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AN INVARIANT DERIVED FROM THE ALEXANDER POLYNOMIAL FOR HANDLEBODY-KNOTS

Shin'ya OKAZAKI

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Abstract

A handlebody-knot is a handlebody embedded in the 3-sphere. We introduce an invariant for handlebody-knots derived from their Alexander polynomials. The value of the invariant is a vertex-weighted graph. As an application, we describe a sufficient condition for a handlebody-knot to be irreducible and a necessary condition for a link to be a constituent link of a handlebody-knot.

1. Introduction

A genus g handlebody-knot is a genus g handlebody embedded in the 3-sphere S^3, denoted by H. Any handlebody-knot can be represented by some connected spatial graph. Two handlebody-knots are equivalent if one can be transformed into the other by an isotopy of S^3. Suzuki [12] introduced the notion of neighborhood congruence for spatial graphs. The neighborhood congruence class of a connected spatial graph corresponds to a handlebody-knot.

In this paper, we introduce an invariant for handlebody-knots whose value is a vertex-weighted graph. We introduce an equivalence relation $\sim$ on the Laurent polynomial ring $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$. We define the vertex-weighted graph G_f for a Laurent polynomial $f \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$ as an invariant for $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]/\sim$. The d-th Alexander polynomial $\Delta^{(d)}_{(H,M)}(t_1, t_2, \ldots, t_g)$ is an invariant for a pair of a genus g handlebody-knot H and its (oriented and ordered) meridian system M. This invariant is in $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_g^{\pm 1}]$. We define an invariant G_H for handlebody-knots as $G_{\Delta^{(g)}_{(H,M)}(t_1, t_2, \ldots, t_g)}$. The invariant G_H does not depend on the choice of the meridian system of H.

In Section 2, we recall the definition of the Alexander polynomial for a pair of a handlebody-knot H and its meridian system M, and we define an invariant G_H for handlebody-knots. As applications of the invariant G_H, we describe a sufficient condition for a handlebody-knot to be irreducible in Section 3 and a necessary condition for a link to be a constituent link of a handlebody-knot in Section 4. In Section 5, we introduce an equivalence class of handlebody-knots, and demonstrate that G_H is an invariant for this equivalence class of handlebody-knots. The appendix contains a table of G_H and $\hat{G}_H$ for handlebody-knots in the table of genus 2 handlebody-knots with up to six crossings in [4].

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2. An invariant for handlebody-knots

Throughout this paper, we work in the PL category. A genus g *handlebody-knot* is a genus g handlebody embedded in the 3-sphere S^3, denoted by H. Any handlebody-knot can be represented by some connected spatial graph. A *diagram* of a handlebody-knot is a diagram of a connected spatial graph that represents the handlebody-knot. Two handlebody-knots are *equivalent* if one can be transformed into the other by an isotopy of S^3.

We recall the definitions of the universal abelian covering spaces and the Alexander polynomial for handlebody-knots [6, 10]. Let H be a genus g handlebody-knot in S^3 and $M = \{m_1, m_2, \dots, m_g\}$ an (oriented and ordered) meridian system of H. Let E be the exterior of H, that is, the closure of $S^3 \setminus H$. Let $G = \pi_1(E)$ be the fundamental group of E. Let t_i be the homology class in the integral homology group $H_1(E)$ represented by m_i for $i = 1, 2, \dots, g$. Then, $H_1(E)$ is a free abelian group of rank g generated by $t_1, \dots, t_g$. Let $\gamma : G \to H_1(E)$ be the Hurewicz epimorphism. The covering space over E corresponding to the subgroup $\mathrm{Ker}(\gamma) = [G, G]$ of G is called the *universal abelian covering space* E_γ of E. Because $H_1(E)$ acts on E_γ as the covering transformation group, $H_1(E_\gamma)$ is regarded as a module over the integral group ring $\mathbb{Z}H_1(E)$ of $H_1(E)$. By regarding $H_1(E)$ as the multiplicative free abelian group $\Pi_{i=1}^{g}\langle t_i\rangle$ with basis $t_1, t_2, \dots, t_g$, we identify $\mathbb{Z}H_1(E)$ with the Laurent polynomial ring $\mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, \dots, t_g^{\pm1}]$ in the variables $t_1, \dots, t_g$. Thus, we can regard $H_1(E_\gamma)$ as a $\mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, \dots, t_g^{\pm1}]$-module. Let $p : E_\gamma \to E$ be the covering projection and b be a point in E. Then, $H_1(E_\gamma, p^{-1}(b))$ can also be regarded as a $\mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, \dots, t_g^{\pm1}]$-module.

Definition 2.1 (Alexander polynomial for handlebody-knots). The *Alexander matrix* A of a pair consisting of a handlebody-knot H and its meridian system M is an $m \times n$ presentation matrix of $H_1(E_\gamma, p^{-1}(b))$. The *d-th Alexander polynomial* $\Delta^{(d)}_{(H,M)}(t_1, t_2, \dots, t_g)$ of (H, M) is defined to be the greatest common divisor of all $(n-d)$-minors of A for $d = 0, 1, \dots, n-1$. For $d \geq n$, we define $\Delta^{(d)}_{(H,M)}(t_1, t_2, \dots, t_g) = 1$.

The Alexander polynomial is an invariant for a basis of $H_1(E)$ up to multiplication by units in $\mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, \dots, t_g^{\pm1}]$. We fix a meridian system M of H. Then, we can assume that basis of $H_1(E)$ is M. Thus, the Alexander polynomial is an invariant for a pair consisting of H and M up to multiplication by units in $\mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, \dots, t_g^{\pm1}]$. For simplicity, we denote $\Delta^{(g)}_{(H,M)}(t_1, t_2, \dots, t_g)$ by $\Delta_{(H,M)}(t_1, t_2, \dots, t_g)$, because $\Delta^{(g)}_{(H,M)}(t_1, t_2, \dots, t_g)$ is useful for genus g handlebody-knots. We can obtain $\Delta^{(d)}_{(H,M)}(t_1, t_2, \dots, t_g)$ using Fox's free calculus [5] or the C-complex for H [10].

