Araujo, V.
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SINKS AND SOURCES FOR C^1 DYNAMICS WHOSE LYAPUNOV EXPONENTS HAVE CONSTANT SIGN

VITOR ARAUJO

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Abstract

Let $f : M \to M$ be a C^1 map of a compact manifold M, with dimension at least 2, admitting some point whose future trajectory has only negative Lyapunov exponents. Then this trajectory converges to a periodic sink. We need only assume that Df is never the null map at any point (in particular, we need no extra smoothness assumption on Df nor the existence of a invariant probability measure), encompassing a wide class of possible critical behavior. Similarly, a trajectory having only positive Lyapunov exponents for a C^1 diffeomorphism is itself a periodic repeller (source).

Analogously for a C^1 open and dense subset of vector field on finite dimensional manifolds: for a flow ϕ_t generated by such a vector field, if a trajectory admits weak asymptotic sectional contraction (the extreme rates of expansion of the Linear Poincaré Flow are all negative), then this trajectory belongs either to the basin of attraction of a periodic hyperbolic attracting orbit (a periodic sink or an attracting equilibrium); or the trajectory accumulates a codimension one saddle singularity. Similar results hold for weak sectional expanding trajectories.

Both results extend part of the non-uniform hyperbolic theory (Pesin's Theory) from the C^{1+} diffeomorphism setting to C^1 endomorphisms and C^1 flows. Some ergodic theoretical consequences are discussed. The proofs use versions of Pliss' Lemma for maps and flows translated as (reverse) hyperbolic times, and a result ensuring that certain subadditive cocycles over C^1 vector fields are in fact additive.

Contents

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1. Introduction and statements of the results

In what follows M is a connected compact finite d-dimensional manifold M, with $d \geq 2$, endowed with a Riemannian metric $\langle \cdot, \cdot \rangle$ which induces a norm $\| \cdot \|$ on the tangent bundle of M and a distance dist on M, and a volume form m that we call Lebesgue measure. For any subset A of M we denote by $\overline{A}$ the (topological) closure of A.

We extend the following well-known result from Nonuniform Hyperbolic (Pesin's) Theory (see e.g. [15, Corollary S.5.2] or [7, Corollary 15.4.2]) to C^1 endomorphisms and C^1 singular vector fields.

Theorem 1.1. *Let f be a Hölder-C^1 diffeomorphism of M and μ a f-invariant ergodic probability measure such that all its Lyapunov exponents are negative (respectively, positive). Then* $\operatorname{supp}\mu$ *is an attracting (respectively, repelling) periodic orbit.*

Results along this line for one-dimensional transformations usually assume at least the same amount of extra smoothness: see e.g. Mañé [19]; Campanino [11] and Przytycki [28]. Other results assume only C^1 smoothness but have dimensional restrictions; see e.g. [6].

In all these results the existence of a invariant probabilty measure is another standing assumption which we mostly avoid in the main statements, but explore some of its consequences in what follows.

As a consequence of these results, points with negative asymptotic rates of expansion belong to the basin of a periodic attracting orbit, which is its stable manifold. In contrast, we note that in [8] the authors show that, for generic C^1-diffeomorphisms, hyperbolic measures having positive and negative Lyapunov exponents do not necessarily admit (un)stable invariant manifolds.

1.1. The discrete time case. For C^1 maps on compact manifolds we obtain a necessary and sufficient condition for a given trajectory to be on the basin of an attracting periodic orbit from asymptotic information on the derivative.

Let $f : M \to M$ be a C^1 map such that $\inf_{x \in M} \|Df(x)\| > 0$. The Subadditive Ergodic Theorem (see e.g. [18, 30]) ensures that the largest asymptotic growth rate

$$\chi(x) = \lim_{n \to +\infty} \ln \|Df^n(x)\|^{1/n}$$

exists for all x on a total probability subset since $\ln^+ \|Df\| = \max\{0, \ln \|Df\|\}$ is μ-integrable for each f-invariant probability measure μ.

In what follows we write $A_k^-(x) = \liminf_{n\to+\infty} n^{-1} \sum_{j=0}^{n-1} \ln \|Df^k(f^{kj}x)\|$.

We recall that $p \in M$ belongs to a periodic orbit (with period τ) if there exists $\tau \in \mathbb{Z}^+$ so that $f^\tau p = p$. This periodic orbit $\mathcal{O}_f(p) = \{p, fp, \dots, f^{\tau-1}p\}$ is attracting (a sink, for short) if there exists a neighborhood V_p of p such that $f^\tau \mid_{V_p}: V_p \to V_p$ is a contraction: there exists $0 < \lambda < 1$ so that $\operatorname{dist}(f^\tau q, f^\tau r) < \lambda \operatorname{dist}(q, r), \forall q, r \in V_p$. Equivalently, $\|Df^\tau(p)\| < \lambda$ for some $\lambda \in (0, 1)$.

The basin of attraction of a sink $\mathcal{O}_f(p)$ is the following subset $B(\mathcal{O}_f(p)) = \{x \in M : \omega(x) = \mathcal{O}_f(p)\}$, where the omega-limit $\omega(x)$ of x is the set of accumulation points of the positive orbit of x: $y \in \omega(x) \iff \exists n_k \nearrow \infty : f^{n_k}x \xrightarrow[k\to\infty]{} y$.

Theorem A. *Let $f : M \to M$ be a C^1 map such that* $\inf_{x\in M} \|Df(x)\| > 0$. *Then $x \in M$ is contained in the basin of attraction of a attracting periodic orbit (a sink) if, and only if, $A_k^-(x) < 0$ for some $k \in \mathbb{Z}^+$.*

Coupling the pointwise result above with the Subadditive Ergodic Theorem and ergodic decomposition, we deduce:

Corollary 1.2. *Let μ be an invariant probability measure with respect to a C^1 map $f : M \to M$ such that* $\inf_{x\in M} \|Df(x)\| > 0$ *and $\chi(x) < 0, \mu$-a.e. $x \in M$. Then μ decomposes as $\tilde{\mu} + \sum_{i\geq 1} \mu_i$, where each μ_i is a Dirac mass equidistributed on a periodic attracting orbit of f (a sink), the sum is over at most countably many such orbits, and $\tilde{\mu}$ (which might be the null measure) satisfies $A_1^-(x) \geq 0, \tilde{\mu}$-a.e. $x \in M$. In addition, if μ is f-ergodic, then μ is concentrated on the orbit of a periodic attractor (sink).*

Remark 1.3.
1. *We do not need Hölder continuity of the derivative* in the arguments proving Theorem A and Corollary 1.2.
2. We do not need that f be a diffeomorphism or local diffeomorphism; compare with Corollary S.5.2 of [15, Supplement] where the usual Hölder condition on the derivative of a diffeomorphism in *Pesin's Theory*, or non-uniform hyperbolic theory, is used to construct *hyperbolic blocks*.
3. We need only to assume that $Df(x)$ is not the null map for all $x \in M$, and this weak condition is compatible with a wide class of critical points of a smooth map $f \in C^1(M, M)$.
4. The previous assumption ensures that $A_k^-(x) > k \ln \inf_{x\in M} \|Df(x)\| > -\infty$ for all $x \in M$ and all $k \geq 1$, and so also $\chi(x) \geq \ln \inf_{x\in M} \|Df(x)\| > -\infty$ on a total probability subset of points x.

1.1.1. **The diffeomorphism case.** If f is a C^1 diffeomorphism, then exchanging f with f^{-1} we have that $\tilde{\chi}(x) = \lim_{n\to+\infty} \ln \|Df^{-n}(x)\|^{1/n} = \lim_{n\to+\infty} \ln \|Df^n(x)^{-1}\|^{1/n}$ exists for x on a total probability subset of M and gives the least asymptotic growth rate. We also write $\tilde{A}_k^-(x) = \liminf_{n\to\infty} n^{-1} \sum_{j=0}^{n-1} \ln \|(Df^k(f^{kj}x))^{-1}\|$. We say that a periodic orbit of f is *repelling* if it is an attracting periodic orbit for f^{-1}.

Theorem B. *Let $f : M \to M$ be a C^1 diffeomorphism. Then $x \in M$ belongs to a repelling periodic orbit (a source) if, and only if, $\tilde{A}_k^-(x) < 0$ for some $k \in \mathbb{Z}^+$.*

We easily deduce the following ergodic consequence from the above pointwise result.

Corollary 1.4. *Let μ be an invariant probability measure with respect to a C^1 diffeomorphism $f : M \to M$. Then it admits a decomposition*[1] $\mu = \tilde{\mu} + \sum_{i\geq 1} \nu_i + \sum_{j\geq 1} \rho_i$, *where each ν_i (respectively, ρ_i) is a Dirac mass equidistributed on a periodic attracting (resp. repelling) orbit of f, both sums are over at most countably many such orbits, and $\tilde{\mu}$ satisfies $\chi(x) \geq 0 \geq -\tilde{\chi}(x)$ for $\tilde{\mu}$-a.e. $x \in M$.*

In particular, if μ is non-atomic, then $\mu = \tilde{\mu}$ and so either μ has some zero exponent, or μ is a hyperbolic measure with exponents of different signs.

1.1.2. **Robustness of negative Lyapunov exponents.** In contrast to the results above, it is well-known that positive Lyapunov exponents in all directions on a total probability set for a C^1 local diffeomorphism of M imply that f is a uniformly expanding map. More precisely, see [1, 12], if $\tilde{\chi}(x) = \lim_{n\to+\infty} \ln \|Df^n(x)^{-1}\|^{1/n} < 0$ for $\mu-$ a.e. $x \in M$ with respect to every f-invariant probability measure μ, then we can find constants $C, \sigma > 0$ so that $\|Df^n(x)^{-1}\| \leq Ce^{-\sigma n}$ for all $x \in M, n \geq 1$. This is a robust situation: the assumptions automatically hold for a C^1-neighborhood of such local diffeomorphisms; see e.g. [30].

In the same setting exchanging positive with negative exponents in all directions we obtain the following.

Theorem C. *If a C^1 map $f : M \to M$ is such that* $\inf_{x\in M} \|Df(x)\| > 0$ *and for all f-invariant probability measures μ we have $\chi(x) < 0, \mu$-a.e. $x \in M$, then there exists a unique periodic attracting orbit $\mathcal{O}_f(p)$ whose basin is M.*

See next Subsection 1.3 for comments and corollaries of this.

1.2. The case of (singular) vector fields. Let $\mathfrak{X}^1(M)$ be the space of C^1 vector fields on M which are inwardly transverse to the boundary endowed with the C^1 topology and ϕ_t be flow generated by $G \in \mathfrak{X}^1(M)$. We denote by $D\phi_t$ *the derivative of ϕ_t with respect to the ambient variable q and set $D_q\phi_t = D\phi_t(q)$.* An analogous Subadditive Ergodic Theorem also holds: $\chi_G(x) = \lim_{T\to+\infty} \ln \|(D\phi_T(x)\|^{1/T}$ exists on a total probability subset.

To state analogous results for vector fields we need some preliminary notions about critical elements of the flow induced by a vector field and, since the vector field direction always has zero Lyapunov exponent for every invariant probability measure, we need to deal with the derivative cocycle of the flow ϕ_t generated by the vector field G restricted to the normal direction to the flow: these notions can be defined for a flow on any finite dimensional Riemannian manifold.

1.2.1. **Some preliminary notions.** Given $G \in \mathfrak{X}^1(M)$, where M is a compact finite dimensional Riemannian manifold with dimension $d \geq 2$, we denote by *DG the derivative of the vector field G* with respect to the ambient variable q, and when convenient we write D_qG for the derivative DG at q, also denoted by DG_q, where $DG_q v = \nabla_v G(y)$ where ∇ is the unique Levi-Civita connection compatible with the Riemannian metric on M. Given $q \in M$ an orbit segment $\{\phi_t q; a \leq t \leq b\}$ is denoted by $\phi_{[a,b]}q$.

Critical elements. An *equilibrium* or *singularity* for G is a point $\sigma \in M$ such that $\phi_y(\sigma) = \sigma$ for all $t \in \mathbb{R}$, i.e. a fixed point of all the flow maps, which corresponds to a zero of the associated vector field G: $G(\sigma) = \vec{0}$. We denote by $\mathrm{Sing}(G) = \{x \in M : G(x) = \vec{0}\}$ the set of singularities of G. Every point $p \in M$ which is not a singularity, that is p satisfies

[1]Some of the summands in the decomposition might be null.

$G(p) \neq 0$, is a *regular* point for G.

An *orbit* of G is a set $\mathcal{O}(q) = \mathcal{O}_G(q) = \{\phi_t q : t \in \mathbb{R}\}$ for some $q \in M$. Hence $\sigma \in M$ is a singularity of G if, and only if, $\mathcal{O}_G(\sigma) = \{\sigma\}$. A *periodic orbit* of G is an orbit $\mathcal{O} = \mathcal{O}_G(p)$ such that $\phi_T p = p$ for some minimal $T > 0$ (equivalently $\mathcal{O}_G(p)$ is compact and $\mathcal{O}_G(p) \neq \{p\}$). We denote by $\mathrm{Per}(G)$ the set of all periodic orbits of G.

A *critical element* of a given vector field G is either a singularity or a periodic orbit. The set $\mathrm{Crit}(G) = \mathrm{Sing}(G) \cup \mathrm{Per}(G)$ is the set of *critical elements* of G.

Limit sets. Attractors. If $q \in M$, we define *omega-limit* set $\omega_G(q)$ as the set of accumulation points of the positive orbit $\{\phi_t q : t \geq 0\}$ of q. We also define the alpha-limit set $\alpha_G(q) = \omega_{-G}$, where $-G$ is the time reversed vector field G, corresponding to the set of accumulation points of the negative orbit of q.

A subset Λ of M is *invariant* for G (or G-invariant) if $\phi_t\Lambda = \Lambda, \forall t \in \mathbb{R}$. We note that $\omega_G(q), \alpha_G(q), \mathrm{Sing}(G)$ and their complements in M are G-invariant.

For every compact invariant set Λ of X we define the *stable set* of Λ

$$W^s_G(\Lambda) = \{q \in M : \omega_G(q) \subset \Lambda\},$$

and also its *unstable set*

$$W^u_G(\Lambda) = \{q \in M : \alpha_G(q) \subset \Lambda\}.$$

A compact invariant subset Λ of G is *attracting* if $\Lambda_G(U) = \cap_{t\geq 0}\phi_t(U)$ equals Λ for some neighborhood U of Λ satisfying $\overline{\phi_t(U)} \subset U, \forall t > 0$. In this case the neighborhood U is called an *isolating neighborhood* of Λ. Analogously, Λ is *repelling* if it is attracting for $-G$. We say Λ is a proper subset if $\emptyset \neq \Lambda \neq M$.

Hyperbolic critical elements. A (hyperbolic) *sink* of G is a singularity which is also an attracting set, it is a trivial attracting set of G. A *source* of G is a trivial repelling subset of G, i.e. a singularity which is attracting for $-G$.

A *singularity σ is hyperbolic* if the eigenvalues of $DG(\sigma)$, the derivative of the vector field at σ, have real part different from zero. In particular, sinks and sources are hyperbolic singularities, since all the eigenvalues of the former have negative real part and those of the latter have positive real part.

A *periodic orbit $\mathcal{O}_G(p)$ of G is hyperbolic* if the eigenvalues of $D\phi_T(p) : T_pM \to T_pM$ (the derivative of the diffeomorphism ϕ_T at p with $T > 0$ the period of p) are all different from 1.

When a critical element is hyperbolic, then its stable and unstable sets have the structure of an immersed manifold (a consequence of the Stable Manifold Theorem, see e.g. [23]), and are known as *stable* and *unstable manifolds*.

In the particular case of attracting critical elements, the corresponding stable set (manifold) is also known as its (topological) *basin*.

Linear Poincaré Flow. If x is a regular point of a C^1 vector field G (i.e. $G(x) \neq \vec{0}$), denote by $N_x = \{v \in T_xM : \langle v, G(x)\rangle = 0\}$ the orthogonal complement of $G(x)$ in T_xM. Denote by $O_x : T_xM \to N_x$ the orthogonal projection of T_xM onto N_x. For every $t \in \mathbb{R}$ define, see Figure 1.2.1

$$P^t_x : N_x \to N_{\phi_t x} \quad \text{by} \quad P^t_x = O_{\phi_t x} \circ D\phi_t(x).$$

It is easy to see that $P = \{P_x^t : t \in \mathbb{R}, G(x) \neq 0\}$ satisfies the cocycle relation $P_x^{s+t} = P_{\phi_s x}^t \circ P_x^s$ for every $t, s \in \mathbb{R}$. The family $P = P_G$ is called the *Linear Poincaré Flow* of G.

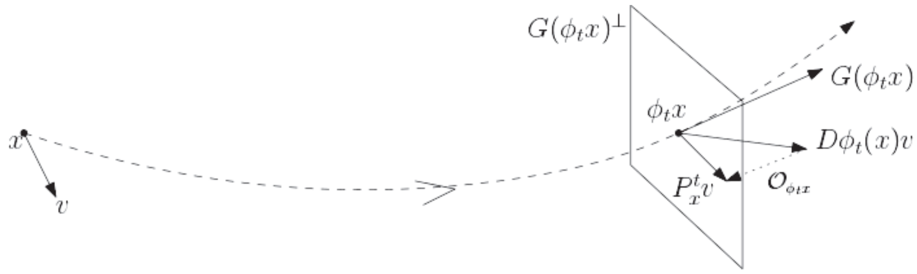

Fig. 1. Sketch of the Linear Poincaré flow P_x^t of a vector $v \in T_x M$ with $x \in M \setminus \mathrm{Sing}(G)$.

REMARK 1.5. The Linear Poincaré Flow does not immediately extends to the smooth semiflow setting and this is an important tool in our proofs; see Conjecture 5.

1.2.2. **Negative (positive) exponents and sinks.** First we consider setting similar to Theorems A and B. In what follows we write $\chi_G^-(x) = \liminf_{T\to+\infty} \ln \|D\phi_T(x)\|^{1/T}$ and $\tilde{\chi}_G^-(x) = \liminf_{T\to+\infty} \ln \|D\phi_T(x)^{-1}\|^{1/T}$.

Theorem D. *Given $G \in \mathfrak{X}^1(M)$ suppose that $x \in M$ satisfies $\chi_G^-(x) < 0$. Then there exists a hyperbolic sink $\sigma \in \mathrm{Sing}(G)$ so that $\phi_t x \to \sigma$ as $t \to \infty$. Otherwise, suppose that $\tilde{\chi}_G^-(x) < 0$. Then x is a repelling equilibrium (a source).*

Since the flow ϕ_t induced by a vector field $G \in \mathfrak{X}^1(M)$ satisfies $D\phi_t(x)G(x) = G(\phi_t x), x \in M, t \in \mathbb{R}$, it is natural to consider trajectories which have asymptotic contraction along all transversal directions to the vector field, which we refer to as *sectional asymptotic contraction*.

1.2.3. **Negative sectional exponents and sinks.** However, weak sectional asymptotic contraction along a given trajectory does not necessarily implies that this trajectory converges to a sink, either a singularity or a periodic orbit, as the following example shows.

