Kato, S. and Takano, K.
Osaka J. Math.
57 (2020), 711–736

ON SOME RELATIVELY CUSPIDAL REPRESENTATIONS: CASES OF GALOIS AND INNER INVOLUTIONS ON GL_n

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(Received January 22, 2019, revised April 11, 2019)

Abstract

Relatively cuspidal representations attached to a p-adic symmetric space G/H are thought of as the building blocks for all the irreducible H-distinguished representations of G. This work provides certain new examples of relatively cuspidal representations. We study three examples of symmetric spaces; $\mathrm{GL}_n(E)/\mathrm{GL}_n(F)$, $\mathrm{GL}_{2m}(F)/\mathrm{GL}_m(E)$, and $\mathrm{GL}_n(F)/(\mathrm{GL}_{n-r}(F) \times \mathrm{GL}_r(F))$ where E/F is a quadratic extension of p-adic fields. Those representations are given by induction from cuspidal distinguished representations of particular kinds of parabolic subgroups stable under the involution.

Introduction

Let G be a reductive p-adic group, H the fixed point subgroup of an involution σ on G, and Z the center of G. A smooth representation π of G is said to be H-distinguished if it carries a non-zero H-invariant linear form. Such representations are the main object of harmonic analysis of the symmetric space G/H, via Frobenius reciprocity

$$\mathrm{Hom}_H(\pi, \mathbf{1}) \simeq \mathrm{Hom}_G(\pi, C^\infty(G/H)).$$

H-matrix coefficients of π are the right H-invariant functions $\varphi_{\Lambda,v}$ on G defined by

$$\varphi_{\Lambda,v}(g) = \langle \Lambda, \pi(g^{-1})v \rangle \quad (g \in G)$$

for $\Lambda \in \mathrm{Hom}_H(\pi, \mathbf{1})$ and $v \in \pi$. We say that an H-distinguished representation π of G is *H-relatively cuspidal* if all the H-matrix coefficients of π are compactly supported modulo ZH.

In our earlier work [13, Theorem 7.1], we gave the following result which might be regarded as a basic theorem towards the classification of irreducible H-distinguished representations of G:

> *For an irreducible H-distinguished representation π of G, there exists a σ-split parabolic subgroup Q of G and an irreducible $L \cap H$-relatively cuspidal representation ρ of $L = Q \cap \sigma(Q)$ such that π is a subrepresentation of* $\mathrm{Ind}_Q^G(\rho)$.

Here, a parabolic subgroup Q of G is said to be σ-split if Q and $\sigma(Q)$ are opposite. This theorem is a symmetric space analogue of Jacquet's subrepresentation theorem [2, 2.5]. Hence, as an analogue of Harish-Chandra's *philosophy of cusp forms*, relatively cuspidal

2010 Mathematics Subject Classification. Primary 22E50; Secondary 11F70, 20G05, 22E35.

This work was supported by Japan Society for the Promotion of Science KAKENHI Grant Number JP26400011.

representations are thought of as "building blocks" for all distinguished representations.

In this paper, we give a construction of relatively cuspidal representations for the following three symmetric spaces:

- $\mathrm{GL}_n(E)/\mathrm{GL}_n(F)$ (in Section 2).
- $\mathrm{GL}_{2m}(F)/\mathrm{GL}_m(E)$ (in Section 3).
- $\mathrm{GL}_n(F)/(\mathrm{GL}_{n-r}(F) \times \mathrm{GL}_r(F))$ (in Section 4).

Here, E/F is a quadratic extension of non-archimedean local fields. The method of construction is the induction from cuspidal distinguished representations of σ-stable parabolic subgroups. So the examples provided in this paper are non-cuspidal but relatively cuspidal ones. Note in particular that there is no irreducible cuspidal distinguished representation for the third case with $n - r \neq r$ [15].

It is known that cuspidal distinguished representations are relatively cuspidal [13, Proposition 8.1]. Such representations have been studied by several authors (e.g., [6], [7], [8]). There were only few examples of non-cuspidal but relatively cuspidal representations. At first, [13, 8.2, 8.3] gave such examples for the symmetric spaces $\mathrm{GL}_{2n}(F)/\mathrm{Sp}_n(F)$ and $\mathrm{GL}_n(F)/(\mathrm{GL}_{n-1}(F) \times \mathrm{GL}_1(F))$. The method employed in this paper is, in a sense, a simplified reformulation of the one in [13]. A part of our results is found also in [18], [20] for the symmetric space $\mathrm{GL}_n(E)/\mathrm{GL}_n(F)$, and in [14], [20] for $\mathrm{GL}_{2m}(F)/(\mathrm{GL}_m(F) \times \mathrm{GL}_m(F))$.

We claim that the representations of the form $\mathrm{Ind}_P^G(\rho)$, for σ-stable parabolic subgroups $P = MU$ and cuspidal $M \cap H$-distinguished representations ρ of M, are H-relatively cuspidal (for the above three symmetric spaces). On the contrary, we believe that H-relatively cuspidal representations will not appear in the composition series of representations $\mathrm{Ind}_Q^G(\rho)$ for any proper σ-split parabolic subgroup Q and any $L \cap H$-distinguished representation ρ of $L = Q \cap \sigma(Q)$ (again from the analogy with the *philosophy of cusp forms*). Some related matter will appear at the end of section 4.

Let us summarize the contents of this paper. Section 1 gives preparations for whole of the paper, including the construction of H-distinguished representations by induction from σ-stable parabolic subgroups (**1.2**) and the criterion of relative cuspidality in terms of Jacquet modules along σ-split parabolic subgroups (**1.4**). From sections 2 to 4, the study for the three cases will be given separately. However, the procedures of these three sections are parallel and constituted from the following issues:

- Determination of σ-stable parabolic subgroups suitable for our construction: **2.2**, **3.2**, and **4.3** or **4.8**.
- Computation of the character $\mu_{M\cap H}$ used in the construction of H-distinguished representations: **2.2**, **3.3**, and **4.9**.
- Statement of the main result: **2.4**, **3.5**, and **4.11**.
- Description of maximal σ'-split parabolic subgroups for a suitable conjugate σ' of σ: **2.5**, **3.6**, and **4.12**.
- Proof of relative cuspidality by studying Jacquet modules along maximal σ'-split parabolic subgroups: **2.6**, **3.7**, and **4.13**.

Section 5 is for an additional discussion on the choice of stable parabolic subgroups used in our construction. We consider some candidates for relevant parabolic subgroups based on maximal σ-split tori which are F-anisotropic modulo the center.

We are grateful to the referee for his or her careful reading, useful comments and sugges-

tions.

1. Preliminaries and preparations

1.1. Let F be a non-archimedean local field, of which the residual characteristic is not equal to 2. Let $\mathbf{G}$ be a connected reductive F-group, σ an F-involution on $\mathbf{G}$, and $\mathbf{H}$ the subgroup of σ-fixed points in $\mathbf{G}$. Let $\mathbf{Z}$ denote the center of $\mathbf{G}$. For an algebraic F-group denoted by a boldface capital letter, the group of its F-points is denoted by the corresponding ordinary capital, such as $G = \mathbf{G}(F)$.

By a representation of G (or a G-module), we always mean a smooth representation of G on a complex vector space. For a representation π of G and an automorphism ϕ on G, ${}^{\phi}\pi$ denotes the representation $\pi \circ \phi^{-1}$ of G on the same vector space. If ρ is a representation of a subgroup M of G and $\phi = \mathrm{Ad}(w)$ for $w \in N_G(M)$, the normalizer of M in G, then ${}^{\phi}\rho$ is denoted also by ${}^{w}\rho$.

For a quasi-character μ of H, a representation π of G is said to be *(H,μ)-distinguished* if $\mathrm{Hom}_H(\pi,\mu) \neq \{0\}$, and *$H$-distinguished* if it is $(H,\mathbf{1})$-distinguished, where $\mathbf{1}$ denotes the trivial character of H.

The following elementary lemma is easy to prove.

Lemma. *Let π be an H-distinguished representation of G.*
(1) *If π has a central character, then it is trivial on $Z \cap H$.*
(2) *Any filtration of π consisting of G-submodules of finite length has an H-distinguished subquotient.*

Right H-invariant functions on G of the form

$$\varphi_{\Lambda,v}(g) = \langle \Lambda, \pi(g^{-1})v \rangle$$

for $v \in \pi$ and $\Lambda \in \mathrm{Hom}_H(\pi,\mathbf{1})$ are called H-matrix coefficients of π. An H-distinguished representation π of G is said to be *H-relatively cuspidal* if the support of $\varphi_{\Lambda,v}$ is compact modulo ZH for any $v \in \pi$ and $\Lambda \in \mathrm{Hom}_H(\pi,\mathbf{1})$.

1.2. If a parabolic F-subgroup $\mathbf{P}$ of $\mathbf{G}$ is σ-stable, then so is its unipotent radical $\mathbf{U} = R_u(\mathbf{P})$. Also, there exists a σ-stable Levi subgroup, say $\mathbf{M}$, of $\mathbf{P}$. We shall call such a pair $(\mathbf{P},\mathbf{M})$ a *σ-stable pair*. For such a pair, we have a Levi decomposition

$$\mathbf{P} \cap \mathbf{H} = (\mathbf{M} \cap \mathbf{H})(\mathbf{U} \cap \mathbf{H}).$$

Let δ_P (resp. $\delta_{P\cap H}$) be the modulus character of P (resp. of $P \cap H$). For each σ-stable pair $(\mathbf{P},\mathbf{M})$, we define the quasi-character $\mu_{M\cap H}$ of $M \cap H$ by

$$\mu_{M\cap H} = \delta_{P\cap H} \cdot \delta_P^{-1/2}|_{M\cap H}.$$

The following is an easy way to construct H-distinguished representation of G.

Proposition. *If ρ is an $(M \cap H, \mu_{M\cap H})$-distinguished representation of M, then the normalized induced representation $\mathrm{Ind}_P^G(\rho)$ is H-distinguished.*

Proof. Given a non-zero $\lambda \in \mathrm{Hom}_{M\cap H}(\rho, \mu_{M\cap H})$, one has a non-zero H-invariant linear form Λ on $\mathrm{Ind}_P^G(\rho)$ defined by

$$\langle \Lambda, \phi \rangle = \int_{P \cap H \backslash H} \langle \lambda, \phi(\dot{h}) \rangle \, d\dot{h}$$

where $d\dot{h}$ denotes a fixed quasi-invariant measure on $P \cap H\backslash H$. See [19] for details. □

We expect that this proposition applied to cuspidal $(M \cap H, \mu_{M\cap H})$-distinguished representations ρ of M provides H-relatively cuspidal representation of G, under certain relevant choices of P and M. In the three examples mentioned in the introduction we show that our expectation is true, under some regularity condition on ρ. The results will be stated in **2.4**, **3.5**, and **4.11**.

1.3. An F-torus $\mathbf{S}$ of $\mathbf{G}$ is said to be *σ-split* if $\sigma(s) = s^{-1}$ for all $s \in \mathbf{S}$. A σ-split torus which is also F-split is said to be (σ, F)-split. It is well-known that maximal σ-split F-tori (resp. maximal (σ, F)-split tori) are mutually conjugate in $\mathbf{G}$. The dimension of maximal σ-split F-tori (resp. maximal (σ, F)-split tori) is called *the rank* (resp. *the F-rank*) of σ, or of $\mathbf{G}/\mathbf{H}$.

A parabolic F-subgroup $\mathbf{Q}$ of $\mathbf{G}$ is said to be *σ-split* if $\mathbf{Q} \cap \sigma(\mathbf{Q})$ is a Levi subgroup of $\mathbf{Q}$. Such subgroups are used in characterizing relative cuspidality of distinguished representations (see the next subsection).