The Alexander polynomial for (H, M) corresponds to the Alexander polynomial for a spatial graph Γ that represents H. In [7] and [8], Kinoshita introduced the Alexander polynomial for spatial graphs. In [9], Kinoshita introduced a basis of $\mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, \dots, t_g^{\pm1}]$ as a basis z of the integral first homology group $H_1(\Gamma)$ of Γ which is a dual basis of $H_1(E)$, and introduced the elementary ideals $E_d(\Gamma, z)$ associated with z as an invariant for spatial graphs. In [12], Suzuki introduced a representation of H as a g-leafed rose C. The basis of $\mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, \dots, t_g^{\pm1}]$ is determined by meridians of the constituent link of C for calculating the one-variable elementary ideal of H.

Let $MCG(\partial H)$ be the mapping class group of ∂H. Let $MCG^*(\partial H)$ be a subgroup of $MCG(\partial H)$ consisting of those homeomorphisms which can be extended to homeomorphisms of H onto itself. Let $\phi \in MCG^*(\partial H)$. Replacing a meridian system M with $\phi(M)$ of H corresponds to a change of basis for $H_1(E)$, which is represented by a matrix in $GL(g, \mathbb{Z})$. That is, there exists a matrix $\begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^g \\ x_2^1 & x_2^2 & \cdots & x_2^g \\ \vdots & \vdots & \ddots & \vdots \\ x_g^1 & x_g^2 & \cdots & x_g^g \end{bmatrix} \in GL(g, \mathbb{Z})$ such that $t_i \in H_1(E)$ is mapped to $t'_i = t_1^{x_1^i} t_2^{x_2^i} \cdots t_g^{x_g^i}$ for $i = 1, 2, \ldots, g$. Here $\{t_1, t_2, \ldots, t_g\}$ and $\{t'_1, t'_2, \ldots, t'_g\}$ are the basis of $H_1(E)$ induced from M and $\phi(M)$, respectively. Throughout this paper, we assume that the action of $GL(g, \mathbb{Z})$ on $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_g^{\pm 1}]$ is as above. Then the following lemma holds.

Lemma 2.2. $\Delta_{(H,\phi(M))}(t_1, t_2, \ldots, t_g) = \Delta_{(H,M)}(t'_1, t'_2, \ldots, t'_g)$.

We introduce an equivalence relation on $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$ as follows: For two Laurent polynomials f_1 and f_2 in $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$, we say that f_1 and f_2 are *equivalent*, denoted by $f_1 \sim f_2$, if f_1 is equal to f_2 up to multiplication by units in $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$ and the action of $GL(n, \mathbb{Z})$ on $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$.

We define an invariant for $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]/\sim$ as follows: For a Laurent polynomial $f = \sum_{i=1}^m c_i t_1^{x_1^i} t_2^{x_2^i} \cdots t_n^{x_n^i} \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$, let $T_i = c_i t_1^{x_1^i} t_2^{x_2^i} \cdots t_n^{x_n^i}$ be the i-th term of f, C_f the set $\{c_i\}$ of coefficients of terms $T_1, T_2, \ldots, T_m$, and P_f the set $\{\boldsymbol{p}^i = (x_1^i, x_2^i, \ldots, x_n^i) \in \mathbb{R}^n\}$ of position vectors determined by the exponents of T_i for $i = 1, 2, \ldots, m$. Note that $x_j^i \in \mathbb{Z}$ for $j = 1, 2, \ldots, n$. The terms $T_1, T_2, \ldots, T_m$ of f are mapped to mutually different terms of $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$ through multiplication by units in $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$ and the action of $GL(g, \mathbb{Z})$. Thus, the following lemmas hold.

Lemma 2.3. *The set C_f, up to multiplication by ± 1 to all elements of C_f, is an invariant for $f \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]/\sim$.*

Lemma 2.4. *The set P_f, up to parallel translation to all elements of P_f and linear transformation by $GL(n, \mathbb{Z})$ on $\mathbb{R}^n$ to all elements of P_f, is an invariant for $f \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]/\sim$.*

DEFINITION 2.5 (VERTEX-WEIGHTED GRAPH G_f). The vertex-weighted graph G_f for $f = \sum_{i=1}^m c_i t_1^{x_1^i} t_2^{x_2^i} \cdots t_n^{x_n^i} \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$ is a simple bipartite graph whose vertex set is a disjoint union of a black vertex set and a white vertex set. The black vertex set consists of black vertices $b_1, b_2, \ldots, b_m$ whose labels are $c_1, c_2, \ldots, c_m$, respectively. For each $(n+1)$-tuple of position vectors $\boldsymbol{p}^{i_0}, \boldsymbol{p}^{i_1}, \ldots, \boldsymbol{p}^{i_n}$ in P_f whose convex hull in $\mathbb{R}^n$ contains no vectors of $P_f \setminus \{\boldsymbol{p}^{i_0}, \boldsymbol{p}^{i_1}, \ldots, \boldsymbol{p}^{i_n}\}$, we take a white vertex labeled by the absolute value of the determinant of

$$\begin{bmatrix} x_1^{i_1} - x_1^{i_0} & x_1^{i_2} - x_1^{i_0} & \cdots & x_1^{i_n} - x_1^{i_0} \\ x_2^{i_1} - x_2^{i_0} & x_2^{i_2} - x_2^{i_0} & \cdots & x_2^{i_n} - x_2^{i_0} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{i_1} - x_n^{i_0} & x_n^{i_2} - x_n^{i_0} & \cdots & x_n^{i_n} - x_n^{i_0} \end{bmatrix}.$$

The white vertex is connected to the $(n+1)$ black vertices $b_{i_0}, b_{i_1}, \ldots, b_{i_n}$ by edges. The simple bipartite graph thus obtained is G_f

