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MORE ABOUT HAMZA'S THEOREM

Wenjie SUN and Jiangang YING

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Abstract

In this short article, we give a necessary and sufficient condition for the form (1.3), like the one in Hamza's theorem, to be closable and prove that every regular and strongly local Dirichlet form is the closure of a form of Hamza's type, which can be represented in terms of effective intervals introduced in [7].

Contents

1. Introduction

A Dirichlet form is a closed and symmetric bilinear form with Markovian property on $L^2(E, m)$ space, where E is a nice topological space and m is a fully supported Radon measure on E. Due to a series of important works by M. Fukushima, M. L. Silverstein in 1970's, a regular Dirichlet form is always associated with a symmetric Markov process. We refer the notions and terminologies in the theory of Dirichlet forms to [2, 4], and a brief introduction will also be given in §2.

A classical method to construct a regular Dirichlet form is to start from a closable and Markovian symmetric form with the domain $C_c^\infty(\mathbb{R})$, where $C_c^\infty(\mathbb{R})$ denotes the function space consisting of all smooth functions with compact support on $\mathbb{R}$. Then the closure of $C_c^\infty(\mathbb{R})$ with respect to this form is a regular Dirichlet form. For example the Dirichlet form associated with Brownian motion is the closure of the following closable form

$$(1.1)\qquad \begin{aligned} \mathscr{D}[\mathscr{E}] &= C_c^\infty(\mathbb{R}), \\ \mathscr{E}(u, v) &:= \frac{1}{2}\int_{-\infty}^{+\infty} u'(x)v'(x)dx. \end{aligned}$$

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In [5], Hamza studied the following form

$$(1.2)\qquad \begin{aligned} &\mathscr{D}[\mathscr{E}] = C_c^\infty(\mathbb{R}),\\ &\mathscr{E}(u,v) := \frac{1}{2}\int_{-\infty}^{+\infty} u'(x)v'(x)\nu(dx), \end{aligned}$$

where ν is a positive Radon measure on $\mathbb{R}$. The Hamza's theorem states that the form $(\mathscr{E}, \mathscr{D}[\mathscr{E}])$ is closable if and only if ν is absolutely continuous and its Radon-Nikodym derivative $a(x) := \frac{d\nu}{dx}$ vanishes a.e. on its singular set $S(a)$, where $S(a)$ is defined as

$$S(a) := \mathbb{R} \setminus R(a), \qquad R(a) := \left\{ x \in \mathbb{R} : \int_{x-\varepsilon}^{x+\varepsilon} \frac{1}{a(\xi)} d\xi < \infty \text{ for some } \varepsilon > 0 \right\}.$$

We refer to [4, Theorem 3.1.6] and [1, Appendix] for the proof of the Hamza's theorem. We also mention that Li et al. in [7] provide a probabilistic proof of the Hamza's theorem, based on a representation theorem of regular and strongly local Dirichlet forms on $\mathbb{R}$ (Cf. [7, Theorem 2.1]), which roughly says that such a Dirichlet form can always be represented by a series of so-called effective intervals $\{(I_n, \mathtt{s}_n) : n \geq 1\}$. This means that the associated process behaves as an irreducible diffusion with scale function $\mathtt{s}_n$ on each I_n, and the points outside these intervals can be regarded as 'traps' in the sense that the process will never move starting from these points. We refer to [9] for the background of diffusion process, and a short review on the representation theorem will be given in §2.

Recently in [6], the authors considered the following class of functions:

$$\mathscr{C}_{\mathfrak{f}} := C_c^\infty \circ \mathfrak{f} = \{\varphi \circ \mathfrak{f} : \varphi \in C_c^\infty(\mathfrak{f}(\mathbb{R}))\}.$$

where $\mathfrak{f} \in \mathbf{S}(\mathbb{R})$, and $\mathbf{S}(\mathbb{R})$ is given by

$$\mathbf{S}(\mathbb{R}) = \{\mathfrak{f} : \mathbb{R} \to \mathbb{R} \mid \mathfrak{f} \text{ is strictly increasing and continuous, } \mathfrak{f}(0) = 0\}.$$

Throughout this paper, given a continuous and strictly increasing function $\mathtt{t}$ on an interval I, $d\mathtt{t}$ represents its induced Lebesgue-Stieltjes measure on I. We also use $\lambda_{\mathtt{t}}$ for $d\mathtt{t}$. If a function u on I can be written as $u = \varphi \circ \mathtt{t}$ for some absolutely continuous function φ, we say $u \ll \mathtt{t}$ (or $u \ll \lambda_{\mathtt{t}}$) and $\frac{du}{d\mathtt{t}} = \frac{du}{d\lambda_{\mathtt{t}}} := \varphi' \circ \mathtt{t}$.

We shall focus on the following form, which is similar to (1.2):

$$(1.3)\qquad \begin{aligned} &\mathscr{D}[\mathscr{E}] = \mathscr{C}_{\mathfrak{f}},\\ &\mathscr{E}(u,v) := \frac{1}{2}\int_{-\infty}^{+\infty} \frac{du}{d\lambda_{\mathfrak{f}}}\frac{dv}{d\lambda_{\mathfrak{f}}} d\nu. \end{aligned}$$

The main results in this paper are the following two theorems.

