c_\tau) - u^n(c_\tau)) \rangle \} (\dot{w}(\tau))^2 (\tau - [\tau]) d\tau \\ &\quad + \int_0^t \{ \langle y^{\epsilon, n}(c'_\tau), a(J_\epsilon a^n(c'_\tau) - au^n(c'_\tau)) \rangle \\ &\quad + \langle y^{\epsilon, n}(c'_\tau), J_\epsilon a^* a (J_\epsilon u^n(c'_\tau) - u^n(c'_\tau)) \rangle \} (\dot{w}(\tau))^2 (\tau - [\tau]) d\tau \\ &\quad + \int_0^t \langle a(J_\epsilon u^n(c'_\tau) - u^n(c'_\tau)), a(J_\epsilon a^n(c'_\tau) - au^n(c'_\tau)) \rangle \\ &\quad \times (\dot{w}(\tau))^2 (\tau - [\tau]) d\tau, \end{aligned}$$

where we have set $L_\epsilon = J_\epsilon a^* A_\epsilon + A_\epsilon a J_\epsilon$, $K_\epsilon = A_\epsilon^* + A_\epsilon a$ and $c_\tau, c'_\tau \in][\tau], \tau[$. Let:

$$z^{\epsilon, n}(t) = \sup_{\tau \leq t} |y^{\epsilon, n}(\tau)|_S^4.$$

Then from the boundedness of the last two operators, martingale and Schwarz inequalities it follows that:

$$\begin{aligned}
 Ez^{\epsilon,n}(t) &\leq CE \int_0^t (z^{\epsilon,n}(\tau) + \sqrt{z^{\epsilon,n}(\tau)}\psi_{\epsilon,n}^1(\tau))d\tau \\
 &\quad + CE \int_0^t (z^{\epsilon,n}(\tau) + \sqrt{z^{\epsilon,n}(\tau)}\psi_{\epsilon,n}^2(\tau))(\dot{w}(\tau))^4(\tau - [\tau])^2 d\tau \\
 &\quad + E \int_0^t \psi_{\epsilon,n}^3(\tau)(\dot{w}(\tau))^4(\tau - [\tau])^2 d\tau \\
 &:= I_1 + I_2 + I_3,
 \end{aligned}$$

where

$$\begin{aligned}
 \psi_{\epsilon,n}^1(t) &= \sup_{\tau \leq t} |J_\epsilon u^n(\tau) - u^n(\tau)|_{s+1}, \\
 \psi_{\epsilon,n}^2(t) &= \sup_{\tau \leq t} |K_\epsilon(J_\epsilon u^n(t) - u^n(t))|_s^2 + |a(J_\epsilon a^n(t) - au^n(t))|_s^2 \\
 &\quad + |J_\epsilon a^* a(J_\epsilon u^n(t) - u^n(t))|_s^2, \\
 \psi_{\epsilon,n}^3(t) &= |a(J_\epsilon u^n(t) - u^n(t))|_s^2 |a(J_\epsilon au^n(t) - au^n(t))|_s^2.
 \end{aligned}$$

Now, using Lemma 2.2 we get:

$$E \sup_{\tau \leq T} (\psi_{\epsilon,n}^1(t))^2 \leq E \sup_{\tau} |J_\epsilon u^n(\tau) - u^n(\tau)|_{s+1}^4 \leq \alpha_1(\epsilon) E \sup_{\tau} |u^n(\tau)|_{s+1}^4,$$

and similarly

$$\begin{aligned}
 E \sup_{\tau \leq T} (\psi_{\epsilon,n}^2(t))^2 &\leq \alpha_2(\epsilon) E \sup_{\tau} |u^n(\tau)|_{s+2}^8, \\
 E \sup_{\tau \leq T} (\psi_{\epsilon,n}^3(t))^2 &\leq \alpha_3(\epsilon) E \sup_{\tau} |u^n(\tau)|_{s+2}^8,
 \end{aligned}$$

with $\alpha_i(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$, $i = 1, 2, 3$. Hence, denoting

$$\phi_{\epsilon,n}(t) = Ez^{\epsilon,n}(t),$$

it follows that

$$I_1 \leq C \int_0^t (\phi_{\epsilon,n}(\tau) + \sqrt{\phi_{\epsilon,n}(\tau)}\alpha_1(\epsilon))d\tau.$$

For the term I_2 , the direct use of the Schwarz inequality does not lead to the good estimate; instead we proceed as follows:

$$\begin{aligned}
 I_2 &\leq 16 \int_0^t (\phi_{\epsilon,n}(\tau) + \sqrt{\phi_{\epsilon,n}(\tau)}(E(\psi_{\epsilon,n}^2)^2)^{1/2}) \frac{(\tau - [\tau])^2}{(\Delta\tau)^4} d\tau \\
 &\quad + \int_0^t \{[(Ez^{\epsilon,n}(\tau)^2)^{1/2} + (E(z^{\epsilon,n}(\tau)))^{1/2}(E(\psi_{\epsilon,n}^2(\tau))^2)^{1/2}] \\
 &\quad \times (E(\Delta w(\tau))^8 1_{\Delta|w(\tau)| \geq 2\Delta\tau})^{1/2} \frac{(\tau - [\tau])^2}{(\Delta\tau)^4}\} d\tau.
 \end{aligned}$$