According to Lemma 2.3, the set of labels of black vertices up to multiplication by ± 1

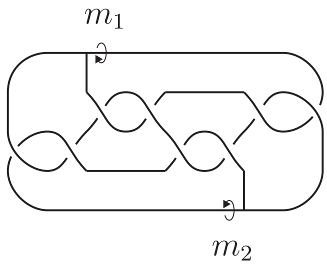

Fig. 1. A handlebody-knot H

are invariants for $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]/\sim$. The label of a white vertex is the n-volume of the n-parallelotope determined by the n vectors $\boldsymbol{p}^{i_1} - \boldsymbol{p}^{i_0}, \boldsymbol{p}^{i_2} - \boldsymbol{p}^{i_0}, \ldots, \boldsymbol{p}^{i_n} - \boldsymbol{p}^{i_0}$ in $\mathbb{R}^n$, which is an invariant for n-parallelotopes up to parallel translation and the linear transformation given by $GL(n, \mathbb{Z})$ on $\mathbb{R}^n$. Thus, according to Lemma 2.4, the labels of white vertices are invariants for $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]/\sim$. Note that if $m < n + 1$, then G_f does not have a white vertex and G_f is not connected.

An *isomorphism* of the vertex-weighted graphs G_f and G'_f is a bijection between the vertex sets of G_f and G'_f that maps the black vertex set and the white vertex set of G_f to the black vertex set and the white vertex set of G'_f, respectively, such that any two vertices of G_f are adjacent if and only if the images of the two vertices are adjacent in G'_f. If an isomorphism exists between G_f and G'_f, then G_f and G'_f are said to be *isomorphic*. The following lemma holds.

Lemma 2.6. *The isomorphism class of the vertex-weighted graph G_f up to multiplication by ± 1 to all labels of the black vertices is an invariant for $f \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]/\sim$.*

We define the vertex-weighted graph G_H for (H, M) as $G_{\Delta_{(H,M)}(t_1,t_2,\ldots,t_g)}$. Note that G_H does not depend on the choice of the meridian system of H. We have the following theorem. This is the main result of this paper.

Theorem 2.7. *The isomorphism class of the vertex-weighted graph G_H up to multiplication by ± 1 to all labels of the black vertices is an invariant for handlebody-knots.*

EXAMPLE 2.8. Let H be a handlebody-knot and M its meridian system as depicted in Fig.1. Then, the Alexander polynomial $\Delta_{(H,M)}(t_1, t_2)$ of (H, M) is $t_1^2 - t_1 + t_2^2 - t_2 + t_1 t_2$. We have $T_1 = t_1^2$, $T_2 = -t_1$, $T_3 = t_2^2$, $T_4 = -t_2$, and $T_5 = t_1 t_2$; $c_1 = 1$, $c_2 = -1$, $c_3 = 1$, $c_4 = -1$, and $c_5 = 1$; and $\boldsymbol{p}^1 = (2, 0)$, $\boldsymbol{p}^2 = (1, 0)$, $\boldsymbol{p}^3 = (0, 2)$, $\boldsymbol{p}^4 = (0, 1)$, and $\boldsymbol{p}^5 = (1, 1)$ in $\mathbb{R}^2$, as depicted in Fig.2. We take the black vertex b_1 labeled with $c_1 = 1$. Similarly, we have b_2, b_3, b_4, and b_5 as depicted in Fig.3. For three tuple of position vectors $\boldsymbol{p}^1$, $\boldsymbol{p}^2$, and $\boldsymbol{p}^3$ in $P_{\Delta_{(H,M)}(t_1,t_2)}$ whose convex hull in $\mathbb{R}^3$, $\boldsymbol{p}^5$ is in the convex hull. Therefore, G_H does not contain a white vertex connected to b_1, b_2, and b_3.

For three tuple of position vectors $\boldsymbol{p}^1$, $\boldsymbol{p}^2$, and $\boldsymbol{p}^4$ in $P_{\Delta_{(H,M)}(t_1,t_2)}$ whose convex hull in $\mathbb{R}^3$ contains no vectors of $P_{\Delta_{(H,M)}(t_1,t_2)} \setminus \{\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{p}^4\}$, we take a white vertex labeled by 1 which is the absolute value of the determinant of $\begin{bmatrix} 1-2 & 0-2 \\ 0-0 & 1-0 \end{bmatrix}$. The white vertex is connected to the three black vertices $\boldsymbol{p}^1$, $\boldsymbol{p}^2$, and $\boldsymbol{p}^4$ by edges. Similarly, we have other white vertices, and we have G_H as depicted in Figure 3.

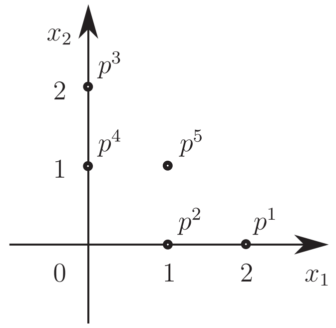

Fig. 2. Position vectors $\boldsymbol{p}^1$, $\boldsymbol{p}^2$, $\boldsymbol{p}^3$, $\boldsymbol{p}^4$, and $\boldsymbol{p}^5$

