

GOERITZ AND SEIFERT MATRICES FROM DEHN PRESENTATIONS

DANIEL S. SILVER, LORENZO TRALDI and SUSAN G. WILLIAMS

(Received September 20, 2018, revised April 1, 2019)

Abstract

The Goeritz matrix of a link is obtained from the Jacobian matrix of a modified Dehn presentation associated to a diagram using Fox’s free differential calculus. When the diagram is special the Seifert matrix can also be determined from the presentation.

1. Introduction

Lebrecht Goeritz introduced an integral matrix associated to knot diagrams in [7]. The Goeritz matrix represents a class of quadratic forms of a link, a class that is invariant under link isotopy. Goeritz showed that the p th Minkowski units of his matrix, for p an odd prime, are invariant. So too is the absolute value of the determinant of the Goeritz matrix.

Herbert Seifert soon recognized the topological importance of the Goeritz matrix [13]. It is a relation matrix for the first homology group of the 2-fold cover of the 3-sphere \mathbb{S}^3 branched over each component of the link, and it determines a linking form on homology classes.

During the succeeding half century, the definition of the Goeritz matrix was extended and modified (see [8, 9, 12, 15]). However, unlike the Alexander matrix (see below), which can be derived from a presentation of the link group, the Goeritz matrix has been defined by combinatorial and topological means. Our purpose is to show how the Goeritz matrix arises directly from a presentation of the link group that is closely related to the well-known Dehn presentation, using the machinery of Fox’s free differential calculus.

The presentation that we give (Theorem 4.1) is obtained from a link diagram. When the diagram is special and not split, the presentation yields a Seifert matrix for the link (Theorem 6.1). Consequently, link invariants that are obtainable from a Seifert matrix can also be found from such a group presentation. For example, it is well known that the Blanchfield pairing of a knot is such an invariant. (See [6] for a new proof of this fact and some of its history.)

The present paper was motivated by [14], in which the first and third authors showed that a Seifert matrix of an oriented link arises as the Laplacian matrix of a directed checkerboard graph with ± 1 -edge weights of a special diagram of the link.

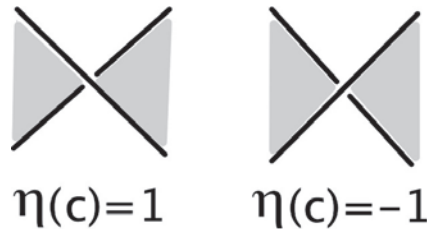


Fig. 1. Goeritz index of crossing c

2. The Goeritz matrix

Consider a link $L \subset \mathbb{S}^3$ with diagram D in the plane. It is well known that the complementary regions of D can be shaded in checkerboard fashion so that each arc of the diagram separates a shaded region from an unshaded one. We adopt the common convention that the unbounded region is unshaded.

The *reduced Goeritz matrix* $G = G(D)$ associated with the checkerboard diagram D is an $n \times n$ integral matrix, where n is the number of bounded unshaded regions. Let U_0 be the unbounded region, let U_1, \dots, U_n be the bounded unshaded regions, and, for $i \neq j \in \{0, \dots, n\}$, let $C_{i,j}$ be the set of crossings of D at which both U_i and U_j are incident. Let $\eta(c) = \pm 1$ denote the *Goeritz index* of c , as in Figure 1. Then for $1 \leq i, j \leq n$ the i, j th entry $G_{i,j}$ is given by

$$(2.1) \quad G_{i,j} = \begin{cases} - \sum_{c \in C_{i,j}} \eta(c) & \text{if } i \neq j \\ \sum_{\substack{k=0 \\ k \neq i}}^n \sum_{c \in C_{i,k}} \eta(c) & \text{if } i = j. \end{cases}$$

We will make frequent use of the *checkerboard graph* $\Gamma = \Gamma(D)$ with vertices corresponding to the shaded regions of D . Two vertices are joined by an edge whenever the corresponding regions meet at a crossing. (If only one shaded region appears at a crossing, the corresponding edge is a loop.) For notational convenience, we use the symbols of the shaded regions of D also for the vertices of Γ . We label each edge of Γ with *weight* $+1$ or -1 according to the Goeritz index of the corresponding crossing (Figure 1). The number of connected components of Γ will be denoted by $\beta = \beta(D)$.

The reduced Goeritz matrix G is a reduced version of the much-studied Laplacian matrix associated with a checkerboard graph defined in the same way, but with a vertex for each unshaded region of D instead.

3. Fox’s differential calculus

The free differential calculus [2, 3, 4, 5] is a standard tool in both knot theory and combinatorial group theory. We briefly review the fundamental ideas.

DEFINITION 3.1. Let $w = w_1^{\epsilon_1} \cdots w_n^{\epsilon_n}$ be a word in symbols w_1, \dots, w_n with exponents $\epsilon_i \in \{1, -1\}$. The symbols w_1, \dots, w_n need not be distinct. For each i , let W_i denote the initial subword $w_1^{\epsilon_1} \cdots w_{i-1}^{\epsilon_{i-1}}$. For $w \in \{w_1, \dots, w_n\}$, the *partial derivative* $\partial W / \partial w$ is the element of the integral group ring of the free group on w_1, \dots, w_n :

$$(3.1) \quad \frac{\partial W}{\partial w} = \sum_{w_i=w} \begin{cases} W_i & \text{if } \epsilon_i = 1 \\ -W_i w^{-1} & \text{if } \epsilon_i = -1. \end{cases}$$

(The empty subword W_i is identified with the identity element 1 of the group ring.)

DEFINITION 3.2. Let $\langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$ be a presentation of a group π . The *Jacobian matrix* of the presentation is the $n \times m$ matrix J with i, j entry equal to $\partial r_i / \partial x_j$.

REMARK 3.3. When $\langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$ is a presentation of the group $\pi = \pi_L$ of a link L , the following specializations of the Jacobian matrix J will be used.