Here we have simply used Schwarz's inequality and $EX(\Delta w(\tau))^4 \leq 16(\Delta\tau)^2E(X) + E(X1_{(\Delta w(\tau))^2 \geq 4\Delta\tau})$ for a random variable X . We recall that $\dot{w}(t) = (\Delta w(t))/\Delta t$ with $\Delta t = T/n$. On the other hand we have:

$$E(\Delta w(t))^8 1_{(\Delta w)^2 \geq 4\Delta t} \leq 2 \int_{2\sqrt{\Delta t}}^{+\infty} \frac{8}{\sqrt{2\pi\Delta t}} e^{-x^2/2(\Delta t)} \leq C e^{-1/\Delta t}.$$

Now, using Lemma 6.2 which implies that $E(z^{\epsilon,n}(T))^2, E z^{\epsilon,n}(T)$ are uniformly bounded (w.r.t. n and ϵ), we get:

$$\begin{aligned} I_2 &\leq C \int_0^t (\phi_{\epsilon,n}(\tau) + \sqrt{\phi_{\epsilon,n}(\tau)} \alpha_2(\epsilon)) \frac{(\tau - [\tau])^2}{(\Delta\tau)^4} d\tau \\ &\quad + C \int_0^t \frac{-1/\Delta\tau}{(\Delta\tau)^2} \int_0^t \alpha_2(\epsilon) (\tau - [\tau])^2 d\tau \\ &\leq C \int_0^t (\phi_{\epsilon,n}(\tau) + \sqrt{\phi(\tau)_{\epsilon,n}} \alpha_2(\epsilon)) \frac{(\tau - [\tau])^2}{(\Delta\tau)^4} d\tau + \beta(\Delta\tau), \end{aligned}$$

with $\beta(\Delta\tau) \rightarrow 0$ as $\Delta\tau \rightarrow 0$. Finally, for the term I_3 we have:

$$\begin{aligned} I_3 &\leq \int_0^t (E(\psi_{\epsilon,n}^3)^2)^{1/2} (E(\Delta w)^8)^{1/2} \frac{(\tau - [\tau])^2}{(\Delta\tau)^4} \\ &\leq C \alpha_3(\epsilon) \int_0^t \frac{(\tau - [\tau])^2}{(\Delta\tau)^2} \leq C \alpha_3(\epsilon). \end{aligned}$$

Now, since we know that $\phi_{\epsilon,n}(t)$ is bounded, we can estimate the terms $\sqrt{\phi_{\epsilon,n}(t)}$ by a constant and we get an estimate of the form:

$$\phi_{\epsilon,n}(t) \leq C \int_0^t \phi_{\epsilon,n}(\tau) \left(1 + \frac{(\tau - [\tau])^2}{(\Delta\tau)^2}\right) d\tau + \alpha(\epsilon)(1 + \beta(\Delta\tau)),$$

and the use of the Gronwall lemma completes the proof of Lemma 4.1. □

6.2.2. Proof of Theorem 4.4. Let J_n be the Friedrichs mollifier $J_{1/n}$ and let us denote by $u(t, \phi)$ the solution to Eq. (4.35) and u_n the solution to:

$$(6.51) \quad du_n = a_t(x, D)J_n u_n \circ dw(t) + b_t(x, D)J_n u_n(t)dt, \quad u_n(0) = u_0$$

To prove Theorem 4.4, we shall reduce its assertion to proving the same limit (4.36) for the solution u_n to Eq. (6.51), for which the operators $a_t(x, D)J_n$ and $b_t(x, D)J_n$ are bounded and this can be done as in finite dimension; this is the purpose of the lemma 6.5 below. The reduction to this case is done via the following two lemmas:

Lemma 6.3. *For $A > 0$ sufficiently large we have:*

$$P_1(\delta, n) := \Pr(\sup_n \sup_{t \in I} |u_n(t)|_s > A \|w\|_\infty < \delta) \leq c \exp(-c'(\log A)^2),$$

(c is independent of $\delta \in]0, 1]$). Also

$$P'_1(\delta) := \Pr(\sup_{t \in I} |u(t)|_s > A \|w\|_\infty < \delta) \leq c \exp(-c'(\log A)^2).$$

Lemma 6.4. *There exists $N \geq 1$ such that:*

$$\limsup_{\delta \rightarrow 0} \Pr(\sup_{t \in I} |u(t, w) - u_N(t, w)|_{s-2} > \eta, \sup_n \sup_t |u_n(t)|_s \leq A, \sup_{t \in I} |u(t)|_s \leq A \|w\|_\infty < \delta) = 0.$$

Lemma 6.5. *Let $\eta > 0$. Then for each $N > 0$ fixed, we have:*

$$\lim_{\delta \rightarrow 0} \Pr(\sup_{t \in I} |u_N(t, w) - u(t, \phi)|_{s-2} > \eta, \sup_n \sup_t |u_n(t)|_s \leq A, \sup_{t \in I} |u(t)|_s \leq A \|w\|_\infty < \delta) = 0.$$

For the sake of simplicity we will prove these lemmas in the case $b \equiv 0$.