Several facts about $H = \mathbf{H}(F)$-conjugacy of σ-split parabolic subgroups are recollected in [13, §2]. Here let us recall the following (from [13, 2.5]):

Lemma. *Let $\mathbf{A}$ be a maximal (σ, F)-split torus of $\mathbf{G}$. If $Z_{\mathbf{G}}(\mathbf{A}) \cap \mathbf{H}$ has trivial Galois cohomology over F, then there exists a minimal parabolic subgroup $\mathbf{P}_0 \supset \mathbf{A}$ such that every σ-split parabolic subgroup is H-conjugate to a σ-split one containing $\mathbf{P}_0$.*

1.4. From now on, we say briefly that P is a parabolic subgroup of G if it is the group of F-points of a parabolic F-subgroup $\mathbf{P}$ of $\mathbf{G}$. For a representation π of G and a parabolic subgroup Q of G, let π_Q denote the normalized Jacquet module of π along Q. In [13, §6] we gave a criterion for relative cuspidality of π in terms of invariant linear forms on Jacquet modules along σ-split parabolic subgroups. For our later use, we record a variant of it in the following form:

Proposition. *Let π be an admissible H-distinguished representation of G. If π_Q is not $L \cap H$-distinguished for every maximal σ-split parabolic subgroup Q of G (where $L = Q \cap \sigma(Q)$), then π is H-relatively cuspidal.*

Proof. This is a direct consequence of [13, 6.9 and 5.9]. □

1.5. For our convenience, we often change the involution by inner automorphisms. For an element $\gamma \in G$, consider

$$\sigma' = \mathrm{Int}(\gamma) \circ \sigma \circ \mathrm{Int}(\gamma^{-1})$$

where $\mathrm{Int}(\gamma)$ denotes the inner automorphism $g \mapsto \gamma g \gamma^{-1}$ on G. Such an involution σ' is said to be $\mathrm{Int}(G)$-conjugate to σ. The σ'-fixed point subgroup H' in G is related to H as

$$H' = \gamma H \gamma^{-1}.$$

In this situation we note the following obvious facts:

(1) A pair (P, M) is σ-stable if and only if $(\gamma P\gamma^{-1}, \gamma M\gamma^{-1})$ is σ'-stable.
(2) For a representation π of G, one has

$$\mathrm{Hom}_H(\pi, \mathbf{1}) \simeq \mathrm{Hom}_{H'}(\pi, \mathbf{1})$$

by $\Lambda \mapsto \Lambda \circ \pi(\gamma^{-1})$. This isomorphism shows that π is H-distinguished (resp. H-relatively cuspidal) if and only if it is H'-distinguished (resp. H'-relatively cuspidal).

1.6. In the following sections we mainly deal with general linear groups and use the notation $G = G_n = \mathrm{GL}_n(F)$ (or $\mathrm{GL}_n(E)$ in Section 2 where E is a quadratic extension of F). Let $B = B_n$ (resp. $D = D_n$) be (the F-points of) the Borel subgroup (resp. maximal torus) consisting of upper triangular (resp. diagonal) matrices in G. A parabolic subgroup is referred to as *standard* if it contains B, and is said to be of type $(n_1, \ldots, n_k)$ (which is a partition of n, i.e., $\sum_{i=1}^{k} n_i = n$) if it is of the form

$$P = \left\{ \begin{pmatrix} x_1 & & * \\ & \ddots & \\ 0 & & x_k \end{pmatrix} \,\middle|\, x_i \in G_{n_i}\ (i = 1, \ldots, k) \right\}.$$

The Levi subgroup

$$M = \left\{ \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_k \end{pmatrix} \,\middle|\, x_i \in G_{n_i}\ (i = 1, \ldots, k) \right\}$$

of P is called a *standard Levi subgroup* of type $(n_1, \ldots, n_k)$. We also say that (P, M) is a standard pair. The modulus character δ_P of the above P is given by

$$\delta_P \begin{pmatrix} x_1 & & * \\ & \ddots & \\ 0 & & x_k \end{pmatrix} = \prod_{1 \leqq i < j \leqq k} \left|\det(x_i)\right|^{n_j} \cdot \left|\det(x_j)\right|^{-n_i}$$

where $\left|\cdot\right|$ denotes the normalized absolute value of the field of entries, either F or E.

1.7. In sections 3 and 4, we study the symmetric spaces of $G = G_n = \mathrm{GL}_n(F)$ defined by inner involutions. Those are given by $\sigma = \mathrm{Int}(\varepsilon)$ where ε^2 is central in G. There are the following two possibilities:

(I) $\varepsilon^2 = \tau \cdot 1_n$ where $\tau \in F^\times$ is not a square in $F^\times$. In this case, the eigenvalues of ε are $\pm\sqrt{\tau}$ only. Put $E = F(\sqrt{\tau})$. The Galois automorphism of E over F permutes the corresponding eigenvectors, hence the multiplicities of $\sqrt{\tau}$ and $-\sqrt{\tau}$ are the same. As a result n is necessarily even, say $n = 2m$. It is easy to see that any such ε is G-conjugate to the element

$$\varepsilon_1 := \begin{pmatrix} 0 & 1_m \\ \tau \cdot 1_m & 0 \end{pmatrix}.$$

We consider $\sigma_1 = \mathrm{Int}(\varepsilon_1)$. For any other ε such that $\varepsilon^2 = \tau \cdot 1_{2m}$, the involution $\sigma = \mathrm{Int}(\varepsilon)$ is $\mathrm{Int}(G)$-conjugate to σ_1. Indeed, if $\varepsilon = \gamma\varepsilon_1\gamma^{-1}$, then $\sigma = \mathrm{Int}(\varepsilon)$ coincides with $\mathrm{Int}(\gamma) \circ \sigma_1 \circ \mathrm{Int}(\gamma^{-1})$.

The σ_1-fixed point subgroup H_1 in G is isomorphic to $\mathrm{GL}_m(E)$ (see **3.1**). So the symmetric space in this case is of the form $\mathrm{GL}_{2m}(F)/\mathrm{GL}_m(E)$.

(II) $\varepsilon^2 = c^2 \cdot 1_n$ for some $c \in F^\times$. Replacing ε by a scalar multiple, we may assume $\varepsilon^2 = 1_n$ in this case. Then ε is G-conjugate to the element

$$\varepsilon_1 = \varepsilon_1^{(r)} := \begin{pmatrix} 1_{n-r} & 0 \\ 0 & -1_r \end{pmatrix},$$

for some r, $0 \leqq r \leqq n$. We consider $\sigma_1 = \sigma_1^{(r)} = \mathrm{Int}(\varepsilon_1^{(r)})$. For any other ε such that $\varepsilon^2 = 1_n$, the involution $\sigma = \mathrm{Int}(\varepsilon)$ is $\mathrm{Int}(G)$-conjugate to $\sigma_1^{(r)}$ if ε is G-conjugate to $\varepsilon_1^{(r)}$.

The σ_1-fixed point subgroup H_1 in G is isomorphic to the direct product $\mathrm{GL}_{n-r}(F) \times \mathrm{GL}_r(F)$ (see **4.1**). Therefore we may (and will) assume that $1 \leqq r \leqq [n/2]$ where $[x]$ for $x \in \mathbb{R}$ denotes the greatest integer less than x.

1.8. As the ingredients for our construction of relatively cuspidal representations, we use cuspidal distinguished representations of stable Levi subgroups. Concretely we need such representations for the symmetric spaces $\mathrm{GL}_n(E)/\mathrm{GL}_n(F)$, $\mathrm{GL}_{2m}(F)/\mathrm{GL}_m(E)$, and $\mathrm{GL}_{2r}(F)/(\mathrm{GL}_r(F) \times \mathrm{GL}_r(F))$. As for the first and third cases, examples of those are constructed in [7]. Examples for the second case is not seen in the literature. However, at least when E/F is unramified, one can obtain such representations by using the method of [6] and the result concerning a similar problem over finite fields: Let $K = \mathrm{GL}_{2m}(\mathcal{O}_F)$ and $K_1 = 1_{2m} + \varpi_F \mathrm{Mat}_{2m}(\mathcal{O}_F)$, so that K/K_1 is isomorphic to GL_{2m} over the residue field of F. Here, $\mathcal{O}_F$ (resp. ϖ_F) denotes the valuation ring (resp. a prime element) of F. If τ in **1.7 (I)** belongs to $\mathcal{O}_F^\times$, then σ leaves K and K_1 stable, inducing the same kind of involution on the finite GL_{2m}. By the result of [1] and [10], one can find irreducible cuspidal distinguished representations for this finite symmetric space. Starting from such representations, inflate these to representations of K and induce up to G by compact-mod-Z induction (with trivial central character). Then we obtain irreducible cuspidal representations of G, and these are actually $\mathrm{GL}_m(E)$-distinguished by the Mackey decomposition theorem in [8, 2.1].

2. The case of Galois involution on GL_n

2.1. Let E be a quadratic extension of F. The Galois automorphism of E over F is denoted by $x \mapsto \overline{x}$. We consider the group $G = G_n = \mathrm{GL}_n(E)$ and the Galois involution $\sigma = \sigma_n$ on G:

$$\sigma(g) = \overline{g} \ \ (= (\overline{g_{ij}}) \ \text{ if } \ g = (g_{ij}) \in G).$$

The subgroup $H = H_n$ of σ-fixed points in G is $\mathrm{GL}_n(F)$. The center $Z = Z_n$ of G consists of scalar matrices, and is identified with $E^\times$. Note also that $Z \cap H$ coincides with the center of H, and is identified with $F^\times$.

2.2. In this Galois case, all standard parabolic subgroups of G (together with standard Levi subgroups) are σ-stable. If P is the standard parabolic subgroup of type $(n_1, \ldots, n_k)$, then $P \cap H$ is the standard parabolic subgroup of $H = \mathrm{GL}_n(F)$ of the same type. The modulus characters δ_P and $\delta_{P\cap H}$ are given by the formula of **1.6**, using the absolute values of E and F respectively. By the relation $\left|\cdot\right|_E^{1/2} = \left|\cdot\right|_F$ on $F^\times$, we have $\delta_P^{1/2}|_{P\cap H} = \delta_{P\cap H}$. Thus the quasi-character $\mu_{M\cap H}$ on $M \cap H$ considered in **1.2** is trivial in this Galois case.

2.3. Let us recall the following result due to Flicker, on irreducible $\mathrm{GL}_n(F)$-distinguished representations of $\mathrm{GL}_n(E)$.

Proposition. *If an irreducible representation π of G is H-distinguished, then ${}^{\sigma}\pi$ is equivalent to the contragredient $\widetilde{\pi}$ of π.*

Proof. In [3, Proposition 12], it was shown that such a π is invariant under the unitary involution $g \mapsto {}^t\overline{g}^{-1}$ on G. Also, by the work of Gel'fand-Kazhdan [4], the orthogonal involution $g \mapsto {}^tg^{-1}$ on G sends π to its contragredient $\widetilde{\pi}$. □

2.4. Now we state the main claim of this section. A part of this result is found also in [18] and [20].

Theorem. *Let $(n_1, \ldots, n_k)$ be a partition of n and (P, M) the corresponding standard σ-stable pair. For each i, take an irreducible cuspidal H_{n_i}-distinguished representation ρ_i of G_{n_i} and form $\rho = \otimes_{i=1}^{k}\rho_i$, which is an irreducible cuspidal $M \cap H$-distinguished representation of M.*

Suppose that $\rho_i \not\simeq \rho_j$ for any $i \neq j$. Then,
(1) *The induced representation $\mathrm{Ind}_P^G(\rho)$ is H-distinguished and irreducible.*
(2) *The induced representation $\mathrm{Ind}_P^G(\rho)$ is H-relatively cuspidal.*

Proof of (1). H-distinction of $\mathrm{Ind}_P^G(\rho)$ was already seen in **1.2**. As is well-known, $\mathrm{Ind}_P^G(\rho)$ is reducible if and only if $\rho_i \simeq \left|\det(\cdot)\right|_E^{\pm 1} \cdot \rho_j$ for some $i \neq j$ (such that $n_i = n_j$) [2]. However, the central characters of ρ_i and ρ_j are both trivial on $Z \cap H \simeq F^{\times}$ by (1) in **1.1**. By comparison of the central characters restricted to $Z \cap H$, we cannot have the reducibility. □

H-relative cuspidality will be seen in **2.6**, using the criterion recorded in the proposition of **1.4**.