EXAMPLE 1. Consider the vector field known as "Bowen example"; see e.g. [29] and Figure 2. This vector field is inwardly transverse to the boundary of $M = \mathbb{S}^1 \times [-1, 1]$. Let $W = \cup_{i=1}^4 W_i$ be the set formed by the heteroclinic connections between and including the equilibria σ_1, σ_2. The future trajectories under the corresponding flow ϕ_t of every $z \in M \setminus W$ accumulates on either side of the heteroclinic connections, as suggested in the figure, if we impose the condition $\lambda_1^- \lambda_2^- > \lambda_1^+ \lambda_2^+$ on the eigenvalues of the saddle equilibria σ_1 and σ_2 (for more specifics on this see [29] and references therein) so that σ_i are area contracting: $|\det D\phi_t(\sigma_i)| \to 0$ exponentially fast with $t > 0$, $i = 1, 2$.

It is well-known (see [29] for more details) that the time taken by the orbit $\phi_t x$ of any point x in the connected components of $\mathbb{S}^2 \setminus W$ containing one of σ_3, σ_4, with exception of the equilibria σ_3, σ_4, while passing through a small neighborhood of either σ_1 or σ_2 is much larger than all the previous history of the orbit. Then the rate $\ln \|P_x^T\|^{1/T}$ oscillates between the value of λ_i^+ (when approaching) and λ_i^- (at departure) at each passage near σ_i, $i = 1, 2$,

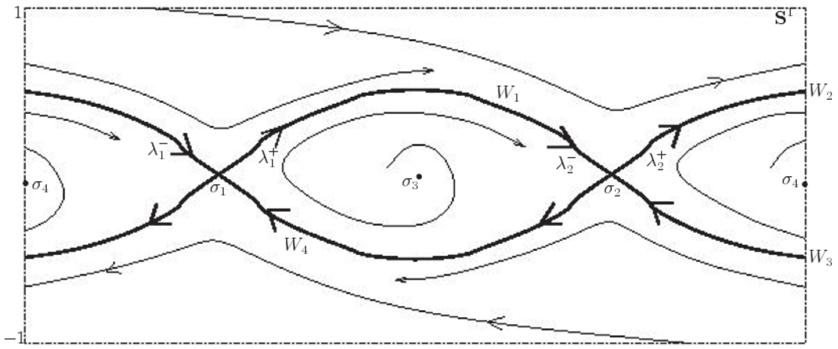

Fig. 2. A sketch of Bowen's example flow.

that is $\liminf_{T\to\infty} \ln \|P_x^T\|^{1/T} < 0 < \limsup_{T\to\infty} \ln \|P_x^T\|^{1/T}$.

All points z on the connected components of $\mathbb{S}^2 \setminus W$ containing the boundary of M also accumulate W and thus satisfy the same asymptotic rates.

The previous Example 1 motivates us to state a partial analogue to Theorem A in the vector field setting.

Theorem E. *Let $G \in \mathfrak{X}^1(M)$ be such that* $\mathrm{Sing}(G)$ *(possibly empty) is hyperbolic. If $x \in M \setminus \mathrm{Sing}(G)$ satisfies $\liminf_{T\to\infty} \ln \|P_x^T\|^{1/T} < 0$, then*

- *either x is contained in the basin of attraction of a sink: either an attracting equilibrium or a hyperbolic periodic attracting orbit;*
- *or the orbit of x accumulates a hyperbolic codimension* 1 *saddle singularity*[2].

To obtain the same conclusion as Theorem A for a sectional contracting trajectory of a vector field, we need to assume a stronger condition on the asymptotic contracting rate.

Theorem F. *Let $G \in \mathfrak{X}^1(M)$ be such that* $\mathrm{Sing}(G)$ *is hyperbolic. If $x \in M \setminus \mathrm{Sing}(G)$ is such that $\limsup_{T\to\infty} \ln \|P_x^T\|^{1/T} < 0$, then x is contained in the basin of attraction of a sink: either an attracting equilibrium or a hyperbolic periodic attracting orbit.*

Remark 1.6. (1) We do not need Hölder continuity of the derivative in the arguments proving Theorems E and F and corollaries.

(2) The condition "$\mathrm{Sing}(G)$ is hyperbolic" imposed on G in the statement of Theorems E and F is satisfied by an open and dense subset of $\mathfrak{X}^1(M)$; see e.g. [23].

(3) In the particular case $\mathrm{Sing}(G) = \emptyset$, Theorems E and F become the direct analogue to Theorem A in the vector field setting: the trajectory of x converges to either an attracting fixed point of the flow or to an attracting periodic orbit, even if asymptotic contraction only holds sectionally.

1.2.4. **Positive sectional exponents and sources.** Akin to expanding maps and expanding measures, for expanding semiflows the asymptotic expansion condition on a given trajectory does not necessarily implies that the trajectory is a (periodic) source.

[2]The stable manifold of the singularity has codimension one as an immersed submanifold of M.

Example 2. The geometrical Lorenz expanding semiflow introduced by Williams [31] exhibits asymptotic expansion in the transversal direction of all positive time trajectories not falling into the singularity, has a dense regular trajectory and a dense subset of periodic expanding trajectories; see [31] for details.

The analogous to Theorem B is also true for sectional expansion.

Theorem G. *Let $G \in \mathfrak{X}^1(M)$ be such that* $\mathrm{Sing}(G)$ *is hyperbolic. If $x \in M \setminus \mathrm{Sing}(G)$ satisfies* $\liminf_{T\to\infty} \ln \|(P_x^T)^{-1}\|^{1/T} < 0$*, then*

(1) *either x belongs to a hyperbolic periodic repelling orbit;*
(2) *or the orbit of x accumulates a hyperbolic saddle singularity of index* 1[3].

If $x \in M \setminus \mathrm{Sing}(G)$ is such that $\limsup_{T\to\infty} \ln \|(P_x^T)^{-1}\|^{1/T} < 0$*, then x satisfies item* (1).

Remark 1.7. Example 1 also provides an instance of item (2) in the statement of Theorem G. This example is easily adapted to higher dimensions: just multiply Bowen's vector field G by a "North-South" vector field in the nth sphere $\mathbb{S}^n, n \geq 1$ to obtain higher dimensional instances of Theorems E and G.

1.3. Comments, corollaries and conjectures. We comment and state some corollaries of the results in what follows, and then some conjectures. The proofs of the corollaries are given later in the text: see next Subsection 1.4 on the organization of this text.

1.3.1. **The C^1 endomorphism setting.**

Negative Lyapunov exponents everywhere.

Corollary 1.8. *Let $K \subset M$ be a compact f-invariant subset such that $\chi^-(x) < 0$ for all $x \in K$. Then K is the union of a finite family of sinks.*

The setting of Theorems A and C is robust: there exists a C^1 neighborhood $\mathcal{U}$ of f such that each $g \in \mathcal{U}$ satisfies the same assumptions and conclusions.

Example 3. An example of a C^1 endomorphism $f : \mathbb{S}^2 \to \mathbb{S}^2$ satisfying the conclusion of Theorem C can be given as follows: consider

- $h : \mathbb{S}^2 \circlearrowleft$ the North-South map on $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, given by the time-1 map of the gradient flow $\dot{w} = \nabla\varphi(w)$ with $\varphi(x, y, z) = z$, where $w = (x, y, z) \in \mathbb{S}^2$;
- P the stereographic projection from $N(0, 0, 1)$ to $\mathbb{S}^2$.

Then let $g : \mathbb{S}^2 \to \mathbb{C}$ be the C^1 map that sends $\mathbb{S}^2$ to the half-sphere $\mathbb{S}^2 \cap \{z < 0\}$ together with the surface R of revolution generated by the three half-circles with diameter $1/3$ drawn in the left hand side of Figure 3. We choose $g\,|_{\mathbb{S}^2\cap\{z<0\}}$ to be the identity and $g\,|_{\mathbb{S}^2\cap\{z\geq 0\}}$ as the vertical projection from $\mathbb{S}^2 \cap \{z \geq 0\}$ to S.

Finally, define f as the composition $h \circ P \circ g$. Note that the image of $f_0 = P \circ g$ is contained in $\mathbb{S}^2 \cap \{z \leq 0\}$ and so the image of f is contained in $\mathbb{S}^2 \cap \{z < 0\}$, hence f is a strict contraction. Moreover, $f_0(N) = S(0, 0, -1)$. Hence $f = h \circ f_0$ contracts distances uniformly and fixes S, which is a sink attracting all points of $\mathbb{S}^2$.

[3]The stable manifold of the singularity has dimension one as an immersed submanifold of M.

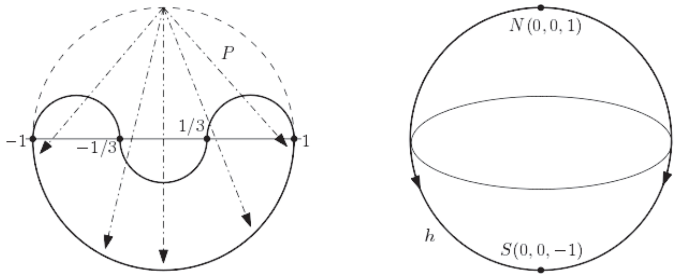

Fig.3. Maps whose composition defines a C^1 transformation with negative Lyapunov exponents everywhere and a unique sink with full basin.

Negative exponents Lebesgue almost everywhere. It is known that there are C^1 open families of local diffeomorphisms satisfying $\tilde{A}_1^-(x) < 0$ for Lebesgue almost points of the ambient manifold and which are not uniformly expanding; see [25] and references therein and also [2, Appendix] for a concrete example of open classes of such local diffeomorphisms.

Remark 1.9. (1) It is well-known that for (expanding maps and) expanding measures there exists a dense subset of periodic sources in its support; see [25].

(2) It is known that $\tilde{\chi} < 0$ for μ-a.e. implies $\tilde{A}_k^-(x) < 0$ μ-a.e. for any given f-invariant measure μ and some $k \in \mathbb{Z}^+$; see e.g. [2]. It is conjectured that $\tilde{A}_1^-(x) < 0$ for m-a.e. x implies the existence of an f-invariant probability measure μ satisfying $\tilde{\chi}(x) < 0, \mu$-a.e. x; see e.g [25] and more recently [26].

In our setting, it is natural to consider C^1 maps satisfying $A_k^- < 0$ for Lebesgue almost all points and some $k \geq 1$.

Corollary 1.10. *Let $f : M \to M$ be a C^1 map such that* $\inf_{x\in M} \|Df(x)\| > 0$. *Then $A_k^-(x) < 0, m$-a.e. $x \in M$ for some $k \in \mathbb{Z}^+$ if, and only if, there exists an at most countable family of Dirac masses concentrated on periodic attracting orbits (sinks) whose basins form an open, dense and also a full Lebesgue measure subset of M.*

Example 4. An example of a diffeomorphism satisfying the conclusion of Corollary 1.10 can be constructed by the direct product of the map from [3, Example 1] and the North-South map, both on the circle.

The latter is represented in Figure 4 given by the time-1 map h of the gradient flow $\dot{z} = \nabla\psi(z)$ with $\psi(x, y) = y$ on $z = (x, y) \in \mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. The former can be seen as the time-1 map g of the gradient flow $\dot{x} = \nabla\varphi(x)$ with $\varphi : [-\pi^{-1}, \pi^{-1}] \to \mathbb{R}, t \mapsto t^4 \sin(1/t)$, (where we identify $\pm\pi^{-1}$ to obtain the circle) exhibiting a countable number of attracting fixed sinks whose basins cover the entire domain of φ with the exception of the countably many local maxima $\{m_k\}$ of φ; see Figure 5.

Then $g \times h : \mathbb{S}^1 \times \mathbb{S}^1 \circlearrowleft$ is a C^1 map having a countable number of attracting fixed sinks whose basins cover the entire space $[-\pi^{-1}, \pi^{-1}] \times \mathbb{S}^1$ with the exception of the Lebesgue null sets $[-\pi^{-1}, \pi^{-1}] \times \{N\}$ and $\{m_k\} \times \mathbb{S}^1$.

This situation is however not robust as the following example shows.

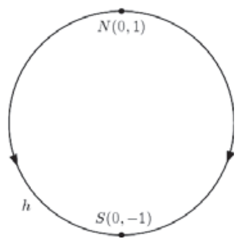

Fig. 4. The North-South map on the circle.

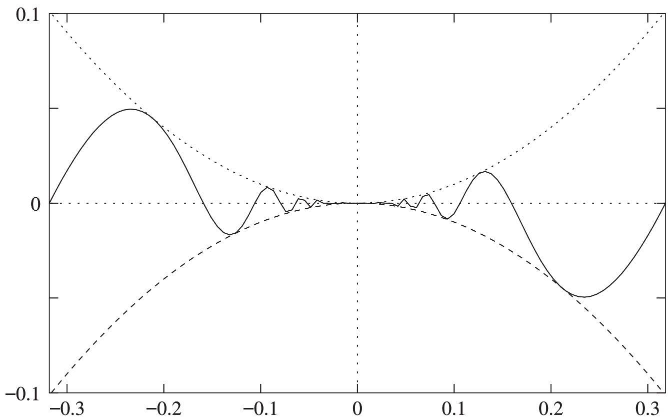

Fig. 5. A map whose gradient flow has infinitely many sinks whose basins form a open, dense and full measure subset of the ambient space.

EXAMPLE 5. There exist one parameter families $\varphi_\mu : \mathbb{S}^2 \to \mathbb{S}^2, -1 \le \mu \le 1$ of smooth diffeomorphisms such that (see e.g. [24, Chapter 5, Section 1 & Chapter 7, Section 2] and references therein)

- there exists a sink $S \in \mathbb{S}^2$ whose basin is an open, dense and full Lebesgue measure subset of $\mathbb{S}^2$ for $-1 \le \mu \le 0$; and
- there are parameters $\mu_n \searrow 0$ so that φ_{μ_n} admits positive Lebesgue measure subsets of points x with a positive Lyapunov exponent (e.g., the ergodic basin of Hénon-like attractors near the generic unfolding of a quadratic homoclinic tangency).

1.3.2. **The singular C^1 vector field setting.** The pointwise statements of the continuous version of the results take advantage of the existence of infinitesimal generators of the cocycles $\ln\|D\phi_t(x)v\|$ and $\ln\|P_x^t v\|$ to avoid assumptions on time averages; see Section 3.2. The cocycle relation for the derivative of the flow of G and for the Linear Poincaré Flow P_G implies that the functions

$$\Gamma(v) = \Gamma_G(v) : \mathbb{R}\times M \to \mathbb{R}, \quad (t,x) \mapsto \Gamma_t(x)v = \ln\|D\phi_t(x)v\|;$$
$$\tilde\Gamma(v) = \tilde\Gamma_G(v) : \mathbb{R}\times M \to \mathbb{R}, \quad (t,x) \mapsto \tilde\Gamma_t(x)v = \ln\|D\phi_t(x)^{-1}v\|;$$
$$\psi(v) = \psi_P(v) : \mathbb{R}\times M \setminus \mathrm{Sing}(G) \to \mathbb{R}, (t,x) \mapsto \psi_t(x)v = \ln\|P_x^t v\| \quad \text{and}$$
$$\tilde\psi(v) = \tilde\psi_P(v) : \mathbb{R}\times M \setminus \mathrm{Sing}(G) \to \mathbb{R}, (t,x) \mapsto \tilde\psi_t(x)v = \ln\|(P_x^t)^{-1}v\|$$

are *additive*: $\Gamma_{t+s}(y)v \leq \Gamma_s(\phi_t y)v + \Gamma_t(y)v$ and $\psi_{t+s}(x)v \leq \psi_s(\phi_t x)v + \psi_t(x)v$ for $y \in M, x \in M \setminus \mathrm{Sing}(G)$ and $t, s \in \mathbb{R}$; and similarly for $\tilde{\Gamma}$ and $\tilde{\psi}$. In Section 3 a more detailed version of the following is stated and proved, mainly as a consequence of the extra smoothness gained along trajectories of the flow generated by a C^1 vector field, since these trajectories become C^2 curves.

Theorem 1.11. *The functions $D_G(v) := \lim_{h\to 0} h^{-1}\Gamma_h(y)(v)$, $\tilde{D}_G(v) := \lim_{h\to 0} h^{-1}\tilde{\Gamma}_h(y)v$, $D(v) := \lim_{h\to 0} h^{-1}\psi_h(x)v$ and $\tilde{D}(v) := \lim_{h\to 0} h^{-1}\tilde{\psi}_h(x)v$ are continuous and uniformly bounded on $T^1_x M$. Moreover $\Gamma_t(y)w = \int_0^t D_G(\Phi_s w)\,ds$ and $\psi_t(x)v = \int_0^t D(\widehat{\Phi_s} v)\,ds$, for $t \in \mathbb{R}$, $y \in M$, $w \in T^1_y M$ and $x \in M \setminus \mathrm{Sing}(G)$, $v \in T^1_x M \cap G^\perp$; and similarly for $\tilde{\Gamma}$ and $\tilde{\psi}$.*

Here T^1M os the unit tangent bundle, $G^\perp$ is the normal bundle to G on $M \setminus \mathrm{Sing}(G)$; Φ_t is the induced flow on the unit tangent bundle $\Phi_t v = \frac{D\phi_t(x)v}{\|D\phi_t(x)v\|}$ and $\widehat{\Phi_t}$ the analogous construction with P^t_x in the place of $D\phi_t(x)$.

Hence, for instance, we can replace $\lim_{T\to\infty} \ln \|P^T_x\| < 0, \mu$-a.e x for an ergodic G-invariant probability measure by the condition $\mu(\sup_{v\in T^1_x M\cap G^\perp} D(v)) < 0$ and so on; see the statement of Corollary 1.13 in what follows.

On sectional Lyapunov exponents. We can also interpret $\mu(D) < 0$ as a condition on the Lyapunov spectrum of μ. The Oseledets Multiplicative Ergodic Theorem states that Lyapunov exponents exist for the cocycle $D\phi_t(x)$ for a total probability subset of points: for any G-invariant probability measure and for μ-a.e. x there exists $k = k(x) \in \{1, \dots, d = \dim(M)\}$, numbers $\chi_1(x) < \cdots < \chi_k(x)$ and a $D\phi_t$-invariant decomposition $T_x M = E^1_x \oplus \cdots \oplus E^k_x$ (i.e., $D\phi_t E^i_x = E^i_{\phi_t x}$) so that

$$\chi(x, v) = \lim_{t\to\pm\infty} \log \|D\phi_t(x) \cdot v\|^{1/t} = \chi_i(x), \quad \forall v \in E^i_x \setminus \{\vec{0}\}, 1 \leq i \leq k(x).$$

Moreover, $\chi(x, G(x)) = 0$ for μ-a.e. $x \in M \setminus \mathrm{Sing}(G)$. In addition, the angles between any two Oseledets subspaces decay sub-exponentially fast along orbits of f (see e.g. [7, Theorem 1.3.11 & Remark 3.1.8]): $\lim_{t\to\pm\infty} \frac{1}{n} \log \sin \angle\left(\bigoplus_{i\in I} E^i_{\phi_t x}, \bigoplus_{j\notin I} E^j_{\phi_t x}\right) = 0$ for any $I \subset \{1, \dots, k(x)\}$ and μ-a.e. x, where for any given pair E, F of complementary subspaces (i.e. $E \oplus F = T_x M$) we set $\cos \angle(E, F) := \inf\{|\langle v, w\rangle| : \|v\| = 1 = \|w\|, v \in E, w \in F\}$. This implies, in particular, that for any 2-dimensional subspace S of $T_x M$ the value of $\lim \ln |\det D\phi_t \mid S|^{1/T}$ equals the sum of the two largest Lyapunov exponents of all basis of S. Since the direction of the flow has zero Lyapunov exponent, the assumptions on Theorems E, F and G can be restated as: $\liminf_{T\to\infty} \ln |\det D\phi_t \mid S| < 0$ for every two-dimensional subspace S of $T_x M$; or with $\limsup$ etc. This is why it is natural to label these conditions on a trajectory of a flow as *asymptotic sectional growth conditions* or *conditions on Lyapunov exponents transverse to the vector field*, since $\mu(\sup_{v\in T^1_x M\cap G^\perp} D) < 0$ for an ergodic G-invariant probability measure μ amounts to say that the the Lyapunov exponents are μ-a.e. equal to $\chi_1 < \cdots < \chi_{k-1} < 0 = \chi_k$ for some $k \leq d$.