Proof of Lemma 6.3. Let $\gamma > 0, n \geq 1$ and $Z_n(t) := |u_n(t)|_s^2$. Then by the Itô formula we have:

$$\begin{aligned} \log(Z_n(t) + \gamma) &= \log(|u_0|_s^2 + \gamma) + \int_0^t \frac{\langle A_n u_n(\tau), u_n(\tau) \rangle_s}{Z_n(\tau) + \gamma} \circ dw(\tau) \\ &= \log(|u_0|_s^2 + \gamma) + \frac{\langle A_n u_n(\tau), u_n(\tau) \rangle_s w(t)}{Z_n(\tau) + \gamma} \\ &\quad - \int_0^t \frac{\langle L_n u_n(\tau), u_n(\tau) \rangle_s}{(Z_n(\tau) + \gamma)} w(\tau) \circ dw(\tau) \\ &\quad + \int_0^t \frac{\langle A_n u_n(\tau), u_n(\tau) \rangle_s^2}{(Z_n(\tau) + \gamma)^2} w(\tau) \circ dw(\tau) \\ &= \log(|u_0|_s^2 + \gamma) + \frac{\langle A_n u_n(\tau), u_n(\tau) \rangle_s w(\tau)}{Z_n(\tau) + \gamma} \\ &\quad - \int_0^t \frac{\langle L_n u_n(\tau), u_n(\tau) \rangle_s}{(Z_n(\tau) + \gamma)} w(\tau) dw(\tau) \\ &\quad + \int_0^t \frac{\langle A_n u_n(\tau), u_n(\tau) \rangle_s^2}{(Z_n(\tau) + \gamma)^2} w(\tau) dw(\tau) \\ &\quad + \frac{1}{2} \int_0^t \left\{ \frac{\langle A_n u_n(\tau), u_n(\tau) \rangle_s^2}{(Z_n(\tau) + \gamma)^2} - \frac{\langle L_n u_n(\tau), u_n(\tau) \rangle_s}{(Z_n(\tau) + \gamma)} \right. \\ &\quad + w(\tau) \left[\frac{1}{(Z_n(\tau) + \gamma)^2} (\langle L_n u_n(\tau), u_n(\tau) \rangle_s \langle A_n(\tau) u_n(\tau), u_n(\tau) \rangle_s \right. \\ &\quad - \langle M_n u_n(\tau), u_n(\tau) \rangle_s (Z_n(\tau) + \gamma)) \\ &\quad + \frac{2}{(Z_n(\tau) + \gamma)^2} \langle A_n u_n(\tau), u_n(\tau) \rangle_s \langle L_n u_n(\tau), u_n(\tau) \rangle_s \\ &\quad \left. \left. - \frac{2}{(Z_n(\tau) + \gamma)^3} \langle A_n u_n(\tau), u_n(\tau) \rangle_s^3 \right] \right\} d\tau \\ &:= \log(|u_0|_s^2 + \gamma) + k_{1,n}(t)w(t) + \int_0^t k_{2,n}(\tau)w(\tau)dw(\tau) \\ &\quad + \int_0^t (k_{3,n}(\tau)w(\tau) + k_{4,n})d\tau, \end{aligned}$$

where we have used the following notation:

$$\begin{aligned} A_n &= aJ_n + J_n a^*, \\ L_n &= A_n a J_n + J_n a^* A_n, \\ M_n &= L_n a J_n + J_n a^* L_n \end{aligned}$$

All these operators form a bounded family in OPS⁰ under the assumptions (iii)–(v), (iv’). Hence, there is a constant K such that $k_{i,n}(t) \leq K$ a.s. for all n and $i = 1, \dots, 4$. Therefore, on the set $\{\|w\|_\infty \leq \delta\}$, there is a constant $M > 0$ such that:

$$\log(Z_n(t)) \leq M + \sup_{t \in I} \left| \int_0^t k_{2,n}(\tau) w(\tau) dw(\tau) \right|$$

Then for A sufficiently large we get:

$$\Pr(\sup_n \sup_{t \in I} \log(|u_n(t)|_s^2) \geq (\log A), |w|_\infty < \delta) \leq c_1 \exp\left(-\frac{c_2(\log(A))^2}{\delta^2}\right)$$

Now, we recall the following estimate of $\Pr(|w|_\infty < \delta)$ (see Ikeda-Watanabe [30], lemma 8.1, p.519): there exist constants c_3, c_4 such that:

$$(6.52) \quad \Pr(|w|_\infty < \delta) \sim c_3 \exp\left(-\frac{c_4}{\delta^2}\right).$$