The abelianization π/π' is a free abelian group of finite rank μ equal to the number of components of the link. An abelianization homomorphism $\alpha : \pi \rightarrow (t_1) \times \dots \times (t_\mu)$ can be defined sending an oriented meridian of the i th component of the link to t_i , and we use α to identify the integral group ring $\mathbb{Z}[\pi/\pi']$ with the Laurent polynomial ring $\mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}]$. (The homomorphism α depends only on the order and orientation of link components. We follow Fox's convention [4, p. 122] that the meridian t_i represents a loop whose linking number with the i th component is -1 .) By extension we have a group ring homomorphism $\alpha : \mathbb{Z}[F(x_1, \dots, x_m)] \rightarrow \mathbb{Z}[\pi/\pi']$, where $F(x_1, \dots, x_m)$ is the free group on x_1, \dots, x_m . The specialization J^α is defined by replacing each entry of J with its image under α .

For any group presentation of π , the matrix J^α represents a homomorphism of free $\mathbb{Z}[\pi/\pi']$ -modules with cokernel isomorphic to the *Alexander module* of L ; that is, the first relative homology group of the universal abelian covering space of $\mathbb{S}^3 \setminus L$ with respect to the preimage of a point. The matrix J^α is also called an *Alexander matrix* of L .

Let τ be the composition of α with the homomorphism $\mathbb{Z}[\pi/\pi'] \rightarrow \mathbb{Z}[t^{\pm 1}]$ sending each t_i to t . The specialization J^τ is defined by replacing each entry of J with its image under τ .

Finally, let ν be the composition of τ with the homomorphism $\mathbb{Z}[t^{\pm 1}] \rightarrow \mathbb{Z}$, where t and t^{-1} are mapped to -1 . We refer to the specialization ν as *2-reduction*. Define J^ν by replacing each entry of J with its image under ν .

4. Statement of main theorem

In this section we state our primary result, which describes a link group presentation such that the 2-reduced Alexander presentation is closely related to the Goeritz matrix. Before stating the result we explain how the presentation is constructed from a non-split diagram D . (The modification needed for a split diagram is not difficult, but we will discuss it later, in the proof of the main theorem.) We recall that a link diagram is *split* if some embedded circle separates it into two nonempty parts. Otherwise the diagram is *non-split*.

Select a shaded region adjacent to the unbounded region and label it S_0 . It will correspond to the generator $x_{n+\beta} = x_{n+1}$ in the statement of the theorem. Recall that the unbounded region is labeled U_0 . Denote the remaining unshaded regions by U_1, \dots, U_n ; they will correspond to the generators x_1, \dots, x_n .

Bounded faces of the plane graph Γ are identified with the regions U_1, \dots, U_n . We orient the boundary ∂U_i of each U_i in the counterclockwise sense and join it to S_0 by a base path in Γ . The homotopy classes of these based boundary loops freely generate the fundamental group $\pi_1(\Gamma, S_0)$.

For each U_i we define a relator r_i in the generators U_1, \dots, U_n, S_0 , as follows. Beginning

at S_0 , we follow the based loop associated to U_i . Each time we traverse an edge, we apply the labeling convention illustrated in Figure 2 to label the next vertex along the based loop in terms of S_0 and the unshaded generators of the regions that border the based loop. As this is done inductively, when returning to S_0 we have a word W_i in U_1, \dots, U_n, S_0 , which we call the *return value* of the based boundary loop. We define r_i to be $W_i \bar{S}_0$, the *boundary relator* of the region U_i . (Here and throughout we often denote the inverse of a group element g by \bar{g} .)

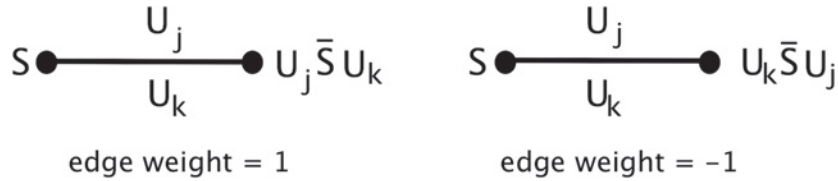


Fig.2. Computing boundary along an edge of Γ

Computation of the return values W is expedited by the following combinatorial process. Travel in the preferred direction along a based boundary loop of Γ that contains ∂U_i and determines the boundary relator r_i . At each edge record a formal fraction $\frac{a}{b}$, where a, b are labels of the regions to either side of each edge; if the edge is weighted $+1$ (resp. -1), then a is the label of the region to the left (resp. right) while b is the label of the region to the right (resp. left).

If the based boundary loop containing ∂U_i has odd length $2l - 1$, then we obtain a sequence:

$$(4.1) \quad \frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_{2l-2}}{b_{2l-2}}, \frac{a_{2l-1}}{b_{2l-1}}$$

The return value W_i has the form $W' \bar{S}_0 W''$, where W' is the zig-zag alternating product of numerators and denominators in the sequence (4.1), working backward from the numerator a_{2l-1} of the last term, and including the inverse of each denominator. W'' is a similar product, working forward from the denominator b_1 of the first term and including the inverse of each numerator. Explicitly,

$$(4.2) \quad W_i = a_{2l-1} \bar{b}_{2l-2} \cdots \bar{b}_2 a_1 \bar{S}_0 b_1 \bar{a}_2 \cdots \bar{a}_{2l-2} b_{2l-1}.$$

If the based boundary loop containing ∂U_i has even length $2l$, then we obtain a sequence:

$$(4.3) \quad \frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_{2l-1}}{b_{2l-1}}, \frac{a_{2l}}{b_{2l}}$$

The return value W_i now has the form $W' S_0 W''$, where W' is the zig-zag alternating product of numerators and denominators in the sequence (4.3), working backward from the numerator a_{2l} of the last term, and including the inverse of each denominator. W'' is again a similar product, beginning with the inverse of the numerator a_1 of the first term, and including the inverse of each numerator. We have

$$(4.4) \quad W_i = a_{2l} \bar{b}_{2l-1} \cdots a_2 \bar{b}_1 S_0 \bar{a}_1 b_2 \cdots \bar{a}_{2l-1} b_{2l}.$$

Example 5.3, involving the Borromean rings, provides an example of computing boundary relators using formal fractions and zig-zag products described above.