2.5. Let $w_0 = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$ be the anti-diagonal permutation matrix in G. By Hilbert's Theorem 90, there exists an element $\gamma \in G$ such that $\gamma\sigma(\gamma)^{-1} = w_0$. We consider the involution

$$\sigma' = \sigma'_n := \mathrm{Int}(w_0) \circ \sigma \ \left(= \mathrm{Int}(\gamma) \circ \sigma \circ \mathrm{Int}(\gamma)^{-1}\right)$$

on G, with the fixed point subgroup $H' = H'_n$. Then, the basis of the root system of (G, D) corresponding to B is a σ'*-basis*, so that every σ'-split parabolic subgroup is G-conjugate to a σ'-split one containing B (see [13, §2] for details).

For each ℓ, $1 \leqq \ell \leqq [n/2]$, let $Q = Q_\ell$ be the standard parabolic subgroup of type $(\ell, n-2\ell, \ell)$. These are maximal σ'-split parabolic subgroups. On the standard Levi subgroup L of Q, the action of σ' is seen as

$$\sigma' \begin{pmatrix} x_1 & & 0 \\ & x_2 & \\ 0 & & x_3 \end{pmatrix} = \begin{pmatrix} \sigma'_\ell(x_3) & & 0 \\ & \sigma'_{n-2\ell}(x_2) & \\ 0 & & \sigma'_\ell(x_1) \end{pmatrix},$$

so that the σ'-fixed point subgroup $L \cap H'$ of L is of the form

$$L \cap H' = \left\{ \begin{pmatrix} x & & 0 \\ & y & \\ 0 & & \sigma'_\ell(x) \end{pmatrix} \,\middle|\, x \in G_\ell,\ y \in H'_{n-2\ell} \right\}.$$

As a maximal (σ, F)-split torus of G, we take

$$A = \left\{ \operatorname{diag}(a_1, \ldots, a_n) \,\middle|\, \begin{array}{c} a_i \in F^\times,\ a_{n-i+1} = a_i^{-1}, \text{ and} \\ a_{[n/2]+1} = 1 \text{ if } n \text{ is odd} \end{array} \right\}.$$

Then the centralizer $Z_G(A)$ is the maximal torus of all diagonal matrices (with entries in $E^\times$). As a result,

$$Z_G(A) \cap H' = \{\operatorname{diag}(t_1, \ldots, t_n) \mid t_i \in E^\times,\ t_{n-i+1} = \overline{t_i} \text{ for all } i\},$$

which is the product of $[n/2]$-copies of $E^\times$, and one more factor $F^\times$ if n is odd. Anyway, the Galois cohomology of $Z_{\mathbf{G}}(\mathbf{A}) \cap \mathbf{H}'$ over F is trivial. By the lemma in **1.3** (where we can take $\mathbf{P}_0 = \mathbf{B}$), it turns out that any maximal σ'-split parabolic subgroup of G is H'-conjugate to one of Q_ℓ, $1 \leqq \ell \leqq [n/2]$.

2.6. Proof of 2.4 (2). $\operatorname{Ind}_P^G(\rho)$ is also H'-distinguished by **1.5** (2). We use the criterion in **1.4** to deduce H'-relative cuspidality. Then we can conclude H-relative cuspidality again by **1.5** (2).

Thus, from the description at the end of **2.5** and the argument in [13, 6.10], it is enough to show the following:

Claim. For each maximal σ'-split parabolic subgroup $Q = Q_\ell$ given in **2.5**, the Jacquet module $(\operatorname{Ind}_P^G(\rho))_Q$ is not $L \cap H'$-distinguished.

Suppose that $(\operatorname{Ind}_P^G(\rho))_Q$ is $L \cap H'$-distinguished for some $Q = Q_\ell$. By the theory of Bernstein-Zelevinsky (so-called *Geometric Lemma*, [2, 2.12 and 2.13 (a)]) applied to the case of irreducible cuspidal ρ, we have a filtration of L-submodules of $(\operatorname{Ind}_P^G(\rho))_Q$ whose successive quotients are isomorphic to

$$\mathcal{F}_w(\rho) := \operatorname{Ind}_{L \cap wPw^{-1}}^L ({}^w\rho)$$

for w in a set of representatives of $Q\backslash G/P$ such that $L \cap wPw^{-1}$ is a proper parabolic subgroup in L. Note that we can take w as a permutation matrix in G. By (2) in **1.1**, there must be a representative w such that $\mathcal{F}_w(\rho)$ is $L \cap H'$-distinguished.

As a representation of $L \simeq G_\ell \times G_{n-2\ell} \times G_\ell$, we may put $\mathcal{F}_w(\rho) = I_1 \otimes I_2 \otimes I_3$ where I_1 and I_3 are representations induced up to G_ℓ, and I_2 is one up to $G_{n-2\ell}$, with the inducing data chosen from $\{\rho_1, \ldots, \rho_k\}$. In particular, there are disjoint sets J_1, $J_3 \subset \{1, \ldots, k\}$ of indices such that $\sum_{i \in J_1} n_i = \sum_{i \in J_3} n_i = \ell$, and that I_1 (resp. I_3) is induced from the data $\{\rho_i \mid i \in J_1\}$ (resp. $\{\rho_i \mid i \in J_3\}$) in a suitable order.

Look at the restriction to the product of the first and the third factors. The $L \cap H'$-distinction implies that $I_1 \otimes I_3$, a representation of $G_\ell \times G_\ell$, is $\Delta^{\sigma'_\ell}(G_\ell)$-distinguished, where

$$\Delta^{\sigma'_\ell}(G_\ell) = \left\{(x, \sigma'_\ell(x)) \in G_\ell \times G_\ell \,\middle|\, x \in G_\ell\right\}.$$

This is equivalent to saying that there is a non-zero G_ℓ-morphism from I_1 to $\widetilde{(I_3)^{\sigma'_\ell}}$. Here, we can see that I_3 is irreducible and H'_ℓ-distinguished, by the same discussion as that of **2.4**. Applying the proposition in **2.3** to I_3, we have

$$\widetilde{(I_3)^{\sigma'_\ell}} \simeq \widetilde{\widetilde{I_3}} \simeq I_3.$$

Hence the inducing data $\{\rho_i \mid i \in J_1\}$ and $\{\rho_i \mid i \in J_3\}$ must be equal up to order, by [2, 2.9]. This contradicts to the assumption that $\rho_i \not\cong \rho_j$ for any $i \neq j$. $\square$

3. The case of inner involutions on GL_n (I)

3.1. Take an element $\tau \in F^\times$ which is not a square and form a quadratic extension $E = F(\sqrt{\tau})$ of F. We consider the group $G = G_{2m} = \mathrm{GL}_{2m}(F)$ and the involution

$$\sigma_1 = \mathrm{Int}(\varepsilon_1), \quad \varepsilon_1 = \begin{pmatrix} 0 & 1_m \\ \tau \cdot 1_m & 0 \end{pmatrix}$$

on G. In the $m \times m$-block form, σ_1 is written as

$$\sigma_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & \tau^{-1}c \\ \tau b & a \end{pmatrix},$$

hence the σ_1-fixed point subgroup $H_1 = H_{1,2m}$ in G is of the form

$$H_1 = H_{1,2m} = \left\{ \begin{pmatrix} a & b \\ \tau b & a \end{pmatrix} \in G \;\middle|\; a, b \in \mathrm{Mat}_{m\times m}(F) \right\},$$

which is isomorphic to the group $\mathrm{GL}_m(E)$ via

$$a + \sqrt{\tau} \cdot b \mapsto \begin{pmatrix} a & b \\ \tau b & a \end{pmatrix}.$$

Note that the restriction of $\det(\,\cdot\,)$ to H_1 gives the relation

$$\det \begin{pmatrix} a & b \\ \tau b & a \end{pmatrix} = \det(a + \sqrt{\tau} \cdot b) \cdot \det(a - \sqrt{\tau} \cdot b)$$

since $\left(\begin{smallmatrix} a & b \\ \tau b & a \end{smallmatrix}\right)$ is conjugate to $\left(\begin{smallmatrix} a+\sqrt{\tau}b & 0 \\ 0 & a-\sqrt{\tau}b \end{smallmatrix}\right)$ over E.

Put

$$X = X_{2m} = \left\{ \varepsilon \in G \;\middle|\; \varepsilon^2 = \tau \cdot 1_{2m} \right\}.$$

Then $\varepsilon_1 \in X$, and any $\varepsilon \in X$ is G-conjugate to ε_1 (see **1.7**). Hence for any $\varepsilon \in X$, the inner involution $\sigma = \mathrm{Int}(\varepsilon)$ is $\mathrm{Int}(G)$-conjugate to σ_1, and the σ-fixed point subgroup H is G-conjugate to H_1.

3.2. Let us describe σ-stable pairs (P, M) among the standard class, to which we can apply the proposition in **1.2**. By **1.5** (1) we observe the following: For a σ_1-stable pair (P_1, M_1) which is not necessarily standard, there is a standard σ-stable pair (P, M) where σ is $\mathrm{Int}(G)$-conjugate to σ_1. So we shall determine standard pairs (P, M) which can be stable under $\sigma = \mathrm{Int}(\varepsilon)$ for some $\varepsilon \in X$.

Let (P, M) be the standard pair in $G = G_{2m}$ of type $(n_1, \dots, n_k)$. If it is a σ-stable pair for $\sigma = \mathrm{Int}(\varepsilon)$ with $\varepsilon \in X$, then each n_i has to be even: Indeed, if P and M are stable under σ, then

$$\varepsilon \in N_G(P) \cap N_G(M) = P \cap N_G(M) = M.$$

Say, $\varepsilon = \begin{pmatrix} \varepsilon^{(1)} & & 0 \\ & \ddots & \\ 0 & & \varepsilon^{(k)} \end{pmatrix}$, $\varepsilon^{(i)} \in G_{n_i}$. Since $\varepsilon^2 = \tau \cdot 1_{2m}$, we must have $\left(\varepsilon^{(i)}\right)^2 = \tau \cdot 1_{n_i}$ for each i, which implies that n_i is even by **1.7**.

Thus, as the ingredients, we use only the standard pairs (P, M) in $G = G_{2m}$ of type $(2m_1, \dots, 2m_k)$ where $m = \sum_{i=1}^{k} m_i$. As for the element $\varepsilon \in X$ such that $\sigma = \mathrm{Int}(\varepsilon)$ leaves (P, M) stable, we shall take $\varepsilon = \begin{pmatrix} \varepsilon^{(1)} & & 0 \\ & \ddots & \\ 0 & & \varepsilon^{(k)} \end{pmatrix}$ where

$$\varepsilon^{(i)} = \begin{pmatrix} 0 & 1_{m_i} \\ \tau \cdot 1_{m_i} & 0 \end{pmatrix} \in G_{2m_i}.$$

In this case, $M \cap H$ is given by

$$M \cap H = \left\{ \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_k \end{pmatrix} \,\middle|\, x_i \in H_{1,2m_i}\ (i = 1, \dots, k) \right\}.$$

3.3. We shall see that the quasi-character $\mu_{M\cap H}$ defined in **1.2** is trivial, for every σ-stable standard pair (P, M) of type $(2m_1, \dots, 2m_k)$.