It follows that for A sufficiently large we have for some $c, c' > 0$

$$\Pr(\sup_n \sup_{t \in I} |u_n(t)|_s > A | |w|_\infty < \delta) \leq c \exp(-c'(\log A)^2).$$

The second estimate of the lemma is proved in the same way as the first one. □

Proof of Lemma 6.4. Let us denote by $k(n)$ a sequence such that $k(n) \rightarrow 0$ and $|J_n v - v|_{s'} \leq k(n)|v|_{s'+1}$, $s' = s - 2, s - 1$ for all v is in $H^{s'+1}$. Then we have: $|u_n - u|_{s-2} \leq |u_n - J_n u|_{s-2} + k(n)|u|_{s-1}$. Hence, to prove the lemma, it suffices to prove that there is $N \geq 1$ such that $\lim_{\delta \rightarrow 0} P(\delta, n, A) = 0$ where:

$$P(\delta, N, A) := \Pr(\sup_{t \in I} |J_N u(t) - u_N(t)|_{s-2} > \eta, \sup_n \sup_t |u_n(t)|_s \leq A, \sup_{t \in I} |u(t)|_s \leq A | |w|_\infty < \delta).$$

Let us set $v_n(t) = u_n(t) - J_n u(t)$. Then $dv_n(t) = a(t)v_n(t) \circ dw(t) + [a, J_n]u(t) \circ dw(t)$ and by the Itô formula, we have:

$$\begin{aligned} \langle v_n(t), v_n(t) \rangle_{s-2} &= |J_n u_0 - u_0|_{s-2}^2 + \langle \langle Av_n(t), v_n(t) \rangle_{s-2} + 2Re \langle v_n(t), [a, J_n]u(t) \rangle_{s-2} \rangle w(t) \\ &\quad - \int_0^t (\langle \langle Av_n(\tau), v_n(\tau) \rangle_{s-2} + 2Re \langle v_n(\tau), [a, J_n]u(\tau) \rangle_{s-2} \rangle w(\tau) \circ dw(\tau) \\ &= |J_n u_0 - u_0|_{s-2}^2 + [\langle \langle Av_n(t), v_n(t) \rangle_{s-2} + 2Re \langle v_n(t), [a, J_n]u(t) \rangle_{s-2} \rangle w(t) \\ &\quad - \int_0^t \langle \langle Av_n(\tau), v_n(\tau) \rangle_{s-2} \rangle (\tau) w(\tau) \circ dw(\tau) - \int_0^t X_1(\tau) w(\tau) d\tau \\ &\quad - \frac{1}{2} \int_0^t X_2(\tau) d\tau, \end{aligned}$$

where:

$$\begin{aligned} X_1(\tau) &= 2Re \langle v_n(\tau), [a, J_n]u(\tau) \rangle_{s-2} \\ X_2(\tau) &= w(t) [2Re \langle A_n v_n(\tau), [a, J_n]u(\tau) \rangle_{s-2} + 2Re \langle A'_n v_n(\tau), v_n(\tau) \rangle_{s-2}] \\ &\quad + 2Re \langle v_n(\tau), [a, J_n]u(\tau) \rangle_{s-2}, \end{aligned}$$

and $A'_n = [a, J_n]a + a^*[a, J_n]$. The idea is that the terms with a multiplicative factor $w(t)$ will be controlled by the condition $|w|_\infty < \delta$ and a factor $k(n)$ which tends to 0, and the terms involving stochastic integrals will be controlled via the exponential inequality of martingales and the estimate (6.52), but to do so we will be led to use two iterations of the Itô formula that mimic the integration by part in stochastic calculus. So we write:

$$\langle v_n(t), v_n(t) \rangle_{s-2} = I_{1,n}(t) + I_{2,n}(t) + I_{3,n}(t)$$

with

$$\begin{aligned} I_{1,n}(t) &= |J_n u_0 - u_0|_{s-2}^2 + (\langle Av_n(t), v_n(t) \rangle_{s-2} + 2Re \langle v_n(t), [a, J_n]u(t) \rangle_{s-2})w(t) \\ &\quad - \frac{1}{2} \int_0^t X_2(\tau) d\tau \\ I_{2,n}(t) &= - \int_0^t \langle Av_n(\tau), v_n(\tau) \rangle_{s-2} (\tau) w(\tau) \circ dw(\tau) \\ I_{3,n}(t) &= - \int_0^t X_1(\tau) w(\tau) dw(\tau) \end{aligned}$$