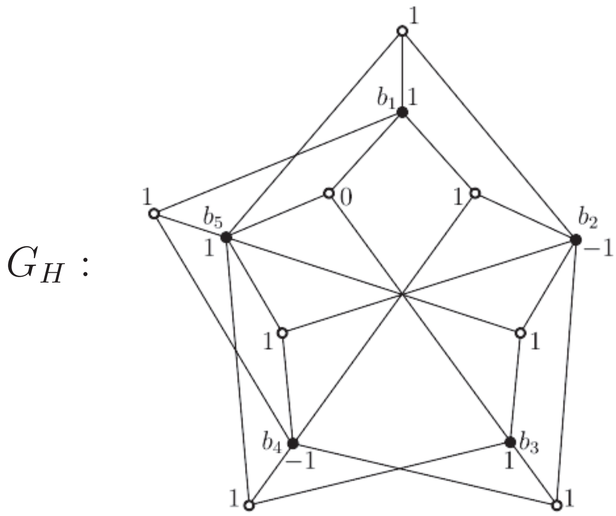

Fig. 3. The vertex-weighted graph G_H

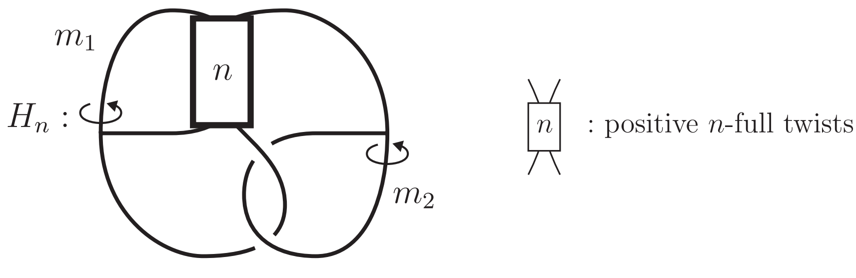

Fig. 4. A handlebody-knot H_n

The following example shows that there exist infinitely many handlebody-knots whose invariants are mutually different.

EXAMPLE 2.9. Let H_n be a handlebody-knot for $n \neq 0$ and M its meridian system, as depicted in Fig.4. We have $\Delta_{(H_n,M)}(t_1,t_2) = t_1^n + t_2 - 1$. The invariant G_{H_n} is as depicted in

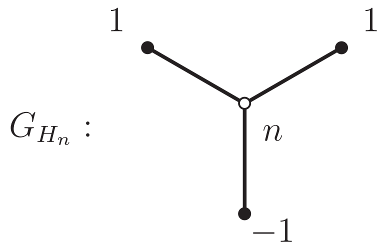

Fig.5. The vertex-weighted graph G_{H_n}

Fig.5.

3. Irreducibility for handlebody-knots

In this section, as an application of Theorem 2.7, we describe a sufficient condition for a handlebody-knot to be irreducible. A handlebody-knot H is *reducible* if there exists a 2-sphere in S^3 such that the intersection of H and the 2-sphere is an essential disk properly embedded in H. A handlebody-knot is *irreducible* if it is not reducible. In [12], Suzuki introduced the irreducibility as the "primeness" of handlebody-knots and demonstrated the uniqueness of the factorization of H. In [3], Ishii and Kishimoto provided methods for detecting the irreducibility using the quandle coloring invariant.

Let B_1 and B_2 be 3-balls in S^3 such that $B_1 \cup B_2 = S^3$ and $B_1 \cap B_2 = \partial B_1 = \partial B_2$. Let H_i be a genus g_i handlebody-knot in B_i for $i = 1, 2$. When $H_1 \cap H_2$ is one disk, $H_1 \cup H_2$ is a genus $g_1 + g_2$ handlebody-knot in S^3. We denote this by $H_1 \sharp H_2$, where we remark that the handlebody-knot $H_1 \sharp H_2$ depends only on the handlebody-knots H_1 and H_2. If a handlebody-knot H is reducible, then there exist handlebody-knots H_1 and H_2 such that $H = H_1 \sharp H_2$. As $S^3 \setminus H$ is the boundary-connected sum of those of H_1 and H_2, the fundamental group of $S^3 \setminus H$ is the free product of those of H_1 and H_2. Thus, the following lemma holds [12].

Lemma 3.1. *For a genus g_1 handlebody-knot H_1 and genus g_2 handlebody-knot H_2 and their meridian systems M_1 and M_2, respectively, The Alexander polynomial $\Delta^{(g_1+g_2)}_{(H_1\sharp H_2, M_1\cup M_2)}(t_1, t_2, \ldots, t_{g_1+g_2})$ of $(H_1 \sharp H_2, M_1 \cup M_2)$ is the product of $\Delta^{(g_1)}_{(H_1,M_1)}(t_1, t_2, \ldots, t_{g_1})$ and $\Delta^{(g_2)}_{(H_2,M_2)}(t_{g_1+1}, t_{g_1+2}, \ldots, t_{g_1+g_2})$.*

Because $\mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, \ldots, t_n^{\pm1}]$ is a unique factorization domain, a Laurent polynomial $f \in \mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, \ldots, t_n^{\pm1}]$ can be uniquely expressed as $cf_1f_2\cdots f_m$, where $f_i \in \mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, \ldots, t_n^{\pm1}]$ is irreducible for $i = 1, 2, \ldots, m$ and $c \in \mathbb{Z}$. For a Laurent polynomial $f \in \mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, \ldots, t_n^{\pm1}]$, we define the set $\hat{G}_f$ as follows:

$$\hat{G}_f = \begin{cases} \{G_f \mid f \in \mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, \ldots, t_n^{\pm1}]\} & \text{if } f = 0 \\ \{G_{f_i} \mid 1 \le i \le m\} & \text{otherwise.} \end{cases}$$

If $f = 0$, then $\hat{G}_f$ is an infinite set. If $f = 1$, then $\hat{G}_f$ is an empty set. From the definition of $\hat{G}_f$, we have the following lemma. We use this to prove Theorem 4.2 in Section 4.