Theorem 4.1. *Assume that D is a checkerboard shaded diagram of a link L . Let n be the number of bounded unshaded regions, and $\beta = \beta(D)$ the number of connected components of the checkerboard graph $\Gamma = \Gamma(D)$. Then $\pi_L = \pi_1(\mathbb{S}^3 \setminus L)$ has a presentation of the form*

$$(4.5) \quad \langle x_1, \dots, x_{n+\beta} \mid r_1, \dots, r_n \rangle$$

such that

- generators x_1, \dots, x_n correspond to the bounded unshaded regions of D ;
- generators $x_{n+1}, \dots, x_{n+\beta}$ correspond to certain shaded regions of D , one for each component of $\Gamma(D)$, with $x_{n+\beta}$ corresponding to a shaded region adjacent to the unbounded region of D ;
- when D is non-split the relators r_i are the boundary relators $W_i \bar{S}_0$ of the unshaded bounded regions U_i of D ; and
- the 2-reduction J' of J is equal to $(G \mathbf{0})$, where G is the reduced Goeritz matrix and $\mathbf{0}$ is the $n \times \beta$ zero matrix.

REMARK 4.2. If the graph $\Gamma(D)$ is connected, then $\beta = 1$ and $x_{n+\beta} = x_{n+1}$ in the statement of the theorem.

Since any region of D can be regarded as the unbounded region via stereographic projection, the following is an immediate consequence of Theorem 4.1.

Corollary 4.3. *Assume that D is a non-split diagram of a link L . The group π_L is generated by m elements, where m is the minimum of the numbers of shaded and unshaded regions of D .*

The number m in Corollary 4.3 is often much smaller than the number of generators required for a Wirtinger presentation. For instance, the Wirtinger presentation of the group of the pretzel link $K(p, q, r)$ has $p + q + r$ generators; but, for any values of p, q and r , the presentation of Theorem 4.1 requires only three generators.

5. Proof of Theorem 4.1

We review the Dehn presentation of a link group π_L (see [10] for further details). Begin with a diagram D of L , a generic projection of L in the plane, using an artistic device to indicate how arcs pass over each other. Let U_0 denote the unbounded region. Choose two basepoints, one above U_0 , the other below U_0 and directly under the first. Each complementary region R of the diagram has an associated element of π_L , denoted also by R , and represented by a loop described as follows. Begin at the upper basepoint, and follow a horizontal path to a point over the interior of R ; then descend along a vertical path through R until reaching the depth of the lower basepoint; travel along a horizontal path to the lower basepoint; finally ascend through U_0 to the upper basepoint. Notice that the element of π_L corresponding to U_0 is the identity.

Defining relators are of the form $R_1 \bar{R}_2 R_3 \bar{R}_4$ (see Figure 3), one for each crossing of D .

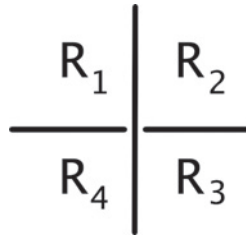


Fig.3. Crossing with Dehn relator $R_1\overline{R_2}R_3\overline{R_4}$

By a *Dehn relator* we will mean any such relator. With these relators, the generators R corresponding to all regions generate the free product $\hat{\pi} \cong \pi_L * \mathbb{Z}$, where the infinite cyclic factor is generated by U_0 . The Dehn presentation of π_L is obtained by including the relator U_0 along with the Dehn relators.

REMARK 5.1. We offer a few general comments about Dehn presentations.

(1) Dehn generators have infinite order in $\hat{\pi}$, a fact that can be seen by mapping $\hat{\pi}$ to the infinite cyclic group \mathbb{Z} , sending each generator to 1.

(2) Unlike the Wirtinger presentation, Dehn presentations do not require arcs of the link diagram to be oriented.

(3) Re-indexing the regions R_i produces equivalent presentations of $\hat{\pi}_L$. (See [10] for these and other facts about Dehn presentations.)

(4) If the overpassing arc in Figure 3 belongs to the j th component of the link and is oriented upward, then $\alpha(R_1\overline{R_2}) = t_j = \alpha(R_4\overline{R_3})$. Similarly, if the underpassing arc belongs to the k th component and is oriented from left to right then $\alpha(R_1\overline{R_4}) = t_k = \alpha(R_2\overline{R_3})$.

We prove Theorem 4.1 first for any non-split link diagram D . For such diagrams every bounded region is homeomorphic to a disk.

According to the construction described in the previous section, each boundary relator r_i is a consequence of the Dehn relators that correspond to the edges of the associated based boundary loop. It follows that adjoining r_1, \dots, r_n to the Dehn presentation does not change the fact that we have a presentation of G .

Since the based boundaries of the regions U_1, \dots, U_n generate the fundamental group of Γ , it follows that $W\overline{S_0}$ is in the normal closure of r_1, \dots, r_n for the boundary of *any* based loop in Γ . In particular, the loop that borders the unbounded region U_0 determines a relation r^{out} that is a consequence of r_1, \dots, r_n .

For any based boundary loop, we can replace the counterclockwise direction of ∂U_i with the clockwise direction. The sequence of formal fractions that we obtain (see Section 4) is a formal inverse: the order of terms is reversed while numerators and denominators are interchanged. Replacing W_i by the new return value produces another word r^* that we also call a boundary relator. It is not difficult to check that r^* is a cyclic permutation of r (resp. r^{-1}) if the loop has odd (resp. even) length.

Next we eliminate all shaded generators except S_0 . For this it is convenient to use a spanning tree T of Γ . Following branches of T from S_0 , we rewrite each shaded generator

(vertex) in terms of U_1, \dots, U_n, S_0 (see Figure 2). At each step we eliminate via a Tietze transformation both a shaded generator (vertex) and a Dehn relation (incident edge).