Asymptotic contraction (Lebesgue almost) everywhere. The setting of Theorems D, E, F and G is robust: on a C^1 neighborhood $\mathcal{U}$ of G in $\mathfrak{X}^1(M)$ we have the same assumptions and conclusions.

If we replace f by the flow generated by $G \in \mathfrak{X}^1(M)$ and the assumptions $\chi^-(x) < 0$ or $A^-_k(x) < 0$ by $\chi^-_G(x)$ on Corollaries 1.8 and 1.10, then we get the same conclusions in the vector field setting. Moreover, since the Linear Poincaré Flow is only defined for regular points, we also have

Corollary 1.12. *Let $G \in \mathfrak{X}^1(M)$ be such that* $\mathrm{Sing}(G)$ *is hyperbolic.*

(1) *If* $\limsup_{T\to\infty} \ln \|P_x^T\|^{1/T} < 0$ *for*[4] *m-a.e.* $x \in M$*, then an at most countable family of periodic attracting orbits or attracting equilibria (i.e., an at most enumerable family of sinks) whose basins form an open, dense and also a full Lebesgue measure subset of* M*.*

(2) *Let* $K \subset M$ *be a compact* G*-invariant subset such that* $\liminf_{T\to\infty} \ln \|P_x\|^{1/T} < 0$ *for all* $x \in K$*. Then* K *is the union of a finite family of sinks.*

There are many classes of examples of vector fields having an open, dense and full Lebesgue measure subset in the basin of attraction of a family of sinks and are arbitrarily C^1 close to a vector field having a positive Lebesgue measure subset of trajectories with some asymptotic expansion; see e.g. [9, Chapter 9]. We outline one of these.

Example 6. Using singular cycles, Morales [22] studied the unfolding of a geometric Lorenz attractor when the singularity contained in this attractor goes through a saddle-node bifurcation. It is shown in [22] that there exist one-parameter families $(G_t)_{t\in[-1,1]}$ of vector field in a 3-manifold M which *unfold a Lorenz attractor directly into a Plykin attractor*. This means that there are $\mu \in (-1, 1)$ and $\delta > 0$ such that

- if $t \in [(\mu - \delta, \mu)$, then G_t has a geometric Lorenz attractor.
- G_μ is a saddle-node Lorenz vector field.
- if $t \in (\mu, \mu + \delta)$, then G_t is an Axiom A vector field (see e.g.[24]).

The vector fields G_t for $t \in (\mu-\delta, \mu]$ satisfy $\chi^-_{G_t}(x) > 0$ for a positive Lebesgue measure subset of points, namely the basin of attraction of the (saddle-node) geometric Lorenz attractor. In contrast, G_t for $t \in (\mu, \mu + \delta)$ has finitely many hyperbolic attractors whose basins form an open, dense and full Lebesgue subset of M; see e.g. [10].

Decomposition of invariant probability meeasures for vector fields. We can obtain ergodic statements similar to Corollaries 1.2 and 1.4.

Corollary 1.13. *Let μ be an invariant probability measure with respect to a C^1 vector field $G \in \mathfrak{X}^1(M)$. Then it admits a decomposition*[5] $\mu = \tilde\mu + \sum_{i\ge1} \nu_i + \sum_{j\ge1} \rho_i$*, where each* ν_i *(respectively,* ρ_i*) is a Dirac mass equidistributed on a periodic attracting (resp. repelling) orbit of* G*, both sums are over at most countably many such orbits, and* $\tilde\mu$ *satisfies* $\chi_G(x) \ge 0$ *and* $\tilde\chi_G(x) \ge 0$ *for* $\tilde\mu$*-a.e.* $x \in M$*.*

In particular, if μ is non-atomic, then $\mu = \tilde\mu$ and so either μ has some zero exponent, or μ is a hyperbolic measure with exponents of different signs.

1.3.3. **Conjectures.** The pointwise statements of the continuous version of the results took advantage of the existence of infinitesimal generators of the cocycles $\ln \|D\phi_t(x)\|$ and $\ln \|P_x^t\|$ to avoid assumptions on time averages, as in the statements of Theorems A and B.

Conjecture 1. In the discrete setting we can argue as in the vector field setting to reduce asymptotic growth conditions to asymptotic average growth condition. That is, replacing the

[4] Since $\mathrm{Sing}(G)$ is hyperbolic, then $m(\mathrm{Sing}(G)) = 0$ because (if non-empty) $\mathrm{Sing}(G)$ is finite.

[5] Again, some of the summands in the decomposition might be null.

assumptions $A^-(x) < 0$ or $\tilde{A}^-(x) < 0$ by $\liminf_{n\to\infty} \ln \|Df^n(x)^{\pm 1}\|^{1/n} < 0$ in the statements of Theorems A and B.

A positive answer to this would be an advance to answer the conjecture mentioned in Remark 1.9(2); see also [26].

We expect the assumption that Df is never the null map is an artifact of our proof and can be bypassed.

Conjecture 2. For maps the result of Theorem A is still valid without any extra assumptions on the derivative.

We should not need to use hyperbolicity assumptions on the vector field G.

Conjecture 3. Theorems E, F and G hold for all vector fields $G \in \mathfrak{X}^1(M)$.

From Remark 1.7 we conjecture that Example 1 is paradigmatic.

Conjecture 4. Let G be a vector field satisfying $\liminf_{T\to\infty} \frac{1}{T} \ln \|(P_x^T)^{-1}\| < 0$ and also $\liminf_{T\to\infty} \frac{1}{T} \ln \|P_x^T\| < 0$ for a open, dense and full Lebesgue measure subset of M. Then G exhibits saddle connections similar to Example 1.

Due to the simple character of the dynamics of sinks and sources, we should be able to obtain similar results in the setting of continuous flows and smooth semiflows, not necessarily generated by vector fields.

Conjecture 5. There exists an open and dense family of continuous flows or smooth semiflows on manifolds where an extended notion of sectional asymptotic expansion or contraction along trajectories ensures the existence of sources or sinks.

1.4. Organization of the text. We prove Theorems A, B and C in Section 2, together with Corollaries 1.2, 1.4, 1.8 and 1.10. We state a version of Pliss' Lemma 2.1 for flows in Subsection 3.1. In Subsection 3.2 we translate the assumptions of Theorems D, E and F in a convenient format. Then we use these results as tools for the proof of the first part of the statement of Theorem D in Subsection 3.3 and the proof of Theorems E and F in the remaining Subsections 3.4, 3.5, 3.6 and 3.7. In Subsection 3.8 we prove the second part of the statement of Theorem D and Theorem G. In the last Subsection 3.9 we prove some technical lemmas.

2. The discrete time case

Here we prove Theorems A, B and C together with their corollaries.

Proof of Theorem A. Exchanging f by f^k in what follows we may assume without loss of generality that $k = 1$. By assumption, we have $\zeta > 0$ and a strictly increasing sequence $m_i \nearrow \infty$ so that $\frac{1}{m_i}\sum_{j=0}^{m_i-1} \ln \|Df(f^j x)\| < -\zeta$ as $i \nearrow \infty$. We can now use the following.

Lemma 2.1 (Pliss Lemma; see e.g. Chapter IV.11 in [20]). *Let $H \geq c_2 > c_1 > 0$ and $\theta = (c_2 - c_1)/(H - c_1)$. Given real numbers $a_1, \dots, a_N$ satisfying $\sum_{j=1}^N a_j \geq c_2 N$ and $a_j \leq H$ for all $1 \leq j \leq N$, there are $\ell > \theta N$ and $1 < n_1 < \dots < n_\ell \leq N$ such that $\sum_{j=n+1}^{n_i} a_j \geq$*

$c_1 \cdot (n_i - n)$ *for each* $0 \le n < n_i,\ i = 1, \dots, \ell$.

We set $c_2 = -\zeta$, $c_1 = c_2/2$, $H = -\ln \inf_{x\in M} \|Df(x)\|$ and $a_j = -\ln \|Df(f^{m_i - j}x)\|$ for $1 < j \le m_i$. Notice that we are inverting the summation order.

Then for $\theta = c_2/(2H - c_2) > 0$ and $N = m_i$ Pliss Lemma ensures that there are $\ell > \theta N$ and $1 < n_1 < \cdots < n_\ell \le m_i$ such that for each $0 \le n < n_k$ and $k = 1, \dots, \ell$

$$\prod_{j=n+1}^{n_k} \|Df(f^{m_i - j}x)\| \le e^{-c_1(n_k - n)}.$$

The iterates $m_i - n_k$ are *reverse hyperbolic times* for the f-orbit of x with respect to m_i; similar times were used in [21] by Mañé and by Liao in [16]. Pliss' Lemma ensures that there are infinitely many reverse hyperbolic times n_i along the f-orbit of x with respect to m_i and, because $\theta > 0$, we can assume that $(m_i - n_i) \nearrow \infty$. Consequently, if h is a reverse hyperbolic time with respect to m_i, then $\|Df^j(f^h x)\| \le \lambda^j$ with $\lambda = e^{-\zeta/2}$ for all $j = 1, \dots, m_i - h$. This uniform contractive property can be extended to a neighborhood using the fact that f is *a C^1 map such that Df is never the null transformation*, as follows.

Lemma 2.2 (Existence of forward contracting balls)**.** *There exist $\delta_1 > 0$ (depending only on f and λ) and $\lambda_1 = \sqrt{\lambda} \in (0,1)$ such that if n is a reverse hyperbolic time for $x \in M$ with respect to $m > n$, then for every $0 < j \le m - n$ there are subsets V_{n+j} containing $f^{n+j}(x)$ such that $V_n = B(f^n(x), \delta_1)$; $f^j(V_n) \subset V_{n+j}$, and $f^j \mid_{V_n}: V_n \to V_{n+j}$ is a λ_1^j-contraction.*

Proof of Lemma 2.2. We basically follow [2, Lemma 5.2] adapting the same ideas to the present setting. Since $\inf_{x\in M} \|Df(x)\| > 0$ we have that the map $\psi : M \times M \to \mathbb{R}, (x, y) \mapsto \|Df(x)\|/\|Df(y)\|$ is uniformly continuous. Hence, we can find $\delta_1 > 0$ so that

$$\operatorname{dist}(x, y) < \delta_1, (x, y) \in M \times M \implies \frac{\|Df(x)\|}{\|Df(y)\|} \le \frac{1}{\lambda_1}.$$

We write $x_j = f^j x$ for $j \ge 0$. We construct the neighborhoods V_{n+j} by induction on j. Note first that

$$y \in V_n \implies \|Df(y)\| \le \lambda_1^{-1} \|Df(x_n)\| \le \lambda_1^{-1}\lambda = \lambda_1.$$

So for every pair $y, z \in V_n$ and a smooth curve $\gamma : [0,1] \to V_n$ connecting $\gamma(0) = z$ to $\gamma(1) = y$ we have

$$\operatorname{dist}(fy, fz) \le |f \circ \gamma| = \int_0^1 \|Df \circ \gamma \cdot \dot\gamma\| \le \lambda_1 \int_0^1 \|\dot\gamma\| = \lambda_1 |\gamma|,$$

where $|\gamma|$ denotes the length of the smooth curve γ. This shows that $f \mid_{V_n}$ is a λ_1-contraction.

Now let us assume that V_{n+i} is already defined for $0 < i \le j < m - n - 1$: V_{n-i} is a set containing x_{n+i} and $f^i \mid_{V_n}: V_n \to V_{n+i}$ is a λ_1^i-contraction. We define $V_{n+j+1} = f^{n+j+1} V_n$ which contains x_{n+j+1} and, since $\operatorname{diam} V_{n+i} \le \lambda_1^i \operatorname{diam} V_n < \delta_1$ for $i = 1, \dots, j$, we can write for each $y_0 \in V_n$

$$\|Df^{j+1}(y_0)\| \le \prod_{i=0}^{j} \|Df(y_i)\| = \prod_{i=0}^{j} \|Df(x_{n+i})\| \frac{\|Df(y_i)\|}{\|Df(x_{n+i})\|} \le \frac{\lambda^{j+1}}{\lambda_1^{j+1}} = \lambda_1^{j+1},$$

so that $f^{j+1} \mid_{V_n}: V_n \to V_{n+j+1}$ is a λ_1^{j+1}-contraction. □

Remark 2.3. If we allow $Df(\bar{x}) \equiv 0$ for some $\bar{x}$, then we might have $\|Df(y_i)\|$ proportionally much larger than $\|Df(x_i)\|$ with both y_i, x_i close to $\bar{x}$ and the larger factors in $\prod_i \|Df(y_i)\|$ may not be compensated.

2.1. Nested contractions argument. Since M is compact and x has infinitely many reverse hyperbolic times $n_1 < n_2 < \dots$ with respect to $m_1 < m_2 < \dots$ so that $(m_i - n_i) \nearrow \infty$, we obtain an accumulation point $\bar{x} = \lim x_{n_{k_j}}$ and we rewrite the subsequence as x_{n_j} in what follows. We let $\xi \in (0,1)$ be such that $4\xi < 1 - \xi - \xi^2$ and assume that $\operatorname{dist}(x_{n_k}, \bar{x}) < \xi\delta_1$ for all $k \geq 1$. Then we choose iterates $n_j > n_2 > n_1$ satisfying $m_j - n_j > n_2 - n_1$, $\lambda_1^{n_2-n_1} < 1/2$ and $\operatorname{dist}(x_{n_j}, \bar{x}) < \xi^2\delta_1$; see Figure 6.

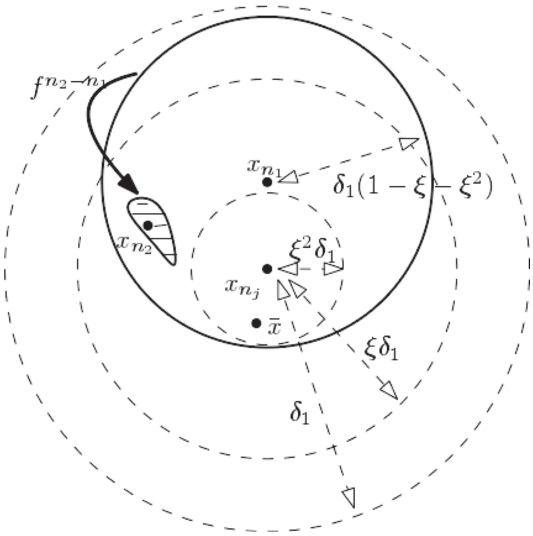

Fig. 6. Relative positions of the iterates of x at reverse hyperbolic times n_1, n_2 and n_j.

Then $\operatorname{dist}(x_{n_1}, x_{n_2}) < 2\xi\delta_1$ and $\operatorname{dist}(x_{n_1}, x_{n_j}) < (\xi + \xi^2)\delta_1$ and also

$$B(x_{n_1}, \delta_1 - (\xi + \xi^2)\delta_1) \subset B(x_{n_j}, \delta_1).$$

Moreover, since $m_j - n_j > n_2 - n_1$ we can write

$$f^{n_2-n_1} B(x_{n_1}, \delta_1(1 - \xi - \xi^2)) \subset B(x_{n_2}, \delta_1(1 - \xi - \xi^2)\lambda_1^{n_2-n_1}).$$

We claim that

$$B(x_{n_2}, \delta_1(1 - \xi - \xi^2)\lambda_1^{n_2-n_1}) \subset B(x_{n_1}, \delta_1(1 - \xi - \xi^2)).$$

Assuming this claim, we have the $\lambda_1^{n_2-n_1}$-contraction

$$f^{n_2-n_1} \mid_{B(x_{n_1}, \delta_1(1-\xi-\xi^2))} : B(x_{n_1}, \delta_1(1 - \xi - \xi^2)) \circlearrowleft$$

and since f is a continuous map, there exists a unique fixed point p for $f^{n_2-n_1}$ in this ball which is in the basin of attraction of p. Since x_{n_1} is in the basin of attraction of p, then x_0 belongs to the basin of attraction of the periodic orbit $p, fp, \dots, f^{n_2-n_1-1}p$.

To complete the proof, we prove the claim. For this it is enough to note that

$$\operatorname{dist}(x_{n_2}, x_{n_1}) + \delta_1(1 - \xi - \xi^2)\lambda_1^{n_2 - n_1} < \delta_1(1 - \xi - \xi^2)$$

if

$$2\xi + (1 - \xi - \xi^2)\lambda_1^{n_2 - n_1} < 1 - \xi - \xi^2$$

which is equivalent to

$$2\xi < (1 - \xi - \xi^2)(1 - \lambda_1^{n_2 - n_1}).$$

This inequality is now a consequence of the choices of ξ and $n_2 - n_1$. □

2.2. Negative Lyapunov exponents for an invariant probability measure. Here we prove Corollary 1.2. Let f be a C^1 map of M and μ an f-invariant probability measure satisfying $\chi(x) < 0, \mu$-a.e. x. The Subadditive Ergodic Theorem guarantees that $\chi(x) = \inf_{n \geq 1} \int \ln \|Df^n\|^{1/n} \, d\mu$ and so there exists $\xi > 0$ and we can find $N > 1$ big enough so that $\int \ln \|Df^N\| \, d\mu < -\zeta$ for $\zeta = \xi N$.

Now we apply the following standard result.

Theorem 2.4 (Ergodic Decomposition Theorem; see e.g. Chapter 2 in [20].)**.** *Let $f : X \to X$ be a measurable (Borelean) invertible transformation on the compact metric space X such that the set of f-invariant probability measures $\mathcal{M}(f, X)$ is non-empty. Then there exists a total probability subset Σ such that*

- *for every $x \in \Sigma$ the weak* limit of $|n|^{-1} \sum_{j=0}^{n-1} \delta_{f^j(x)}$ when $n \to \pm\infty$ exists and equals an f-ergodic probability measure μ_x;*
- *for every $\mu \in \mathcal{M}(f, X)$ and every μ-integrable $\varphi : X \to \mathbf{R}$, φ is μ_x-integrable for μ-almost every x and $\int \varphi \, d\mu = \int \left(\int \varphi \, d\mu_x \right) d\mu(x)$.*

By the ergodic decomposition of the f^N-invariant measure μ, we have $-\zeta > \int \ln \|Df^N\| \, d\mu = \int \int \ln \|Df^N\| \, d\mu_x \, d\mu(x)$ and so the subset $U = \{x \in M : \mu_x(\ln \|Df^N\|) < 0\}$[6] satisfies $\mu U > 0$. Hence, since μ_x is f^N-ergodic, x satisfies

$$A_N^-(x) = \liminf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \ln \|Df^N(f^{Nj}x)\| = \mu_x(\ln \|Df^N\|) < 0.$$

By Theorem A we conclude that μ-a.e. $x \in U$ belongs to the open basin of attraction of some sink. Hence, for μ-a.e. $x \in U$ we have that $x \in \operatorname{supp} \mu$ and there exists an open neighborhood V_x of x so that $\mu V_x > 0$ and V_x is contained in the basin of attraction of some periodic attracting orbit $p = p(x) \in M$ of f^N, which is also a sink for f.