Lemma 3.2. *For Laurent polynomials f and f' in $\mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, \ldots, t_n^{\pm1}]$, if $f | f'$, then $\hat{G}_f \subset \hat{G}_{f'}$.*

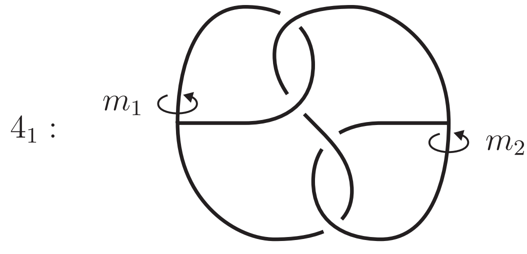

Fig. 6. The handlebody-knot 4_1

We define the set $\hat{G}_H^{(d)}$ by $\hat{G}_{\Delta^{(d)}_{(H,M)}(t_1,t_2,\dots,t_g)}$ for the d-th Alexander polynomial $\Delta^{(d)}_{(H,M)}(t_1,t_2,\dots,t_g)$ of (H,M). By Theorem 2.7, $\hat{G}_H^{(d)}$ is an invariant for handlebody-knots. For simplicity, we denote $\hat{G}_H^{(g)}$ by $\hat{G}_H$ for genus g handlebody-knots. The following theorem gives a sufficient condition for a handlebody-knot to be irreducible.

Theorem 3.3. *For a handlebody-knot H, if there exists $G_f \in \hat{G}_H$ that has a white vertex with a nonzero label, then H is irreducible.*

Proof. Let H_i be a genus g_i handlebody-knot and M_i its meridian system for $i = 1, 2$. Set $H = H_1 \sharp H_2$ and $M = M_1 \cup M_2$. Note that H is a genus $g_1 + g_2$ handlebody-knot and M is its meridian system. We show that, for any $G_f \in \hat{G}_H^{(g_1+g_2)}$, all labels of white vertices of G_f are equal to zero. By Lemma 3.1, $\Delta^{(g_1+g_2)}_{(H,M)}(t_1,t_2,\dots,t_{g_1+g_2})$ is equal to the product of $\Delta^{(g_1)}_{(H_1,M_1)}(t_1,t_2,\dots,t_{g_1})$ and $\Delta^{(g_2)}_{(H_2,M_2)}(t_{g_1+1},t_{g_1+2},\dots,t_{g_1+g_2})$. Let $f \in \mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, \dots, t_{g_1+g_2}^{\pm1}]$ be an irreducible polynomial that is a factor of $\Delta^{(g_1+g_2)}_{(H,M)}(t_1,t_2,\dots,t_{g_1+g_2})$. Then, f is a factor of $\Delta^{(g_1)}_{(H_1,M_1)}(t_1,t_2,\dots,t_{g_1})$ or $\Delta^{(g_2)}_{(H_2,M_2)}(t_{g_1+1},t_{g_1+2},\dots,t_{g_1+g_2})$, because $\mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, \dots, t_{g_1+g_2}^{\pm1}]$ is a unique factorization domain.

If f is a factor of $\Delta^{(g_1)}_{(H_1,M_1)}(t_1,t_2,\dots,t_{g_1})$, then all labels of white vertices of G_f are zero, because the label of the white vertex of G_f is the absolute value of the (g_1+g_2)-volume of the degenerated (g_1+g_2)-parallelotope determined by the g_1+g_2 vectors in the g_1-dimensional vector space $\mathbb{R}^{g_1} \subset \mathbb{R}^{g_1+g_2}$. Similarly, if f is a factor of $\Delta^{(g_2)}_{(H_2,M_2)}(t_{g_1+1},t_{g_1+2},\dots,t_{g_1+g_2})$, then all labels of white vertices of G_f are zero. Thus, Theorem 3.3 holds. $\square$

EXAMPLE 3.4. The handlebody-knot depicted in Fig.6 is 4_1 in the table of genus 2 handlebody-knots with up to six crossings in [4]. Let M be a meridian system of 4_1, as depicted in Fig.6. We have $\Delta_{(4_1,M)}(t_1,t_2) = t_1 + t_2 - 1$, and G_{4_1} is as depicted in Fig.7. As $t_1 + t_2 - 1$ is irreducible, we have $\hat{G}_{4_1} = \{G_{4_1}\}$. Because G_{4_1} has a white vertex whose label is 1, 4_1 is irreducible by Theorem 3.3.

The following example shows that there exists an irreducible genus g handlebody-knot for each genus g.

EXAMPLE 3.5. Let H_g be a genus g handlebody-knot and M a meridian system of H_g, as depicted in Fig.8. Note that H_2 is 4_1. Taniyama showed that H_g is irreducible as a spatial graph [13]. We show that H_g is irreducible as a handlebody-knot using Theorem 3.3.

We have

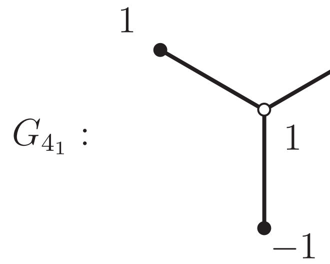

Fig. 7. The vertex-weighted graph G_{4_1}

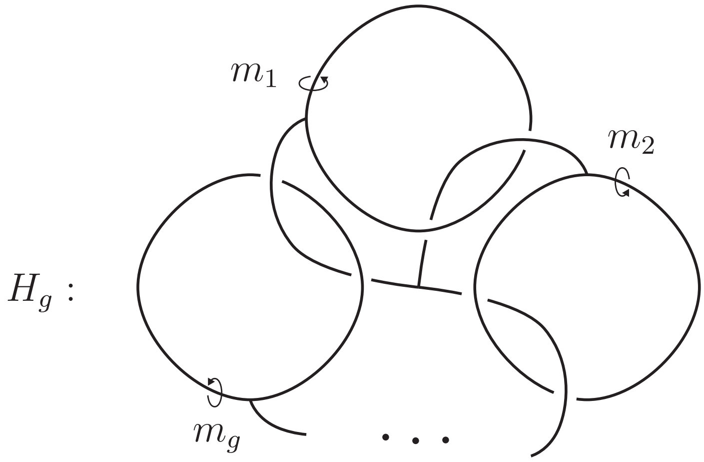

Fig. 8. A handlebody-knot H_g

$$\begin{aligned}\Delta_{(H_g,M)}(t_1,t_2,\ldots,t_g) &= \prod_{i=1}^{g}(t_i-1)-\prod_{i=1}^{g}t_i\\ &= \sum_{i=1}^{g-1}\left((-1)^{g+i}\sum_{j_1,j_2,\ldots,j_i\in S}t_{j_1}t_{j_2}\cdots t_{j_i}\right)+(-1)^g,\end{aligned}$$