At first, δ_P on $M \cap H$ is computed as

$$\delta_P \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_k \end{pmatrix} = \prod_{1 \leqq i < j \leqq k} |\det(x_i)|_F^{2m_j} \cdot \left|\det(x_j)\right|_F^{-2m_i}$$

by the formula in **1.6**. Further, if $\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_k \end{pmatrix} \in M \cap H$, then $x_i \in H_{1,2m_i}$ is of the form

$$x_i = \begin{pmatrix} a_i & b_i \\ \tau b_i & a_i \end{pmatrix}, \quad a_i + \sqrt{\tau} b_i \in \mathrm{GL}_{m_i}(E).$$

By the remark in **3.1**, we have

$$\left|\det(x_i)\right|_F = \left|\det(a_i + \sqrt{\tau} b_i) \det(a_i - \sqrt{\tau} b_i)\right|_F = \left|\det(a_i + \sqrt{\tau} b_i)\right|_E,$$

hence

$$\delta_P^{1/2} \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_k \end{pmatrix} = \prod_{1 \leqq i < j \leqq k} \left|\det(a_i + \sqrt{\tau} b_i)\right|_E^{m_j} \cdot \left|\det(a_j + \sqrt{\tau} b_j)\right|_E^{-m_i}.$$

Next we compute $\delta_{P\cap H}$. To determine the elements $\begin{pmatrix} 1_{2m_1} & & b_{ij} \\ & \ddots & \\ 0 & & 1_{2m_k} \end{pmatrix}$ of $U \cap H$, look at the relation

$$\begin{pmatrix} \varepsilon^{(1)} & & 0 \\ & \ddots & \\ 0 & & \varepsilon^{(k)} \end{pmatrix} \begin{pmatrix} 1_{2m_1} & & b_{ij} \\ & \ddots & \\ 0 & & 1_{2m_k} \end{pmatrix} \begin{pmatrix} \varepsilon^{(1)} & & 0 \\ & \ddots & \\ 0 & & \varepsilon^{(k)} \end{pmatrix}^{-1} = \begin{pmatrix} 1_{2m_1} & & b_{ij} \\ & \ddots & \\ 0 & & 1_{2m_k} \end{pmatrix}.$$

We must have $\varepsilon^{(i)} b_{ij} \varepsilon^{(j)^{-1}} = b_{ij}$ for each upper right block b_{ij}. Write $b_{ij} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where A, B, C, D are $m_i \times m_j$-matrices. By the relation

$$\begin{pmatrix} 0 & 1_{m_i} \\ \tau 1_{m_i} & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & \tau^{-1} 1_{m_j} \\ 1_{m_j} & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

the block b_{ij} is of the form $\left(\begin{smallmatrix} A & B \\ \tau B & A \end{smallmatrix}\right)$, which can be identified with the element $A + \sqrt{\tau} B \in \mathrm{Mat}_{m_i \times m_j}(E)$.

In the adjoint action of $M \cap H$ on $U \cap H$, the part

$$b_{ij} \ \mapsto \ x_i \cdot b_{ij} \cdot x_j^{-1}$$

can be regarded as the left action of $a_i + \sqrt{\tau} b_i$, and the right action of $(a_i + \sqrt{\tau} b_i)^{-1}$, on the matrix $A + \sqrt{\tau} B \in \mathrm{Mat}_{n_i \times n_j}(E)$: Indeed, the operation

$$\begin{pmatrix} a_i & b_i \\ \tau b_i & a_i \end{pmatrix} \begin{pmatrix} A & B \\ \tau B & A \end{pmatrix} = \begin{pmatrix} a_i A + \tau b_i B & a_i B + b_i A \\ \tau b_i A + \tau a_i B & a_i A + \tau b_i B \end{pmatrix}$$

from the left corresponds to the relation

$$(a_i + \sqrt{\tau} b_i) \cdot (A + \sqrt{\tau} B) = (a_i A + \tau b_i B) + \sqrt{\tau}(b_i A + a_i B),$$

and similarly from the right. Gathering all parts $1 \leqq i < j \leqq k$, we have

$$\delta_{P \cap H} \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_k \end{pmatrix} = \prod_{1 \leqq i < j \leqq k} \left|\det(a_i + \sqrt{\tau} b_i)\right|_E^{m_j} \cdot \left|\det(a_j + \sqrt{\tau} b_j)\right|_E^{-m_i}.$$

As a consequence we have $\delta_P^{1/2}|_{M \cap H} = \delta_{P \cap H}$, hence $\mu_{M \cap H} \equiv 1$ on $M \cap H$.

3.4. We recall the following result due to Guo [5] on irreducible $\mathrm{GL}_m(E)$-distinguished representations of $\mathrm{GL}_{2m}(F)$.

Proposition. *If an irreducible representation π of G is H-distinguished, then π is self-contragredient.*

3.5. Now we state the main claim of this section.

Theorem. *Let $(m_1, \dots, m_k)$ be a partition of m, (P, M) the standard pair in $G = G_{2m}$ of type $(2m_1, \dots, 2m_k)$, and $\sigma = \mathrm{Int}(\varepsilon)$ given at the end of* **3.2**. *For each i, take an irreducible cuspidal $H_{1,2m_i}$-distinguished representation ρ_i of G_{2m_i} and form $\rho = \otimes_{i=1}^k \rho_i$, which is an irreducible cuspidal $M \cap H$-distinguished representation of M.*

Suppose that $\rho_i \not\simeq \rho_j$ for any $i \neq j$. Then,
(1) *The induced representation $\mathrm{Ind}_P^G(\rho)$ is H-distinguished and irreducible.*
(2) *The induced representation $\mathrm{Ind}_P^G(\rho)$ is H-relatively cuspidal.*

Proof of (1). H-distinction of $\mathrm{Ind}_P^G(\rho)$ can be seen by **1.2** and **3.3**. Since inner involutions are trivial on the center, the central characters of ρ_i are all trivial, hence the reducibility condition $\rho_i \simeq \left|\det(\cdot)\right|_F^{\pm 1} \cdot \rho_j$ of [2] cannot be satisfied for any $i \neq j$. □

H-relative cuspidality will be seen in **3.7**.

3.6. We consider the involution

$$\sigma' = \sigma'_{2m} = \mathrm{Int}(\varepsilon'), \quad \varepsilon' = \begin{pmatrix} 0 & w_0 \\ \tau \cdot w_0 & 0 \end{pmatrix}$$

on $G = G_{2m}$, where w_0 denotes the $m \times m$ anti-diagonal permutation matrix. Since $(\varepsilon')^2 = \tau \cdot 1_{2m}$, the involution σ' is $\mathrm{Int}(G)$-conjugate to σ_1, and also to σ in **3.2**. Let $H' = H'_{2m}$ be the σ'-fixed point subgroup in G. By the same reason as that in **2.5**, every σ'-split parabolic subgroup is G-conjugate to a standard σ'-split one.

Maximal σ'-split parabolic subgroups among standard ones are given by $Q = Q_\ell$, the standard one of type $(\ell, 2m-2\ell, \ell)$, for $1 \leqq \ell \leqq m$. On the standard Levi subgroup L of Q, the action of σ' is given by

$$\sigma'\begin{pmatrix} x_1 & & 0 \\ & x_2 & \\ 0 & & x_3 \end{pmatrix} = \begin{pmatrix} w_0 x_3 w_0^{-1} & & 0 \\ & \sigma'_{2m-2\ell}(x_2) & \\ 0 & & w_0 x_1 w_0^{-1} \end{pmatrix},$$

where w_0 is now of size $\ell \times \ell$. Hence

$$L \cap H' = \left\{ \begin{pmatrix} x & & 0 \\ & y & \\ 0 & & w_0 x w_0^{-1} \end{pmatrix} \;\middle|\; x \in G_\ell,\ y \in H'_{2m-2\ell} \right\}.$$

As a maximal (σ, F)-split torus of G, we may take

$$A = \{\mathrm{diag}(a_1, \ldots, a_{2m}) \mid a_i \in F^\times,\ a_{n-i+1} = a_i^{-1}\ (1 \leqq i \leqq m)\}.$$

Then $Z_G(A)$ is the maximal torus of all diagonal matrices and

$$Z_G(A) \cap H' = \{\mathrm{diag}(t_1, \ldots, t_{2m}) \mid t_i \in F^\times,\ t_{n-i+1} = t_i\ \ (1 \leqq i \leqq m)\}.$$

As an algebraic F-group, the Galois cohomology of $Z_{\mathbf{G}}(\mathbf{A}) \cap \mathbf{H}'$ over F is trivial. So, by the lemma in **1.3**, we can assert that any maximal σ'-split parabolic subgroup of G is H'-conjugate to one of Q_ℓ, $1 \leqq \ell \leqq m$.

3.7. Proof of 3.5 (2). By the same discussion as in **2.6**, it is enough to prove the following:

Claim. For each maximal σ'-split parabolic subgroup $Q = Q_\ell$ given in **3.6**, the Jacquet module $(\mathrm{Ind}_P^G(\rho))_Q$ is not $L \cap H'$-distinguished.

Suppose the contrary. As in **2.6**, at least one of

$$\mathcal{F}_w(\rho) := \mathrm{Ind}^L_{L \cap wPw^{-1}}({}^w\rho)$$

has to be $L \cap H'$-distinguished, where w is in a set of representatives of $Q \backslash G/P$ such that $L \cap wPw^{-1}$ is a proper parabolic subgroup in L. Put $\mathcal{F}_w(\rho) = I_1 \otimes I_2 \otimes I_3$ where I_1 and I_3 are representations induced up to G_ℓ, and I_2 is one up to $G_{2m-2\ell}$. There are disjoint sets J_1, $J_3 \subset \{1, \ldots, k\}$ of indices such that $\sum_{i \in J_1} 2m_i = \sum_{i \in J_3} 2m_i = \ell$, and that I_1 (resp. I_3) is induced from the data $\{\rho_i \mid i \in J_1\}$ (resp. $\{\rho_i \mid i \in J_3\}$) in a suitable order. It is irreducible from the same discussion as that in **3.5**. By the form of $L \cap H'$ in **3.6**, the $L \cap H'$-distinction of $\mathcal{F}_w(\rho)$ implies that $I_1 \simeq \widetilde{I_3}$. Here $\widetilde{I_3}$ is induced from $\{\widetilde{\rho_i} \mid i \in J_3\}$, and for each i we have $\widetilde{\rho_i} \simeq \rho_i$ by **3.4**. As a result we must have $I_1 \simeq I_3$, which contradicts to the assumption that $\rho_i \not\simeq \rho_j$ for any $i \neq j$. □

4. The case of inner involutions on GL_n (II)

4.1. Put

$$X = \{\varepsilon \in G_n \mid \varepsilon^2 = 1_n\}.$$

In this section we consider the inner involution $\sigma = \mathrm{Int}(\varepsilon)$ on $G = G_n$ for $\varepsilon \in X$. For each r, $0 \leqq r \leqq n$, consider the element

$$\varepsilon_1 = \varepsilon_1(n-r,r) = \begin{pmatrix} 1_{n-r} & 0 \\ 0 & -1_r \end{pmatrix}$$

and the corresponding involution $\sigma_1 = \sigma_1(n-r,r) = \mathrm{Int}(\varepsilon_1(n-r,r))$. The σ_1-fixed point subgroup $H_1 = H_1(n-r,r)$ is given by

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \,\middle|\, a \in G_{n-r},\ b \in G_r \right\} \simeq G_{n-r} \times G_r.$$

For each r, Let $X(n-r,r)$ be the set of all elements of X which are G-conjugate to $\varepsilon_1(n-r,r)$. Then X can be decomposed as the disjoint union $\bigsqcup_{0\leqq r\leqq n} X(n-r,r)$. If $\varepsilon \in X(n-r,r)$, then the fixed point subgroup H of $\sigma = \mathrm{Int}(\varepsilon)$ is G-conjugate to $H_1(n-r,r) \simeq G_{n-r} \times G_r$. Thus, in studying H-distinguished representations it is enough to consider the case $1 \leqq r \leqq [n/2]$. In such a case, it is well-known that the rank and the F-rank of $G/H_1(n-r,r)$ are both equal to r.

As a particular case, we call the pair (G_n, σ) *even type* if n is even, say $n = 2m$, and $\sigma = \mathrm{Int}(\varepsilon)$ for $\varepsilon \in X(m,m)$. Hence the corresponding symmetric space (of even type) is isomorphic to $\mathrm{GL}_{2m}(F)/\left(\mathrm{GL}_m(F) \times \mathrm{GL}_m(F)\right)$.

