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EVOLUTION OF RELATIVE YAMABE CONSTANT UNDER RICCI FLOW

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Abstract

Let W be a manifold with boundary M given together with a conformal class $\bar{C}$ which restricts to a conformal class C on M. Then the relative Yamabe constant $Y_{\bar{C}}(W, M; C)$ is well-defined. We study the short-time behavior of the relative Yamabe constant $Y_{[\bar{g}_t]}(W, M; C)$ under the Ricci flow $\bar{g}_t$ on W with boundary conditions that mean curvature $H_{\bar{g}_t} \equiv 0$ and $\bar{g}_t|_M \in C = [\bar{g}_0]$. In particular, we show that if the initial metric $\bar{g}_0$ is a Yamabe metric, then, under some natural assumptions, $\frac{d}{dt}\big|_{t=0} Y_{[\bar{g}_t]}(W, M; C) \geq 0$ and is equal to zero if and only the metric $\bar{g}_0$ is Einstein.

1. Introduction

In this short note, we analyze the behavior of the relative Yamabe constant of manifolds with boundary under the Ricci flow with appropriate boundary conditions. It turns out that the evolution equation of the relative Yamabe constant is analogous to the one on closed manifolds, given in [4].

1.1. Relative Yamabe constant. Let W be a compact manifold with boundary $M = \partial W \neq \emptyset$ and $\dim W = n \geq 3$. For a metric $\bar{g}$ on W we denote by $g = \bar{g}|_M$ and by $H_{\bar{g}}$ the mean curvature along the boundary M with respect to the outward unit normal vector. We also denote by $[\bar{g}]$ and $[g]$ the corresponding conformal classes, and by $C(W)$ and $C(M)$ the spaces of conformal classes on W and M, respectively. Given $\bar{C} \in C(W)$ we define $C = \partial\bar{C}$ to be the restriction $\bar{C}|_M$ and we denote by $C(W, M)$ the space $\{(\bar{C}, C) \mid \bar{C} \in C(W),\ C = \partial\bar{C}\ \}$. We also consider a normalized conformal class $\bar{C}^{(0)} = \{\ \bar{g} \in \bar{C} \mid H_{\bar{g}} \equiv 0\ \}$. It is easy to observe that the normalized class $\bar{C}^{(0)} \subset \bar{C}$ is always non-empty, and there is a natural bijection between the spaces $C(W, M)$ and $C^{(0)}(W, M) = \{(\bar{C}^{(0)}, C) \mid \bar{C} \in C(W),\ C = \partial\bar{C}\ \}$ (see [5, (1.4)]).

Let $\mathcal{R}(W)$ stand for the space of Riemannian metrics on W. Fix a conformal class $C \in C(M)$, we need the following spaces of metrics:

$$\mathcal{R}_C(W, M) = \{\bar{g} \in \mathcal{R}(W) \mid \partial[\bar{g}] = C\ \}, \quad \mathcal{R}^{(0)}_C(W, M) = \{\bar{g} \in \mathcal{R}_C(W, M) \mid H_{\bar{g}} \equiv 0\ \}.$$

Consider the normalized Einstein-Hilbert functional $I_C : \mathcal{R}^{(0)}_C(W, M) \to \mathbb{R}$ given by

$$I_C(\bar{g}) = \frac{\int_W R_{\bar{g}} d\sigma_{\bar{g}}}{\mathrm{Vol}_{\bar{g}}(W)^{\frac{n-2}{n}}},$$

where $R_{\bar{g}}$ is the scalar curvature and $d\sigma_{\bar{g}}$ is the volume element. Similarly to the case of closed manifolds, the Einstein-Hilbert functional I_C is not bounded for any manifold of

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dimension ≥ 3 with any fixed conformal class on the boundary.

Denote by $\mathrm{Crit}(I_C)$ the set of *critical metrics* of the functional I_C. It is well-known that the set $\mathrm{Crit}(I_C) \subset \mathcal{R}_C^{(0)}(W, M)$ coincides with the set of Einstein metrics $\bar{g}$ on W such that $\partial[\bar{g}] = C$ and the mean curvature $H_{\bar{g}} \equiv 0$ on M, see [1].