Using the boundedness of the operators A and L (and noting that for $v \in H^s$ we have: $||J_n, a]v|_{s-2} \leq C_1 k(n)|v|_s$ and $|J_n v - v|_{s-2} \leq k(n)|v|_s$ with $k(n) \rightarrow 0$), it follows that on the set $\{\sup_n \sup_{t \in I} |u_n(t)|_s \leq A, \sup_{t \in I} |u(t)|_s \leq A\}$, there exist constants $C_1(A), C_2(A), C_3(A)$ such that

$$I_{1,n}(t) \leq C_1(A)|w|_\infty + k(n)C_2(A).$$

This implies that for n sufficiently large and δ sufficiently small ($n \geq N_1, \delta \leq \delta_1$), we will have $C_1(A)|w|_\infty + k(n)C_2(A) \leq \eta/2$ and then

$$\begin{aligned} P(n, \delta, A) &\leq \Pr(\sup_{t \in I} |I_{1,n}(t)| > \eta/2, \sup_n \sup_{t \in I} |u_n(t)|_s \leq A, \sup_{t \in I} |u(t)|_s \leq A | |w|_\infty < \delta) \\ &\quad + \Pr(\sup_{t \in I} (|I_{2,n}(t)| + |I_{3,n}(t)|) > \eta/2, \sup_n \sup_{t \in I} |u_n(t)|_s \leq A, \\ &\quad \sup_{t \in I} |u(t)|_s \leq A | |w|_\infty < \delta) \\ &\leq \Pr(\sup_{t \in I} |I_{3,n}(t)| > \eta/4, \sup_n \sup_{t \in I} |u_n(t)|_s \leq A, \sup_{t \in I} |u(t)|_s \leq A | |w|_\infty < \delta) \\ &\quad + \Pr(\sup_{t \in I} |I_{2,n}(t)| > \eta/4, \sup_n \sup_{t \in I} |u_n(t)|_s \leq A, \sup_{t \in I} |u(t)|_s \leq A | |w|_\infty < \delta) \\ &:= P_1(\delta, n, A) + P_2(\delta, n, A). \end{aligned}$$

Using the fact that $|2Re \langle v_n(\tau), [a, J_n]u(\tau) \rangle_{s-2}| \leq k(n)C_6(A)$, it follows that

$$P_1(\delta, n, A) \leq C_7 \exp -\frac{1}{C_6^2(A)k(n)^2\delta^2} C_8 \exp \frac{C_9}{\delta^2}$$

where we have used (6.52) and the exponential inequality (6.45). Hence for large N , ($N \geq N_2 \geq N_1$) we have $P_1(\delta, n, A) \leq C_{10} \exp(-C_{11}/\delta^2)$, where $C_{10}, C_{11} > 0$ depend only on N_2, A . This implies that for all $n \geq N_2$ we have $\lim_{\delta \rightarrow 0} P_1(\delta, n, A) = 0$. Now we shall show the same result for $P_2(\delta, n, A)$. By the Itô formula we have:

$$\begin{aligned} I_{2,n}(t) &= \langle Av_n(t), v_n(t) \rangle_{s-2} w(t)^2 \\ &\quad - \int_0^t \langle Av_n(\tau), v_n(\tau) \rangle_{s-2} w(\tau) \circ dw(\tau) \\ &\quad - \int_0^t \langle Lv_n(\tau), v_n(\tau) \rangle_{s-2} \circ w(\tau)^2 \circ dw(\tau) \\ &\quad - \int_0^t 2Re \langle v_n(\tau), A[a, J_n]u(\tau) \rangle_{s-2} w(\tau)^2 \circ dw(\tau) \end{aligned}$$

$$\begin{aligned}
 & - \frac{1}{2} \int_0^t 2\operatorname{Re} \langle v_n(\tau), A[a, J_n]u(\tau) \rangle_{s-2} 2w(\tau) d\tau \\
 & - \frac{1}{2} \int_0^t 2w(\tau)^2 [\operatorname{Re} \langle av_n(\tau), A[a, J_n]u(\tau) \rangle_{s-2} + \langle [a, J_n]u(\tau), A[a, J_n]u(\tau) \rangle_{s-2} \\
 & + \langle v_n(\tau), A[a, J_n]au(\tau) \rangle_{s-2}] d\tau
 \end{aligned}$$

which yields

$$\begin{aligned}
 I_{2,n}(t) &= \frac{1}{2} \left[- \int_0^t (\langle Lv_n(\tau), v_n(\tau) \rangle_{s-2} + 2\operatorname{Re} \langle v_n(\tau), A[a, J_n]u(\tau) \rangle_{s-2}) w(\tau)^2 dw(\tau) \right. \\
 & \quad + \langle Av_n(t), v_n(t) \rangle_{s-2} w(t)^2 \\
 & \quad - \frac{1}{2} \int_0^t [2 \langle Lv_n(\tau), v_n(\tau) \rangle_{s-2} w(\tau) + \langle Mv_n(\tau), v_n(\tau) \rangle_{s-2} w(\tau)^2] d\tau \\
 & \quad - \frac{1}{2} \int_0^t [2\operatorname{Re} \langle v_n(\tau), A[a, J_n]u(\tau) \rangle_{s-2} 2w(\tau) d\tau \\
 & \quad + 2w(\tau)^2 \operatorname{Re} \langle av_n(\tau), A[a, J_n]u(\tau) \rangle_{s-2} + 2\operatorname{Re} \langle [a, J_n]au(\tau), A[a, J_n]u(\tau) \rangle_{s-2} \\
 & \quad + \langle av_n(\tau), A[a, J_n]au(\tau) \rangle_{s-2}] d\tau \\
 & := I_{4,n}(t) + I_{5,n}(t),
 \end{aligned}$$

where $I_{4,n}(t)$ is the stochastic integral term (we note in this term a factor $w(\tau)^2$ which will be used below) and $I_{5,n}(t)$ contains all the other terms.