4.2. We recall the following result on cuspidal distinguished representations due to Matringe.

Proposition. *Suppose that $n > 1$. Let $\sigma = \mathrm{Int}(\varepsilon)$ be the inner involution on $G = G_n$ with $\varepsilon \in X(n-r,r)$, $1 \leqq r \leqq [n/2]$, and H the σ-fixed point subgroup in G. Let μ be any quasi-character of H. Then there exists an irreducible cuspidal (H,μ)-distinguished representation of G only if (G,σ) is of even type.*

Proof. This is given in [15] for trivial μ. The proof in [15] also works for a general μ by [16, Proposition 3.1]. □

4.3. Let (P,M) be the standard pair in G of type $(n_1,\dots,n_k)$ and $\sigma = \mathrm{Int}(\varepsilon)$ with $\varepsilon \in X(n-r,r)$, $1 \leqq r \leqq [n/2]$. If σ leaves both P and M stable, then we must have $\varepsilon \in M$ as in **3.2**. So we may put $\varepsilon = \begin{pmatrix} \varepsilon^{(1)} & & 0 \\ & \ddots & \\ 0 & & \varepsilon^{(k)} \end{pmatrix}$ where $\varepsilon^{(i)} \in G_{n_i}$. Since $\varepsilon \in X(n-r,r)$, we have $\varepsilon^{(i)} \in X(n_i - r_i, r_i)$ for some r_i such that $\sum_{1\leqq i\leqq k} r_i = r$. The pair $(M, \sigma|_M)$ can be regarded as the direct product of $(G_{n_i}, \sigma^{(i)})$ where $\sigma^{(i)} = \mathrm{Int}(\varepsilon^{(i)})$, $1 \leqq i \leqq k$.

The following is an immediate consequence of the previous proposition.

Corollary. *Let μ be any quasi-character of $M \cap H$. If there exists an irreducible cuspidal $(M \cap H, \mu)$-distinguished representation of M, then for all $1 \leqq i \leqq k$, either $(G_{n_i}, \sigma^{(i)})$ is of even type or $n_i = 1$.* □

4.4. According to the above corollary, we may only consider the standard pairs of which the sizes n_i of diagonal blocks are even or equal to 1. In such cases, put $\sum_{n_i:\,\mathrm{even}} n_i = 2r'$. Then we may further suppose that $r \geqq r'$ by the above corollary, where r is the rank of G/H.

Our main target is the case where $r = r'$, dealt with in **4.8** – **4.14**. As for the treatment in the case that $r > r'$, see **4.15**. Also, for the proof of the main result of this section we prepare one lemma (in **4.7**) concerning the excluded case that $r < r'$.

4.5. We shall give some description of representatives of the double cosets $P\backslash G/H_1$ which is needed for the proof of Lemma in **4.7**.

Let r be a fixed integer with $1 \leqq r \leqq [n/2]$ and consider the involution $\sigma_1 = \sigma_1(n-r, r)$. Put

$$\widetilde{Y} = \{y \in G \mid \sigma_1(y) = y^{-1}\},$$

on which G acts from the left by

$$g * y := gy\sigma_1(g)^{-1} \quad (g \in G,\ y \in \widetilde{Y}).$$

Then G/H_1 is identified with the G-orbit $Y := G * 1_n \subset \widetilde{Y}$ through the orbit map $g \mapsto g * 1_n$. The description of $P\backslash G/H_1$ is equivalent to that of P-orbits in Y under the $*$-action.

Let W be the group of permutation matrices in G (which is identified with the Weyl group of the maximal torus D consisting of diagonal matrices) and put

$$W_{(2)} = \{w \in W \mid w^2 = 1_n\}.$$

Note that $w\varepsilon_1 \in \widetilde{Y}$ if $w \in W_{(2)}$. Regarding $w \in W_{(2)}$ as a permutation, let $p(w)$ be the number of pairs of indices interchanged by w. For each $w \in W_{(2)}$, put

$$D^{-w} = \{t \in D \mid wtw^{-1} = t^{-1}\}$$

and consider the homomorphism

$$\tau_w : D \to D, \quad \tau_w(t) = twt^{-1}w^{-1}$$

on D. Then, the image $\tau_w(D)$ is of finite index in D^{-w}. It is easy to see that

$$D^{-w}/\tau_w(D) \simeq \{\pm 1\}^{n-2p}$$

where $p = p(w)$. As representatives for $D^{-w}/\tau_w(D)$, we may take $\delta \in D^{-w}$ whose diagonal entries are ± 1, and in particular the i-th diagonal entry is 1 if i is an index interchanged by w.

Now, the B-orbit decomposition of $\widetilde{Y}$ and Y are given as follows:

Lemma. (1) $\displaystyle \widetilde{Y} = \bigsqcup_{w \in W_{(2)}} \Big(\bigsqcup_{\delta \in D^{-w}/\tau_w(D)} B * (\delta w \varepsilon_1) \Big).$

(2) $\displaystyle Y = \bigsqcup_{\substack{w \in W_{(2)},\\ p(w) \leqq r}} \Big(\bigsqcup_{\substack{\delta \in D^{-w}/\tau_w(D),\\ \delta w \text{ is conjugate to } \varepsilon_1}} B * (\delta w \varepsilon_1) \Big).$

Proof. (2) readily follows from (1) if one considers the condition for $\delta w \varepsilon_1$ to belong to Y. For (1), first we have

$$\widetilde{Y} = \bigsqcup_{w \in W_{(2)}} \Big(Bw\varepsilon_1 B \cap \widetilde{Y} \Big) \quad \text{(disjoint)}$$

by Bruhat decomposition intersected with $\widetilde{Y}$. We consider B-orbit decomposition of $Bw\varepsilon_1 B \cap \widetilde{Y}$ for eacn $w \in W_{(2)}$. By a standard argument as in [9, 6.6], we obtain

$$Bw\varepsilon_1 B \cap \widetilde{Y} = U * (Dw\varepsilon_1 \cap \widetilde{Y})$$

where U is the unipotent radical of B. Next, it is easy to see that

$$Dw\varepsilon_1 \cap \widetilde{Y} = D^{-w} \cdot w\varepsilon_1$$

and the right hand side is decomposed as

$$D^{-w} \cdot w\varepsilon_1 = \bigsqcup_{\delta \in D^{-w}/\tau_w(D)} D * (\delta w \varepsilon_1)$$

for, if we write $t = \delta\tau_w(t_0)$ with $t_0 \in D$, one has $tw\varepsilon_1 = t_0 * (\delta w\varepsilon_1)$. This completes the proof of (1). □

Corollary. *Let P be a parabolic subgroup containing B. Then, as a complete set $\{\xi\}$ of representatives for $P\backslash G/H_1$, one can take those ξ such that*

$$\xi\sigma_1(\xi)^{-1} = \delta w \varepsilon_1$$

with (i) $w \in W_{(2)}$, $p(w) \leqq r$, (ii) $\delta \in D^{-w}/\tau_w(D)$, *and* (iii) δw *is conjugate to* ε_1 *in* G. □

4.6. Let $P = MU$ be any standard parabolic subgroup of G and take a representative ξ of $P\backslash G/H_1$ as in the above corollary. Put

$$\sigma_\xi = \mathrm{Int}(\xi) \circ \sigma_1 \circ \mathrm{Int}(\xi^{-1}).$$

Note that $\sigma_\xi = \mathrm{Int}(\delta w)$, since $\xi\sigma_1(\xi)^{-1} = \delta w\varepsilon_1$. Let M_ξ be the σ_ξ-fixed point subgroup in $M \cap \sigma_\xi(M)$ and consider the quasi-character μ_ξ of M_ξ defined by

$$\mu_\xi = \delta_P^{-1/2}|_{M_\xi} \cdot \delta_{P_\xi},$$

where $P_\xi = P \cap \xi H \xi^{-1}$. The following proposition is a refinement of *Mackey theory* for $P\backslash G/H_1$ due to Offen.

Proposition. *Let ρ be an irreducible cuspidal representation of M. If the induced representation* $\mathrm{Ind}_P^G(\rho)$ *is H_1-distinguished, then there exists a representative ξ of $P\backslash G/H_1$ such that $\sigma_\xi(M) = M$ and ρ is (M_ξ, μ_ξ)-distinguished.*

Proof. This is just an adaptation of [19, Corollary 5.2] to our case. □

4.7. Using the above proposition we have the following assertion, which will be used in the proof of the main theorem of this section.

Lemma. *Let $P = MU$ be the standard parabolic subgroup of G of type $(n_1, \ldots, n_k)$ where each n_i is even or equal to* 1. *Suppose that $\sum_{n_i:\, even} n_i > 2r$ where r is the rank of G/H_1. Then, the induced representation* $\mathrm{Ind}_P^G(\rho)$ *is not H_1-distinguished for any irreducible cuspidal representation ρ of M.*

Proof. As a representation of $M \simeq G_{n_1} \times \cdots \times G_{n_k}$ we write $\rho = \rho_1 \otimes \cdots \otimes \rho_k$ where ρ_i is an irreducible cuspidal representation of G_{n_i}. Suppose that $\mathrm{Ind}_P^G(\rho)$ is H_1-distinguished. Let ξ be a representative of $P\backslash G/H_1$ obtained by the proposition of **4.6** with $\xi\sigma_1(\xi)^{-1} = \delta w\varepsilon_1$ as in the corollary in **4.5**. Then, $\sigma_\xi|_M$ is given by $\mathrm{Int}(\delta w)$, with $(\delta w)^2 = 1$. Since $\sigma_\xi(M) = wMw^{-1} = M$, the blocks of M are either stable or interchanged by $\mathrm{Int}(w)$. Let us look at only even-size blocks of M. Put

$$J = \left\{ 1 \leqq i \leqq k \;\middle|\; \begin{array}{c} n_i \text{ is even and the } i\text{-th block} \\ \text{is stable under } \operatorname{Int}(w) \end{array} \right\}$$

and

$$J' = \left\{ 1 \leqq i \leqq k \;\middle|\; \begin{array}{c} n_i \text{ is even and the } i\text{-th block} \\ \text{is interchanged by } \operatorname{Int}(w) \end{array} \right\}.$$

For $i \in J$, σ_ξ is an inner involution on the i-th diagonal block G_{n_i} whose fixed point subgroup is of the form $G_{a_i} \times G_{b_i}$, $a_i + b_i = n_i$. The irreducible cuspidal representation ρ_i of G_{n_i} has to be distinguished with respect to the subgroup $G_{a_i} \times G_{b_i}$ and some quasi-character arising from μ_ξ. By the result of Matringe in **4.2**, we must have $a_i = b_i$. As a result, the rank of $(G_{n_i}, \sigma_\xi|_{G_{n_i}})$ is equal to $\frac{1}{2}n_i$.

Next, let i, $j \in J'$ be indices such that the i-th block G_{n_i} and the j-th block G_{n_j} are interchanged by $\operatorname{Int}(w)$. By the remark on the diagonal entries of δ mentioned before the lemma in **4.5**, we have $\operatorname{Int}(\delta w)|_{G_{n_i} \times G_{n_j}} = \operatorname{Int}(w)|_{G_{n_i} \times G_{n_j}}$, hence the rank of $(G_{n_i} \times G_{n_j}, \sigma_\xi|_{G_{n_i} \times G_{n_j}})$ is equal to $n_i\ (= n_j)$.