Fix $(\bar{C}, C) \in \mathcal{C}(W, M)$, the *relative Yamabe constant* $Y_{\bar{C}}(W, M; C)$ is defined as

$$Y_{\bar{C}}(W, M; C) = \inf_{\bar{g}\in\bar{C}^{(0)}} I_C(\bar{g}).$$

The functional $I_C|_{\bar{C}^{(0)}} : \bar{g} \mapsto I(\bar{g})$ from restriction is called the *Yamabe functional*.

Let ν be the outward unit normal vector along the boundary M. Fix a $\bar{g} \in \bar{C}^{(0)}$, then any metric in $\bar{C}^{(0)}$ can be written as $u^{\frac{4}{n-2}}\bar{g}$ where $u \in C_+^\infty(W)$ is a smooth positive function satisfying $\partial_\nu u \equiv 0$ on M, see [5, (1.4)].

Hence the relative Yamabe constant $Y_{[\bar{g}]}(W, M; [g])$ can be written as

$$Y_{[\bar{g}]}(W, M; [g]) = \inf_{\substack{u \in C_+^\infty(W) \\ \partial_\nu u \equiv 0 \text{ on } M}} \frac{\int_W \left(\frac{4(n-1)}{n-2}\left|\nabla_{\bar{g}} u\right|^2 + R_{\bar{g}} u^2\right) d\sigma_{\bar{g}}}{\left(\int_W u^{\frac{2n}{n-2}} d\sigma_{\bar{g}}\right)^{\frac{n-2}{n}}}, \tag{1}$$

where $\nabla_{\bar{g}}$ is the Riemannian connection of the metric $\bar{g}$.

The Euler-Lagrange equation for any minimizer u of the functional (1) is

$$\begin{aligned} -\frac{4\,(n-1)}{n-2}\Delta_{\bar{g}} u + R_{\bar{g}} u &= Y_{[\bar{g}]}(W, M; [g]) u^{\frac{n+2}{n-2}} && \text{on } W, \\ \partial_\nu u &\equiv 0 && \text{on } M. \end{aligned} \tag{2}$$

We assume that the minimizer u is normalized as

$$\int_W u^{\frac{2n}{n-2}} d\sigma_{\bar{g}} = 1. \tag{3}$$

It is well-known that there exists a solution of (2) in a generic case due to Escobar, see [5] for details. Actually Escobar's results have been generalized in many ways, for example, see [3, 8]. It is also well-known that the normalized solution u is unique if $Y_{[\bar{g}]}(W, M; [g]) \leq 0$ and there are examples of multiple solutions if $Y_{[\bar{g}]}(W, M; [g]) > 0$. For a normalized minimizer u, the metric $u^{\frac{4}{n-2}}\bar{g}$ has volume one, constant scalar curvature $Y_{[\bar{g}]}(W, M; [g])$, and zero mean curvature along the boundary. The metric is called a *Yamabe metric*.

There is a remarkable property of the relative Yamabe constant $Y_{\bar{C}}(W, M; C)$, namely, we have the inequality $Y_{\bar{C}}(W, M; C) \leq Y_{\bar{C}_{st}}(D^n, S^{n-1}; C_{st})$, where $\bar{C}_{st}$ is conformal classes of the standard metric on the hemisphere D^n with totally geodesic boundary S^{n-1}. Furthermore, the equality holds if and only if the pair $((W, \bar{C}), (M, C))$ is conformally equivalent to $((D^n, \bar{C}_{st}), (S^{n-1}, C_{st}))$. This leads to the definition of the *Yamabe invariant* $Y(W, M; C)$:

$$Y(W, M; C) = \sup_{\bar{C},\, \partial\bar{C}=C} Y_{\bar{C}}(W, M; C). \tag{4}$$

REMARK. Assume that a conformal metric $\tilde{g} = \tilde{u}^{\frac{4}{n-2}}\bar{g}$ has zero mean curvature on M and has constant scalar curvature $R_{\tilde{g}} = \tilde{R}$. Then the function $\tilde{u}$ satisfies the Euler-Lagrange equations (2), where the relative Yamabe constant $Y_{[\bar{g}]}(W, M; [g])$ is replaced by $\tilde{R}$. Hence the metric $\tilde{u}^{\frac{4}{n-2}}\bar{g}$ is a critical metric, but not necessarily a minimum of the functional $I_C|_{\bar{C}^{(0)}}$.