Each remaining Dehn relator r corresponds to an edge e of Γ not contained in the spanning tree T . Consider the unique based loop in $T \cup e$ with arbitrary orientation. The boundary relator of the loop is an element of the normal closure of the set of Dehn relators associated to the edges of the loop. All but the relator corresponding to e are trivial when rewritten in terms of U_1, \dots, U_n, S_0 . Since the boundary relator is a consequence of r_1, \dots, r_n , so is the Dehn relator r corresponding to e . We discard r .

It follows now that the link group π_L has a presentation with generators U_1, \dots, U_n, S_0 and boundary relators of the based boundaries of the regions U_1, \dots, U_n . As discussed in Section 3, it follows that if J is the Jacobian matrix of this presentation, then J^α is an Alexander matrix for L .

We proceed with a proof that the 2-reduced Jacobian matrix J' coincides with $(G \mathbf{0})$, where G is the reduced Goeritz matrix $G = G(D)$ and $\mathbf{0}$ is a column of zeroes.

Recall that the rows and columns of G correspond to the bounded unshaded regions U_1, \dots, U_n . Each U_i corresponds to a bounded region of the graph Γ , and we can obtain the entries of the i th row of the reduced Goeritz matrix by following the boundary of this region in the counterclockwise direction. With respect to this counterclockwise orientation, the region U_i will be on the left of each edge and an unshaded region U_j will be on the right. If $i \neq j$ and the edge has positive weight, then we record $+1$ (resp. -1) in the ii (resp. ji) entry. If $i \neq j$ and the edge has negative weight, then we record -1 (resp. $+1$) in the ii (resp. ji) entry.

Lemma 5.2. *The homomorphism τ applied to any shaded generator yields an odd power of t , while τ applied to any unshaded generator yields an even power of t . It follows that every shaded generator S has 2-reduction $\nu(S) = -1$, while every unshaded generator U_i has 2-reduction $\nu(U_i) = +1$.*

Proof. The lemma is verified recursively using part (4) of Remark 5.1, starting with $U_0 = 1$. □

Recall that the boundary relator $r_i = W_i \overline{S}_0$ is constructed by following the based boundary of U_i , taking into account the Dehn relation at each crossing. It follows that W_i consists almost completely of unshaded generators; the only exception is a single appearance of either S_0 or \overline{S}_0 . Regardless of whether it is S_0 or \overline{S}_0 that appears in W_i , Definition 3.1 and Lemma 5.2 imply that $\nu(\partial r_i / \partial S_0) = 0$.

Now consider an edge e of the boundary ∂U_i , and give e the edge direction consistent with the counterclockwise orientation of ∂U_i . Suppose e has positive weight. Then the associated Dehn relation has the form $S' = U_i \overline{S} U_j$, where S, S' are the initial and terminal vertices, respectively. It follows that the relator r_i will have either the form $A U_i B S_0 C U_j D \overline{S}_0$ or the form $A \overline{U}_j B \overline{S}_0 C \overline{U}_i D \overline{S}_0$, where A, B, C, D include only unshaded generators. Either way, Definition 3.1 and Lemma 5.2 tell us that the contribution of the indicated appearance of U_i to the value of $\nu(\partial r_i / \partial U_i)$ is $+1$, and the contribution of the indicated appearance of U_j to the value of $\nu(\partial r_i / \partial U_j)$ is -1 . These contributions are the same as the contributions of the edge e to entries of the i th row of the Goeritz matrix G .

Similarly, if e has negative weight then the relator r_i will have either the form

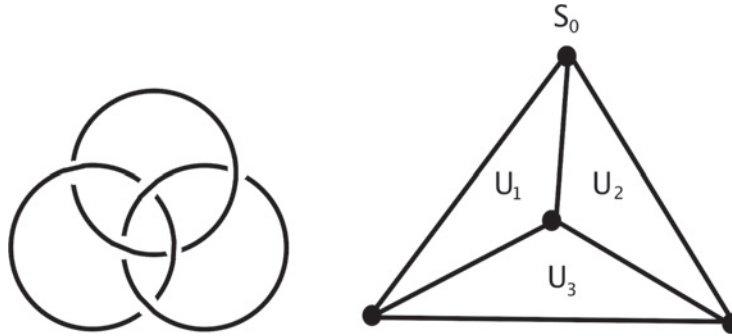


Fig.4. Diagram D of Borromean rings L (left) and its checkerboard graph Γ (right)

$AU_jBS_0CU_iD\bar{S}_0$ or the form $A\bar{U}_iB\bar{S}_0C\bar{U}_jD\bar{S}_0$, where A, B, C, D include only unshaded generators. Once again, the contributions of the indicated appearances of U_i and U_j to the i th row of J' are equal to the contributions of e to the i th row of G .

This completes the proof of Theorem 4.1 for non-split diagrams.

EXAMPLE 5.3. Consider the diagram D of the Borromean rings and associated checkerboard graph $\Gamma = \Gamma(D)$ in Figure 4. All edges have weight $+1$. The based boundary loop of U_1 yields the sequence of formal fractions $\frac{U_1}{U_0}, \frac{U_1}{U_3}, \frac{U_1}{U_2}$ and the associated return value W_1 is the final element of the sequence:

$$U_1\bar{S}_0U_0, \quad U_1\bar{U}_0S_0\bar{U}_1U_3, \quad U_1\bar{U}_3U_1\bar{S}_0U_0\bar{U}_1U_2.$$

The relator r_1 is $U_1\bar{U}_3U_1\bar{S}_0U_0\bar{U}_1U_2\bar{S}_0$. Similarly, r_2 and r_3 are, respectively,

$$U_2\bar{U}_3U_2\bar{S}_0U_1\bar{U}_2U_0\bar{S}_0$$

$$U_1\bar{U}_2U_3\bar{U}_1U_2\bar{S}_0U_1\bar{U}_3U_0\bar{U}_3U_2\bar{S}_0.$$

In order to get a presentation for π_L we must delete the occurrences of U_0 .