Putting this all together, the rank of σ_ξ restricted to $\prod_{n_i:\text{even}} G_{n_i}$ is equal to

$$\sum_{i \in J} \frac{1}{2} n_i + \frac{1}{2} \sum_{i \in J'} n_i = \frac{1}{2} \sum_{n_i:\,\text{even}} n_i,$$

which is strictly greater than r by assumption. However, the rank of σ_ξ on G is equal to r by the choice of ξ in **4.5**. This leads to a contradiction. □

4.8. Let us go back to **4.3** and now take the involution into account. Consider the standard pair (P, M) of type $(n_1, \ldots, n_k)$ where each n_i is even or equal to 1 and put $\sum_{n_i:\,\text{even}} n_i = 2r'$. Let $\sigma = \operatorname{Int}(\varepsilon)$ be as in **4.3** where $\varepsilon \in M \cap X(n-r, r)$. As our main target we study the case that $r' = r$. So, we consider standard pairs of type

$$(2r_1, \ldots, 2r_k, \underbrace{1, \ldots, 1}_{n-2r})$$

where $\sum_i r_i = r$. We put $s = n - 2r$. Note that $s \geqq 0$ in general, and $s = 0$ means that the symmetric space is of even type. We may further assume that $\varepsilon \in X(n-r, r)$ is of the form

$$\varepsilon = \begin{pmatrix} \varepsilon^{(1)} & & & 0 \\ & \ddots & & \\ & & \varepsilon^{(k)} & \\ 0 & & & 1_s \end{pmatrix}, \quad \varepsilon^{(i)} = \varepsilon_1(r_i, r_i) = \begin{pmatrix} 1_{r_i} & 0 \\ 0 & -1_{r_i} \end{pmatrix}.$$

In such a case $M \cap H$ consists of matrices of the form

$$\begin{pmatrix} x_1 & & & & & \\ & \ddots & & & 0 & \\ & & x_k & & & \\ & & & t_1 & & \\ & 0 & & & \ddots & \\ & & & & & t_s \end{pmatrix}, \quad \begin{array}{l} x_i = \begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix}, \ a_i, b_i \in G_{r_i}, \\ \\ t_j \in F^\times. \end{array}$$

4.9. Let us compute the character $\mu_{M \cap H}$ in **1.2** for the above P, M and σ. At first, by the formula in **1.6** we have

$$\delta_P\begin{pmatrix} x_1 & & & & & \\ & \ddots & & & 0 & \\ & & x_k & & & \\ & & & t_1 & & \\ & 0 & & & \ddots & \\ & & & & & t_s \end{pmatrix} = \prod_{1\leqq i<j\leqq k}\left|\det(a_i)\det(b_i)\right|^{2r_j}\left|\det(a_j)\det(b_j)\right|^{-2r_i}$$

$$\times\prod_{1\leqq i\leqq k}\left|\det(a_i)\det(b_i)\right|^{s}\prod_{1\leqq j\leqq s}\left|t_j\right|^{-2r}\prod_{1\leqq i<j\leqq s}\left|t_i\right|\left|t_j\right|^{-1}.$$

Next, by considering conjugation from $\varepsilon_1(n-r,r)$ to ε, we may regard $P\cap H$ as

$$\begin{pmatrix}\text{parabolic of type}\\ (r_1,\dots,r_k,\underbrace{1,\dots,1}_{s})\end{pmatrix}\times\begin{pmatrix}\text{parabolic of type}\\ (r_1,\dots,r_k)\end{pmatrix}$$

in $H\simeq \mathrm{GL}_{n-r}(F)\times\mathrm{GL}_r(F)$. Under this identification, $\delta_{P\cap H}$ is computed as follows:

$$\delta_{P\cap H}\begin{pmatrix} x_1 & & & & & \\ & \ddots & & & 0 & \\ & & x_k & & & \\ & & & t_1 & & \\ & 0 & & & \ddots & \\ & & & & & t_s \end{pmatrix} = \prod_{1\leqq i<j\leqq k}\left|\det(a_i)\right|^{r_j}\left|\det(a_j)\right|^{-r_i}\prod_{1\leqq i\leqq k}\left|\det(a_i)\right|^{s}$$

$$\times\prod_{1\leqq j\leqq s}\left|t_j\right|^{-r}\prod_{1\leqq i<j\leqq s}\left|t_i\right|\left|t_j\right|^{-1}\prod_{1\leqq i<j\leqq k}\left|\det(b_i)\right|^{r_j}\left|\det(b_j)\right|^{-r_i}.$$

As a consequence, $\mu_{M\cap H}=\delta_P|_{P\cap H}^{-1/2}\cdot\delta_{P\cap H}$ is given as follows:

$$\mu_{M\cap H}\begin{pmatrix} x_1 & & & & & \\ & \ddots & & & 0 & \\ & & x_k & & & \\ & & & t_1 & & \\ & 0 & & & \ddots & \\ & & & & & t_s \end{pmatrix} = \prod_{1\leqq i\leqq k}\left|\det(a_i)\right|^{s/2}\left|\det(b_i)\right|^{-s/2}$$

$$\times\left(\prod_{1\leqq i<j\leqq s}\left|t_i\right|\left|t_j\right|^{-1}\right)^{1/2}.$$

Here, $x_i=\begin{pmatrix} a_i & 0\\ 0 & b_i\end{pmatrix}\in G_{2r_i}$, $a_i, b_i\in G_{r_i}$, and $t_i\in F^\times$.

Note that $\mu_{M\cap H}$ is trivial if $s=0$, i.e., if (G,σ) is of even type.

4.10. We recall the following result due to Jacquet and Rallis in [11].

Proposition. *Let H be the fixed point subgroup of $\sigma=\mathrm{Int}(\varepsilon)$ in $G=G_n$, for $\varepsilon\in X$. If an irreducible representation π of G is H-distinguished, then π is self-contragredient.* □

In the case of even type, let $\mu_{r,z}$ be the quasi-character of the subgroup $H_1(r,r)$ of G_{2r} defined by

$$\mu_{r,z}\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}=\left|\det(a)\right|^{z}\left|\det(b)\right|^{-z}\quad(z\in\mathbb{C}).$$

We also notice the following lemma shown by using the results of [12] and [11]:

Lemma. *Let π be an irreducible cuspidal representation of G_{2r}. If π is $H_1(r,r)$-distinguished, then it is also $(H_1(r,r),\mu_{r,z})$-distinguished for any $z\in\mathbb{C}$.*

Proof. By [12, Theorem 5.6], an irreducible cuspidal $H_1(r,r)$-distinguished representation π has a non-zero *Shalika functional*, say ℓ. Using this, [11, §6] gives a linear form $I(\cdot,z)$ on π by the integral

$$I(v,z)=\int_{G_r}\langle \ell,\pi\begin{pmatrix}a & 0\\ 0 & 1_r\end{pmatrix}v\rangle\cdot\left|\det(a)\right|^{z-1/2}da$$

for $\mathrm{Re}(z)$ sufficiently large. If π is cuspidal, then $I(\cdot,z)$ can be extended to an entire function on $z\in\mathbb{C}$, and is non-zero as a linear form on π (see [11, p. 117]). This shows that π is $(H_1(r,r),\mu_{r,z})$-distinguished for any $z\in\mathbb{C}$. □

4.11. We state the main claim of this section. For the even type case $s=0$, the same result is found also in [14] and [20].

Theorem. *Let $(r_1,\dots,r_k)$ be a partition of r where $1\leqq r\leqq[n/2]$ and put $s=n-2r$. Let (P,M) be the standard pair of type $(2r_1,\dots,2r_k,\underbrace{1,\dots,1}_{s})$ and consider the involution $\sigma=\mathrm{Int}(\varepsilon)$ given in* **4.8**. *For each i, take an irreducible cuspidal $H_1(r_i,r_i)$-distinguished representation ρ_i of G_{2r_i} and form*

$$\rho=\rho_1\otimes\cdots\otimes\rho_k\otimes\delta_{B_s}^{1/2}$$

(where we understand that $\rho=\rho_1\otimes\cdots\otimes\rho_k$ if $s=0$).

Suppose that $\rho_i\not\cong\rho_j$ for any $i\neq j$. Then,
(1) *The induced representation $\mathrm{Ind}_P^G(\rho)$ is H-distinguished.*
(2) *If $s\leqq 1$, then $\mathrm{Ind}_P^G(\rho)$ is irreducible.*
(3) *The induced representation $\mathrm{Ind}_P^G(\rho)$ is H-relatively cuspidal.*

Proof of (1) and (2). By **4.9** and the lemma in **4.10** for $z=s/2$, we can see that ρ is $(M\cap H,\mu_{M\cap H})$-distinguished. Hence $\mathrm{Ind}_P^G(\rho)$ is H-distinguished by **1.2**. The irreducibility in the case $s\leqq 1$ can be shown exactly in the same way as **3.5**. □

The proof of (3) will be given in **4.13**.

4.12. We consider the involution

$$\sigma'=\sigma'_{(n,r)}=\mathrm{Int}(\varepsilon'),\quad \varepsilon'=\varepsilon'_{(n,r)}=\begin{pmatrix} & & w_0\\ & 1_s & \\ w_0 & & \end{pmatrix}$$

where w_0 denotes the $r\times r$ anti-diagonal permutation matrix. Note that $\varepsilon'\in X(n-r,r)$, so σ' is $\mathrm{Int}(G)$-conjugate to $\sigma_1=\sigma_1(n-r,r)$. Let $H'=H'_{(n,r)}$ be the σ'-fixed point subgroup in G. As in **2.5**, every σ'-split parabolic subgroup is G-conjugate to a standard σ'-split one.

Maximal σ'-split parabolic subgroups among standard ones are given by $Q=Q_\ell$, the standard one of type $(\ell,n-2\ell,\ell)$, for $1\leqq\ell\leqq r$. On the standard Levi subgroup L of Q, σ' acts as

$$\sigma'\begin{pmatrix} x_1 & & 0 \\ & x_2 & \\ 0 & & x_3 \end{pmatrix} = \begin{pmatrix} w_0 x_3 w_0^{-1} & & 0 \\ & \sigma'_{(n-2\ell,r-\ell)}(x_2) & \\ 0 & & w_0 x_1 w_0^{-1} \end{pmatrix}$$

where w_0 is now of size $\ell \times \ell$. So we have

$$L \cap H' = \left\{ \begin{pmatrix} x & & 0 \\ & y & \\ 0 & & w_0 x w_0^{-1} \end{pmatrix} \,\middle|\, x \in G_\ell,\ y \in H'_{(n-2\ell,r-\ell)} \right\}.$$

The subtorus

$$A = \{\mathrm{diag}(a_1, \ldots, a_r, \underbrace{1, \ldots, 1}_{s}, a_r^{-1}, \ldots, a_1^{-1}) \mid a_i \in F^\times\}$$

gives a maximal σ'-split torus of G which is also F-split. One has

$$Z_G(A) \cap H' = \left\{ \begin{pmatrix} t & & 0 \\ & x & \\ 0 & & w_0 t w_0^{-1} \end{pmatrix} \,\middle|\, t_i \in D_r,\ x \in G_{n-2r} \right\} \simeq D_r \times G_{n-2r}.$$

Hence, as an algebraic F-group, $Z_{\mathbf{G}}(\mathbf{A}) \cap \mathbf{H}'$ has trivial Galois cohomology over F. From the lemma in **1.3**, any maximal σ'-split parabolic subgroup of G is H'-conjugate to one of Q_ℓ, $1 \leqq \ell \leqq r$.

4.13. Proof of 4.11 (3). As in the proof of **2.6** and **3.7**, it is enough to show that the Jacquet module $\left(\mathrm{Ind}_P^G(\rho)\right)_Q$ is not $L \cap H'$-distinguished for each $Q = Q_\ell$, $1 \leqq \ell \leqq r$. So we show that the pieces $\mathcal{F}_w(\rho) = \mathrm{Ind}_{L \cap wPw^{-1}}^L({}^w\rho)$ are not $L \cap H'$-distinguished for all w in a set of representatives of $Q \backslash G / P$ as in **2.6**.

Suppose that some of $\mathcal{F}_w(\rho)$ is $L \cap H'$-distinguished. Write $\mathcal{F}_w(\rho)$ as $I_1 \otimes I_2 \otimes I_3$ where I_1 and I_3 are representations induced up to G_ℓ, and I_2 is one up to $G_{n-2\ell}$. Looking at the form of $L \cap H'$ in **4.12**, the $L \cap H'$-distinction implies the following two conditions:

(i) I_1 and I_3 share a common irreducible subquotient.

(ii) I_2 is $H'_{(n-2\ell,r-\ell)}$-distinguished.