1.2. The Result. We consider the Ricci flow with the following boundary conditions:

$$\partial_t \bar{g}_t = -2\mathrm{Ric}_{\bar{g}_t} \quad \text{with mean curvature } H_{\bar{g}_t} \equiv 0 \text{ and } \bar{g}_t|_M \in [\bar{g}_0|_M] = C_0. \tag{5}$$

Here is our main result:

Theorem A. *Let $\bar{g}_t$ be the solution of Ricci flow* (5) *on W with initial metric $\bar{g}_0$ being a Yamabe metric. Assume that there is a C^1-family of positive smooth functions $u(t)$, $t \in [0, \epsilon)$ for some $\epsilon > 0$, with $u(0) = 1$ such that the metric $u(t)^{\frac{4}{n-2}}\bar{g}_t$ is a Yamabe metric (with unit volume and constant scalar curvature $Y_{[\bar{g}_t]}(W, M; C_0)$). Then $\frac{d}{dt}\Big|_{t=0} Y_{[\bar{g}_t]}(W, M; C_0) \geq 0$ and the equality holds if and only if $\bar{g}_0$ is an Einstein metric.*

In fact, we will compute the evolution equation for the Yamabe constant and the corresponding sub-critical constant under Ricci flow (5), see Theorem B below. If the initial metric $\bar{g}_0$ is a Yamabe metric, then the formula is very simple:

$$\frac{d}{dt}\Big|_{t=0} Y_{[\bar{g}_t]}(W, M; C_0) = 2\int_W |\mathrm{Ric}^0_{\bar{g}_0}|^2 d\sigma_{\bar{g}_0} \geq 0, \tag{6}$$

where $\mathrm{Ric}^0_{\bar{g}_0}$ is the norm of the traceless Ricci tensor. The proof is given at the end of §2.

1.3. Ricci Flow on manifolds with boundary. Since we have assumed the existence of the Ricci flow (5) on manifolds with boundary in Theorem A, here we briefly review some results due to Gianniotis [6] on the initial value problem of (5).

Let $(W, \bar{g}^{(0)})$ be a Riemannian manifold with the mean curvature $H_{\bar{g}^{(0)}} \equiv 0$ along M. Choose conformal class $C_0 = [\bar{g}^{(0)}|_M]$ on boundary M. According to [6, Theorem 1.2], there exists a short-time solution $\bar{g}_t$ of (5) in the space

$$C^\infty(W \times (0, T]) \cap C^{1+\alpha,(1+\alpha)/2}(W \times [0, T])$$

with the property that $\bar{g}_t$ converges to the initial metric $\bar{g}^{(0)}$ in the $C^{1+\alpha}(W)$-Cheeger-Gromov topology and C^∞-topology away from the boundary M as $t \to 0^+$, (i.e. it converges to $\bar{g}^{(0)}$ up to diffeomorphism).

To get better regularity of the short-time solution $\bar{g}_t$ of (5), the initial metric $\bar{g}^{(0)}$ has to satisfy the following compatibility condition, namely,

$$(\mathrm{Ric}_{\bar{g}^{(0)}})^T = \tilde{f} \cdot (\bar{g}^{(0)})^T, \tag{7}$$

see [6, Theorem 1.1]. Here $\tilde{f}$ is a smooth positive function and S^T denotes the tangential to the boundary part of the tensor S, see [6, p.314]. Then the solution $\bar{g}_t$ is in

$$C^{2+\alpha,(2+\alpha)/2}(W \times [0, T])$$

and converges to the initial metric $\bar{g}^{(0)}$ in $C^{2+\alpha}(W)$ ([6, Theorem 4.2]).

From above discussion we conclude that the relative Yamabe constant $Y_{[\bar{g}_t]}(W, M; [\bar{g}^{(0)}|_M])$ is differentiable for $t > 0$. Furthermore, if the compatibility condition (7) is satisfied, then relative Yamabe constant $Y_{[\bar{g}_t]}(W, M; [\bar{g}^{(0)}|_M])$ is continuous at $t = 0$.