$$\pi_L \cong \left\langle U_1, U_2, U_3, S_0 \left| \begin{array}{l} U_1\bar{U}_3U_1\bar{S}_0\bar{U}_1U_2\bar{S}_0, \quad U_2\bar{U}_3U_2\bar{S}_0U_1\bar{U}_2\bar{S}_0, \\ U_1\bar{U}_2U_3\bar{U}_1U_2\bar{S}_0U_1\bar{U}_3\bar{U}_3U_2\bar{S}_0 \end{array} \right. \right\rangle$$

The 2-reductions of the boundary relators of r_1, r_2, r_3 can also be computed using above sequences of fractions. For r_1 we have:

$$U_1 - U_0, \quad U_1 - U_0 + U_1 - U_3, \quad U_1 - U_0 + U_1 - U_3 + U_1 - U_2 = 3U_1 - U_0 - U_2 - U_3.$$

Similarly, r_2 and r_3 yield, respectively, $3U_2 - U_1 - U_3 - U_0$ and $3U_3 - U_1 - U_0 - U_2$. To construct the 2-reduction of the Jacobian matrix for the presentation of π_L , we delete occurrences of U_0 . The result is

$$\begin{pmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & -1 & 0 \\ -1 & -1 & 3 & 0 \end{pmatrix},$$

with the last column corresponding to S_0 . This is the same $(G \mathbf{0})$ matrix that results from the definition of G (see Section 2).

Finally, we consider a general diagram D of any link L . When D is split, the checkerboard graph $\Gamma = \Gamma(D)$ is combinatorially well defined but does not contain complete information about L . (Consider, for example, unlink diagrams consisting of μ circles, some of which may be concentric.) As before, label the unbounded unshaded region U_0 and the remaining ones U_1, \dots, U_n . For each component Γ_λ of the graph Γ , $\lambda = 0, \dots, \beta - 1$, we choose a shaded region S_λ corresponding to a vertex of Γ_λ . We identify $U_0, U_1, \dots, U_n, S_0, \dots, S_{\beta-1}$ and remaining shaded regions of Σ with the generators of the Dehn presentation for $\hat{\pi}_L$ arising from the diagram D .

Consider the surface Σ in the plane consisting of the shaded regions of D together with marked bands replacing the crossings between adjacent regions. Each marking is $+1, -1$ according to the Goeritz index of the crossing, as in Figure 1. Each component Σ_λ corresponds to a graph component Γ_λ . The group $\pi_1(\Sigma_\lambda, S_\lambda)$ is free, generated by embedded loops based at S_λ that run counter-clockwise around the holes which correspond to the unshaded regions $U_{\lambda,1}, \dots, U_{\lambda,n_\lambda}$ exterior to the surface. Each loop determines a directed cycle graph with vertices corresponding to S_λ and other shaded regions, and edges corresponding to traversed bands. As before, we label edges with weights $+1, -1$ according to the Goeritz index of the crossing. We also label the left- and right-hand sides of each directed edge with symbols of the unshaded regions U_j, U_k that appear on the those sides of the band. Then using Figure 2 we define the *return value* $W_{\lambda,i}$ of the based loop to be the word in $U_0, \dots, U_n, S_\lambda$ obtained by following the loop around. (If the loop avoids crossings then the return value is 1.) We define the *boundary relator* $r_{\lambda,i}$ to be $W_{\lambda,i}\overline{S_\lambda}$. And as before, the boundary relator of any based loop of Σ_λ is contained in the normal closure of the $r_{\lambda,i}$.

Each component Σ_λ has a combinatorially defined checkerboard graph Γ_λ with vertices and edges corresponding to shaded regions and crossings. We select a spanning tree T_λ for Γ_λ and use it and Tietze transformations to eliminate shaded generators other than S_λ as well as the Dehn relators corresponding to its edges. The same argument as in the case of non-split diagrams shows that the remaining Dehn relators, rewritten in terms of $U_1, \dots, U_n, S_0, \dots, S_{\beta-1}$, are consequences of the boundary relators $r_{\lambda,1}, \dots, r_{\lambda,n_\lambda}$. We delete them from the presentation.

We have shown that $\hat{\pi}_L$ has a presentation with generators $U_0, U_1, \dots, U_n, S_0, \dots, S_{\beta-1}$ and relators $r_{0,1}, \dots, r_{0,n_0}, \dots, r_{\beta-1,1}, \dots, r_{\beta-1,n_{\beta-1}}$. Adjoining the relator U_0 yields a presentation for π_L .

The unshaded regions of D that are non-simply connected border different components of the surface S , and hence the 2-reduced Jacobian of the presentation for π_L that we have described can differ from the reduced Goeritz matrix G in the corresponding rows and columns.

We rectify the problem by adjusting our presentation of π_L . Consider a bounded non-simply connected unshaded region. It is a simply-connected region for some component S_{λ_0} , and it contains one or more components $S_{\lambda_1}, \dots, S_{\lambda_s}$. Replace the boundary relator r_{λ_0} in the presentation just obtained with $r_{\lambda_0}\tilde{r}_{\lambda_1}^* \cdots \tilde{r}_{\lambda_s}^*$. (The order of the relations will not matter. Recall that \tilde{r} is the boundary relation of the outermost loop of the component, and $*$ indicates that the loop is traversed in the clockwise direction.) We repeat the procedure for each bounded non-simply connected unshaded region of D .