Now, if $s \leqq 1$, then the inducing data (for I_1, I_2 and I_3) are $\rho_1, \ldots, \rho_k$, and possibly one more character. So the condition (i) leads to a contradiction as in the argument of **3.7** together with the proposition in **4.10**. For the rest of the proof, we suppose that $s \geqq 2$. In this case the condition (i) can be achieved by choosing suitable characters coming from $\delta_{B_s}^{1/2}$ for the inducing data of I_1 and I_3. So we look at the condition (ii) in such a case. The inducing data of I_2 must contain all of $\rho_1, \ldots, \rho_k$. However, the rank of $G_{n-2\ell}/H'_{(n-2\ell,r-\ell)}$ is $r - \ell$, which is less than r. Hence I_2 cannot be $H'_{(n-2\ell,r-\ell)}$-distinguished by the lemma in **4.7**, a contradiction. □

4.14. In the case where $s \geqq 2$, the representation $\mathrm{Ind}_P^G(\rho)$ in Theorem **4.11** is reducible. However, we can show the following.

Corollary. *Let P, M, ρ and $\sigma = \mathrm{Int}(\varepsilon)$ be as in* **4.11** *and suppose that $s = n - 2r \geqq 2$. Let (P', M') be the standard parabolic and Levi subgroup of type $(2r_1, \ldots, 2r_k, s)$. Then, the induced representation*

$$\pi' := \mathrm{Ind}_{P'}^{G}(\rho_1 \otimes \cdots \otimes \rho_k \otimes \mathbf{1}_{G_s})$$

is an irreducible quotient of $\mathrm{Ind}_P^G(\rho)$ *which is* H*-distinguished and* H*-relatively cuspidal.*

Proof. By induction in stages, $\mathrm{Ind}_P^G(\rho)$ is identified with

$$\mathrm{Ind}_{P'}^{G}\left(\rho_1 \otimes \cdots \otimes \rho_k \otimes \mathrm{Ind}_{B_s}^{G_s}(\delta_{B_s}^{1/2})\right).$$

The induced representation $\mathrm{Ind}_{B_s}^{G_s}(\delta_{B_s}^{1/2})$ has one dimensional trivial quotient, hence π' is a quotient of $\mathrm{Ind}_P^G(\rho)$. Irreducibility of π' can be shown by the discussion on *segments* in [21, Theorem 4.2]. Furthermore, π' is H-distinguished, as will be seen below. Note that (P', M') is a σ-stable pair. The σ-fixed point subgroup $M' \cap H$ of M' consists of matrices of the form

$$\begin{pmatrix} x_1 & & & 0 \\ & \ddots & & \\ & & x_k & \\ 0 & & & y \end{pmatrix}, \quad x_i = \begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix}, \; a_i, b_i \in G_{r_i}, \; y \in G_s.$$

One can compute the character $\mu_{M'\cap H}$ in the same way as **4.9**. The result is:

$$\mu_{M'\cap H}\begin{pmatrix} x_1 & & & 0 \\ & \ddots & & \\ & & x_k & \\ 0 & & & y \end{pmatrix} = \prod_{1\leqq i\leqq k} \left|\det(a_i)\right|^{s/2}\left|\det(b_i)\right|^{-s/2}.$$

Thus the representation $\rho_1 \otimes \cdots \otimes \rho_k \otimes \mathbf{1}_{G_s}$ of M' is $(M' \cap H, \mu_{M'\cap H})$-distinguished, and in turn, π' is H-distinguished by **1.2**. Finally, all the H-matrix coefficients of π' are given by those of $\mathrm{Ind}_P^G(\rho)$ since π' is a quotient of $\mathrm{Ind}_P^G(\rho)$. Hence the claim readily follows by the result of **4.11** and the definition of relative cuspidality. □

Remark. The above corollary is a generalization of [13, 8.2] where only the case $r = 1$ was treated.

4.15. As a final remark of this section, we mention the case where $r > r'$ noticed in **4.4**. Let (P, M) be the standard pair of type

$$(2r_1, \ldots, 2r_k, \underbrace{1, \ldots, 1}_{n-2r'}), \quad \text{where} \sum_{1\leqq i\leqq k} r_i =: r' < r.$$

We consider the involution $\sigma = \mathrm{Int}(\varepsilon)$ with

$$\varepsilon = \begin{pmatrix} \varepsilon^{(1)} & & & & 0 \\ & \ddots & & & \\ & & \varepsilon^{(k)} & & \\ & & & 1_{n-r-r'} & \\ 0 & & & & -1_{r-r'} \end{pmatrix}, \quad \varepsilon^{(i)} = \varepsilon_1(r_i, r_i) = \begin{pmatrix} 1_{r_i} & 0 \\ 0 & -1_{r_i} \end{pmatrix}.$$

Then σ is of rank r, leaving the pair (P, M) stable, and there is an irreducible cuspidal $(M \cap H, \mu_{M\cap H})$-distinguished representation ρ of M. However, the proof in **4.13** does not work for this case. Actually we do not expect that $\mathrm{Ind}_P^G(\rho)$ is relatively cuspidal according to the following observation (see a comment in Introduction):

There exists an involution σ'' which is Int(G)*-conjugate to σ, a proper σ''-split parabolic subgroup Q of G, and an $L\cap H''$-relatively cuspidal representation ρ' of $L = Q \cap \sigma''(Q)$ where H'' denotes the σ''-fixed point subgroup, such that* $\mathrm{Ind}_P^G(\rho) = \mathrm{Ind}_Q^G(\rho')$.

To observe this, regard M as the product $G_{2r_1} \times \cdots \times G_{2r_k} \times D_s \times D_{r-r'} \times D_{r-r'}$. Then $M \cap H$ consists of

$$\begin{pmatrix} x_1 & & & & & \\ & \ddots & & & 0 & \\ & & x_k & & & \\ & & & t_1 & & \\ & 0 & & & t_2 & \\ & & & & & t_3 \end{pmatrix}, \qquad \begin{array}{l} x_i = \begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix}, \ a_i,\ b_i \in G_{r_i}, \\ \\ t_1 \in D_s,\ t_2,\, t_3 \in D_{r-r'}. \end{array}$$

The computation of $\mu_{M\cap H}$ can be carried out as in **4.9**. We have

$$\mu_{M\cap H}\begin{pmatrix} x_1 & & & & & \\ & \ddots & & & 0 & \\ & & x_k & & & \\ & & & t_1 & & \\ & 0 & & & t_2 & \\ & & & & & t_3 \end{pmatrix} = \prod_{1\leqq i\leqq k} \mu_{r_i,s/2}(x_i)\cdot \delta_{B_s}^{1/2}(t_1)\cdot \chi(t_2)\cdot \check{\chi}^{-1}(t_3),$$

where χ is a suitable character of $D_{r-r'}$ and $\check{\chi} = \chi \circ \mathrm{Int}(w_0)$. So, an irreducible cuspidal $(M \cap H, \mu_{M\cap H})$-distinguished representation ρ of M is of the form

$$\rho = \rho_1 \otimes \cdots \otimes \rho_k \otimes \delta_{B_s}^{1/2} \otimes \chi \otimes \check{\chi}^{-1},$$

where ρ_i is an irreducible cuspidal $H_1(r_i, r_i)$-distinguished representation of G_{2r_i}. Now let Q be the parabolic subgroup of type $(2r' + s, \underbrace{1, \ldots, 1}_{2(r-r')})$. Regard its Levi subgroup L as the product $G_{2r'+s} \times D_{r-r'} \times D_{r-r'}$ and let π' be the representation of $G_{2r'+s}$ induced from $\rho_1 \otimes \cdots \otimes \rho_k \otimes \delta_{B_s}^{1/2}$. Then, by induction in stages, we have

$$\mathrm{Ind}_P^G(\rho) = \mathrm{Ind}_Q^G(\pi' \otimes \chi \otimes \check{\chi}^{-1}).$$

There is an involution σ'' which is Int(G)-conjugate to σ such that Q is σ''-split and further,

$$\sigma''\begin{pmatrix} g' & & 0 \\ & t_1 & \\ 0 & & t_2 \end{pmatrix} = \begin{pmatrix} \varepsilon' g' \varepsilon'^{-1} & & 0 \\ & t_2 & \\ 0 & & t_1 \end{pmatrix}$$

where $\varepsilon' = \begin{pmatrix} \varepsilon^{(1)} & & & 0 \\ & \ddots & & \\ & & \varepsilon^{(k)} & \\ 0 & & & 1_s \end{pmatrix}$. The representation π' of $G_{2r'+s}$ is distinguished (and is relatively cuspidal) with respect to the fixed point subgroup of Int(ε') by **4.11**, hence $\rho' := \pi' \otimes \chi \otimes \check{\chi}^{-1}$ is $L \cap H''$-distinguished and $L \cap H''$-relatively cuspidal.

5. Remarks on the stable pairs

5.1. In sections 2-4, we have obtained relatively cuspidal representations of the form $\mathrm{Ind}_P^G(\rho)$ starting from σ-stable pairs (P, M). Especially in Section 4, we have excluded the case where M has no irreducible cuspidal $M \cap H$-distinguished representations. But, such a limitation seems to be not enough as was suggested by **4.15**. There is a more apparent case, *the group case*. Let G be the direct product $G_0 \times G_0$ where G_0 is a reductive p-adic group, σ the involution given by $\sigma(g_1, g_2) = (g_2, g_1)$. Then, an irreducible H-distinguished representation of G is of the form $\pi_0 \otimes \widetilde{\pi}_0$ where π_0 is an irreducible representation of G_0. It is H-relatively cuspidal if and only if π_0 is cuspidal (see [13, 1.5]). Now, σ-stable pairs are of the form $(P, M) = (P_0 \times P_0, M_0 \times M_0)$ where $P_0 = M_0 U_0$ is any parabolic subgroup of G_0. One can take an irreducible cuspidal $M \cap H$-distinguished representation of M in the form $\rho = \rho_0 \otimes \widetilde{\rho}_0$ where ρ_0 is an irreducible cuspidal representation of M_0. However, the representation $\mathrm{Ind}_P^G(\rho) \simeq \mathrm{Ind}_{P_0}^{G_0}(\rho_0) \otimes \mathrm{Ind}_{P_0}^{G_0}(\widetilde{\rho}_0)$ is not relatively cuspidal since $\mathrm{Ind}_{P_0}^{G_0}(\rho_0)$ is not cuspidal.

5.2. We have an expectation that there is a relationship between relatively cuspidal representations and maximal σ-split tori which are F-anisotropic modulo the center. Based on such a point of view, we consider some candidates for σ-stable pairs suitable for the construction of relatively cuspidal representations as follows.

Suppose that we have a maximal σ-split F-torus $\mathbf{S}$ of $\mathbf{G}$ which is F-anisotropic modulo $\mathbf{Z}$ (or equivalently, modulo the σ-split part of $\mathbf{Z}$). Take a maximal F-torus $\mathbf{T}$ of $\mathbf{G}$ containing $\mathbf{S}$. Then $\mathbf{T}$, as well as its F-split part $\mathbf{T}_d$, is σ-stable. Put

$$\mathbf{M} = Z_{\mathbf{G}}(\mathbf{T}_d),$$

the centralizer of $\mathbf{T}_d$ in $\mathbf{G}$. We have a decomposition $\mathbf{T}_d = (\mathbf{T}_d \cap \mathbf{H})^0 \cdot \mathbf{S}_d$, where $(\,\cdot\,)^0$ stands for the identity component. Since $\mathbf{S}_d$ is central in $\mathbf{G}$ by assumption on $\mathbf{S}$, we can see that

$$\mathbf{M} = Z_{\mathbf{G}}((\mathbf{T}_d \cap \mathbf{H})^0).$$

Hence, by [9, the proof of 3.4], there is a σ-stable parabolic F-subgroup $\mathbf{P}$ of $\mathbf{G}$ having $\mathbf{M}$ as a Levi subgroup. We consider the parabolic and Levi subgroups arising in this way are relevant ones for the construction of relatively cuspidal representations applied to the proposition in **1.2**.