This means that $\mu_y = \delta_{p(x)}$ for μ-a.e. $y \in V_x$. Since X is a compact metric space and sinks are isolated orbits, it follows that $\mu = \tilde{\mu} + \sum_{i \geq 1} \delta_{p_i}$ where $\tilde{\mu}$ is the restriction of μ to $M \setminus U$, which may be null measure.

Finally, if we assume that μ is f-ergodic, then we conclude that $\mu = \delta_p$ for some periodic sink p for f. The proof of Corollary 1.2 is complete.

2.3. Negative Lyapunov exponents on total probability. We now prove Theorem C and Corollary 1.10.

[6] In what follows, we write $\mu(\varphi) = \int \varphi \, d\mu$ for any integrable function $\varphi : M \to \mathbb{R}$.

Proof of Theorem C. We first claim that the assumption on f implies

$$\forall x \in M\ \exists k \in \mathbb{Z}^+ : A_k^-(x) < 0. \tag{1}$$

To prove the claim we argue by contradiction: let us assume that there exists $x \in M$ satisfying $A_k^-(x) \geq 0$ for all $k \in \mathbb{Z}^+$ and let μ be some weak* accumulation point of $\mu_n = n^{-1}\sum_{j=1}^{n-1}\delta_{f^j x}$. Since the assumptions on f give

$$\inf_{k\geq 1}\frac{1}{k}\int \ln\|Df^k\|\,d\mu = \lim_{n\to\infty}\frac{1}{n}\int \ln\|Df^n\|\,d\mu = \int \lim_{n\to\infty}\frac{1}{n}\ln\|Df^n\|\,d\mu < 0,$$

we obtain $\mu(\ln\|Df^k\|) < 0$ for some $k \in \mathbb{Z}^+$. Because $\ln\|Df^k\|$ is continuous, by definition of weak* convergence we get

$$\mu(\log\|Df^k\|) = \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\ln\|Df^k(f^{kj}x)\| < 0$$

in direct contradiction with the choice of x. This contradiction proves the claim (1).

Now Theorem A ensures that all points x belong to the basin of some attracting periodic orbit.

We claim that there exists only one such orbit. Otherwise, let $\mathcal{O}(p_n)$ be the (at most denumerable) collection of attracting periodic orbits of f and let B_n the collection of its basins, that is $B_n = \{x \in M : \omega(x) = \mathcal{O}(p_n)\}$.

The continuity of f guarantees that each B_n is an open subset of M: if $f^m x \in V_n$ for some $m \in \mathbb{Z}^+$, where V_n is a neighborhood of p_n such that $\overline{f^{\tau_n}V_n} \subset V_n$; $f^{\tau_n}p_n = p_n$ and $f^{\tau_n}\mid_{V_n}: V_n \to V_n$ is a contraction; then there exists a neighborhood U_x so that $f^m U_x \subset V_n$ and so $U_x \subset B_n$. Clearly $(B_n)_n$ is a pairwise disjoint collection of subsets.

From (1) and Theorem A we have[7] $M = \sum_n B_n$. Since these are distinct attracting periodic orbits, there exists a pair m, n such that $\overline{B_n} \cap \overline{B_m} \neq \emptyset$. Otherwise, we would have $M = \sum_n \overline{B_n}$ and each B_n becomes simultaneously open and closed, contradicting the connectedness of M.

Let us take $x \in \overline{B_n} \cap \overline{B_m}$. By (1) and the previous continuity argument, a neighborhood V_x of the point x belongs to the basin of some attracting periodic orbit. Since $V_x \cap B_n \neq \emptyset \neq V_x \cap B_m$, we deduce that the attracting periodic orbits $\mathcal{O}(p_n)$ and $\mathcal{O}(p_m)$ must be the same. This contradiction proves the claim and completes the proof of Theorem C. □

Now it is easy to prove Corollaries 1.8 and 1.10.

Proof of Corollary 1.8. Using the notation introduced in the proof of Theorem C, we replace M by the compact f-invariant subset K and obtain $K = \sum_n K_n$, where each $K_n = K \cap B_n$ is relatively open in K. This open cover of K must admit a finite subcover, so the number of sinks is finite. Moreover by f-invariance we get $K_n \subset \cap_{i\geq 0} f^i(B_n) = \mathcal{O}(p_n)$ where $\mathcal{O}(p_n)$ is the periodic attracting orbit whose basin is B_n. So K is a finite collection of sinks, completing the proof. □

Proof of Corollary 1.10. Using again the notation introduced in the proof of Theorem C we have, by Theorem A, that $M = \sum_n B_n, m$ mod 0. Moreover, each B_n is an open subset of

[7] From now on, the sum of sets denotes disjoint union.

M. Hence $\sum_n B_n$ is an open, dense and full Lebesgue measure subset of M. □

2.4. Positive Lyapunov exponents for a C^1 diffeomorphism. We are now ready to prove Theorem B and Corollary 1.4.

Proof of Theorem B. On the one hand, clearly $\tilde{A}_k^-(x) < 0$ if x belongs to a repelling periodic orbit with period $k \in \mathbb{Z}^+$.

On the other hand, if $\tilde{A}_k^-(x) < 0$ for some $k \in \mathbb{Z}^+$, then exchanging f by f^k we assume $k = 1$ without loss of generality. We get $\zeta > 0$ and a strictly increasing sequence $m_i \nearrow \infty$ so that $\frac{1}{m_i}\sum_{j=0}^{m_i-1} \ln \|Df(f^j x)^{-1}\| < -\zeta$ as $i \nearrow \infty$. We apply Lemma 2.1 with $c_2 = -\zeta$, $c_1 = c_2/2$ and $H = -\ln \inf_{x\in M} \|Df(x)^{-1}\|$ and also $a_j = \ln \|Df(f^j x)^{-1}\|$ for $0 \le j < m_i$, to obtain $\ell > \theta N$ with $\theta = c_2/(2H - c_2) > 0$ and $1 < n_1 < \cdots < n_\ell \le N$ so that $\|Df^{n_k-n}(f^{n+1}x)^{-1}\| \le \prod_{j=n+1}^{n_k} \|Df(f^j x)^{-1}\| \le e^{-c_1(n_k-n)}$, for each $0 \le n < n_k$ and $k = 1, \ldots, \ell$. Each n_i is a hyperbolic time for x and we can prove the following with $\lambda = e^{-c_1} = e^{-\zeta/2}$.

Lemma 2.5 (Existence of backward contracting balls)**.** *There exists $\delta_1 > 0$ (depending only on f and λ) such that if n is a hyperbolic time for x, then for every $0 < j \le m - n$ there are neighborhoods V_j of $f^j x$ in M for which*

(1) *f^{n-j} maps V_j diffeomorphically onto the ball of radius δ_1 around $f^n x$;*
(2) *for $1 \le j < n$ and $y, z \in V_0$,* $\operatorname{dist}(f^{n-j}(y), f^{n-j}(z)) \le \lambda^{j/2} \operatorname{dist}(f^n(y), f^n(z))$.

Proof. Just follow [2, Lemma 5.2] and notice that it is enough to have

$$\operatorname{dist}(x, y) < \delta_1, (x, y) \in M \times M \implies \frac{\|Df(x)^{-1}\|}{\|Df(y)^{-1}\|} \le \frac{1}{\sqrt{\lambda}}$$

for this proof to go through. □

Hence we can repeat the nested contracting argument from Subsection 2.1 in this setting obtaining a periodic point p for some power f^k of f such that $p \in B(f^n x, \delta_1)$ for some hyperbolic time n of x. Thus, by Lemma 2.5 and since f is invertible, we get $\operatorname{dist}(x, f^{-n}p) \le \delta_1 \lambda^{n/2}$ and we can take n larger than any predetermined quantity. We conclude that x belongs to the f-orbit of p, concluding the proof of Theorem B. □

Proof of Corollary 1.4. Consider the measurable subsets $E = \{x \in M : \chi(x) < 0\}$ and $\tilde{E} = \{x \in M : \tilde{\chi}(x) < 0\}$ and note that $E + \tilde{E} + M \setminus (E + \tilde{E})$ is a measurable partition of M formed by f-invariant subsets. Moreover, if $A_k^-(x) < 0$ (respectively, $\tilde{A}_k^-(x) < 0$) for some $k \in \mathbb{Z}^+$, then $x \in E$ (resp., $x \in \tilde{E}$). In addition, if $\mu(E) > 0$, then by Ergodic Decomposition[8] $\mu(1_E \cdot \chi) = \int_E \int \chi \, d\mu_x \, d\mu(x)$ and so $\inf_{n\ge1} \int_E \ln \|Df^n\|^{1/n} \, d\mu_x = \mu_x(1_E \cdot \chi) < 0$ for μ-a.e. $x \in E$ by the Subadditive Ergodic Theorem applied to the invariant subset E. The Ergodic Theorem now gives $\mu_x(1_E \cdot A_k^-) \le \mu_x(1_E \cdot \chi) < 0$ for some $k = k(x) \in \mathbb{Z}^+$ (respectively, $\mu_x(1_{\tilde{E}} \cdot \tilde{A}_{\tilde{k}}^-) \le \mu_x(1_{\tilde{E}} \cdot \tilde{\chi}) < 0$ for some $\tilde{k} = \tilde{k}(x) \in \mathbb{Z}^+$ if $\mu(\tilde{E}) > 0$).

From Theorem A we deduce that μ-a.e. $x \in E$ belongs to the basin of a sink $\mathcal{O}_f(p) = \{p, fp, \ldots, f^{\tau-1}p\}$ for some period $\tau \in \mathbb{Z}^+$, and since μ_x is ergodic and f an invertible map, we get $\mu_x = \tau^{-1}\sum_{j=0}^{\tau-1} \delta_{f^j p}$ (resp., by Theorem B μ-a.e. $x \in \tilde{E}$ belongs to some periodic repelling orbit $\mathcal{O}_f(q) = \{q, fq, \ldots, f^{\tau-1}q\}$ and so $\mu_x = \tau^{-1}\sum_{j=0}^{\tau-1} \delta_{f^j q}$).

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[8]We write 1_A for the indicatior function: $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \in M \setminus A$ for $A \subset M$.

Finally $\chi(x) \geq 0$ and $\tilde{\chi}(x) \geq 0$ for $x \in M \setminus (E + \tilde{E})$ by construction. Hence, since attracting and repelling periodic orbits are isolated in M, they form an at most enumerable subset and so we decompose μ as in the statement of Corollary 1.4. $\square$

3. The flow case

We now prove Theorems D, E and F. We fix $G \in \mathfrak{X}^1(M)$ and state a version of Pliss' Lemma 2.1 for flows in the next subsection, and then translate the assumptions of Theorems D, E and F in a convenient format in Subsection 3.2 to be used in the following subsections.

3.1. Pliss lemma for flows. Following Arroyo-Hertz [5, Theorem 3.5] we state and prove for completeness the following version of Pliss' Lemma for differentiable functions instead of sequences (whose statement and proof can be found in [27] and [17]).

Theorem 3.1. *Given $\varepsilon > 0, A, c \in \mathbb{R}, c > A$, if $H : [0,T] \to \mathbb{R}$ is differentiable, $H(0) = 0, H(T) < cT$ and $c + \varepsilon > \inf(H') > A$, then the set*

$$\mathcal{H}_\varepsilon = \{\tau \in [0,T] : H(s) - H(\tau) < (c+\varepsilon)(s-\tau), \forall \tau \leq s \leq T\}$$

has Lebesgue measure greater than θT, where $\theta = \varepsilon/(c + \varepsilon - A)$.

Remark 3.2. (1) This result ensures that there exists $\tau \in \mathcal{H}_\varepsilon$ such that $T - \tau > \theta T$.
(2) For given fixed $0 < \eta < \bar{\varepsilon}$, since $H(\tau+\eta) - H(\tau) = \int_\tau^{\tau+\eta} H' \geq A\eta$, we can write

$$\begin{aligned} H(s) - H(\tau+\eta) &= (c+\varepsilon)(s-\tau) - (H(\tau+\eta) - H(\tau)) \\ &\leq (c+\varepsilon)(s-(\tau+\eta)) + (c+\varepsilon)\eta - A\eta \\ &= \left(c + \varepsilon + \eta\frac{c+\varepsilon-A}{s-(\tau+\eta)}\right)(s-(\tau+\eta)) \\ &< (c+\hat{\varepsilon})(s-(\tau+\eta)) \end{aligned}$$

for all $s > \tau + \eta$, where $\hat{\varepsilon} > \varepsilon$ with $\hat{\varepsilon} - \varepsilon$ as small as needed, if $\bar{\varepsilon}$ is small enough.
So we have $\tau + \eta \in \mathcal{H}_{\hat{\varepsilon}}$ for small $\eta, \hat{\varepsilon} - \varepsilon > 0$ whenever $\tau \in \mathcal{H}_\varepsilon$.

We postpone the proof of Theorem 3.1 to Section 3.9 and use it as a tool in what follows.

3.2. Linear Poincaré Flow and differentiability. We start the proof of Theorems E and F by expressing the assumptions in their statements in a form suitable to apply the previous Theorem 3.1.

We fix $G \in \mathfrak{X}^1(M)$ and let P_G be the Linear Poincaré Flow of G.

The cocycle relation for the derivative of the flow and for the Linear Poincaré Flow implies that the functions

$$\Gamma(v) = \Gamma_G(v) : \mathbb{R} \times M \to \mathbb{R}, \quad (t,x) \mapsto \Gamma_t x = \ln \|D\phi_t(x) \cdot v\|, \quad v \in T_x^1 M \quad \text{and}$$
$$\psi(v) = \psi_P(v) : \mathbb{R} \times M \setminus \mathrm{Sing}(G) \to \mathbb{R}, (t,x) \mapsto \phi_t x = \ln \|P_x^t \cdot v\|, \quad v \in T_x^1 M, \langle v, G(x)\rangle = 0,$$

are *subadditive*, where T^1M is the unit tangent bundle: for $t, s \in \mathbb{R}$

$$\Gamma_{t+s}(v)y \leq \Gamma_s(v)(\phi_t y) + \Gamma_t(v)y, \quad y \in M, \quad \text{and}$$
$$\psi_{t+s}(v)x \leq \psi_s(v)(\phi_t x) + \psi_t(v)x, \quad x \in M \setminus \mathrm{Sing}(G).$$

The following result provides a sufficient condition to ensure the existence of the time derivative of a subadditive cocycle over a C^1 vector field.

Lemma 3.3. *Let $\psi : \mathbb{R} \times U \to \mathbb{R}$ be a subadditive function for the flow of $G \in \mathfrak{X}^1(M)$ on the invariant subset U of M. If $\psi_0 x = 0$ for all $x \in U$, $D_+(x) := \limsup_{h\to 0} \frac{1}{h}\psi_h x < \infty$ and $D_-(x) := \liminf_{h\to 0} \frac{1}{h}\psi_h x < \infty$ are continuous functions of $x \in U$, then $\partial_h \psi_h x \mid_{h=0\pm} = D_\pm(x) = \lim_{h\to 0\pm} \frac{1}{h}\psi_h x$ and the derivative exists if $D_-(x) = D_+(x)$.*

Proof. See [4, Lemma 4.12] and its proof, where it is implicitly assumed that $D_-(x) = D_+(x)$ but the existence of lateral limits and derivatives is addressed. □

Now we take advantage of the fact that both $\Gamma_G(x) = \sup_{v\in T_x^1 M} \Gamma_G(v)(x)$ and $\psi_P(x) = \sup_{v\in T_x^1 M} \psi_P(v)(x)$ are continuously generated subadditive cocycles over a C^1 vector field: the following results shows that they are bounded by additive cocycles and provides useful continuity properties of their infinitesimal generators.

Lemma 3.4. *Define $D_{G\pm}(x) := \pm \limsup_{h\to 0\pm} \frac{\pm 1}{h}\Gamma_h x$ and $D_\pm(x) := \pm \limsup_{h\to 0\pm} \frac{\pm 1}{h}\psi_h x$ and set $L = \sup_{x\in M} \|DG_x\|$. Then*

(1) *$|D_{G\pm}(y)| \le L$ for all $y \in M$ and $|D_\pm(x)| \le L$ for all $x \in M \setminus \mathrm{Sing}(G)$;*

(2) *$y \in M \mapsto D_{G\pm}(y)$ and $x \in M \setminus \mathrm{Sing}(G) \mapsto D_\pm(x)$ are continuous functions: in local coordinates[9] we have*

$$|D_{G\pm}(y) - D_{G\pm}(y')| \le \dim M \cdot \|DG_y - DG_{y'}\| \quad \text{and}$$
$$|D_\pm(x) - D_\pm(y)| \le \dim M \cdot \|\mathcal{O}_x DG_x - \mathcal{O}_y DG_y\|;$$

(3) *in addition*

$$\int_0^t D_{G-}(\phi_s x)\,ds \le \Gamma_t(x) \le \int_0^t D_{G+}(\phi_s x)\,ds \quad \text{and} \tag{2}$$
$$\int_0^t D_-(\phi_s x)\,ds \le \psi_t(x) \le \int_0^t D_+(\phi_s x)\,ds;$$

and both $t \mapsto \Gamma_t y$ and $t \mapsto \psi_t x$ are bounded above and below by additive functions for any fixed $y \in M$ and $x \in M \setminus \mathrm{Sing}(G)$.

Now we translate the assumptions of Theorems D, F and E using these infinitesimal generators. We define the maps

$$\widehat{\Phi_t} : T^1 M \cap G^\perp \to T^1 M \cap G^\perp, v \mapsto \frac{P^t v}{\|P^t v\|} \quad \text{and} \quad \Phi_t : T^1 M \to T^1 M, v \mapsto \frac{D\phi_t v}{\|D\phi_t v\|}.$$

Observe that the lateral limits exist and are equal for each $\psi_t(v)$ and $\Gamma_t(v)$ since

$$\ln \|P_x^{s+t} \cdot v\| = \ln \|P^t_{\phi_s x} \cdot (P^s_x \cdot v)\| = \ln \frac{\|P^t_{\phi_s x} \cdot (P^s_x \cdot v)\|}{\|P^s_x \cdot v\|} + \ln \|P^s_x \cdot v\|;$$

and so we can write $\Gamma_{s+t}(v) = \Gamma_t(\widehat{\Phi_s} v) + \Gamma_s(v)$. We similarly obtain additivity for ψ_t with respect to Φ_t.

[9]More precisely $|D_\pm(x) - D\pm(y)| \le \dim M \cdot \|\mathcal{O}_x DG_x - D(\exp_x)^{-1}_{\exp_x^{-1} y} \circ \mathcal{O}_y DG_y \circ D(\exp_x)_{\exp_x^{-1} y}\|$ for y in the range of $\exp_x$.