That the new presentation is equivalent to the one with which we began can easily be seen by considering the relations from innermost components of Σ and working outward. Any

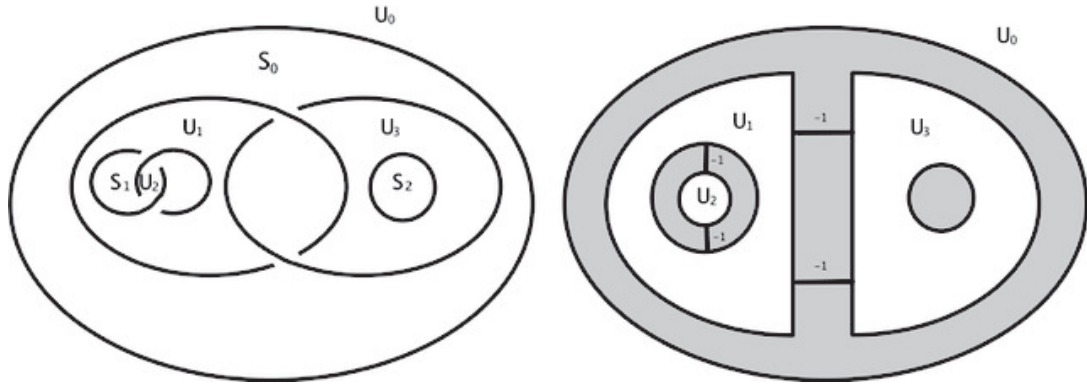


Fig.5. Split diagram D of link L (left) and surfaces $\Sigma_0, \Sigma_1, \Sigma_2$ (right)

relation that we append is a consequence of a previous relator.

It is straightforward to see that the trailing principal minor of J^v of the new presentation coincides with the reduced Goeritz matrix.

EXAMPLE 5.4. Consider the split diagram D of the 6-component link L in Figure 5. The surface Σ has three components Σ_0, Σ_1 and Σ_2 containing shaded regions S_0, S_1 and S_2 , respectively. The fundamental group $\pi_1(\Sigma_0, S_0)$ is freely generated by two based loops, one running around the left-hand side and the other along the right. Their return values are easily computed using cycle graphs, each of length two and edges with weight -1 . The first boundary is $U_3\bar{U}_1S_0\bar{U}_3U_1$. The second is $U_1\bar{U}_3S_0\bar{U}_1U_3$. The surface Σ_1 has an infinite cyclic fundamental group, and the boundary of a based loop generator is $U_1\bar{U}_2S_1\bar{U}_1U_2$. The surface Σ_2 is simply connected, and we do not need to compute any boundary for it. Putting all of this together, we have:

$$\pi_L \cong \langle U_1, U_2, U_3, S_0, S_1, S_2 \mid U_3\bar{U}_1S_0\bar{U}_3U_1\bar{S}_0, U_1\bar{U}_3S_0\bar{U}_1U_3\bar{S}_0, U_1\bar{U}_2S_1\bar{U}_1U_2\bar{S}_1 \rangle.$$

The unshaded region U_1 is non-simply connected. We modify its associated relator in order to produce a presentation that will yield the reduced Goeritz matrix.

The unshaded region U_3 is also non-simply connected. However, its associated relator needs no modification since the inner boundary of the region has trivial boundary.

The modified presentation of π_L is:

$$\pi_L \cong \left\langle U_1, U_2, U_3, S_0, S_1, S_2 \mid \begin{array}{l} U_3\bar{U}_1S_0\bar{U}_3U_1\bar{S}_0U_2\bar{U}_1S_1\bar{U}_2U_1\bar{S}_1, U_1\bar{U}_3S_0\bar{U}_1U_3\bar{S}_0, \\ U_1\bar{U}_2S_1\bar{U}_1U_2\bar{S}_1 \end{array} \right\rangle.$$

Direct computation using the free differential calculus shows that the 2-reduced version of the Jacobian matrix of the last presentation is

$$J^v = \begin{pmatrix} -4 & 2 & 2 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The submatrix consisting of the first three columns is the reduced Goeritz matrix G associated to the shaded diagram D .

6. Special diagrams

Consider an oriented link L with diagram D . As before, we checkerboard shade the diagram so that the unbounded region remains unshaded. In this section we assume that the diagram is *special*, that is, its shaded regions form an oriented spanning surface F for the link. (Every link has a special diagram. See, for example, [1].) Using arc orientations and the right-hand rule, we label each shaded region by $+$ or $-$, regarding regions labeled $+$ as one side of the surface, regions labeled $-$ as the other.

We assume also that D is a non-split diagram. As in Section 5, we consider the plane checkerboard graph Γ , identifying its vertices with shaded regions of D and its bounded faces with the bounded unshaded regions U_1, \dots, U_n of D .

Let S_0 be a shaded vertex labeled $+$. For each U_i we select a base path from S_0 to a vertex labeled $+$ on the boundary ∂U_i . By Theorem 4.1 the link group π_L has a presentation of the form

$$(6.1) \quad \langle S_0, U_1, \dots, U_n \mid W_1 \bar{S}_0, \dots, W_n \bar{S}_0 \rangle.$$

Since D is special, the length of every boundary ∂U_i is even. The return value W_i has the form $A_i S_0 \bar{B}_i$, where A_i, B_i are words in U_1, \dots, U_n , each having even length, and the presentation (6.1) can be rewritten as:

$$(6.2) \quad \langle S_0, U_1, \dots, U_n \mid \bar{S}_0 A_1 S_0 = B_1, \dots, \bar{S}_0 A_n S_0 = B_n \rangle.$$

As in Section 5 the words A_i, B_i can be read from the sequence of formal fractions recorded as we travel along the based boundary of U_i . Regard the A_i, B_i as words in the free group F_n on the generating set $\{U_1, \dots, U_n\}$, and let $U_1^{a_{i1}} \dots U_n^{a_{in}}$ be the image of A_i in the abelianization F_n/F'_n . Define A to be the integral $n \times n$ matrix $(a_{i,j})$. Define B similarly as $(b_{i,j})$.

Consider the Seifert matrix H^+ with i, j -entry equal to the linking number $Lk(\partial U_i, \partial U_j^+)$. Here we regard U_i, U_j as oriented curves in the surface F , and ∂U_j^+ as a copy of ∂U_j pushed off the surface in the direction of the positive normal vector. Contributions to linking numbers by base paths cancel and so we ignore base paths. We consider also the Seifert matrix H^- similarly defined but with i, j -entry $Lk(\partial U_i, \partial U_j^-)$, where ∂U_j^- is obtained by pushing off in the negative normal direction.