REMARK. Even though the boundary conditions given in (5) are not as general as the ones studied by Gianniotis [6], they define the Ricci flow on the space $\mathcal{R}^{(0)}_C(W, M)$. Furthermore, the space $\mathcal{R}^{(0)}_C(W, M)$ is suitable for defining a relative version of the Perelman's functional to provide a gradient flow equivalent to the Ricci flow (5).

2. Proofs

2.1. Subcritical Yamabe problem. In order to analyze the Yamabe problem, it is important to consider the *sub-critical regularization* of the Yamabe functional, namely, the functional $\mathsf{Y}_p : \mathcal{R}^{(0)}(W) = \{\bar{g} \in \mathcal{R}(W) \mid H_{\bar{g}} \equiv 0\} \to \mathbb{R}$ defined by

$$\mathsf{Y}_p(W,\bar{g}) := \inf_{\substack{u \in C_+^\infty(W) \\ \partial_\nu u \equiv 0 \text{ on } M}} \frac{\int_W \left(\frac{4(n-1)}{n-2}\left|\nabla_{\bar{g}} u\right|^2 + R_{\bar{g}} u^2\right) d\sigma_{\bar{g}}}{\left(\int_W u^{p+1} d\sigma_{\bar{g}}\right)^{\frac{2}{p+1}}}$$

for $p \in [1, \frac{n+2}{n-2})$. Note that the Yamabe constant $Y_{[\bar{g}]}(W, M; [\bar{g}|_M])$ equals to the constant $\mathsf{Y}_p(W,\bar{g})$ if p attains the critical value $p = \frac{n+2}{n-2}$.

Clearly, the corresponding Euler-Lagrange equation of functional Y_p is

$$\begin{aligned} -\frac{4\,(n-1)}{n-2}\Delta_{\bar{g}} u + R_{\bar{g}} u &= \mathsf{Y}_p(W,\bar{g}) u^p && \text{on } W, \\ \partial_\nu u &\equiv 0 && \text{on } M. \end{aligned} \tag{8}$$

Again, we assume the following normalization condition:

$$\int_W u^{p+1} d\sigma_{\bar{g}} = 1. \tag{9}$$

REMARK. It should be noted that the existence of solution u of (8) and (9) follows from the direct method in the calculus of variation because p is sub-critical exponent. In that case the constants $\mathsf{Y}_p(W,\bar{g})$ have the same sign as the relative Yamabe constant $Y_{[\bar{g}]}(W, M; [\bar{g}|_M])$, however, the value of $\mathsf{Y}_p(W,\bar{g})$ depends on the metric $\bar{g}$, not only on the conformal class $[\bar{g}]$. In particular, if $p = 1$, $\mathsf{Y}_p(W,\bar{g})$ coincides with the principal eigenvalue of the conformal Laplacian with minimal boundary condition, see [5] for details.

2.2. Evolution equations for the constant $\mathsf{Y}_p(W,\bar{g})$. Here is our main technical result:

Theorem B. *Let $\bar{g}_t$ be a solution of the Ricci flow* (5) *on W for $t \in [0,T)$. Denote $g_t = \bar{g}_t|_M$. Given $p \in (1, \frac{n+2}{n-2}]$, assume that there is a C^1-family of smooth positive functions $u(t)$, $t \in [0,T)$, which satisfy*

$$-\frac{4(n-1)}{n-2}\Delta_{\bar{g}_t} u(t) + R_{\bar{g}_t} u(t) = \mathsf{Y}_p(W,\bar{g}_t) u(t)^p \quad \textit{on } W, \tag{10}$$

$$\int_W u(t)^{p+1} d\mu_{\bar{g}_t} = 1, \tag{11}$$

$$\partial_{\nu_t} u(t) = 0 \quad \textit{on } M, \tag{12}$$

where ν_t is the outward unit normal vector with respect to metric $\bar{g}_t$. Then