We can now define the functions $H_G(t,v) = \int_0^t D_G(\Phi_s v)\,ds, v \in T^1M$ and $H(t,v) = \int_0^t D(\widehat{\Phi_s v})\,ds, v \in T^1M \cap G^\perp$, where $G^\perp$ is the normal bundle to G on $M \setminus \mathrm{Sing}(G)$ and $D_G(v), D(w)$ are the infinitesimal generators of $\Gamma_G(v)$ and $\psi(w)$ respectively, for $v \in T^1_xM, w \in T^1_xM \cap G^\perp$. We also get equalities in (2) with these generators.

Lemma 3.5. *The assumption of Theorem D implies*

$$\liminf_{T\to\infty} \frac{1}{T} \sup_{v\in T^1_xM} H_G(T,v) < 0; \tag{3}$$

and the assumptions of Theorems F and E respectively imply

$$\liminf_{T\to\infty} \frac{1}{T} \sup_{v\in T^1_xM\cap G^\perp} H(T,v) < 0 \quad \text{and} \quad \limsup_{T\to\infty} \frac{1}{T} \sup_{v\in T^1_xM\cap G^\perp} H(T,v) < 0. \tag{4}$$

We postpone the proofs of these technical lemmas to Section 3.9.

3.3. Asymptotic contraction in all directions. We are ready to start the proof of Theorem D. If[10] $x \in M \setminus \mathrm{Sing}(G)$ satisfies (3), then there exists $\zeta > 0$ and $T_n \nearrow \infty$ so that $\sup_{v\in T^1_xM} H_G(T_n,v) \le -\zeta T_n$. In addition, we observe that

$$-\zeta T_n > \sup_{v\in T^1_xM} \int_0^{T_n} D_G(\Phi_s v)\,ds \ge T_n \cdot \sup_{v\in T^1_xM} \inf_{0\le s\le T_n} D_G(\Phi_s v)$$

and so $A = \sup_{v\in T^1_xM} \inf_{0\le t\le T_n} D_G(\Phi_s v) < -\zeta$ as required to apply Theorem 3.1 to $H_G(\cdot,v)$ with $c = -\zeta, \varepsilon > 0$ and fixed $v \in T^1_xM$, as long as A is a real number, which is guaranteed by Lemma 3.4. So we apply Theorem 3.1 with $\varepsilon = \zeta/4$, $v \in T^1_xM$ to obtain times $0 < \tau = \tau(v) < T_n$ so that $(T_n - \tau) \ge \theta T_n$ with $\theta \in (0,1)$ satisfying for $s \in [\tau, T_n]$

$$\ln \|D\phi_{s-\tau}(\phi_\tau x)\Phi_\tau v\| = \int_\tau^s D_G(\Phi_u v)\,du \le -\frac{\zeta}{2}(s-\tau). \tag{5}$$

We can replace v by any $u \in V \cap T^1_xM$ for some open neighborhood V of v in the unit sphere at T_xM. By compactness of the unit sphere, we obtain a finite cover $V_1,\dots,V_k$ of T^1_xM with associated times $\tau_1,\dots,\tau_k$ so that $T_n - \tau_i \ge \theta T_n$ and $u \in V_i$ satisfy (5) in the place of v for $T_n - \tau_i \le s \le T_n$, for each $i = 1,\dots,k$. Hence for $\tau_n = \min_{i=1,\dots,k} \tau_i$ we obtain (5) for all $v \in T^1_xM$ and $T_n - \tau_n \le s \le T_n$, and also $T_n - \tau_n \ge \theta T_n$.

In particular, since Φ_t is a bijection from T^1_xM to $T^1_{\phi_t x}M$, we have that 0 is a reverse hyperbolic time of $f = \phi_1$ for the point $x(\tau_n) = \phi_{\tau_n} x$ with respect to[11] $[T_n - \tau_n]$, i.e. $\|Df^k(x(\tau_n))\| \le \prod_{j=0}^{k-1} \|Df(f^j x(\tau_n))\| \le e^{-\zeta k/2}, 1 \le k \le [T_n - \tau_n]$.

Applying Lemma 2.2 we obtain δ_1, λ_1 and $W_n = B(x(\tau_n), \delta_1)$ such that $f^j\mid_{W_n}\colon W_n \to f^j W_n = W_{n+j}$ is a λ_1^j-contraction for $0 < j \le [T_n - \tau_n]$.

We are in the exact same setting of Subsection 2.1. Hence, we can find $m \in \mathbb{Z}^+$ so that f^m has a contracting fixed point p whose basin contains $x(\tau_n)$ for some large $n \in \mathbb{Z}^+$. Thus $p = \sigma \in \mathrm{Sing}(G)$ is a hyperbolic attracting singularity (a sink) for the vector field G and $\phi_t x \xrightarrow[t\to\infty]{} \sigma$. This completes the proof of the first part of the statement of Theorem D.

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[10] There is no loss in generality to assume that $x \in M \setminus \mathrm{Sing}(G)$, for otherwise there is nothing to prove.

[11] Here $[t] = \sup\{n \in \mathbb{Z}^+ : n \le t\}$ is the integer part of $t \in \mathbb{R}$.

3.4. Asymptotic sectional contraction along a trajectory. If the trajectory of $x \in M \setminus \mathrm{Sing}(G)$ satisfies the left hand side of (4), then there exists $c = -\zeta$ and $T_n \nearrow \infty$ so that $\sup_{v \in T_x^1 M \cap G^\perp} H(T_n, v) \le cT_n$ and moreover

$$cT_n > \int_0^{T_n} D(\widehat{\Phi_s v})\, ds \ge T_n \cdot \sup_{v \in T_x^1 M \cap G^\perp} \inf_{0 \le s \le T_n} D(\widehat{\Phi_s v}) = T_n \cdot A$$

and so $A < c$. We apply Theorem 3.1 to $H(\cdot, v)$ with $\varepsilon = -c/2$ and fixed $v \in T_x^1 M \cap G^\perp$ to obtain, reasoning as in the previous subsection, times $0 < \tau_n < T_n$ so that $(T_n - \tau_n) \nearrow \infty$ satisfying for $s \in [\tau_n, T_n]$

$$\ln \|P^{s-\tau_n}_{\phi_{\tau_n} x}\| = \sup_{v \in T_x^1 M \cap G^\perp} \int_{\tau_n}^{s} D(\widehat{\Phi_u v})\, du \le -\frac{\zeta}{2}(s - \tau_n). \tag{6}$$

We say that τ_n is an $e^{-\zeta/2}$-reverse hyperbolic time with respect to T_n.

We divide the proof of Theorem E in two main cases presented in the following Subsection 3.5, for trajectories not accumulating any equilibrium; and Subsection 3.6 for trajectories which accumulate some equilibrium.

Afterwards, we complete the proof of Theorem F in Subsection 3.7.

REMARK 3.6. For $\tau_n < T_n$ as in (6), any $t > \eta > 0$ and assuming without loss of generality that $\zeta < 4L$, we get $\int_{\tau_n+\eta}^{t} D(\widehat{\Phi_s v})\, ds = \int_{\tau_n}^{t} D(\widehat{\Phi_s v})\, ds - \int_{\tau_n}^{\tau_n+\eta} D(\widehat{\Phi_s v})\, ds \le -\zeta t/2 + L\eta$ for each $v \in T_x^1 M \cap G^\perp$ (recall that $|D| \le L$) which is bounded by $(-\zeta/2 + L\eta/t)t < -\zeta t/4 < -\zeta(t-\eta)/4$ since $L\eta/t < \zeta/4 \iff \eta < t\zeta/4L$ and $\eta < \eta\zeta/4L < t\zeta/4L$. This shows that if τ_n is an $e^{-\zeta/2}$-reverse hyperbolic time w.r.t. T_n, then $\ln \|P^t_{\phi_{\tau_n+\eta}}\| = \sup \int_{\tau_n+\eta}^{t} D(\widehat{\Phi_s v})\, ds < -\zeta(t-\eta)/4$ and so any $s \in (\tau_n, T_n)$ becomes a $e^{-\zeta/4}$-reverse hyperbolic time w.r.t. T_n.

3.5. Trajectory away from equilibria. First, we assume that $\omega_G(x) \cap \mathrm{Sing}(G) = \emptyset$ and that $\mathrm{Sing}(G)$ is a finite subset, so that there exists $d_0 > 0$ such that $\mathrm{dist}(\phi_t x, \mathrm{Sing}(G)) \ge d_0, \forall t \ge 0$ and also $\mathrm{dist}(\omega_G(x), \mathrm{Sing}(G)) \ge d_0$.

We show that (6) implies that the flow contracts distances uniformly in the transverse direction to the vector field along longer and longer orbit segments of the positive orbit of x. Compacteness of M then guarantees, by an argument similar to the one presented in Section 2, the existence of a Poincaré section of the flow, together with a neighborhood of a hitting point of the orbit of x, which is sent inside itself by some Poincaré return map. This provides a sink for that Poincaré return map which, as is well-known, gives a periodic attracting orbit for the flow containing x in its basin.

3.5.1. **Forward sectional contracting balls.** The uniform bound on the distance away from equilibria ensures that there exists $0 < \rho \le d_0$ such that for each $y \in \mathcal{O}_G^+(x) \cup \omega_G(x)$[12] we can construct a Poincaré cross-section of G through y as

$$S_y = \exp_y(B(0,\rho) \cap G(y)^\perp), \tag{7}$$

where $\exp_y : T_y M \to M$ is the standard exponential map induced by the Riemannian structure on M; $B(0,\rho)$ is the ρ-neighborhood of the origin in the tangent space $T_y M$ with the distance induced by $\|\cdot\|_y = \langle \cdot, \cdot \rangle^{1/2}$; and $G(y)^\perp$ is the subspace of $T_y M$ orthogonal to $G(y)$.

[12] We write $\mathcal{O}_G^+(x) = \{\phi_t x : t \ge 0\}$ and note that both $\omega_G(x)$ and $\mathrm{Sing}(G)$ are compact.

We also write $S_y(\xi) = \exp_y(B(0, \xi\rho) \cap G(y)^\perp)$ for $\xi \in (0, 1]$.

By uniform continuity of $\eta : (M \setminus B(\mathrm{Sing}(G), d_0))^2 \to \mathbb{R}, (z, w) \mapsto \|P^1_z\|/\|P^1_w\|$ together with the subadditivity of ψ, we can find $\xi_0 > 0$ so that

$$\phi_{[-2,2]} S_y(\xi_0) \cap \mathrm{Sing}(G) = \emptyset \quad \text{and} \quad z, w \in S_y(\xi_0), 0 \le t \le 1 \implies \frac{\|P^t_z\|}{\|P^t_w\|} \le e^{\zeta/4} = \lambda_1^{-1}. \tag{8}$$

Similarly to Section 2, we write $x(t) = \phi_t x$ for $t \in \mathbb{R}$ in what follows.

Proposition 3.7 (Existence of forward sectional contracting balls)**.** *Let $\tau_n < T_n$ be the pair of strictly increasing sequences obtained before satisfying* (6)*. For every $\delta_0 > 0$ there exists $\xi_0 > 0$ satisfying* (8) *such that, if* $\mathrm{dist}(\phi_t x, \mathrm{Sing}(G)) \ge d_0, \forall t \in [\tau_n, T_n]$*, then for each $s \in (\tau_n, T_n]$ there exists a C^1 smooth well-defined diffeomorphism with its image $R_s : S_{x(\tau_n)}(\xi_0) \to S_{x(s)}(\xi_0)$ such that R_s is a Poincaré map, $R_s(x(\tau_n)) = x(s)$ and R_s is an $e^{-\frac{\zeta}{4}(s-\tau_n)}$-contraction.*

This result is the analogous to Lemma 2.2 in the flow setting with $\xi_0\rho$ playing the role of δ_1; see Figure 7.

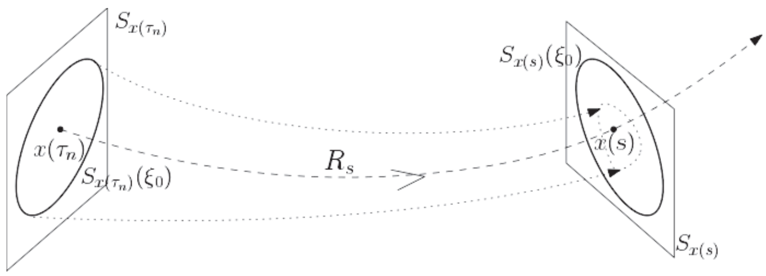

Fig. 7. Sketch of a sectional contracting ball.

Proof of Proposition 3.7. Using $\mathrm{dist}(\phi_{[\tau_n,T_n]}(x), \mathrm{Sing}(G)) \ge d_0 > 0$ we can find ξ_0 satisfying (8) as explained before the statement of the Proposition.

We note that, by construction, $T_{\tau_n} S_{x(\tau_n)}(\xi_0) = G(x(\tau_n))^\perp$ so, from the choice of ξ_0 and for $s = \tau_n + 1$ there exists a well-defined Poincaré map $R_s : W_s \to S_{x(s)}$ from a neighborhood W_s of $x(\tau_n)$ in $S_{x(\tau_n)}(\xi_0)$. We also have

$$\begin{aligned} DR_s(x(\tau_n)) &= \mathcal{O}_{T_{x(s)} S_{x(s)}} \circ D\phi_{s-\tau_n}(x(\tau_n)) \mid_{T_{x(\tau_n)} S_{x(\tau_n)}(\xi_0)} \\ &= \mathcal{O}_{G(x(s))^\perp} \circ D\phi_{s-\tau_n}(x(\tau_n)) \mid_{G(x(\tau_n))^\perp} = P^1_{\phi_{\tau_n} x}. \end{aligned}$$

This together with (8) and Remark 3.6 ensures that $\|DR_s(z)\| < e^{-\frac{\zeta}{4}(s-\tau_n)} = \lambda_1$ for all $z \in W_s$. Then the map R_s is a diffeomorphism with its image which contracts distances at a rate λ_1.

We claim that we may take $W_s = S_{x(s)}(\xi_0)$. To prove this, we fix a direction $v \in G(x(\tau_n))^\perp$ with $\|v\| = 1$ and consider the set

$$\begin{aligned} E = \{t \in (0, \xi_0) : R_s(\exp_{x(\tau_n)}(\xi v)) \in S_{x(s)}(\lambda_1) \quad &\text{and} \\ \|DR_s(\exp_{x(\tau_n)}(\xi v))\| < \lambda_1, \forall 0 \le \xi \le t\}&. \end{aligned}$$

Clearly $E \subset (0, \xi_0)$ and we have already shown that $\sup E > 0$. We note that the claim follows if we prove that $\sup E = \xi_0$. Indeed, since the unit vector v was arbitrarily chosen in $G(x(\tau_n))^\perp$, then $\sup E = \xi_0$ implies that the Poincaré map R_s is well-defined and a λ_1-

contraction on the whole of $S_{x(\tau_n)}(\xi_0)$.

To prove that $\sup E = \xi_0$ we argue by contradiction: let us assume that $0 < \alpha = \sup E < \xi_0$. Then the curve $\gamma(t) = \exp_{x(\tau_n)}(tv), t \in [0, \alpha]$ is sent to a curve $R_s(\gamma) \subset S_{x(s)}(\xi_0)$ with length[13]

$$|R_s \circ \gamma| = \int_0^\alpha \|DR_s \circ \gamma \cdot \dot\gamma(t)\| \, dt \le \lambda_1 \int_0^\alpha \|\dot\gamma(t)\| \, dt = \lambda_1 \alpha < \lambda_1 \xi_0.$$

Hence, on the one hand, we have for each $0 < t < \alpha$

$$\operatorname{dist}_S(R_s(\gamma(t)), x(s)) \le |R_s \circ \gamma| \le \lambda_1 \xi_0 < \xi_0,$$

where dist_S is the induced distance on $S_{x(s)}(\xi_0)$ by the Riemannian distance of M. But $\phi_{[-2,2]} S_{x(\tau_n)}(\xi_0)$ is a flow box, thus $\phi_{[-2,2]} x(\gamma(\alpha)) \cap S_{x(s)}(\xi_0) = \lim_{t \to \alpha} R_s(\gamma(t))$ and we can extend R_s from $\gamma([0, \alpha))$ to $\gamma(\alpha)$. Then $\operatorname{dist}_S(R_s(\gamma(\alpha)), x(s)) \le \lambda_1 \xi_0 < \xi_0$ and this enables us to use the flow box again to extend R_s to a neighborhood of $\gamma(\alpha)$ in $S_{x(\tau_n)}(\xi_0)$. This shows that there is $t \in E$ with $t > \alpha$ and this contradiction completes the proof of claim that $\sup E = \xi_0$.

We observe that the argument above is valid for any $s \in (\tau_n, \tau_n + 1]$ replacing the rate λ_1 by $\lambda_1^{s-\tau_n}$.

Let now $s \in (\tau_n, T_n]$ be given and let us write $s - \tau_n = k + \xi$ with $k \in \mathbb{Z}^+$ and $\xi \in [0, 1)$. Then we divide the interval $[\tau_n, s]$ into $\{[\tau_n + i, \tau_n + i + 1)\}_{i=0,\dots,k-1}$ together with $[\tau_n + k, s]$ and consider the Poincaré maps $R_i : S_{x(\tau_n+i)}(\xi_0) \to S_{x(\tau_n+i+1)}(\xi_0)$ for $i = 0, \dots, k-1$ and $R_k : S_{x(\tau_n+k)}(\xi_0) \to S_{x(T_n)}(\xi_0)$.

Finally, since the image of R_i is contained in the $\lambda_1\rho$-ball around $x(\tau_n + i + 1)$ in $S_{x(\tau_n+i+1)}(\xi_0)$ for $i = 0, \dots, k-1$ and the image of R_k is inside the λ_1^ξ-neighborhood of $x(s)$ in $S_{x(s)}(\xi_0)$, then the composition $R_k \circ R_{k-1} \circ \cdots \circ R_0$ is well-defined and a $\lambda_1^{k+\xi}$-contraction from $S_{x(\tau_n)}(\xi_0)$ to $S_{x(s)}(\xi_0)$. Since $\lambda_1^{k+\xi} = e^{-\frac{\zeta}{4}(s-\tau_n)}$, the proof of the proposition is complete. $\square$

3.5.2. **Infinitely many contracting flow boxes with arbitrary long size and uniform domains.** We follow the same strategy as in Subsection 2.1 replacing δ_1-balls by $\xi_0\rho$-neighborhoods on local cross-sections: by compactness we fix a subsequence of τ_n, which we denote by the same letters, such that $x(\tau_n) \xrightarrow[n\to\infty]{} \bar x \in \omega(x)$ and $(T_n - \tau_n) \nearrow \infty$ together with $S_{\bar x} = S_{\bar x}(1)$ and (using Proposition 3.7) a collection of Poincaré maps with domains in a neighborhood $S_{\bar x}(\xi_0)$ of $\bar x$ in $S_{\bar x}$, as follows.

We start by fixing the local Poincaré map[14] $\Phi : \phi_{[-2,2]} S_{\bar x}(\xi_0) \to S_{\bar x}(\xi_0)$ and $\xi \in (0, 1)$ such that $4\xi < 1 - \xi - \xi^2$. We assume without loss of generality that $\Phi x(\tau_k) \subset S_{\bar x}(\xi\xi_0/2)$ for all $k \ge 1$ and that $\Phi x(\tau_k) = x(\tau_k + \eta_k)$ for some $0 < \eta_k < \bar\varepsilon$ and $\bar\varepsilon > 0$ small, since $\omega(x)$ is an invariant set under the action of the flow. Then we choose j so that $T_j - \tau_j > \tau_2 - \tau_1$, $\lambda_1^{\tau_2-\tau_1} < 1/2$ and $\operatorname{dist}(\Phi x(\tau_j), \bar x) < \xi^2 \xi_0 \rho/2$; see Figure 6 again setting $x_{n_i} = \Phi x(\tau_i + \eta_i)$, $i = 1, 2, j$.