Theorem 6.1. *Let D be a non-split, special diagram of a link L . Then π_L has a presentation of the form $\langle S_0, U_1, \dots, U_n \mid \bar{S}_0 A_1 S_0 = B_1, \dots, \bar{S}_0 A_n S_0 = B_n \rangle$, where the matrix B (resp. A) is equal to the Seifert matrix H^+ (resp. H^-) of the diagram D . If additionally D is alternating, then the presentation describes π_L as an HNN extension with stable letter S_0 .*

Proof. The Seifert matrix H^+ can be computed directly from the diagram D . Begin by placing a dot in the corners of unshaded regions if they appear on the left of an undercrossing arc with respect to its preferred orientation, as illustrated in Figure 6. At each crossing c of the diagram a dot will appear in exactly one unshaded region. Define $\epsilon_i(c) = 1$ if the dot appears in U_i , zero otherwise. We write $c \in \partial U$ if the crossing is incident to the region U . Then

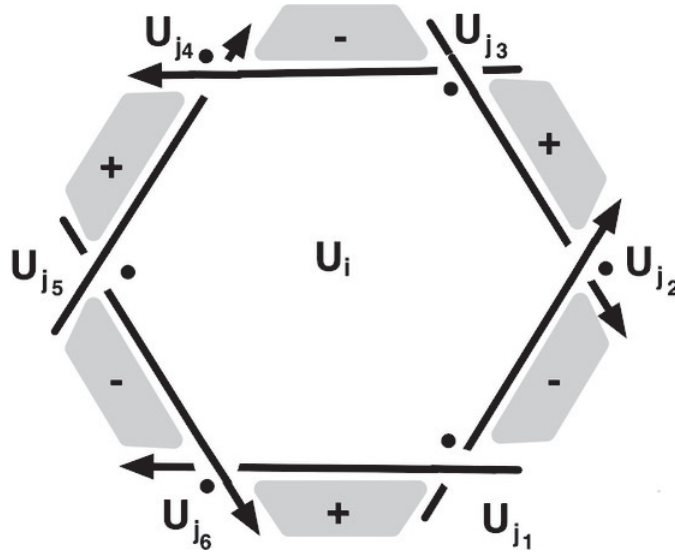


Fig.6. Unshaded region U_i of special alternating diagram

$$(6.3) \quad H_{i,i}^+ = \sum_{c \in \partial U_i} \eta(c)\epsilon_i(c), \quad H_{i,j}^+ = - \sum_{c \in \partial U_i \cap \partial U_j} \eta(c)\epsilon_j(c), \quad (i \neq j).$$

(See page 231 of [1]. The reader is warned that the second summation there is missing the negative sign. The proof, however, is correct.)

We can use formulas (6.3) to find the i th row of the Seifert matrix H^+ by imagining that we are standing in the center of U_i . The diagonal term $H_{i,i}^+$ is the number of dotted corners that we see, each weighted by the Goeritz index $\eta(c) = \pm 1$ of the nearby crossing. Each undotted corner, diagonally across from some region U_j , contributes $\eta(c)$ to the j th column. In Figure 6, for example, where all Goeritz indices are 1, we have $H_{i,i}^+ = 3$ and $H_{i,j_k}^+ = -1$ if $k = 2, 4, 6$; other entries $H_{i,j}^+$ are zero.

Now consider the based boundary loop of U_i . First assume that all Goeritz indices are 1. Beginning at a + vertex and traveling around the loop, we record a sequence of formal fractions

$$(6.4) \quad \frac{\dot{U}_i}{U_{j_1}} \frac{U_i}{\dot{U}_{j_2}} \dots \frac{\dot{U}_i}{U_{j_{2l-1}}} \frac{U_i}{\dot{U}_{j_{2l}}},$$

where l is the length of ∂U_i and $\dot{}$ indicates that a dot is found in that region of the corner. The word A_i is the zig-zag alternating product of numerators and denominators, beginning with the numerator of the last term:

$$A_i = U_i \bar{U}_{j_{2l-1}} U_i \bar{U}_{j_{2l-3}} \dots U_i \bar{U}_{j_1}.$$

Likewise, B_i is the zig-zag alternating product of numerators and denominators, beginning with the inverse of the denominator of the last term:

$$B_i = \bar{U}_{j_{2l}} U_i \bar{U}_{j_{2l-2}} U_i \dots \bar{U}_2 U_i.$$

When we construct B_i from the based boundary of U_i , each crossing from the base path is encountered twice, once before the based boundary loop traverses ∂U_i and once after. The

two encounters are in opposite directions, so according to formula (4.4), the two encounters contribute opposite powers of the same U_j to the word B_i . It follows that the contributions from the base path to B_i cancel in the abelianization of the free group F_n , and we see immediately that the contributions to the i th row of H^+ agree with those given by formulas (6.3).

We have considered only diagrams with crossings having Goeritz index 1. Changing a crossing flips the corresponding numerator and denominator (but leaves the dot in place). It is easy to see that the two methods of computation continue to agree. Hence B is equal to the Seifert matrix H^+ .

If we reverse the orientation of the diagram D , then the new Seifert matrix that we obtain is H^- . It is equal to transpose of H^+ . The effect on the sequence of formal fractions arising from the checkerboard graph is to move each dot, from numerator to denominator or vice versa. We see that A is equal to H^- .

This completes the proof of the first statement of Theorem 6.1. For a proof of the second statement assume that D is a special alternating diagram. (A special diagram is alternating if and only if all Goeritz indices have the same value.) The sets $\{A_1, \dots, A_n\}$ and $\{B_1, \dots, B_n\}$ generate subgroups $gp(A_1, \dots, A_n)$ and $gp(B_1, \dots, B_n)$ of the free group F_n , respectively. In order to prove that the presentation (5.3) expresses π_L as an HNN extension with stable letter S_0 , we must show that the homomorphism ϕ taking A_i to B_i , for each i , is an isomorphism. It suffices to show that A_1, \dots, A_n and B_1, \dots, B_n freely generate $gp(A_1, \dots, A_n)$ and $gp(B_1, \dots, B_n)$, respectively.