We remark that on the set $\{\sup_n \sup_{t \in I} |u^n(t)|_s \leq A, \sup_{t \in I} |u(t)|_s \leq A\}$, we have

$$\sup_{t \in I} |I_{5,n}(t)| \leq C_{12}(A)(|w| + |w|^2),$$

which implies that for δ sufficiently small, we have:

$$P_2(\delta, n, A) = \Pr(\sup_{t \in I} |I_{4,n}(t)| > \eta, \sup_n \sup_{t \in I} |u^n(t)|_s \leq A, \sup_{t \in I} |u(t)|_s \leq A |w|_\infty < \delta)$$

Then, since $|\langle Lv_n(\tau), v_n(\tau) \rangle_{s-2} + 2\operatorname{Re} \langle v_n(\tau), A[a, J_n]u(\tau) \rangle_{s-2}|$ is bounded on the set $\{\sup_n \sup_{t \in I} |u^n(t)|_s \leq A, \sup_{t \in I} |u(t)|_s \leq A\}$, it follows from (6.52) and the exponential inequality that:

$$P_2(\delta, n, A) \leq C_{13} \exp\left(-\frac{C_{14}}{\delta^4}\right) \exp\left(\frac{C_{15}}{\delta^2}\right)$$

which implies that for all $n, \lim_{\delta \rightarrow 0} P_2(\delta, n, A) = 0$. This completes the proof of Lemma 6.4. □

Proof of Lemma 6.5. Since the operators $a_t(x, D)J_n$ and $a_t(x, D)J_n$ are bounded, the proof of this lemma can be done exactly as in the finite-dimensional case, see Ikeda-Watanabe [30], Theorem 8.2, p.419. □

ACKNOWLEDGEMENTS. I am grateful to the anonymous Referee for valuable remarks, corrections and suggestions that lead to clarify the paper and for helpful advice about its presentation.

References

- [1] A. Aboulalaa: *Propagation of microlocal singularities for stochastic partial differential equations*, arXiv:1912.13104.
- [2] S. Aida: *Support theorem for diffusion processes on Hilbert spaces*, Publ. Res. Inst. Math. Sci. **26** (1990), 947–965.
- [3] R. Azencott: *Grandes déviations et applications*, Eighth Saint Flour Probability Summer School–1978 (Saint Flour, 1978), Lecture Notes in Math. 774, Springer, Berlin, 1980.
- [4] A. Ascanelli, S. Coriasco and A. Süß: *Solution theory to Semilinear Hyperbolic Stochastic Partial Differential Equations with polynomially bounded coefficients*, arXiv:1610.01208.
- [5] A. Ascanelli, A. Süß: *Random-field solutions to linear hyperbolic stochastic partial differential equations with variable coefficient*, Stochastic Process. Appl. **128** (2018), 2605–2641.
- [6] I. Bihari: *A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations*, Acta Math. Hungar. **7** (1956), 81–94.
- [7] C. Bauzet, G. Vallet and P. Wittbold: *A degenerate parabolic-hyperbolic Cauchy problem with a stochastic force*, J. Hyperbolic Differ. Equ. **12** (2015), 501–533.
- [8] Z. Brzeźniak, F. Flandoli: *Almost sure approximation of Wong-Zakai type for stochastic partial differential equations*, Stochastic Process. Appl. **55** (1995), 329–358.
- [9] A. Budhiraja, P. Dupuis and V. Maroulas: *Large deviations for infinite-dimensional stochastic dynamical systems*, Ann. Probab. **36** (2008), 1390–1420.
- [10] R. Carmona and D. Nualart: *Random nonlinear wave equations: propagation of singularities*, Ann. Probab. **16** (1988) 730–751.
- [11] P.L. Chow: *Large deviations for some parabolic Itô equations*, Comm. Pure and Appl. Math. **45** (1992), 97–120.
- [12] P.-L. Chow: *Stochastic Partial Differential Equations*, Chapman Hall, Boca Raton, FL, 2007.
- [13] H.O. Cordes: *The Technique of Pseudodifferential Operators*, London Mathematical Society Lecture Notes Series **202**, Cambridge Univ. Press, Cambridge, 1994.
- [14] J.D. Deuschel and D.W. Stroock: *Large Deviations*, Academic Press, Boston, MA, 1989.
- [15] G. Da Prato and J. Zabczyk: *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 2014.
- [16] R.C. Dalang and L. Quer-Sardanyons: *Stochastic integrals for spde's: a comparison*, Expo. Math. **29** (2011), 67–109.
- [17] H. Doss: *Liens entre équations différentielles stochastiques et ordinaires*, Ann. Inst. H. Poincaré Sect. B(N.S) **13** (1977), 99–125.
- [18] M.I. Freidlin and A.D. Wentzell: *Random Perturbations of Dynamical systems*, Springer-Verlag, New York, 1984.
- [19] K.O. Friedrichs: *Symmetric hyperbolic linear differential equations*, Comm. Pure Appl. Math. **7** (1954), 345–392.
- [20] T. Funaki: *Construction of a solution of random transport equation with boundary conditions*, J. Math. Soc. Japan **31** (1979), 719–744.
- [21] B. Gaveau: *The Cauchy problem for the stochastic wave equation*, Bull. Sci. Math. **119** (1995), 381–407.
- [22] L. Gawarecki and V. Mandrekar: *Stochastic Differential Equations in Infinite Dimensions with Applications to Stochastic Partial Differential Equations*, Springer, Heidelberg, 2010.
- [23] B. Gess and P.E. Souganidis: *Stochastic non-isotropic degenerate parabolic-hyperbolic equations*, Stochastic Process Appl. **127** (2017), 2961–3004.
- [24] I. Gyöngy: *On the approximation of stochastic partial differential equations I and II*, Stochastics **25** (1988), 59–85, *ibid.* **26** (1989), 129–164.
- [25] W. Grecksch and C. Tudor: *Parabolic regularization of a first order stochastic partial differential equation*, Stochastic Anal. Appl. **18** (2000), 397–416.
- [26] I. Gyöngy: *The stability of stochastic partial differential equations II*, Stochastics Stochastics Rep. **27** (1989), 189–233.
- [27] B. Hajek: *Stochastic equations of hyperbolic type and a two-parameter Stratonovich calculus*, Ann. Probab. **10** (1982), 451–463.
- [28] M. Hairer and E. Pardoux: *A Wong-Zakai theorem for stochastic PDEs*, J. Math. Soc. Japan **67** (2015), 1551–1604.
- [29] L. Hörmander: *The Analysis of Linear Partial Differential Operators*, Vol. 3 Springer-Verlag, Berlin, 1984.