Note that, from Remark 3.2, the times $\tau_i + \eta_i$ also satisfy the conclusion of Proposition 3.7, since we may take $\bar\varepsilon > 0$ as small as needed.

Now we just repeat the arguments in Subsection 2.1 with $\delta_1 = \xi_0\rho/2$ and $f^{n_2-n_1} = \Phi \circ \phi_{\tau_2+\eta_2-(\tau_1-\eta_1)}$ to obtain an attracting fixed point p for this last map whose basin in $S_{\bar x}(\xi_0)$

[13]By construction γ is a curve with unit speed.

[14]That is, $\Phi(\phi_t x) = x$ for all $x \in S(\xi_0)$ and $-2 \le t \le 2$.

contains x_{n_1}. Then the orbit $\mathcal{O}_G(p)$ is periodic and since p is a sink for $f^{n_2-n_1}$, we conclude that $\mathcal{O}_G(p)$ is a periodic (hyperbolic) sink for G and x belongs to its basin of attraction.

This completes the proof of Theorem E in this case.

3.6. Trajectory accumulating some equilibrium. Alternatively, we assume that the orbit of x accumulates some singularity, that is $\sigma \in \omega(x) \cap \mathrm{Sing}(G)$ and, from now on, we assume that each element of $\mathrm{Sing}(G)$ is hyperbolic. Then

- either $\omega(x) = \{\sigma\}$ and so $x(t) \to \sigma$ when $t \to +\infty$ and
  - if σ is a sink, then x belongs to its basin and we have nothing to prove;
  - if σ is a source, then $\phi_t x \in U$ for some small neighborhood U of σ and arbitrarily large values of $t > 0$. Hence $x \in \phi_{-t}U$ and $\operatorname{diam}(\phi_{-t}U) \to 0$ when $t \nearrow \infty$ so $x = \sigma$, a contradiction. So we are left with
  - σ is a hyperbolic saddle and x belongs to its stable manifold.
- or $\omega(x) \supsetneq \{\sigma\}$ and then σ is again a hyperbolic saddle equilibrium, since
  - if σ is a sink, then because $x(t) \in W^s_G(\sigma)$ for some $t > 0$, we conclude that $x(t) \to \sigma$ when $t \to +\infty$ and so $\omega(x) = \{\sigma\}$, a contradiction; otherwise
  - σ is a source, and then $x = \sigma$ which is a contradiction again.

Next we argue that such accumulation can only happen if σ is a codimension 1 saddle, completing the proof of Theorem E.

3.6.1. **Trajectory in the stable manifold of some equilibrium.** In case $x(t) \to \sigma$ when $t \nearrow \infty$, then $x \in W^s(\sigma)$ and we prove the following for later use.

Lemma 3.8. *Let $\sigma \in \mathrm{Sing}(G)$ be a hyperbolic equilibrium and $q \in W^s(\sigma) \setminus \{\sigma\}$ such that $\liminf_{t\to\infty} \ln \|P^t_q\|^{1/t} < 0$. Then σ is a sink.*

Applying the lemma shows that $\omega(x) = \{\sigma\}$ can only happen if σ is a sink.

We need the following consequence of Gronwall's Inequality in several arguments in what follows, so we state here for later use and present a proof in Section 3.9.

Lemma 3.9. *Let $q_n \in M$, $\sigma \in \mathrm{Sing}(G)$ and $t > 0$ be such that*[15] $\operatorname{dist}(\phi_{[0,t]}q_n, \sigma) \xrightarrow[n\to\infty]{} 0$. *Then $\|P^t_{q_n} - \mathcal{O}_{\phi_t q_n} e^{tDG_\sigma}\| \le \|D\phi_t(q_n) - e^{tDG_\sigma}\| \le \bar\delta_n t e^{Lt}$ where*

$$\bar\delta_n = \sup_{0\le s\le t} \|\mathcal{O}_{\phi_s q_n} DG_{\phi_s q_n} - \mathcal{O}_{\phi_s q_n} DG_\sigma\| \le \sup_{0\le s\le t} \|DG_{\phi_s q_n} - DG_\sigma\| \xrightarrow[n\to\infty]{} 0.$$

Now we can present the proof of the previous lemma.

Proof of Lemma 3.8. Since $\liminf_{t\to\infty} \sup_{v\in T^1_q M\cap G^\perp} H(t,v)/t < 0$ for $H(t,v) = \int_0^t D(\widehat{\Phi_s v})\, ds$, we can find $\zeta > 0$ and reverse hyperbolic times τ_n associated to $T_n \nearrow \infty$ so that $T_n - \tau_n > \theta T_n$ as in (6).

We take $\tau_n > 0$ large enough so that $\operatorname{dist}(\phi_{[\tau_n,T_n]}q, \sigma) \xrightarrow[n\to\infty]{} 0$ and for any given fixed $0 < t < T_n - \tau_n$ we apply Lemma 3.9 with $q_n = q(\tau_n) = \phi_{\tau_n} q$ to get $\|P^t_{q(\tau_n)} - \mathcal{O}_{x(\tau_n+t)} e^{tDG_\sigma}\| \le \bar\delta_n t e^{Lt} \xrightarrow[n\to\infty]{} 0$.

Finally, since $\|P^t_{q(\tau_n)}\| \le e^{-\zeta t/2}$ for all $0 < t < T_n - \tau_n$, by the definition of τ_n as a reverse hyperbolic time, we conclude that for any given fixed $t > 0$, non-zero vectors in $G(q(\tau_n))^\perp$ are contracted by e^{tDG_σ} at a positive rate. But for any vector $v \in E^u_\sigma$ we have

[15] As usual $\operatorname{dist}(A, B) = \inf\{\operatorname{dist}(a, b) : a \in A, b \in B\}$ for $A, B \subset M$.

$\|\mathcal{O}_{q(\tau_n+t)}v\| \geq \|v\|/2$ for all n large enough and so by invariance of E^u_σ we deduce

$$\frac{1}{2}\|e^{tDG_\sigma}v\| \leq \|\mathcal{O}_{x(\tau_n+t)}e^{tDG_\sigma}v\| \leq \bar{\delta}_n\|v\| + \|P^t_{q(\tau_n)}v\| \leq (\bar{\delta}_n + e^{-\zeta t/2})\|v\|$$

and since $t > 0$ is arbitrary, we conclude that $v = \vec{0}$, that is, $E^u_\sigma = \{\vec{0}\}$ and σ is a sink. □

3.6.2. **Trajectory accumulates but does not converge to an equilibrium.** If $\omega(x) \supsetneq \{\sigma\}$, then we again separate the argument into different cases, as follows.

(1) *The orbits segments* $x([\tau_n, T_n])$ *are away from* Sing(G).

If there exists a subsequence $n_k \nearrow \infty$ such that the family of orbit segments $\{x([\tau_{n_k}, T_{n_k}])\}_k$ does not accumulate Sing(G), then we can argue just as in the previous Subsection 3.5. That is, we consider $\bar{x}$ an accumulation point of $x(\tau_{n_k})$ and the cross-section $S_{\bar{x}}$ with a size ρ given by at most $\inf_k\{\text{dist}\,(x([\tau_{n_k}, T_{n_k}]), \text{Sing}(G))\} > 0$, and repeat the same reasoning in Subsection 3.5.2 to obtain a sink in $S_{\bar{x}}$ for some Poincaré return map. This is a contradiction with the assumption that $\omega(x) \neq \{\sigma\}$ and we conclude that this case cannot happen.

(2) *Alternatively*, since Sing(G) is finite, *there exists* $\sigma_0 \in$ Sing(G) *and* $s_{n_k} \in [\tau_{n_k}, T_{n_k}]$ *so that* $x(s_{n_k}) \to \sigma_0$.

In what follows we reindex the sequences to τ_k, T_k and s_k to simplify the notation. We note that σ_0 must be a hyperbolic saddle; for otherwise σ_0 would be a sink and then $\omega(x) = \{\sigma_0\}$.

Lemma 3.10 (convergence to stable manifold of σ_0)**.** *Let $\tau_n < T_n$ be such that $\tau_k \nearrow \infty$, $(T_k - \tau_k) \nearrow \infty$ and satisfy the left hand side of* (6). *Assume that there exists a hyperbolic saddle $\sigma_0 \in$* Sing(G) *and $s_n \in [\tau_k, T_k]$ so that $x(s_k) \to \sigma_0$. If there exists $q \in M \setminus$* Sing(G) *and (perhaps for a subsequence) $x(\tau_k) \to q$, then $q \in W^s(\sigma_0) \setminus \{\sigma_0\}$ and for $t \geq 0$ we have $\|P^t_q\| \leq e^{-\frac{\zeta}{4}t}$.*

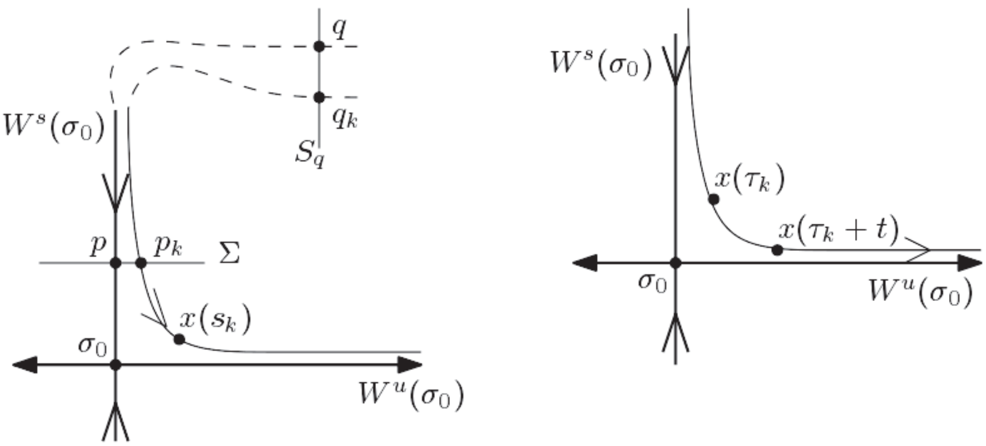

Fig. 8. Relative positions of $p, p_k = x(\tau_k) \in \Sigma$ and $x(s_k)$ close to σ_0 on the left hand side; and of $x(\tau_k), x(\tau_k + t)$ and σ_0 on the right hand side.

Proof. We observe that we can assume without loss of generality that the segment $x([\tau_k, s_k])$ does not accumulate Sing(G) $\setminus \{\sigma_0\}$. For otherwise, because Sing(G) is finite, we would replace s_k by another sequence $\tau_k < s'_k < s_k$ satisfying this property.

Then we can find a cross-section Σ of G at $p \in W^s(\sigma) \setminus \{\sigma\}$ close to σ so that the segment $x([\tau_k, s_k])$ crosses Σ at a point $p_k = x(\tau_k + \upsilon_k)$ and $p_k \xrightarrow[k\to\infty]{} p$; see the left hand side of Figure 8.

Hence, we may apply Proposition 3.7 to the pair $\tau_k, \tau_k+\upsilon_k$ after choosing $S_q(\xi_0)$ the cross-section of G through q with uniform size, where the value of ξ_0 depends on the distance between $\{x([\tau_k, \tau_k + \upsilon_k])\}_k$ and $\mathrm{Sing}(G)$. Since $q_k = x(\tau_k + \eta_k) = \Phi x(\tau_k) \in S_q(\xi_0)$ is such that $q_k \to q$ (and so $\eta_k \to 0$), we have that $q_k \in S_q(\xi_0/2)$ for all big enough k, and obtain a Poincaré map $R : S_q(\xi_0/2) \to \Sigma$ so that $p_k = Rq_k$. Hence, $p = Rq$ and thus $q \in W^s(\sigma)$ as claimed.

Moreover, $\sup_{0\le t\le \upsilon_k} |\phi_t q - x(\tau_k + t)| \xrightarrow[k\to\infty]{} 0$ and also by construction we obtain that for any given $\varepsilon_0 > 0$ there exists $m \in \mathbb{Z}^+$ such that[16] $\sup_{0\le t\le\upsilon_k} \|\mathcal{O}_{\phi_t q} DG_{\phi_t q} - \mathcal{O}_{x(\tau_k+t)} DG_{x(\tau_k+t)}\| < \zeta/4$ for all $k > m$. Thus for any unit vector v not parallel to $G(q)$ and $G(x(\tau_k))$ we obtain from the expressions for $H(v,t)$ at q and $x(\tau_k)$ that $\ln \frac{\|P^t_q v\|}{\|P^t_{x(\tau_k)} v\|} \le \int_0^t |D(\widehat{\Phi^q_s} v) - D(\widehat{\Phi^{x(\tau_k)}_s} v)|\, ds \le \frac{\zeta}{4} t$, where $\widehat{\Phi^z_s} v = \frac{P^s_z v}{\|P^s_z v\|}$ for any $z \in M \setminus \mathrm{Sing}(G)$ and $v \in T^1_z M$ not parallel to $G(z)$. This ensures that $\ln \|P^t_q v\| \le -\zeta t/4$ for all $v \in T^1_q M \cap G^\perp$ which implies last inequality in the statement of the lemma for $0 \le t \le \upsilon_k$.

Finally, note that we may take $\Sigma = \Sigma_k$ closer to σ_0 and obtain $\upsilon_k \nearrow \infty$. This completes the proof of the lemma. □

3.6.3. **Conclusion of the proof of Theorem E.** Now we subdivide the argument to conclude the proof of Theorem E into the following cases according to the accumulation points of $x(\tau_k)$.

- **Case** $x(\tau_k) \to q \notin \mathrm{Sing}(G)$**:** from Lemma 3.10 we have $q \in W^s(\sigma_0) \setminus \{\sigma_0\}$ and also $\|P^t_q\| \le e^{-\frac{\zeta}{4}t}, t \ge 0$. From Lemma 3.8 we conclude that σ_0 is a sink.
- **Otherwise,** $x(\tau_k) \to \sigma \in \mathrm{Sing}(G)$**:** clearly σ is again a saddle. By the local linearization given by the Hartman-Grobman Theorem, for any given fixed $t > 0$ we have $\mathrm{dist}\left(x(\tau_k, T_k], \sigma)\right) \xrightarrow[k\to\infty]{} 0$; see e.g. [23] and the right hand side of Figure 8.

  We now use Lemma 3.9 to obtain $\|P^t_{x(\tau_k)} - \mathcal{O}_{x(\tau_k+t)} e^{tDG_\sigma}\| \le \bar{\delta}_k t e^{Lt} \xrightarrow[k\to\infty]{} 0$.

  Now, since $\|P^t_{x(\tau_k)}\| \le e^{-\zeta t/2}$ for all $0 < t < T_k - \tau_k$ by the definition of τ_k as a reverse hyperbolic time, then non-zero vectors in $G(x(\tau_k))^\perp$ are contracted by e^{tDG_σ} at a positive rate for any given fixed $t > 0$. This shows that σ is a saddle with contracting direction of dimension at least $\dim \mathcal{O}_{x(\tau_k)} = \dim M - 1$.

We have shown that σ_0 is a codimension 1 saddle singularity. The proof of Theorem E is complete.

Remark 3.11. The argument in Remark 3.6 would allow us to replace τ_k by any $s_k \in (\tau_k, T_k)$ and so we would replace σ_0 by σ in the previous argument, but not more, since we do not know the distance between $x(\tau_k)$ and $x(T_k)$.

3.7. The strong sectional asymptotic contracting case. We now use the previous arguments to complete the proof of Theorem F. For if we assume the stronger asymptotic contracting condition on the right hand side of (4), then we can perform all the arguments in

[16] We can assume that all linear maps are comparable in local charts given by the exponential map.

Subsections 3.5 and 3.6, and we are left to show that the positive orbit of x is not allowed to accumulate saddle equilibrium points.

If $x \in M \setminus \mathrm{Sing}(G)$ is such that $\omega(x) \supsetneq \{\sigma\}$ for some $\sigma \in \mathrm{Sing}(G)$, then σ must be a hyperbolic codimension one saddle by the previous arguments, that is, $\dim E^u_\sigma = 1$. Using the Hartman-Grobman Theorem again, we find ourselves in a situation similar to the one on the left hand side of Figure 8.

More precisely, we choose a smooth manifold Σ' with a cusp at σ according to the following; see Figure 9.

Lemma 3.12. *Given a codimension one saddle singularity σ of a C^1 vector field G, there exists a smooth hypersurface Σ' of M tangent to $E^s(\sigma)$ at σ so that $\cos \angle(G(z), E^u_\sigma) \xrightarrow[z\to\sigma]{z\in\Sigma'} 0$ and $\Sigma' \setminus \{\sigma\}$ is a Poincaré section of the flow: that is, in a neighborhood V of σ, for all $p \in V$, we have only one of the following*

- *either $\phi_t p \in V$ for all $t \geq 0$ and $\phi_t p \xrightarrow[t\to+\infty]{} \sigma$ (i.e. $p \in W^s(\sigma)$);*
- *or $\phi_t p \in V$ for all $t \leq 0$ and $\phi_t p \xrightarrow[t\to-\infty]{} \sigma$ (i.e. $p \in W^u(\sigma)$);*
- *or $\exists t_0 \in \mathbb{R} : \phi_{t_0} p \in \Sigma'$ and $\phi_{[0,t_0]} p \subset V$ if $t_0 \geq 0$; or $\phi_{[-t_0,0]} p \subset V$ if $t_0 < 0$.*

We again postpone the proof of this result to Subsection 3.9.

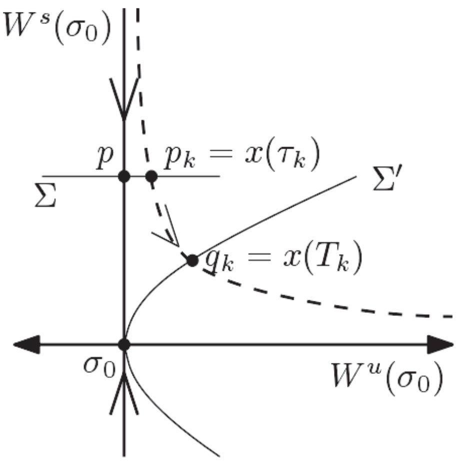

Fig. 9. The strong assymptotic contracting case near a saddle singularity.

From Lemma 3.12 and since the trajectory of x satisfies the strong asymptotic contraction condition in the right hand side of (4), we can find real valued sequences $\tau_k, T_k \nearrow \infty$ such that $q_k = x(T_k) \to \sigma$ and $q_k \in \Sigma'$; and also $\ln \|P_x^{T_k}\| < -\xi T_k$ and $\tau_k < T_k$ is an $e^{-\zeta/2}$-reverse hyperbolic time for x with respect to T_n. We consider two cases:

Case A: either (perhaps for some subsequence) $p_k = x(\tau_k) \to p \in M \setminus \mathrm{Sing}(G)$: in this case we get $p \in W^s(\sigma)$ by Lemma 3.10 and then conclude that σ is a sink by Lemma 3.8.