$$\begin{aligned} \frac{d}{dt}\mathsf{Y}_p(W,\bar{g}_t) = &\int_W \left(\frac{8(n-1)}{n-2}\overline{\mathrm{Ric}}^0(\bar{\nabla} u, \bar{\nabla} u) + 2\left|\overline{\mathrm{Ric}}^0\right|^2 u^2\right) d\sigma_{\bar{g}_t} \\ &+ \left(\frac{2}{n} - \frac{p-1}{p+1}\right)\int_W \left(\frac{4(n-1)}{n-2}\bar{R}\left|\bar{\nabla} u\right|^2 + \bar{R}^2 u^2\right) d\sigma_{\bar{g}_t} \\ &+ \int_W \left(u^2 \bar{\Delta}\bar{R} - \frac{4(n-1)}{(p+1)(n-2)}\bar{R}\bar{\Delta} u^2\right) d\sigma_{\bar{g}_t} \\ &- \frac{8(n-1)}{n-2}\int_M \left(2u\overline{\mathrm{Ric}}(\bar{\nabla} u, \nu_t) + u\partial_{\nu_t} h\right) d\sigma_{g_t}, \end{aligned} \tag{13}$$

where $u = u(t)$, $h = \frac{\partial u}{\partial t}$, and $\overline{\mathrm{Ric}}^0$, $\bar{\nabla}$, $\bar{\Delta}$, and $\bar{R}$ are the traceless Ricci tensor, the Riemann connection, the Laplace-Beltrami operator, and the scalar curvature of metric $\bar{g}_t$, respectively.

Proof. We use a short-hand notation $\mathsf{Y}_p(t) = \mathsf{Y}_p(W, \bar{g}(t))$. First we note that

$$\mathsf{Y}_p(t) = \int_W \left(\frac{4(n-1)}{n-2} \left|\bar{\nabla} u\right|^2 + \bar{R}u^2 \right) d\sigma_{\bar{g}_t}.$$

We compute

$$\begin{aligned} \frac{d}{dt}\mathsf{Y}_p(t) &= \int_W \left(\frac{8(n-1)}{n-2}\overline{\mathrm{Ric}}(\bar{\nabla}u, \bar{\nabla}u) + \frac{8(n-1)}{n-2}\langle \bar{\nabla}u, \bar{\nabla}h\rangle \right) d\sigma_{\bar{g}_t} \\ &+ \int_W \left(\left(\bar{\Delta}\bar{R} + 2\left|\overline{\mathrm{Ric}}\right|^2 \right) u^2 + 2\bar{R}uh \right) d\sigma_{\bar{g}_t} \\ &- \int_W \left(\frac{4(n-1)}{n-2}\left|\bar{\nabla}u\right|^2 + \bar{R}u^2 \right) \bar{R}\, d\sigma_{\bar{g}_t}, \end{aligned} \tag{14}$$

where we have used $\frac{\partial \left|\bar{\nabla}u\right|^2}{\partial t} = 2\overline{\mathrm{Ric}}(\bar{\nabla}u, \bar{\nabla}u) + 2\langle \bar{\nabla}u, \bar{\nabla}h\rangle$, $\frac{\partial \bar{R}}{\partial t} = \bar{\Delta}\bar{R} + 2\left|\overline{\mathrm{Ric}}\right|^2$ and $\frac{\partial d\sigma_{\bar{g}_t}}{\partial t} = -\bar{R}\, d\sigma_{\bar{g}_t}$.

To eliminate the terms containing h in formula (14), we use integration by parts to rewrite

$$\begin{aligned} &\int_W \left(\frac{8(n-1)}{n-2}\langle \bar{\nabla}u, \bar{\nabla}h\rangle + 2\bar{R}uh \right) d\sigma_{\bar{g}_t} \\ &= \int_W \left(-\frac{8(n-1)}{n-2}u\bar{\Delta}h + 2\bar{R}uh \right) d\sigma_{\bar{g}_t} + \frac{8(n-1)}{n-2}\int_M u\partial_{\nu_t} h d\sigma_{g_t}. \end{aligned} \tag{15}$$

Taking derivative $\frac{d}{dt}$ of both sides of the equation (10) and then multiplying the result by $2u$ as in [4, p.149], we get

$$\begin{aligned} -\frac{8(n-1)}{n-2}u\bar{\Delta}h + 2\bar{R}uh = &\frac{16(n-1)}{n-2}u\langle \overline{\mathrm{Ric}}, \bar{\nabla}\bar{\nabla}u\rangle - 2\left(\bar{\Delta}\bar{R} + 2\left|\overline{\mathrm{Ric}}\right|^2 \right) u^2 \\ &+ 2\left(\frac{d}{dt}\mathsf{Y}_p(t) \right) u^{p+1} + 2p\mathsf{Y}_p(t)u^p h. \end{aligned}$$