Remark. (i) The stable pair (P, M) in **4.15** will not arise in the above way, since the rank of $\sigma|_M$ is less than that of σ on G. In *the group case*, it is easy to see that there is no proper stable pair arising in the above way.
(ii) Murnaghan's recent work [17] seems to have a similar point of view. In the terminology of [17], a maximal F-torus $\mathbf{T}$ is said to be *σ-elliptic in* $\mathbf{G}$ if its (σ, F)-split part is central in $\mathbf{G}$. Our $\mathbf{T}$'s are σ-elliptic ones which further contain some maximal σ-split F-torus.

Also, in [20], a σ-stable F-Levi subgroup is called a *σ-elliptic Levi subgroup* if it is not contained in any proper σ-split parabolic subgroup of G (or equivalently, if its (σ, F)-split component is central in G). Our Levi subgroups $M = Z_G(T_d)$ arising in the above way are particular ones among σ-elliptic Levi subgroups.

For the rest of this section we shall verify that the stable pairs we have used in sections 2, 3, and **4.8** actually arise from suitable tori $\mathbf{S}$ and $\mathbf{T}$ as above.

5.3. *The case of Galois involution.*
(1) At first, take an extension $\mathbf{k}/F$ of degree n such that the compositum $\mathbb{K} := \mathbf{k} \cdot E$ is a quadratic extension of $\mathbf{k}$. Then the degree of $\mathbb{K}$ over E is n, hence the multiplicative group $\mathbb{K}^\times$ can be embedded in $G = \mathrm{GL}_n(E)$. The image, say $T = T_n$, is a maximal F-torus. The embedding can be chosen so that T is σ-stable and $\sigma|_T$ coincides with the Galois

involution on $\mathbb{K}^\times$ over $\mathbf{k}$. Let $S = S_n$ be the subtorus of T corresponding to the norm kernel $\mathbb{K}_1^\times := \ker[N_{\mathbb{K}/\mathbf{k}} : \mathbb{K}^\times \to \mathbf{k}]$. Then S is F-anisotropic, σ-split, of dimension n over F, hence is maximal σ-split. The F-split part T_d of T is the scalars from $F^\times$, hence this only gives $Z_G((T)_d) = G$ in the procedure of **5.2**.
(2) In general, take an arbitrary partition $(n_1, \dots, n_k)$ of n and let M be the corresponding standard Levi subgroup, isomorphic to the product $\prod_{i=1}^k G_{n_i}$. For each i, take an extension $\mathbf{k}_i/F$ of degree n_i such that $\mathbb{K}_i := \mathbf{k}_i \cdot E$ is a quadratic extension of $\mathbf{k}_i$. Then we obtain a maximal torus $T_{n_i} \simeq \mathbb{K}_i^\times$ of G_{n_i}, containing a maximal σ_{n_i}-split torus $S_{n_i} \simeq (\mathbb{K}_i^\times)_1$ of G_{n_i}. As subgroups of $M = \prod_{i=1}^k G_{n_i}$, put

$$T = \prod_{i=1}^{k} T_{n_i}, \quad S = \prod_{i=1}^{k} S_{n_i}.$$

Then,

$$\dim_F(S) = \sum_i \dim_F((\mathbb{K}_i^\times)_1) = \sum_i n_i = n,$$

so S is a maximal σ-split torus of G which is F-anisotropic. The F-split part T_d of T is the product of scalars from $F^\times$ in G_{n_i}, hence $Z_G(T_d)$ coincides with M. In this way, any standard pair (P, M) of type $(n_1, \dots n_k)$ arises as a particular one in **5.2**.

5.4. *The case of inner involutions (I).*
(1) First we deal with the case where $m = 1$ in Section 3, that is, $G = \mathrm{GL}_2(F)$, $\sigma_1 = \mathrm{Int}\left(\begin{smallmatrix} 0 & 1 \\ \tau & 0 \end{smallmatrix}\right)$ and $H_1 \simeq E^\times$. The norm image in $F^\times$ from $E^\times$ contains a non-square in $F^\times$. Take one such τ' and form $E' = F(\sqrt{\tau'})$. (Note that we may take $\tau' = \tau$ and $E' = E$ if -1 is in the norm image from E.) Since $\tau' \cdot (-\tau)$ is in the norm image from $E^\times$, we can write

$$\tau' \cdot (-\tau) = x^2 - \tau y^2$$

by some $x, y \in F$. We must have $x \neq 0$, for τ' is not a square. Now put

$$T = T_2 := \gamma \cdot \left\{ \begin{pmatrix} a & b \\ \tau' b & a \end{pmatrix} \in G_2 \,\middle|\, a, b \in F \right\} \cdot \gamma^{-1}, \quad \gamma = \begin{pmatrix} 1 & y \\ 0 & x \end{pmatrix}.$$

This is a maximal F-torus of G_2 isomorphic to $(E')^\times$. By a direct computation it is seen that σ_1 acts on T as the Galois involution of E'/F. So, the σ_1-split part $S = S_2$ of T corresponds to the norm kernel of E'/F, hence is F-anisotropic (and is maximal σ_1-split).
(2) Next we consider $G = G_{2m} = \mathrm{GL}_{2m}(F)$ and $\sigma_1 = \mathrm{Int}\left(\begin{smallmatrix} O & 1_m \\ \tau 1_m & O \end{smallmatrix}\right)$ for a general m. Take an extension $\mathbf{k}/F$ of degree m so that $\mathbb{K} := \mathbf{k} \cdot E$ is a quadratic extension of $\mathbf{k}$. Regarding the multiplication of $\mathbf{k}$ on $\mathbf{k} \simeq F^m$ as an F-linear action, we have an embedding $\mathbf{k} \overset{\iota}{\hookrightarrow} \mathrm{Mat}_m(F)$ and further, $\mathrm{Mat}_2(\mathbf{k}) \overset{\iota}{\hookrightarrow} \mathrm{Mat}_{2m}(F)$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \iota(a) & \iota(b) \\ \iota(c) & \iota(d) \end{pmatrix} \in \mathrm{Mat}_2(\mathrm{Mat}_m(F)) \simeq \mathrm{Mat}_{2m}(F).$$

Thus we may regard $\mathrm{GL}_2(\mathbf{k})$ as a subgroup of $G = G_{2m}$. The restriction of σ_1 to the image of $\mathrm{GL}_2(\mathbf{k})$ is given by $\mathrm{Int}\left(\begin{smallmatrix} 0 & 1 \\ \tau & 0 \end{smallmatrix}\right)$ on $\mathrm{GL}_2(\mathbf{k})$. By using the discussion in **(1)** we can obtain a maximal torus $T_2 \simeq (\mathbb{K}')^\times$ of $\mathrm{GL}_2(\mathbf{k})$ where $\mathbb{K}' = \mathbf{k}(\sqrt{\tau'})$ for a non-square $\tau' \in \mathbf{k}$ in the norm image from $\mathbb{K}$, and also a maximal $\mathrm{Int}\left(\begin{smallmatrix} 0 & 1 \\ \tau & 0 \end{smallmatrix}\right)$-split torus $S_2 \subset T_2$ of $\mathrm{GL}_2(\mathbf{k})$ which is

F-anisotropic. Let us put

$$T = \iota(T_2) \simeq (\mathbb{K}')^\times, \quad S = \iota(S_2) \simeq \ker(N_{\mathbb{K}'/\mathbf{k}}).$$

Then T (resp. S) has dimension $2m$ (resp. m) over F, hence is a maximal torus (resp. a maximal σ_1-split torus) of G_{2m}. Since the F-split part T_d of T is the scalars from $F^\times$, this only gives $Z_G\big((T)_d\big) = G$.

(3) Now, for any partition $(m_1, \dots, m_k)$ of m, let (P, M) be the standard pair of type $(2m_1, \dots, 2m_k)$ which is stable under $\sigma = \mathrm{Int}(\varepsilon)$ in **3.2**. Putting $\sigma^{(i)} = \mathrm{Int}(\varepsilon^{(i)})$, we may regard $(M, \sigma|_M)$ as the direct product of $(G_{2m_i}, \sigma^{(i)})$, $1 \leqq i \leqq k$. By **(2)**, we can take a maximal F-torus $T^{(i)}$ of G_{2m_i} which is F-anisotropic modulo the center of G_{2m_i}, and a maximal $\sigma^{(i)}$-split torus $S^{(i)} \subset T^{(i)}$ of G_{2m_i} which is F-anisotropic. Let T (resp. S) be the product of all the $T^{(i)}$ (resp. $S^{(i)}$). Then S is an F-anisotropic σ-split torus of G, and is maximal σ-split since its dimension over F is $\sum_i m_i = m$. Also, T is a maximal torus of G containing S. The F-split part T_d is the product of scalar matrices in G_{2m_i}, hence $Z_G(T_d) = M$.

5.5. *The case of inner involutions (II).*

(1) First we deal with the case of even type, that is, $n = 2r$ and $G = G_{2r}$, $\sigma_1 = \mathrm{Int}(\varepsilon_1)$ where $\varepsilon_1 = \left(\begin{smallmatrix} 1_r & 0 \\ 0 & -1_r \end{smallmatrix}\right)$. Take an arbitrary quadratic extension $E = F(\sqrt{\tau})$ of F, and an extension $\mathbf{k}$ of F of degree r such that the compositum $\mathbb{K} := E \cdot \mathbf{k}$ is a quadratic extension of $\mathbf{k}$. There are natural embeddings

$$\mathbb{K}^\times \hookrightarrow \mathrm{GL}_r(E) \hookrightarrow \mathrm{GL}_{2r}(F) = G,$$

where the latter one is given in **3.1**. Notice that σ_1 acts on $\mathrm{GL}_r(E)$ as

$$\sigma_1 \left(\begin{smallmatrix} a & b \\ \tau b & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & -b \\ -\tau b & a \end{smallmatrix}\right),$$

hence is the same as the Galois automorphism of E/F on $\mathrm{GL}_r(E)$. Furthermore, the restriction of σ_1 to the image of $\mathbb{K}^\times$ coincides with the Galois automorphism of $\mathbb{K}/\mathbf{k}$. Let T be the image of $\mathbb{K}^\times$ in G. It is F-anisotropic modulo the center of G. Also, by the above observation, the σ_1-split part S of T is identified with the norm kernel of $\mathbb{K}/\mathbf{k}$, which is F-anisotropic. We have $\dim_F(T) = 2r$, $\dim_F(S) = r$, hence T (resp. S) is a maximal F-torus (resp. maximal σ_1-split torus) of G. Since T_d is the center of G, we only have $Z_G\big((T)_d\big) = G$.

(2) Let us turn to the general case, $G = G_n$ and $\sigma_1 = \mathrm{Int}(\varepsilon_1)$ where $\varepsilon_1 = \left(\begin{smallmatrix} 1_{n-r} & 0 \\ 0 & -1_r \end{smallmatrix}\right)$. Let $(r_1, \dots, r_k)$ be any partition of r and consider the standard pair (P, M) of type $(2r_1, \dots, 2r_k, \underbrace{1, \dots, 1}_{s})$ where $s = n - 2r$, and the involution $\sigma = \mathrm{Int}(\varepsilon)$ in **4.8** which leaves (P, M) stable. By **(1)** we can take a maximal $\sigma^{(i)}$-split F-torus S_i of G_{2r_i} which is F-snisotropic, and a maximal F-torus T_i of G_{2r_i} containing S_i. Set

$$S := \left(\prod_{i=1}^{k} S_i\right) \times \underbrace{\{1\} \times \cdots \times \{1\}}_{s}, \quad T := \left(\prod_{i=1}^{k} T_i\right) \times \underbrace{F^\times \times \cdots \times F^\times}_{s}.$$

Then, the dimension of S is $\sum_{i=1}^{k} r_i = r$, hence S is a maximal σ-split F-torus of G which is F-anisotropic. Also, T is a maximal torus of G which gives $Z_G(T_d) = M$.

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