where $S=\{1,2,\ldots,g\}$ and $j_1,j_2,\ldots,j_i$ are mutually different elements in S. By induction on g, we can check $\Delta_{(H_g,M)}(t_1,t_2,\ldots,t_g)$ is irreducible. Hence, we have $\hat{G}_{H_g}=\{G_{H_g}\}$. The set $P_{\Delta_{(H_g,M)}(t_1,t_2,\ldots,t_g)}$ has unit vectors $\boldsymbol{e}_1,\boldsymbol{e}_2,\ldots,\boldsymbol{e}_g$ and the zero vector $\mathbf{0}$ in $\mathbb{R}^g$. For each (g+1)-tuple of position vectors $\boldsymbol{e}_1,\boldsymbol{e}_2,\ldots,\boldsymbol{e}_g$ and $\mathbf{0}$ in $P_{\Delta_{(H_g,M)}(t_1,t_2,\ldots,t_g)}$ whose convex hull in $\mathbb{R}^g$ contains no vectors of $P_f\setminus\{\boldsymbol{e}_1,\boldsymbol{e}_2,\ldots,\boldsymbol{e}_g,\mathbf{0}\}$, we take a white vertex labeled by 1. Thus, G_{H_g} has a white vertex whose label is 1, and H_g is irreducible by Theorem 3.3.

4. Constituent links of a handlebody-knot

In this section, as an application of Theorem 2.7, we describe a necessary condition for a link to be a constituent link of a handlebody-knot. In [12], Suzuki introduced a g-leafed rose that is a connected spatial graph as follows: A *g-leafed rose* $C=K_1\cup K_2\cup\cdots\cup K_g\cup T$ consists of a g-component link $L=K_1\cup K_2\cup\cdots\cup K_g$ and a star graph T, as depicted in Fig.9.

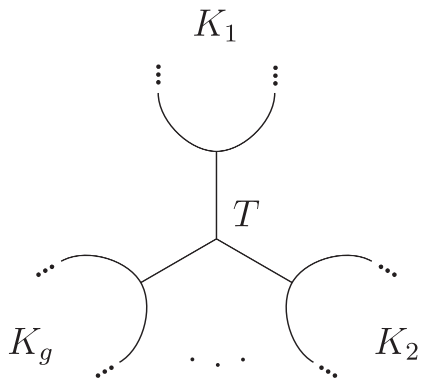

Fig.9. A g-leafed rose

We call L the constituent link of C. For a genus g handlebody-knot, there exist infinitely many g-leafed roses representing the handlebody-knot. We define a *constituent link* of H as the constituent link of a g-leafed rose that represents H. Therefore, there exist infinitely many constituent links of H.

Let M be the meridian system of the constituent link L of C. Let $E_d(C, M)$ and $E_d(L)$ be the d-th elementary ideals of (C, M) and L, respectively; that is, the ideal of $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_g^{\pm 1}]$ generated by all $(n-d)$-minors of the Alexander matrix of C and L, respectively. The following theorem was proved by Suzuki in [11].

Theorem 4.1. [11] $E_{d+g-1}(C, M) \supset E_d(L)$.

Let $\Delta_L^{(d)}(t_1, t_2, \ldots, t_g)$ be the d-th Alexander polynomial of L. We define the set $\hat{G}_L^{(d)}$ as $\hat{G}_{\Delta_L^{(d)}(t_1,t_2,\ldots,t_g)}$. By Theorem 2.7, $\hat{G}_L^{(d)}$ is an invariant for links. The following theorem describes a necessary condition for a link to be a constituent link of a handlebody-knot.

Theorem 4.2. *For a constituent link L of a genus g handlebody-knot H, $\hat{G}_H^{(d+g-1)} \subset \hat{G}_L^{(d)}$.*

Proof. Let C be a g-leafed rose that represents H and M the meridian system of the constituent link L of C. Then, we have the $(d+g-1)$-th Alexander polynomial $\Delta_{(C,M)}^{(d+g-1)}(t_1, t_2, \ldots, t_g)$ of the pair (C, M) and the d-th Alexander polynomial $\Delta_L^{(d)}(t_1, t_2, \ldots, t_g)$ of L.

The Alexander polynomials $\Delta_{(C,M)}^{(d+g-1)}(t_1, t_2, \ldots, t_g)$ and $\Delta_L^{(d)}(t_1, t_2, \ldots, t_g)$ can be uniquely expressed as $uf_1 f_2 \cdots f_n$ and $u' f_1' f_2' \cdots f_m'$, respectively, because $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_g^{\pm 1}]$ is a unique factorization domain. Here, u and u' are units of $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_g^{\pm 1}]$, and f_i, f_j' are irreducible in $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_g^{\pm 1}]$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. We have $\hat{G}_H^{(d+g-1)} = \hat{G}_{\Delta_{(C,M)}^{(d+g-1)}(t_1,t_2,\ldots,t_g)}$.

By Theorem 4.1, $E_{d+g-1}(C, M) \supset E_d(L)$. If $\Delta_{(C,M)}^{(d+g-1)}(t_1, t_2, \ldots, t_g) \neq 0$, then $\Delta_{(C,M)}^{(d+g-1)}(t_1, t_2, \ldots, t_g) \mid \Delta_L^{(d)}(t_1, t_2, \ldots, t_g)$. By Lemma 3.2, $\hat{G}_H^{(d+g-1)} \subset \hat{G}_L^{(d)}$. If $\Delta_{(C,M)}^{(d+g-1)}(t_1, t_2, \ldots, t_g) = 0$, then $\Delta_L^{(d)}(t_1, t_2, \ldots, t_g) = 0$ by Theorem 4.1. Thus, we have $\hat{G}_H^{(d+g-1)} \subset \hat{G}_L^{(d)}$. □

For simplicity, we denote $\hat{G}_L^{(1)}$ as $\hat{G}_L$. By Theorem 4.2, for a constituent link L of a genus g handlebody-knot H, $\hat{G}_H \subset \hat{G}_L$.