Plugging this formula into equation (15) and then plugging the resulting equation into the formula (14) for $\frac{d}{dt}\mathsf{Y}_p(t)$ to eliminate the terms containing h, we have

$$\begin{aligned} \frac{d}{dt}\mathsf{Y}_p(t) = &\int_W \left(\frac{8(n-1)}{n-2}\overline{\mathrm{Ric}}(\bar{\nabla}u, \bar{\nabla}u) - \left(\bar{\Delta}R + 2\left|\overline{\mathrm{Ric}}\right|^2 \right) u^2 \right) d\sigma_{g_t} \\ &+ \int_W \left(\frac{16(n-1)}{n-2}u\langle \overline{\mathrm{Ric}}, \bar{\nabla}\bar{\nabla}u\rangle - \left(\frac{4(n-1)}{n-2}\left|\bar{\nabla}u\right|^2 + \bar{R}u^2 \right) \bar{R} \right) d\sigma_{\bar{g}_t} \\ &+ \frac{2p}{p+1}\mathsf{Y}_p(t)\int_W u^{p+1}\bar{R} d\sigma_{\bar{g}(t)} + 2\frac{d}{dt}\mathsf{Y}_p(t) + \frac{8(n-1)}{n-2}\int_M u\partial_{\nu_t} h d\sigma_{g_t}, \end{aligned}$$

where we have used

$$\int_W u^p h d\sigma_{\bar{g}_t} = \frac{1}{p+1}\int_W u^{p+1}\bar{R} d\sigma_{\bar{g}_t}$$

which is obtained by taking derivative $\frac{d}{dt}$ of the constraint (11).

To simplify further the above formula for $\frac{d}{dt}\mathsf{Y}_p(t)$, we compute using integration by parts

$$\begin{aligned}&\int_W u\langle\overline{\mathrm{Ric}}, \bar\nabla\bar\nabla u\rangle d\sigma_{\bar g_t}\\ =&-\frac{1}{2}\int_W u\langle\bar\nabla\bar R, \bar\nabla u\rangle d\sigma_{\bar g_t} - \int_W \overline{\mathrm{Ric}}(\bar\nabla u, \bar\nabla u) d\sigma_{\bar g_t} + \int_M u\overline{\mathrm{Ric}}(\bar\nabla u, \nu_t) d\sigma_{g_t}\\ =&\frac{1}{4}\int_W \bar R\bar\Delta u^2 d\sigma_{\bar g_t} - \int_W \overline{\mathrm{Ric}}(\bar\nabla u, \bar\nabla u) d\sigma_{g_t} + \int_M u\overline{\mathrm{Ric}}(\bar\nabla u, \nu_t) d\sigma_{g_t},\end{aligned}$$

where we have used $\partial_{\nu_t} u = 0$ to get the last equality. Hence we obtain

$$\begin{aligned}\frac{d}{dt}\mathsf{Y}_p(t) =& \int_W \left(\frac{8\,(n-1)}{n-2}\overline{\mathrm{Ric}}(\bar\nabla u, \bar\nabla u) + \left(\bar\Delta\bar R + 2\left|\overline{\mathrm{Ric}}\right|^2\right)u^2\right) d\sigma_{\bar g_t}\\ &+ \int_W \left(-\frac{4(n-1)}{n-2}\bar R\bar\Delta u^2 + \left(\frac{4(n-1)}{n-2}\left|\bar\nabla u\right|^2 + \bar R u^2\right)\bar R\right) d\sigma_{\bar g_t}\\ &- \frac{2p}{p+1}\mathsf{Y}_p(t)\int_W u^{p+1}\bar R d\sigma_{\bar g_t} - \frac{16(n-1)}{n-2}\int_M u\overline{\mathrm{Ric}}(\bar\nabla u, \nu_t) d\sigma_{g_t}\\ &- \frac{8(n-1)}{n-2}\int_M u\partial_{\nu_t} h d\sigma_{g_t}.\end{aligned}\tag{16}$$