- [30] N. Ikeda and S. Watanabe: *Stochastic differential equations and diffusion processes*, second edition, North Holland, Amsterdam, 1989.
- [31] N. Jacob, A. Potrykus and J.-L. Wu: *Solving a non-linear stochastic pseudo-differential equation of Burgers type*, *Stochastic Process. Appl.* **120** (2010), 2447–2467.
- [32] O. Kallenberg: *Fondations of Modern Probability*, Springer, Berlin, 2002.
- [33] G. Kallianpur and J. Xiong: *Large deviation for a class of stochastic differential equations*, *Ann. Probab.* **24** (1996), 320–345.
- [34] J.U. Kim: *On the stochastic quasi-linear symmetric hyperbolic system*, *J. Differential Equations* **250** (2011), 1650–1684.
- [35] P. Kotelenetz: *Existence, uniqueness and smoothness for a class of function valued stochastic partial differential equations*, *Stochastics* **41** (1992), 177–199.
- [36] P. Kotelenetz: *Stochastic Ordinary and Stochastic Partial Differential Equations: Transition from Microscopic to Macroscopic Equations*: Springer, 2007.
- [37] H. Kumano-Go: *Remarks on pseudo-differential operators*, *J. Math. Soc. Japan* **21** (1969), 413–439.
- [38] H. Kumano-go: *Pseudo-Differential Operators*, MIT Press, Boston, 1981.
- [39] H. Kunita: *First order stochastic partial differential equations*; *Stochastic analysis*: in *Proceedings of Taniguchi International Symposium on Stochastic Analysis Katata and Kyoto, 1982*, North-Holland (1984), 249–269.
- [40] H. Kunita: *Stochastic differential equations and stochastic flow of diffeomorphisms*: in *École d'été de probabilités de Saint-Flour, XII–1982*, *Lecture Notes in Math.* **1097**, Springer 1984, 143–303.
- [41] N. Krylov and B. Rozovskii: *Stochastic evolution equations*, *J. Soviet Math.* **16** (1981), 1233–1277.
- [42] P. Lax: *Hyperbolic Partial Differential Equations*, *Courant Lecture Notes in Mathematics* **14**, American Mathematical Soc. Providence, Ri, 2006.
- [43] P.-L. Lions, B. Perthame and P.E. Souganidis: *Scalar conservation laws with rough (stochastic) fluxes*, *Stoch. Partial Differ. Equ. Anal. Comput.* **1** (2013), 664–686.
- [44] X. Liu and X. Zhang: *The Theory of Stochastic Pseudo-differential Operators and Its Applications I*, arXiv:1103.0138.
- [45] G.J. Lord, C.E. Powell and T. Shardlow: *An Introduction to Computational Stochastic PDEs*, Cambridge University Press, New York, 2014.
- [46] S.V. Lototsky and B.L. Rozovsky: *Stochastic partial differential equations*, Springer, Cham, 2017.
- [47] P. Malliavin: *Géométrie différentielle stochastique*. (French), Presses de l'Université de Montréal, Montréal, Que., 1978.
- [48] M. Métivier: *Semi-martingales*, Walter de Gruyter, Berlin-New York, 1983.
- [49] M. Métivier and J. Pellaumail: *Stochastic Integration*, Academic Press, New York-London-Toronto, Out., 1980.
- [50] P.A. Meyer: *Un cours sur les intégrales stochastiques*: in *Séminaire de Probabilités X*, *Lecture Notes in Math.* **511**, Springer, 1976, 245–400.
- [51] A. Millet and M. Sanz-Solé: *The support of the solution to hyperbolic SPDEs*, *Probab. Theory Related Fields* **98** (1994), 361–387.
- [52] M. Mohammed and M. Sango: *Homogenization of linear hyperbolic stochastic partial differential equation with rapidly oscillating coefficients: The two scale convergence method*, *Asymptot. Anal.* **91** (2015), 341–371.
- [53] S. Nakao and N. Yamato: *Approximation theorem on stochastic differential equations*: in *Proceedings of the International Symposium on Stochastic Differential Equations (Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1976)*, North-Holland, 1978, 283–296.
- [54] T. Nakayama: *Support Theorem for Mild Solutions of SDE's in Hilbert Spaces*, *J. Math. Sci. Univ. Tokyo* **11** (2004), 245–312.
- [55] D. Nualart: *The Malliavin Calculus and Related Topics*, Springer Verlag, New York, 1995.
- [56] S. Ogawa: *A partial differential equation with the noise as a coefficient*, *Zeit. Wahr. Geb.* **28** (1973) 53–71.
- [57] E. Pardoux: *Stochastic partial differential equations and filtering of diffusion processes*, *Stochastics* **3** (1979), 127–167.
- [58] E. Pardoux and A. Rascanu: *Stochastic Differential Equations, Backward SDEs, Partial Differential Equations*, Springer, Cham, 2014.
- [59] S. Peszat: *Large deviation principle for stochastic evolution equations*, *Probab. Theory Related Fields* **98** (1994), 113–136.
- [60] P. Priouret: *Remarques sur les petites perturbations des systèmes dynamiques*: in *Séminaire Probab. XVI*, *Lecture Notes in Math.* **920**, Springer, Berlin-New York, 1982, 184–200.