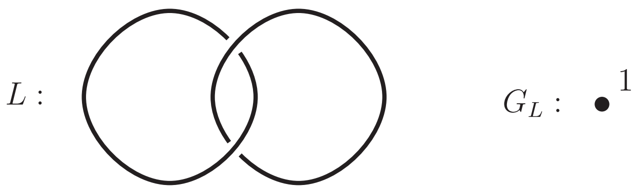

Fig. 10. The Hopf link L and the vertex-weighted graph G_L

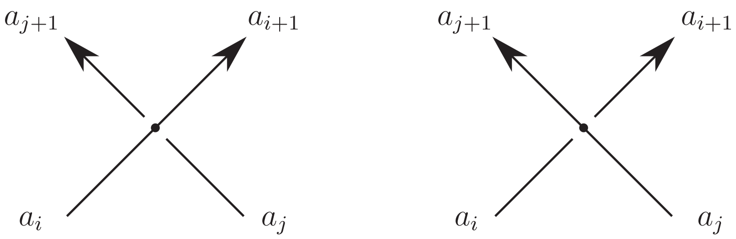

Fig. 11. crossing points of G_+ and G_-

Example 4.3. In Example 3.4, we used G_{4_1} for 4_1. The 1st Alexander polynomial of the Hopf link L is 1, and G_L is as depicted in Fig.10. We have $\hat{G}_L = \emptyset$. As $\hat{G}_{4_1} \not\subset \hat{G}_L$, the Hopf link is not a constituent link of 4_1 by Theorem 4.2.

5. An equivalence class of handlebody-knots

In this section, we introduce an equivalence class of handlebody-knots. A handlebody-knot is represented by a connected spatial graph. A *crossing change* of a handlebody-knot H is a crossing change of a connected spatial graph that represents H. For a handlebody-knot H, a crossing change between two edges of H whose meridians are null-homologous in $S^3 \backslash H$ is called an *N-crossing change*. We say that handlebody-knots H_1 and H_2 are *N-equivalent* if they are transformed into each other by a finite sequence of N-crossing changes and an isotopy of S^3.

The following proposition shows that the Alexander polynomial is an invariant for N-equivalence classes of handlebody-knots up to multiplication by units in $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_g^{\pm 1}]$. This proposition is thought to be mathematical folklore.

Proposition 5.1. *The Alexander polynomial of a spatial graph* Γ *does not change under the N-crossing change on* Γ.

Proof. Spatial graphs Γ_+ and Γ_-, which are as depicted in Fig.11, are identical outside a small 3-ball. Let G_+ and G_- be the fundamental groups of $S^3 \setminus \Gamma_+$ and $S^3 \setminus \Gamma_-$, respectively. We take the generators a_i, a_{i+1}, a_j, and a_{j+1} of the Wirtinger presentation of G_+ and G_- around the crossing point, as depicted in Fig.11.

We have $r_1 : a_i = a_{i+1}$ and $r_2 : a_j a_i = a_{i+1} a_{j+1}$ as relaters of G_+, and $r_1' : a_j = a_{j+1}$ and $r_2 : a_j a_i = a_{i+1} a_{j+1}$ as relaters of G_-. Therefore, we have the following presentations of G_+ and G_-:

$$G_+ = \langle a_1, a_2, \dots, a_m | r_1, r_2, \dots, r_n \rangle, \; G_- = \langle a_1, a_2, \dots, a_m | r_1', r_2, \dots, r_n \rangle.$$

We assume that the generators a_i, a_{i+1}, a_j, and a_{j+1} are mapped to 1 by the abelianizer α_+ of G_+ and α_- of G_-. Let A_+ and A_- be the Alexander matrices of G_+ and G_-, respectively.

$$
\begin{array}{ccl}
 & & \begin{array}{ccccccccccc} & & i & i+1 & & & & j & j+1 & & \\ & & \vee & \vee & & & & \vee & \vee & & \end{array} \\
A_+ & \sim & \begin{bmatrix} \cdots & 0 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & -1 & 0 & \cdots & 0 & 1 & -1 & 0 & \cdots \\ \cdots & * & \boldsymbol{c}_i & \boldsymbol{c}_{i+1} & * & \cdots & * & \boldsymbol{c}_j & \boldsymbol{c}_{j+1} & * & \cdots \end{bmatrix} \\
 & \sim & \begin{bmatrix} \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 1 & -1 & 0 & \cdots \\ \cdots & * & \boldsymbol{c}_i & \boldsymbol{c}_i + \boldsymbol{c}_{i+1} & * & \cdots & * & \boldsymbol{c}_j & \boldsymbol{c}_{j+1} & * & \cdots \end{bmatrix} \\
 & \sim & \begin{bmatrix} \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & -1 & 0 & \cdots \\ \cdots & * & \boldsymbol{c}_i + \boldsymbol{c}_{i+1} & * & \cdots & * & \boldsymbol{c}_j & \boldsymbol{c}_{j+1} & * & \cdots \end{bmatrix} \\
 & \sim & \begin{bmatrix} \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots \\ \cdots & * & \boldsymbol{c}_i + \boldsymbol{c}_{i+1} & * & \cdots & * & \boldsymbol{c}_j & \boldsymbol{c}_j + \boldsymbol{c}_{j+1} & * & \cdots \end{bmatrix} \\
 & \sim & \begin{bmatrix} \cdots & * & \boldsymbol{c}_i + \boldsymbol{c}_{i+1} & * & \cdots & * & \boldsymbol{c}_j + \boldsymbol{c}_{j+1} & * & \cdots \end{bmatrix}
\end{array}
$$

Similarly, $A_- \sim \begin{bmatrix} \cdots & * & \boldsymbol{c}_i + \boldsymbol{c}_{i+1} & * & \cdots & * & \boldsymbol{c}_j + \boldsymbol{c}_{j+1} & * & \cdots \end{bmatrix}$. Thus, we have $A_+ \sim A_-$. □

By Proposition 5.1, we have the following corollary of Theorem 2.7.