Next we eliminate $\mathsf{Y}_p(t)$ from the right-hand side of (16). Multiplying (10) by $\bar R u$ and integrating we get

$$\mathsf{Y}_p(t)\int_W u^{p+1}\bar R d\sigma_{\bar g_t} = \int_W \left(-\frac{2(n-1)}{n-2}\bar R\bar\Delta u^2 + \left(\frac{4(n-1)}{n-2}\left|\bar\nabla u\right|^2 + \bar R u^2\right)\bar R\right) d\sigma_{\bar g_t},$$

hence

$$\begin{aligned}\frac{d}{dt}\mathsf{Y}_p\,(t) =& \int_W \left(\frac{8\,(n-1)}{n-2}\overline{\mathrm{Ric}}(\bar\nabla u, \bar\nabla u) + 2\left|\overline{\mathrm{Ric}}\right|^2 u^2\right) d\sigma_{\bar g_t}\\ &- \frac{p-1}{p+1}\int_W \left(\frac{4\,(n-1)}{n-2}\left|\bar\nabla u\right|^2 + \bar R u^2\right)\bar R d\sigma_{\bar g_t}\\ &+ \int_W \left(u^2\bar\Delta\bar R - \frac{4(n-1)}{(p+1)(n-2)}\bar R\bar\Delta u^2\right) d\sigma_{\bar g_t}\\ &- \frac{8(n-1)}{n-2}\int_M \left(2u\overline{\mathrm{Ric}}(\nabla u, \nu_t) + u\partial_{\nu_t} h\right) d\sigma_{g_t}.\end{aligned}$$

As calculated in [4, p.151] by using $\overline{\mathrm{Ric}} = \overline{\mathrm{Ric}}^0 + \frac{\bar R}{n}\bar g$ and $\left|\overline{\mathrm{Ric}}\right|^2 = \left|\overline{\mathrm{Ric}}^0\right|^2 + \frac{\bar R^2}{n}$, we get (13) from the formula above. □

REMARK. Concerning the assumption in Theorem A, that there are functions $u(t)$ such that the metrics $u\,(t)^{\frac{4}{n-2}}\,\bar g_t$ are a C^1-family of smooth Yamabe metrics, there are two cases here.

Case 1: The relative Yamabe constant $Y_{[\bar g_0]}(W, M; [g_0]) \le 0$. Then there is a unique solution $u(0)$ of the Yamabe problem, and it could be shown that for small t the Yamabe metrics $u(t)^{\frac{4}{n-2}}\bar g_t$ is a smooth in t family of metrics, this is a consequence of recent result due to S. Hamanaka ([2]) who proved a relevant Koiso's decomposition type result for manifolds with boundary (cf. to the Koiso's decomposition theorem [7, Corollary 2.9] for closed manifolds).

Case 2: The relative Yamabe constant $Y_{[\bar{g}_0]}(W, M; [g_0]) > 0$. Then, in general, the corresponding Yamabe metric is not unique. Thus in this case it is not clear whether there exists a C^1-family of smooth functions $u(t)$ which satisfy the assumption even for a short time.

2.3. Boundary terms. Below we compute the boundary terms appearing in (13) using local coordinates. Fix a boundary point $p \in M$ and a time t, we choose local coordinates $(x^1, \cdots, x^n)$ on W around p such that $(\bar{g}_t)_{ij}(p) = \delta_{ij}$ and the outward unit normal vector $\nu_t(p) = \partial_n = \frac{\partial}{\partial x^n}$.

Part A, $\partial_t \nu_t$. The outward unit normal vector at time $\tilde{t}$ near t can be written as

$$\nu_{\tilde{t}}(p) = \frac{1}{b(\tilde{t})}\left(\sum_{i'=1}^{n-1} a^{i'}(\tilde{t})\partial_{i'} + \partial_n\right),$$

where $a^{i'}(t) = 0$ and

$$b(\tilde{t}) = \sqrt{(\bar{g}_{\tilde{t}})_{nn}(p) + 2\sum_{i'=1}^{n-1}(\bar{g}_{\tilde{t}})_{i'n}(p)a^{i'}(\tilde{t}) + \sum_{i',j'=1}^{n-1}(\bar{g}_{\tilde{t}})_{i'j'}(p)a^{i'}(\tilde{t})a^{j'}(\tilde{t})}.$$