- [61] P. Protter: *Stochastic Integration and Differential Equations*, second edition, Springer, Berlin, 2005.
- [62] J. Ren and X. Zahng: *Freidlin-Wentzell's large deviations for stochastic evolution equations*, *J. Funct. Anal.* **254** (2008), 3148–3172.
- [63] D. Revuz and M. Yor: *Continuous Martingales and Brownian Motion*, Third Edition, Springer-Verlag, Berlin, 1999.
- [64] B.L. Rozovskii: *Stochastic Evolution Systems*, Kluwer Academic Publishers Group, Dordrecht, 1990.
- [65] C. Roth: *A combination of finite difference and Wong-Zakai methods for hyperbolic stochastic partial differential equations*, *Stoch. anal. appl.* **24** (2006), 221–240.
- [66] M. Taylor: *Pseudo differential Operators*, Princeton Univ. Press, 1980.
- [67] D. Stroock and S.R.S. Varadhan: *On the support of diffusion processes with application to the strong maximum principle*: in *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* (Univ. California, Berkeley, Calif., 1970/1971), 333–359.
- [68] S. Tindel: *SPDEs with pseudo-differential generators: the existence of a density*, *Appl. Math. (Warsaw)* **27** (2000), 287–308.
- [69] F. Trèves: *Introduction to Pseudo-differential and Fourier Integral Operators*, Vol. 1, Plenum Press, New York-London, 1981.
- [70] K. Twardowska: *An extension of the Wong-Zakai theorem for stochastic evolution equations in Hilbert spaces*, *Stochastic Anal. Appl.* **10** (1992), 471–500.
- [71] K. Twardowska and A. Nowak: *On the relation between the Itô and Stratonovich integral in Hilbert spaces*, *Ann. Math. Sil.* **18** (2004) 49–63.
- [72] J.B. Walsh: *An introduction to stochastic partial differential equations*: in *École d'été de probabilités de Saint-Flour, XIV, 1984*, *Lecture Notes in Math.* **1180**, Springer, Berlin, 1986, 265–437.
- [73] E. Wong and M. Zakai: *On the relation between ordinary and stochastic differential equations*, *Internat J. Engrg. Sci.* **3** (1965), 213–229.
- [74] T. Yastrzhembskiy: *Wong-Zakai approximation and support theorem for semilinear SPDEs with finite dimensional noise in the whole space*, arXiv:1808.07584.

BP. 2119 Rabat Hay Riyad
Rabat
Morocco
e-mail: adnan.aboulalaa@polytechnique.org