Since D is a special alternating diagram, the determinants of H^+ and H^- are nonzero (see Prop. 13.24 of [1]). Hence A_1, \dots, A_n generate a subgroup of $F_n/F'_n \cong \mathbb{Z}^n$ with finite index (equal to the absolute value of the determinant). It follows that A_1, \dots, A_n must freely generate $gp(A_1, \dots, A_n)$. The same argument applies to B_1, \dots, B_n . □

Here is a direct consequence of Theorem 6.1, analogous to the last assertion of Theorem 4.1.

Corollary 6.2. *Let D be a non-split, special diagram of a link L , let H^+, H^- be the corresponding Seifert matrices defined above, and let $\tilde{H} = H^- - tH^+$. If τ is the map defined in Remark 3 then L has an Alexander matrix J such that $J^r = (\tilde{H} \mathbf{0})$, where $\mathbf{0}$ is a column of zeroes.*

Proof. Theorem 6.1 tells us that π_L has a presentation $\langle U_1, \dots, U_n, S_0 \mid r_1, \dots, r_n \rangle$, where $r_i = A_i S_0 \overline{B_i} \overline{S_0}$. Since D is special and $U_0 = 1$, part (4) of Remark 5.1 tells us that every Dehn generator U corresponding to an unshaded region has $\tau(U) = 1$. Also, the Dehn generator S corresponding to a shaded region labeled $+$ (resp. $-$) has $\tau(S) = t$ (resp. $\tau(S) = t^{-1}$). In particular, $\tau(S_0) = t$.

Now, let J be the Alexander matrix obtained from the presentation $\langle U_1, \dots, U_n, S_0 \mid r_1, \dots, r_n \rangle$ using the free differential calculus, as in Section 3. For $1 \leq i \leq n$ the image under τ of the i th entry of the last column (the column corresponding to S_0) is $\tau(A_i) - \tau(A_i S_0 \overline{B_i} \overline{S_0}) = 1 - tt^{-1} = 0$. The fact that the first n columns of J^r are the same as the columns of \tilde{H} follows from the equalities $A = H^-, B = H^+$ of Theorem 6.1. □

REMARK 6.3. More can be said about a link with special non-split alternating diagram. It is known that the Seifert surface F formed by its shaded regions has minimal genus, and splitting \mathbb{S}^3 along F produces a handlebody \mathcal{H} of genus n [11]. The boundary of \mathcal{H} contains two copies F_+, F_- of F with $F_+ \cap F_- = L$. The infinite cyclic cover of $\mathbb{S}^3 \setminus L$ corresponding to the homomorphism $\tau : \pi_1(\mathbb{S}^3 \setminus L) \rightarrow \mathbb{Z} \cong \langle t \mid \rangle$ sending each oriented meridian of L to t can be constructed by gluing countably many copies \mathcal{H}_ν of the handlebody end-to-end, matching $F_+ \subset \mathcal{H}_\nu$ with $F_- \subset \mathcal{H}_{\nu+1}$. With appropriate choice of basepoint and base paths the gluing map induces a monomorphism of fundamental groups that corresponds to the HNN amalgamation map ϕ in the proof of Theorem 6.1.

References

- [1] G. Burde and H. Zieschang: *Knots*, 2nd ed., Walter de Gruyter, Berlin, 2003.
- [2] R.H. Fox: *Free differential calculus. I: Derivation in the free group ring*, Ann. of Math. (2) **57** (1953), 547–560.
- [3] R.H. Fox: *Free differential calculus. II: The isomorphism problem of groups*, Ann. of Math. (2) **59** (1954), 196–210.
- [4] R.H. Fox: *A quick trip through knot theory*; in *Topology of 3-manifolds and related topics*, Prentice-Hall, New Jersey, 1962, 120–167; reprinted by Dover Publications, 2010.
- [5] R.H. Crowell and R.H. Fox: *An introduction to knot theory*, Ginn and Co. 1963, or: Grad. Texts in Math. **57**, Springer-Verlag, Berlin-Heidelberg-New York, 1977.
- [6] S. Friedl and M. Powell: *A calculation of Blanchfield pairings of 3-manifolds and knots*, Mosc. Math. J. **17** (2017), 59–77.
- [7] L. Goeritz: *Knoten und quadratische Formen*, Math. Z. **36** (1933), 647–654.
- [8] C.McA. Gordon and R.A. Litherland: *On the signature of a link*, Invent. Math. **47** (1978), 53–69.
- [9] R.H. Kyle: *Branched covering spaces and the quadratic forms of links*, Ann. of Math. (2) **59** (1954), 539–548.
- [10] P.C. Lyndon and P.E. Schupp: *Combinatorial group theory*, Springer Verlag, Berlin-Heidelberg-New York, 1977.
- [11] W. Menasco: *Closed incompressible surfaces in alternating knot and link complements*, Topology **23** (1984), 37–44.
- [12] K. Murasugi: *On the Minkowski unit of slice links*, Trans. Amer. Math. Soc. **114** (1965), 377–383.
- [13] H. Seifert: *Die Verschlingungsinvarianten der zyklischen Knotenüberlagerungen*, Abh. Math. Sem. Univ. Hamburg **11** (1935), 84–101.
- [14] D.S. Silver and S.G. Williams: *Knot invariants from Laplacian matrices*, J. Knot Theory Ramifications **28** (2019), 1950059, 9pp.
- [15] L. Traldi: *On the Goeritz matrix of a link*, Math. Z. **188** (1985), 203–213.

Daniel S. Silver
Department of Mathematics and Statistics
University of South Alabama
Mobile, AL 36688
USA
e-mail: silver@southalabama.edu

Lorenzo Traldi
Department of Mathematics
Lafayette College
Easton PA 18042
USA
e-mail: traldil@lafayette.edu

Susan G. Williams
Department of Mathematics and Statistics
University of South Alabama
Mobile, AL 36688
USA
e-mail: swilliam@southalabama.edu