From

$$0 = b(\tilde{t}) \cdot \bar{g}_{\tilde{t}}(\nu_{\tilde{t}}, \partial_{i'})(p) = \sum_{j'=1}^{n-1} a^{j'}(\tilde{t})(\bar{g}_{\tilde{t}})_{i'j'}(p) + (\bar{g}_{\tilde{t}})_{i'n}(p)$$

for each $i' = 1, 2, \cdots, n-1$, by taking the derivative $\frac{d}{d\tilde{t}}\big|_{\tilde{t}=t}$ of the equality above we have

$$\sum_{i'=1}^{n-1}\left(\frac{da^{j'}}{d\tilde{t}}\bigg|_{\tilde{t}=t}\delta_{i'j'} + a^{j'}(t)(-2(\mathrm{Ric}_{\bar{g}_t})_{i'j'}(p))\right) - 2(\mathrm{Ric}_{\bar{g}_t})_{i'n}(p) = 0.$$

Hence

$$\frac{da^{i'}}{d\tilde{t}}\bigg|_{\tilde{t}=t} = 2(\mathrm{Ric}_{\bar{g}_t})_{i'n}(p), \qquad \frac{db}{d\tilde{t}}\bigg|_{\tilde{t}=t} = -(\mathrm{Ric}_{\bar{g}_t})_{nn}$$

and

$$\partial_t \nu_t(p) = 2\sum_{i'=1}^{n-1}(\mathrm{Ric}_{\bar{g}_t})_{i'\nu_t}(p)\partial_i + (\mathrm{Ric}_{\bar{g}_t})_{\nu_t\nu_t}(p)\partial_{\nu_t}. \tag{17}$$

Part B, $\partial_{\nu_t} h$. Write $\nu_t = \nu_t^i \partial_i$. From $\partial_n u = \partial_{\nu_t} u = 0$ on M, we have by (17)

$$\begin{aligned} 0 &= \partial_t(\nu_t^i \partial_i u) = (\partial_t \nu_t^i)\partial_i u + \nu_t^i \partial_t \partial_i u \\ &= 2\sum_{i'=1}^{n-1}(\mathrm{Ric}_{\bar{g}_t})_{i'n}\partial_{i'} u + (\mathrm{Ric}_{\bar{g}_t})_{nn}\partial_n u + \nu_t^i \partial_i h. \end{aligned}$$

We have established

$$\partial_{\nu_t} h = -2\sum_{i'=1}^{n-1}(\mathrm{Ric}_{\bar{g}_t})_{i'\nu_t}\partial_{i'} u. \tag{18}$$

Part C, $\overline{\mathrm{Ric}}(\bar\nabla u, \nu)$. This term is

$$\bar R_{ij}\bar\nabla_i u\nu_j = \sum_{i'=1}^{n-1} \bar R_{i'n}\partial_{i'}u + \bar R_{nn}\bar\nabla_n u = \sum_{i'=1}^{n-1} (\mathrm{Ric}_{\bar g_t})_{i'\nu_t}\partial_{i'}u. \tag{19}$$

Finally we give a proof of Theorem A. When $p = \frac{n+2}{n-2}$ in Theorem 2.2, we have that $\mathsf{Y}_p(t) = Y_{[\bar g_t]}(W, M; [g_t])$ and

$$\begin{aligned}\frac{d}{dt}Y_{[\bar g_t]}(W, M; [g_t]) &= \int_W \left(\frac{8(n-1)}{n-2}\overline{\mathrm{Ric}}^0(\bar\nabla u, \bar\nabla u) + 2\left|\overline{\mathrm{Ric}}^0\right|^2 u^2\right) d\sigma_{\bar g_t} \\ &+ \int_W \left(u^2\bar\Delta \bar R - \frac{2(n-1)}{n}\bar R\bar\Delta u^2\right) d\sigma_{\bar g_t} \\ &- \frac{8(n-1)}{n-2}\int_M \left(2u\overline{\mathrm{Ric}}(\nabla u, \nu) + u\partial_{\nu_t}h\right) d\sigma_{g_t}.\end{aligned} \tag{20}$$

Note that if the initial metric $\bar g_0$ is a Yamabe metric with constant scalar curvature $R_{\bar g_0} = Y_{[\bar g_0]}(W, M; [g_0])$, then $u(0) = 1$ and $\partial_{\nu_t}h\big|_{t=0} = 0$ by (18), thus the formula (20) evidently reduces to (6).

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