Case B: or, $p_k \to \mathrm{Sing}(G)$.

If $p_k \to \sigma_0 \neq \sigma$, then σ_0 is again a saddle and we use Remark 3.6 to replace τ_k by $s_k \in (\tau_k, T_k)$ so that s_k is a $e^{-\zeta/4}$-reverse hyperbolic time w.r.t. T_n; and $p_k = x(s_k) \in \Sigma$ for a cross-section Σ to G through $p \in W^s(\sigma) \setminus \{\sigma\}$; see Figure 9. In addition $p_k \to p$

and we have reproduced Case A. Then σ is a sink.

Otherwise, we have $p_k \to \sigma$ and so the segment $x([\tau_k, T_k])$ tends to σ when $k \nearrow \infty$.

We are left with $P^{t-\tau_k}_{p_k}$ a $e^{-\zeta t/2}$-contraction for all $0 < t \le T_k - \tau_k$. Since $G(q_k)^\perp$ is very close to the expanding direction E^u_σ, we obtain a contradiction as in the previous Subsection 3.6.3.

More precisely, for any given fixed $t > 0$ we have $\operatorname{dist}(x([\tau_k \tau_k + t]), \sigma) \xrightarrow[k\to\infty]{} 0$ so we again apply Lemma 3.9 to get $\|P^t_{x(\tau_k)} - \mathcal{O}_{x(\tau_k+t)} e^{tDG_\sigma}\| \le \bar{\delta}_k t e^{Lt} \xrightarrow[k\to\infty]{} 0$. By the choice of T_k and Σ', we also have $\|\mathcal{O}_{x(T_k)} v\| \ge \|v\|/2$ for all $v \in E^u_\sigma$, because $\angle(G(x(T_k)), E^u) \to \frac{\pi}{2}$; see Figure 9. Then we conclude that $E^u_\sigma = \{\vec{0}\}$ as in the proof of Lemma 3.8.

This shows that σ is a sink and completes the proof of Theorem F.

3.8. Weak (sectional) asymptotic expansion. Now we outline the proofs of the second part of Theorem D and of Theorem G since they follow very similar lines to the asymptotic contracting case.

First we note that for the cocycles $\tilde{\Gamma}(t,x)v = \ln\|D\phi_t(x)^{-1}v\|$ and $\tilde{\psi}(t,x) = \ln\|(P^t_x)^{-1}v\|$ a result similar to Lemma 3.4 holds: they admit infinitesimal generators $\tilde{D}_G(v)$ and $\tilde{D}(v)$ respectively, which are continuous functions of $v \in T^1_{M\setminus \mathrm{Sing}(G)}M$ and globally bounded. So the assumptions of the second part of Theorem D and Theorem G imply

$$\liminf_{T\to\infty} \frac{1}{T} \sup_{v\in T^1_x M} \int_0^T \tilde{D}_G(\Phi_s v)\, ds \quad \text{and}$$

$$\liminf_{T\to\infty} \frac{1}{T} \sup_{v\in T^1_x M\cap G^\perp} \int_0^T \tilde{D}(\widehat{\Phi_s x})\, ds \ \text{ or } \ \limsup_{T\to\infty} \sup_{v\in T^1_x M\cap G^\perp} \frac{1}{T}\int_0^T \tilde{D}(\widehat{\Phi_s x})\, ds,$$

are negative. We then apply the same arguments as in the proof of Theorems D, E and F because the analogous to Lemma 2.2 and Proposition 3.7 for the (sectional) expanding case at hyperbolic times are also true. We state the results below and explain what we mean by hyperbolic times in this setting.

In any of the lim inf assumptions above, we can find $\zeta > 0$ and $T_n \nearrow \infty$ so that

$$\sup_{v\in T^1_x M} \int_0^{T_n} \tilde{D}_G(\Phi_s v)\, ds \le -\frac{\zeta T_n}{2} \quad \text{or} \quad \sup_{v\in T^1_x M\cap G^\perp} \int_0^{T_n} \tilde{D}(\widehat{\Phi_s x})\, ds \le -\frac{\zeta T_n}{2}$$

and apply Theorem 3.1 to the functions $\tilde{H}_G(t,v) = \int_{T_n-t}^{T_n} \tilde{D}_G(\Phi_s v)\, ds$ or $\tilde{H}(t,v) = \int_{T_n-t}^{T_n} \tilde{D}(\widehat{\Phi_s v})\, ds$, respectively. We obtain $\tau_n < T_n$ with $\tau_n \nearrow \infty$ and $T_n - \tau_n \nearrow \infty$ such that for $0 < t < \tau_n$

$$\ln\|(D\phi_{\tau_n - t}(\phi_t x))^{-1}\| = \sup_{v\in T^1_x M} \int_t^{\tau_n} \tilde{D}_G(\Phi_s v)\, ds \le -\frac{\zeta}{4}(\tau_n - t) \ \text{ or} \tag{9}$$

$$\ln\|(P^{\tau_n - t}_{\phi_t x})^{-1}\| = \sup_{v\in T^1_x M\cap G^\perp} \int_t^{\tau_n} \tilde{D}_G(\widehat{\Phi_s v})\, ds \le -\frac{\zeta}{4}(\tau_n - t), \tag{10}$$

respectively. These are reverse hyperbolic times, where we have uniform infinitesimal contractions from $\phi_{\tau_n} x$ to $\phi_t(x)$ for $0 < t < \tau_n$ with respect to the flow in (9) or the Linear Poincaré Flow in (10).

Proof of the second part of Theorem D. In the case (9), $[\tau_n]$ is a hyperbolic time for $y_n =$

$\phi_{\tau_n-[\tau_n]}(x)$ with respect to the C^1 diffeomorphism $f = \phi_1$. So we can apply Lemma 2.5 to get infinitely many backward contracting balls to which we can apply the nested contractions argument from Subsection 2.1. We obtain a repelling periodic orbit p for f such that $p \in B(x(\tau_n), \delta_1)$; thus also a repelling periodic orbit $\mathcal{O}_G(p)$ for G. But $G(p) = D\phi_k(p) \cdot G(p) = Df^k(p) \cdot G(p)$ for some $k \geq 1$ and so 1 would be an eigenvalue of $Df^k(p)$ if $G(p) \neq \vec{0}$, contradicting the expansion of the derivative map at repelling periodic points. This shows that $G(p) = \vec{0}$, hence $p = \sigma \in \mathrm{Sing}(G)$ is a repelling equilibrium (a source).

Moreover, by the properties of backward contracting balls, we get $\mathrm{dist}(y_n, \sigma) \leq e^{-\zeta\tau_n/4}$ where we can take n larger than any predetermined quantity. Hence the distance between $\phi_{[0,1]}(x)$ and $\mathcal{O}_G(p)$ is zero and x is a source. This completes the proof of Theorem D. $\square$

Proof of Theorem G. In the case (10) we use the following, whose proof is left to the reader.

Proposition 3.13 (Existence of backward contracting balls)**.** *Let $\tau_n < T_n$ be the pair of strictly increasing sequences obtained above. For every $\delta_0 > 0$ there exists $\xi_0 > 0$ satisfying* (8) *such that, if* $\mathrm{dist}(\phi_t x, \mathrm{Sing}(G)) \geq d_0, \forall t \in [0, \tau_n]$*, then for each $s \in (0, \tau_n]$ there exists a C^1 smooth well-defined diffeomorphism with its image $R_s : S_{x(\tau_n)}(\xi_0) \to S_{x(s)}(\xi_0)$ such that R_s is a Poincaré map (for the time-reversed flow), $R_s(x(\tau_n)) = x(s)$ and R_s is an $e^{-\frac{\zeta}{4}(\tau_n - s)}$-contraction.*

We have now all the tools to apply the same arguments in Subsections 3.5, 3.6 and 3.7 to conclude the proof of Theorem G. $\square$

3.9. Proofs of Lemmata. Now we present the proofs of the technical result previously used as tools in this section.

Proof of Lemma 3.4. We prove items (1-3) for $D_\pm(x)$ only since for $D_{G\pm}(x)$ the arguments are completely analogous, but much simpler, and are left to the reader.

To prove the continuity of $x \in M \setminus \mathrm{Sing}(G) \mapsto D_\pm(x)$ note that $\|P^h_y\| \xrightarrow[h\to 0]{} 1$ and

$$\frac{1}{h}\ln\frac{\|P^h_x\|}{\|P^h_y\|} = \frac{1}{2h}\ln\frac{\|P^h_x\|^2}{\|P^h_y\|^2} = \frac{1}{2h}\ln\left(1 + \frac{\|P^h_x\| - \|P^h_y\|^2}{\|P^h_y\|^2}\right) \leq \frac{\|P^h_x\|^2 - \|P^h_y\|^2}{2h\|P^h_y\|^2}. \tag{11}$$

We express the time derivative of $\|P^h_x\|^2 = \sup_{\|u\|=1}\langle P^h_x u, P^h_x u\rangle$ as follows. On the one hand, writing $\hat{G} = G/\|G\|$ and $\phi_h z = z_h$ for any $z \in M, h \in \mathbb{R}$, we get

$$\begin{aligned}
(P^h_x)' &= (\mathcal{O}_{\phi_h x} D\phi_x(x))' = (D\phi_x(x) - \langle D\phi_x(x), \hat{G}(x_h)\rangle \hat{G}(x_h))' \\
&= DG_{x_h} D\phi_h(x) - \langle DG_{x_h} D\phi_h(x), \hat{G}(x_h)\rangle \hat{G}(x_h) \\
&\quad - \langle D\phi_h(x), \hat{G}(x_h)'\rangle \hat{G}(x_h) - \langle D\phi_h(x), \hat{G}(x_h)\rangle \hat{G}(x_h)'
\end{aligned}$$

and since $\hat{G}(x_h)' = \mathcal{O}_{x_h} DG_{x_h} \hat{G}(x_h)$ we finally obtain $\langle P^h_x, P^h_x\rangle' = 2\langle \mathcal{O}_{x_h} DG_{x_h} P^h_x, P^h_x\rangle$.

Along this proof, we are implicitly assuming that $x_h = \phi_h x$ is in the range of $\exp_x$, identifying the tangent spaces $T_x M$ and $T_{x_h} M$ through $D(\exp_x)_{\exp_x^{-1} x_h}$ and writing $DG_y v$ for $\nabla_v G(y)$ with $y \in M, v \in T_y M$, where ∇ is the Levi-Civita connection associated to the Riemannian metric of M.

On the one hand, for a singular vector $u_h \in G(x)^\perp$ corresponding to the largest singular value[17] of P^h_x and $|h|$ sufficiently small, we have for some intermediate value $s = s(h)$ so that $0 < |s(h)| < |h|$

$$\|P^h_x\|^2 = 1 + 2\int_0^h \langle \mathcal{O}_{x_s} DG_{x_s} P^s_x u_h, P^s_x u_h\rangle\, ds = 1 + 2h\langle \mathcal{O}_{x(s)} DG_{x(s)} P^{s(h)}_x u_h, P^{s(h)}_x u_h\rangle.$$

Therefore $P^{s(h)}_x u_h$ is an eigenvector associated to the largest eigenvalue of[18] $(\mathcal{O}_{x(s)} DG_{x(s)} + (\mathcal{O}_{x(s)} DG_{x(s)})^*)/2$. On the other hand, for $v_h \in G(y)^\perp$ corresponding to the largest singular value of P^h_y we can find some intermediate value $\bar{s} = \bar{s}(h)$ so that $0 < |\bar{s}(h)| < |h|$

$$(11) = \frac{\langle \mathcal{O}_{x_s} DG_{x_s} P^{s(h)}_x u_h, P^{s(h)}_x u_h\rangle - \langle \mathcal{O}_{\bar{y}_s} DG_{\bar{y}_s} P^{\bar{s}(h)}_y v_h, P^{\bar{s}(h)}_y v_h\rangle}{\|P^h_y\|^2}.$$

Hence, when $h \to 0$, using the compactness of the unit sphere, we get u, v accumulation unit vectors of the families $(u_h), (v_h)$ so that

$$\lim_{\delta\searrow 0} \sup_{0<|h|<\delta} \left|\frac{1}{h}\ln\frac{\|P^h_x\|}{\|P^h_y\|}\right| \le \left|\langle \mathcal{O}_x DG_x u, u\rangle - \langle \mathcal{O}_y DG_y v, v\rangle\right| \tag{12}$$

and both u, v are eigenvectors associated to the largest eigenvalues of the operators $(\mathcal{O}_x DG_x + (\mathcal{O}_x DG_x)^*)/2$ and $(\mathcal{O}_y DG_y + (\mathcal{O}_y DG_y)^*)/2$, respectively. Since these are symmetric operators, we apply the following.

Lemma 3.14 ([14, Theorem III.6.11]). *Let A, B be selfadjoint operators of $\mathbb{R}^d$ and $C = A - B$, whose eigenvalues repeated with multiplicities we write as $\alpha_1 \le \cdots \le \alpha_d$, $\beta_1 \le \cdots \le \beta_d$ and $\gamma_1 \le \cdots \le \gamma_d$ respectively. Then $\sum_{i=1}^d |\alpha_i - \beta_i| \le \sum_{i=1}^d |\gamma_i|$.*

As a direct consequence of this result, recall the standard bound for the spectral radius

$$r(C) = \sup\{|\lambda| : \lambda \in \mathrm{sp}(C)\} \le \|C\| = \|A - B\|$$

and so $|\alpha_i - \beta_i| \le d r(C) \le d\|A - B\|$ for each $1 \le i \le d$.

Going back to (12) writing $A = (\mathcal{O}_x DG_x + (\mathcal{O}_x DG_x)^*)/2$ and $B = (\mathcal{O}_y DG_y + (\mathcal{O}_y DG_y)^*)/2$ we get[19] $(12) \le \dim M \cdot \|\mathcal{O}_x DG_x - \mathcal{O}_y DG_y\|$. This together with (11) implies

$$\begin{aligned} D_+(x) = \lim_{\delta\searrow 0} \sup_{0<h<\delta} \frac{\ln\|P^h_x\|}{h} &\le \lim_{\delta\searrow 0}\left(\sup_{0<h<\delta} \frac{\ln\|P^h_y\|}{h} + \sup_{0<h<\delta}\left|\frac{1}{h}\ln\frac{\|P^h_x\|}{\|P^h_y\|}\right|\right) \\ &= D_+(y) + \dim M \cdot \|\mathcal{O}_x DG_x - \mathcal{O}_y DG_y\|) \end{aligned}$$

and since we can exchange x and y this completes the proof of the continuity of $D_+(x)$. Moreove, we can argue with $h \nearrow 0$ using the same inequality (12), thus obtaining continuity for $D_-(x)$ also . This completes the proof of item (2).

For item (1), we note that $\|P^h_x v\| \ge \|(P^h_x)^{-1}\|^{-1}\|v\|, v \in G(x)^\perp$ and that

$$(P^h_x)^{-1} = \mathcal{O}_x \circ D\phi_h(x)^{-1} \circ \mathcal{O}_{\phi_h x} = \mathcal{O}_x \circ D\phi_{-h}(\phi_h x) \circ \mathcal{O}_{\phi_h x}$$

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[17] The largest coefficient of the orthogonal diagonalization of the quadratic form $v \mapsto \langle (P^h_x)^* P^h_x v, v\rangle$, where $*$ denotes the adjoint operator with respect to the inner product.

[18] This is given by $\sup\{\Re\lambda : \lambda \in \mathrm{sp}(\mathcal{O}_{x(s)} DG_{x(s)})\}$ which is the largest real coefficient in the orthogonal diagonalization of the quadratic form $\langle \mathcal{O}_{x(s)} DG_{x(s)} v, v\rangle = \sum_i a_i X_i^2$ where $v = \sum_i X_i e_i$ for some orthonormal basis $e_1, \ldots, e_d$ of $T_x M$.

[19] Recall that we are implicitly assuming that y is in the range of $\exp_x$.

so $\|(P_x^h)^{-1}\| \le \|D\phi_{-h}(\phi_h x)\| \le e^{|h|L}$ from Gronwall's Inequality and consequently $\ln\|P_x^h\|^{1/h} \ge -L$ for all $h \in \mathbb{R}, x \in M \setminus \mathrm{Sing}(G)$.

Hence $D_\pm(x) \ge -L$ and analogously $D_\pm(x) \le L$ for all $x \in M \setminus \mathrm{Sing}(G)$.

For item (3): from the continuity of $x \mapsto D(x)$ and Lemma 3.3 we deduce the relation (2), since $t \mapsto \psi_t x$ satisfies

$$\limsup_{h\to 0+} \frac{\psi_{t+h}x - \psi_t x}{h} \le \limsup_{h\to 0+} \frac{\psi_h(\phi_t x) + \psi_t x - \psi_t x}{h} = D_+(x) \quad \text{and}$$
$$\liminf_{h\to 0-} \frac{\psi_{t+h}x - \psi_t x}{h} \ge \liminf_{h\to 0-} \frac{\psi_h(\phi_t x) + \psi_t x - \psi_t x}{h} = D_-(x).$$

Hence, for any partition $0 = t_0 < t_1 < \cdots < t_k = T$ of the interval $[0, T]$ with width $\delta = \sup_{1\le i\le k}(t_{i+1} - t_i)$ we get on the one hand

$$\psi_T x = \sum_{i=1}^{k} \frac{\psi_{t_{i+1}}x - \psi_{t_i}x}{t_{i+1} - t_i}(t_{i+1} - t_i) \le \sum_{i=1}^{k} \sup_{0<h<\delta}\left(\frac{\psi_h(\phi_{t_i}x)}{h}\right)(t_{i+1} - t_i)$$

and since $\lim_{\delta\searrow 0}\sup_{0<h<\delta}\left(\frac{\psi_h(\phi_{t_i}x)}{h}\right) = D_+(\phi_{t_i}x)$ we obtain $\psi_T x \le \int_0^T D_+(\phi_s x)\,ds$. On the other hand

$$\psi_T x = \sum_{i=1}^{k} \frac{\psi_{t_i}x - \psi_{t_{i+1}}x}{t_i - t_{i+1}}(t_{i+1} - t_i) \ge \sum_{i=1}^{k} \inf_{-\delta<h<0}\left(\frac{\psi_h(\phi_{t_{i+1}}x)}{h}\right)(t_{i+1} - t_i)$$

and since $\lim_{\delta\searrow 0}\inf_{-\delta<h<0}\left(\frac{\psi_h(\phi_{t_{i+1}}x)}{h}\right) = D_-(\phi_{t_{i+1}}x)$ we also get $\psi_T x \ge \int_0^T D_-(\phi_s x)\,ds$ and obtain (2). This completes the proof of the lemma. □

Proof of Lemma 3.5. Let us assume that $-\xi = \liminf_{T\to+\infty}\ln\|D\phi_T(x)\|^{1/T} < 0$. Then for each $\varepsilon > 0$ there exists a sequence $T_n \nearrow \infty$ so that for all $v \in T_x^1 M$ we have from Lemma 3.3

$$H_G(T_n, v) = \int_0^{T_n} D_G(\Phi_s v)\,ds = \ln\|D\phi_{T_n}(x)\cdot v\| \le \ln\|D\phi_{T_n}(x)\| \le -\xi T_n.$$

This proves the first statement of the lemma. The proof of the other statements is similar and left to the reader. □

Proof of Lemma 3.9. We can assume without loss of generality that $\phi_{[0,t]}q_n$ is in the range of $\exp_\sigma$ for all $n \ge 1$ so that we can identify all tangent spaces. Then we note that

$$\|P_{q_n}^h - \mathcal{O}_{\phi_h q_n} D\phi_h\sigma\| = \|\mathcal{O}_{\phi_h q_n} D\phi_h q_n - \mathcal{O}_{\phi_h q_n} e^{hDG_\sigma}\| \le \|D\phi_h q_n - e^{hDG_\sigma}\| \tag{13}$$

so we need only estimate the last norm. For that we use Gronwall's Inequality as follows: $D\phi_h(z)$ is the solution of the Linear Variational Equation $\dot{Z} = DG_{\phi_h z}\cdot Z$ with $Z(0) = Id$, $z = q_n$ or σ, for $h \in [-\varepsilon, \varepsilon]$ and $\varepsilon > 0$ small, in the coordinates of a local chart of M containing both q_n and σ. Then we can write

$$\begin{aligned} D\phi_h(q_n) - e^{hDG_\sigma} &= \int_0^h \left(DG_{\phi_s q_n}\cdot D\phi_s(q_n) - DG_\sigma\cdot e^{sDG_\sigma}\right)ds \\ &= \int_0^h DG_{\phi_s q_n}\cdot\left(D\phi_s(q_n) - e^{sDG_\sigma}\right)ds + \int_0^h \left(DG_{\phi_s q_n} - DG_\sigma\right)\cdot e^{sDG_\sigma}\,ds \end{aligned}$$

and taking norms we obtain $\beta(h) \le \alpha(h) + \int_0^h \gamma(s)\beta(s)\,ds$ where we set $\gamma(s) = \|DG_{\phi_s q_n}\|$; $\alpha(h) = \int_0^h \left\|(DG_{\phi_s q_n} - DG_\sigma)\right\| \cdot \|e^{sDG_\sigma}\|\,ds$ and $\beta(h) = \|D\phi_h(q_n) - e^{hDG_\sigma}\|$. We conclude[20] $\beta(h) \le \alpha(h)\exp\int_0^h \gamma(s)\,ds$.