Corollary 5.2. *The isomorphism class of the vertex-weighted graph G_H, up to multiplication by ± 1 to all labels of the black vertices, is an invariant for the N-equivalence classes of handlebody-knots.*

EXAMPLE 5.3. The handlebody-knot depicted in Fig.12 is 5_4 in the table of genus 2 handlebody-knots with up to six crossings in [4]. It is clear that 5_4 is N-equivalent to the trivial handlebody-knot 0_1. Thus, $\Delta_{(5_4,M)}(t_1, t_2) = \Delta_{(0_1,M)}(t_1, t_2) = 1$.

5_4 :

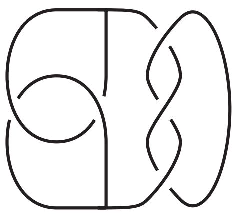

Fig. 12. The handlebody-knot 5_4

Appendix A Table of G_H and $\hat{G}_H$

In this appendix, we present the table of $\Delta^{(2)}_{(H,M)}(t_1, t_2)$, G_H, and $\hat{G}_H$ for handlebody-knots in the table of genus 2 handlebody-knots with up to six crossings in [4]. Here, M is a meridian system of H. Let G_1, G_2, G_3, G_4, and G_5 be the vertex-weighted graphs depicted in Fig.13. Then, we have Table 1.

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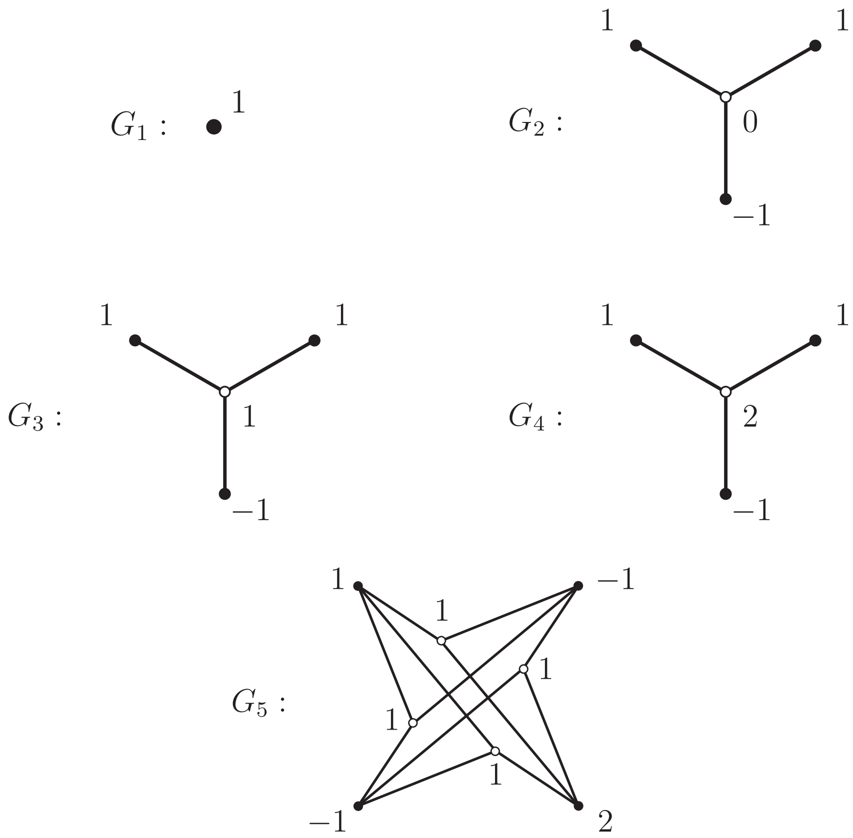

Fig. 13. The vertex-weighted graph G_1, G_2, G_3, G_4, and G_5

Table 1. Table of $\varDelta^{(2)}_{(H,M)}(t_1,t_2)$, G_H, and $\hat{G}_H$.

H	$\varDelta^{(2)}_{(H,M)}(t_1,t_2)$	G_H	$\hat{G}_H$
0_1	1	G_1	$\emptyset$
4_1	t_1+t_2-1	G_3	$\{G_3\}$
5_1	1	G_1	$\emptyset$
5_2	1	G_1	$\emptyset$
5_3	1	G_1	$\emptyset$
5_4	1	G_1	$\emptyset$
6_1	$t_1^2+t_2-1$	G_4	$\{G_4\}$
6_2	1	G_1	$\emptyset$
6_3	1	G_1	$\emptyset$
6_4	1	G_1	$\emptyset$
6_5	1	G_1	$\emptyset$
6_6	1	G_1	$\emptyset$
6_7	$t_1t_2-t_1-t_2+2$	G_5	$\{G_5\}$
6_8	1	G_1	$\emptyset$
6_9	1	G_1	$\emptyset$
6_{10}	1	G_1	$\emptyset$
6_{11}	1	G_1	$\emptyset$
6_{12}	1	G_1	$\emptyset$
6_{13}	1	G_1	$\emptyset$
6_{14}	$t_1^2-t_1+1$	G_2	$\{G_2\}$
6_{15}	$t_1^2-t_1+1$	G_2	$\{G_2\}$
6_{16}	1	G_1	$\emptyset$

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Osaka City University
Advanced Mathematical Institute
3–3–138 Sugimoto, Sumiyoshi-ku Osaka 558
Japan

e-mail: sokazaki@sci.osaka-cu.ac.jp