Now we have $\gamma(h) \le L$ and $\|e^{hDG_\sigma}\| \le e^{hL}$, so we arrive at

$$\alpha(h) \le he^{hL} \sup_{0\le s\le h} \left\|DG_{\phi_s q_n} - DG_\sigma\right\| = \bar{\delta}_n h e^{hL}.$$

ensuring that we can bound (13) by $\bar{\delta}_n hLe^{hL}$.

Finally, if we know that the trajectory $\phi_{[0,t]}q_n$ is close to σ, then we can perform the above integrations and estimations for $h = t$ and complete the proof of the lemma. □

We now prove a second technical lemma.

Proof of Lemma 3.12. We can obtain Σ' simply writting $T_\sigma M = E^s(\sigma) \oplus E^u(\sigma) \cong \mathbb{R}^{\dim M-1} \times \mathbb{R}$ and setting $\Sigma' = \exp_\sigma \Sigma_0$ where $\Sigma_0 = \{(u,v) \in E^s \times E^u : v = \|u\|^2\}$. Indeed, the linear vector field $w = (w_s, w_u) \in T_\sigma M \mapsto DG_\sigma w = (Aw_s, \xi w_u)$, for fixed $\xi > 1$ and $A \in GL(E^s(\sigma))$ with $\Im\,\mathrm{sp}(A^s) \subset \mathbb{R}^-$, over $z \in \Sigma_0$ has angle with the vertical direction $(0,1)$ which tends to zero when $z \to 0$. In fact,

$$\cos\angle((Az_s, \xi z_u)), (0,1)) = \frac{\xi z_u}{\|(Az_s, \xi z_u)\|} \xrightarrow[z\to 0]{z\in\Sigma'} 0 \iff \frac{|z_u|}{\|z_s\|} \xrightarrow[z\to 0]{z\in\Sigma'} 0$$

is a consequence of $z_u = \|z_s\|^2$ (that is, $z \in \Sigma'$) together with $z \to 0$. Since the vector $(0,1)$ is the direction of E^u_σ, we have

$$\cos\angle(G(z), E^u_\sigma) \xrightarrow[z\to\sigma]{z\in\Sigma'} 0 \tag{14}$$

in the linearized case. Hence, G and Σ' satisfy (14) in a small enough neighborhood of σ, because the vector field $\tilde{G} = D(\exp_\sigma)^{-1}G$ on a neighborhood of 0 in $T_\sigma M$ satisfies for each $v \in T_\sigma M$

$$D\tilde{G}_0 v = \lim_{t\to 0} \frac{D(\exp_\sigma)^{-1} G(\exp_\sigma(tv))}{t} = D(\exp_\sigma)^{-1} \lim_{t\to 0} \frac{G(\exp_\sigma(tv))}{t} = DG_\sigma v.$$

Consequently $\|DG_\sigma w - D(\exp_\sigma)_w^{-1} G(\exp_\sigma w)\|/\|w\| \xrightarrow[w\to 0]{} 0$ and thus

$$\left\langle \frac{G(\exp_\sigma w)}{\|G(\exp_\sigma w)\|}, D(\exp_\sigma)_0 \cdot (0,1) \right\rangle = \left\langle \frac{D(\exp_\sigma)_w \tilde{G}(w)}{\|D(\exp_\sigma)_w \tilde{G}(w)\|}, D(\exp_\sigma)_0 \cdot (0,1) \right\rangle$$

has the same limit as $w \to 0$ along Σ_0 as (14), since $D(\exp_\sigma)_0 = Id$ and $\Sigma' = \exp_\sigma \Sigma_0$.

For the sectional property of Σ, by the Hartman-Grobman Linearization Theorem, the flow of DG_σ in a neighborhood V of 0 in $T_\sigma M$ is topologically conjugated to the flow of G is a neighborhood U of σ in M: for any given $\delta > 0$ we can find a homeomorphism $h : V \to U$ such that $\|Id - h^{-1}\exp_\sigma\| < \delta$ and $\phi_t h(w) = h(e^{tDG_\sigma} w)$ for $w \in T_\sigma M$ satisfying $e^{sDG_\sigma} w \in V, \forall 0 \le s \le t$.

It is thus enough to prove that $\tilde{\Sigma} = h^{-1}(\Sigma \setminus \{\sigma\} = h^{-1}\exp_\sigma(\Sigma_0 \setminus \{0\})$ is a Poincaré section of the linearized flow. Since $h^{-1}\exp_\sigma$ is close to the identity map, without loss of generality we can assume that coordinates have been chosen on $T_\sigma M$ so that

[20] See e.g.[23, Lemma 4.7] and [13, Theorem 2.1].

- $\tilde{\Sigma}$ is a graph of a Lipschitz function $g : E^s \cap B(0,2) \to E^u \cong \mathbb{R}$ satisfying $\mathrm{Lip}(g) < 1$, $g(u) \geq 0$ and $g(u) = 0 \implies u = 0$. We set $a = \inf\{g(u) : u \in E^s, \|u\| = 1\}$.
- for $z = (z_s, z_u)$ with $\|z_s\| = 1$ and $0 < z_u < g(z_s), t > 0$, from $e^{tDG_\sigma} z = (e^{At} z_s, e^{\xi t} z_u)$ we deduce

$$a = e^{\xi t} z_u \iff t = \ln(a/z_u)^{1/\xi} \quad \text{and} \quad \|e^{At} z_s\| \leq e^{-\lambda t} = (z_u/a)^{\lambda/\xi} \xrightarrow[z_u \to 0]{} 0,$$

where $\lambda > 0$ is such that $-\lambda \geq \Im(\alpha), \forall \alpha \in \mathrm{sp}(A)$.

Thus the function $F(x,y) = g(x) - y, (x,y) \in \mathbb{R}^{\dim M - 1} \times \mathbb{R}$ is such that $F(z_s, z_u) = g(z_s) - z_u > 0$ and $F(e^{tDG_\sigma} z) = F(e^{At} z_s) - a < 0$ for all z_u sufficiently close to 0, showing that there exists $s = s(z_s, z_u) \in (0, t)$ such that $F(e^{sDG_\sigma} z) = 0$, that is, $e^{sDG_\sigma} z \in \tilde{\Sigma}$.

Moreover, if $F(e^{\bar{s}DG_\sigma} z) = 0$ for some $\bar{s} > s$, then

$$z_u e^{\xi s}(1 - e^{\xi(\bar{s}-s)}) = g(e^{A\bar{s}} z_s) - g(e^{As} z_s) \leq \mathrm{Lip}(g) \|e^{As}(e^{A(\bar{s}-s)} - I) z_s\|$$

which implies for some $0 < \zeta < \bar{s} - s$ by the Mean Value Inequality

$$z_u \leq \mathrm{Lip}(g) \frac{e^{-(\lambda+\xi)s}}{1 - e^{\xi(\bar{s}-s)}} \|e^{A(\bar{s}-s)} - I\| \leq \mathrm{Lip}(g) \frac{(z_u/a)^{1+\lambda/\xi}}{1 - e^{\xi(\bar{s}-s)}} \|A e^{A\zeta}(\bar{s} - s)\|$$

and so we arrive at

$$0 < a \leq \mathrm{Lip}(g) \left(\frac{z_u}{a}\right)^{\lambda/\xi} \frac{\xi(\bar{s} - s)}{1 - e^{\xi(\bar{s}-s)}} \frac{\|A\|}{\xi} e^{-\lambda(\bar{s}-s)} \leq Const \cdot z_u^{\lambda/\xi}$$

yielding a contradiction for all small enough $z_u > 0$. We conclude that there exists a unique $s = s(z)$ so that the future trajectory of z under the flow of G intersects Σ' for all z in a small enough neighborhood of σ.

Hence $\Sigma' \setminus \{\sigma\}$ is a Poincaré section for the flow G, completing the proof of the lemma. □

Finally, we prove the Lemma of Pliss for flows.

Proof of Theorem 3.1. Observe first that we can assume without loss of generality that H is of class C^2. Indeed, let H be differentiable satisfying $H(0) = 0, H(T) < cT$ and $\inf(H') > A$.

If the statement of the theorem is true for any $\tilde{H} : [0,T] \to \mathbb{R}$ of class C^2, then we choose such $\tilde{H}$ so that $\tilde{H}(0) = 0$ and

$$\sup_{t \in [0,T]} \{|H(t) - \tilde{H}(t)|, |H'(t) - \tilde{H}'(t)|\} < \tilde{\varepsilon}$$

for some small $0 < \tilde{\varepsilon} < \varepsilon$. We obtain

$$\tilde{H}(T) = (\tilde{H} - H)(T) + H(T) < \tilde{\varepsilon} + cT \quad \text{and}$$
$$\inf \tilde{H}' = \inf(H' - (\tilde{H}' - H')) \geq A - \tilde{\varepsilon},$$

and writting for $\delta > 0$

$$\tilde{\mathcal{H}}_\delta = \{\tau \in [0,T] : \tilde{H}(s) - \tilde{H}(\tau) < (c+\delta)(s-\tau), \text{ for all } s \in [\tau, T]\}.$$

then we get $|\tilde{\mathcal{H}}_\varepsilon| \geq T\tilde{\theta}$ with

$$\tilde{\theta} = \frac{\varepsilon}{c + \tilde{\varepsilon}/T + \varepsilon - (A - \tilde{\varepsilon})} = \frac{\varepsilon}{c + \varepsilon - A + \tilde{\varepsilon}(1 + 1/T)}$$

and also $\tilde{\mathcal{H}}_\varepsilon \subset \mathcal{H}_{\varepsilon+\tilde{\varepsilon}(1+1/T)}$.

Therefore, since $\mathcal{H}_\varepsilon = \cup_{n\geq 1}\mathcal{H}_{\varepsilon+1/n}$, we conclude that for each small enough $\tilde{\varepsilon} > 0$ we get $\tilde{H}$ of class C^2 which is $\tilde{\varepsilon}$-C^1-close to H and $|\mathcal{H}_\varepsilon| \geq |\tilde{\mathcal{H}}_\varepsilon| \geq \tilde{\theta}T$. Letting $\tilde{\varepsilon} \to 0$ we obtain $\tilde{\theta} \to \theta = \varepsilon/(c + \varepsilon - A)$ as we need.

Let now $\varepsilon > 0, A, c$ and H be as in the statement of the theorem with H of class C^2, and define $G(s) = H(s) - (c + \varepsilon)s$. Since we have already shown that approximating H in the C^1 topology does not change the conclusions of the statement of the theorem, we may also assume without loss of generality that G does not have degenerate critical points; that is, $G'(x) = 0$ if and only if $G''(x) \neq 0$; and, moreover, that its critical values are all distinct. This can be done replacing H by a C^2-close Morse function in what follows.

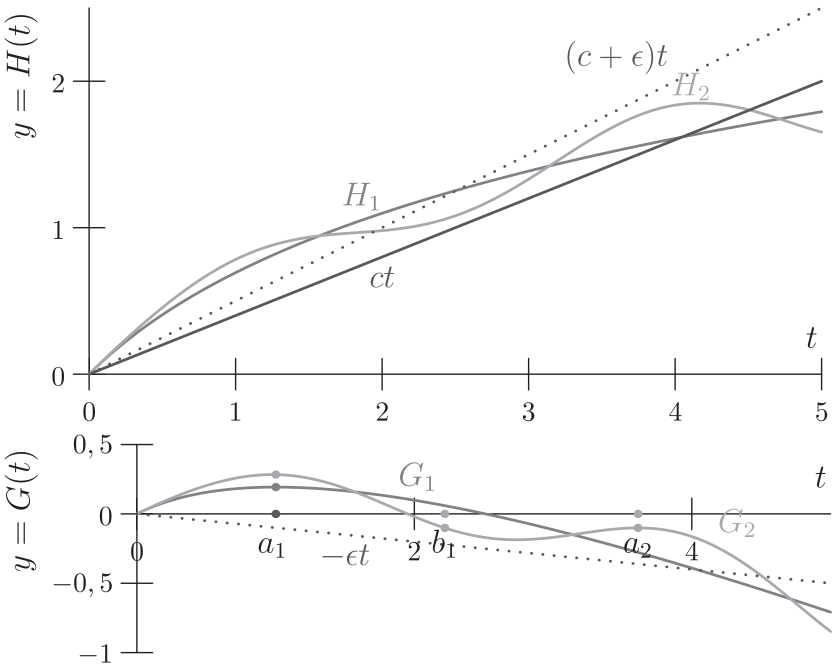

Fig. 10. Illustrative example with the graphs of $H_1(t) = \log(t + 1)$, $H_2(t) = (1 + \sin(2t)/7) \cdot H_1(t)$, $c = 1/2$ and $\varepsilon = 1/10$ above; and also $G_i(t) = H_i(t) - (c + \varepsilon)t, i = 1, 2$ and the points a_i, b_i below. For G_1 we only have $a_1 = 1$; but for G_2 we have $a_1 < b_1 < a_2$.

Now $G(0) = 0$ and $G(T) < -\varepsilon T$, so it is possible to define two (perhaps finite) increasing sequences, say $(a_i)_{i=1}^n$ consisting of critical points of G such that $G(x) < G(a_i)$ *for every* $x > a_i$ (if this sequence is finite, we set the last point $a_n = T$; otherwise $a_i \to T$) and $(b_i)_{i=1}^n$ as the smallest $b > a_i$ such that $G(b_i) = G(a_{i+1})$; see Figure 10.

More precisely, we define the sequence (a_i) and (b_i) recursively: $a_1 = 0$ if $G(t) < 0$ for $t > 0$; otherwise $a_1 = \inf\{s > 0 : G'(s) = 0 \quad \text{and} \quad G(s) > G(t), \forall t > s\}$; now inductively for $i \geq 1$

$$a_{i+1} = \inf\{s > a_i : G'(s) = 0 \quad \text{and} \quad G(s) > G(t), \forall t > s\} \quad \text{and}$$

$$b_i = \inf\{s > a_i : G(s) = G(a_{i+1})\}.$$

Clearly $b_i \le a_{i+1}$, and

- a_1 is a global maximum of G and $G(a_1) \ge 0$;
- each a_i is a local maximum of G;
- $G'(t) \ne 0$ for $a_i < t \le b_i$, otherwise there would be a critical point $\xi < b_i < a_{i+1}$ with the properties of a_{i+1}, contradicting the inductive definition. In addition,
- $G(t) \le G(a_{i+1})$ for $b_i < t < a_{i+1}$ for otherwise there would be a critical point $\xi \in (b_i, a_{i+1})$ with the properties of a_{i+1}, again contradicting the inductive definition.

Letting $B = -\inf G'$, then the Mean Value Theorem ensures that

$$\frac{G(a_i) - G(b_i)}{B} \le b_i - a_i$$

and we claim that the union $\cup_i(a_i, b_i)$ is contained in $\mathcal{H}_\varepsilon$. Indeed

$$\begin{aligned} H(s) - H(\tau) &= H(s) - (c+\varepsilon)s - [H(\tau) - (c+\varepsilon)\tau] + (c+\varepsilon)(s-\tau) \\ &= G(s) - G(\tau) + (c+\varepsilon)(s-\tau). \end{aligned}$$

and so $H(s) - H(\tau) < (c+\varepsilon)(s-\tau)$ if, and only if, $G(s) < G(\tau)$.

Now we let $\tau \in (a_i, b_i)$ for some i and argue by contradiction: let us assume that for a given $t > \tau$ we have $G(t) \ge G(\tau)$. Since there are no critical points in $(\tau, b_i]$, we must have $t \ge b_i$. But this is impossible, because $G(t) \le G(a_{i+1}) = G(b_i) < G(\tau)$ for $b_i < t < a_{i+1}$ and $G(t) \le G(a_{i+1})$ for all $t \ge a_{i+1}$ by construction. This contradiction shows that $\tau \in \mathcal{H}_\varepsilon$, as claimed. Therefore

$$\begin{aligned} |\mathcal{H}_\varepsilon| &\ge \sum_{i=1}^n (b_i - a_i) \ge \frac{1}{B}\sum_{i=1}^n [G(a_i) - G(b_i)] \\ &= \frac{1}{B}\sum_{i=1}^n [G(a_i) - G(a_{i+1})] = \frac{1}{B}[G(a_1) - G(T)]. \end{aligned}$$

Since $G(a_1) \ge 0$ we obtain $|\mathcal{H}_\varepsilon| \ge -\frac{G(T)}{B} \ge \frac{\varepsilon T}{B}$. Notice that since $G'(t) = H'(t) - (c+\varepsilon)$, to get a non-trivial result we need $B > 0$, that is, $A < \inf H' < c + \varepsilon$. Finally

$$B = -\inf G' = \sup(c + \varepsilon - H') = c + \varepsilon - \inf H' \le c + \varepsilon - A$$

and so $|\mathcal{H}_\varepsilon| \ge \frac{\varepsilon}{B} T \ge \frac{\varepsilon}{c+\varepsilon-A} T$ completing the proof of the theorem, by setting $\theta = \frac{\varepsilon}{c+\varepsilon-A}$. □

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Instituto de Matemática e Estatística
Universidade Federal da Bahia
Av. Ademar de Barros s/n
40170–110 Salvador
Brazil
e-mail: vitor.araujo.im.ufba@gmail.com
URL: https://sites.google.com/site/vdaraujo99/