Theorem 1.1. *The form* (1.3) *is closable on* $L^2(\mathbb{R}, m)$ *if and only if the following conditions are satisfied:*

(1) *ν is absolutely continuous with respect to $\lambda_{\mathfrak{f}}$;*

(2) *the Radon-Nikodym derivative $a := \frac{d\nu}{d\lambda_{\mathfrak{f}}}$ vanishes $\lambda_{\mathfrak{f}}$-a.e. on $S_{\mathfrak{f}}(a)$, where $S_{\mathfrak{f}}(a) := \mathbb{R} \setminus R_{\mathfrak{f}}(a)$, and*

(1.4) $$R_{\mathfrak{f}}(a) := \left\{ x \in \mathbb{R} : \int_{x-\varepsilon}^{x+\varepsilon} \frac{1}{a(\xi)} d\lambda_{\mathfrak{f}}(\xi) < \infty \text{ for some } \varepsilon > 0 \right\}$$

are called the singular and regular set of a with respect to $\mathfrak{f}$*, respectively.*

According to this theorem, once we have a function $\mathfrak{f} \in \mathbf{S}(\mathbb{R})$ and a Radon measure ν satisfying the conditions in Theorem 1.1, the form defined by (1.3) is closable and called a form of Hamza's type, and its closure with respect to $\mathscr{E}_1$ is a regular Dirichlet form. We will call it a Dirichlet form *generated by* $(\mathfrak{f}, a)$, where $a = \frac{d\nu}{d\lambda_{\mathfrak{f}}}$. Throughout this paper, when we say a Dirichlet form *generated by* $(\mathfrak{f}, a)$, we always assume that $\mathfrak{f} \in \mathbf{S}(\mathbb{R})$, and $\nu = a \cdot \lambda_{\mathfrak{f}}$ is a Radon measure satisfying the conditions in Theorem 1.1. Such a pair will be called a Hamza pair.

The Dirichlet form generated by a Hamza pair $(\mathfrak{f}, a)$ is regular and strongly local. What about the converse? Is every regular and strongly local Dirichlet form on $L^2(\mathbb{R}, m)$ generated by a Hamza pair $(\mathfrak{f}, a)$? The answer is yes. The following theorem is a representation theorem of another type.

Theorem 1.2. *Every regular and strongly local Dirichlet form on* $L^2(\mathbb{R}, m)$ *is generated by a Hamza pair* $(\mathfrak{f}, a)$ *or equivalently, the closure of a form of Hamza's type.*

Though this conclusion is almost included in the proof of Theorem 1.1, we would still like to write it as a theorem. Comparing to [7, Theorem 2.1], which states that every regular and strongly local Dirichlet form can be represented by effective intervals, this representation is not only more compact, but more importantly, it may be easier to be generalized to create a representation theorem of this type for higher dimensional regular and strongly local Dirichlet forms. Besides, a result about how these two representations are related is given in Theorem 4.3.

The paper is organized as follows. In §2, we shall review some terminologies and theorems in [6], which will play an important role in this paper. The section §3 is mainly devoted to the proof of Theorem 1.1 and Theorem 1.2. In §4, given a Dirichlet form generated by $(\mathfrak{f}, a)$, we shall formulate effective intervals of the Dirichlet form in terms of $(\mathfrak{f}, a)$. This will be used to present conditions under which two pairs $(\mathfrak{f}_1, a_1)$ and $(\mathfrak{f}_2, a_2)$ generate the same Dirichlet form.

2. Preliminaries

In this section we shall review some terminologies and theorems which will be heavily used in this paper. These terminologies and theorems are originally due to [4], [7] and [6], and for the sake of completeness and convenience, we present them here in a concise manner.

Suppose $(\mathscr{E}, \mathscr{D}[\mathscr{E}])$ is a densely defined, symmetric and non-negative definite bilinear form on a Hilbert space $L^2(E, m)$, where E is a locally compact separable metric space and m is a fully supported Radon measure on E. $(\mathscr{E}, \mathscr{D}[\mathscr{E}])$ is called closable if for any $\{u_n\} \subset \mathscr{D}[\mathscr{E}]$ satisfying $\mathscr{E}(u_n - u_m, u_n - u_m) \to 0$ and $\|u_n\|_{L^2} \to 0$ as $n, m \to \infty$, it always holds that $\mathscr{E}(u_n, u_n) \to 0$. It is called closed if $\mathscr{D}[\mathscr{E}]$ is a real Hilbert space under the norm $\|\cdot\|_{\mathscr{E}_1} := \sqrt{\mathscr{E}(\cdot,\cdot) + \|\cdot\|_{L^2}}$. And it is called Markovian if the following property holds: for any $\varepsilon > 0$, there exists a function $\phi_\varepsilon(t)$ satisfying

$$\text{(2.1)}\qquad \begin{aligned} &\phi_\varepsilon(t) = t, \quad \forall t \in [0,1], \quad -\varepsilon \le \phi_\varepsilon(t) \le 1+\varepsilon, \quad \forall t \in \mathbb{R}, \\ &0 \le \phi_\varepsilon(t') - \phi_\varepsilon(t) \le t' - t, \quad \text{whenever } t < t', \end{aligned}$$

such that for every $u \in \mathscr{D}[\mathscr{E}]$,

$$\phi_\varepsilon(u) \in \mathscr{D}[\mathscr{E}], \quad \mathscr{E}(\phi_\varepsilon(u), \phi_\varepsilon(u)) \le \mathscr{E}(u,u).$$

If $(\mathscr{E}, \mathscr{D}[\mathscr{E}])$ is both closed and Markovian, we say that $(\mathscr{E}, \mathscr{D}[\mathscr{E}])$ is a Dirichlet form. We usually use $(\mathscr{E}, \mathscr{F})$ to denote a Dirichlet form. For a Dirichlet form $(\mathscr{E}, \mathscr{F})$, a subset $\mathscr{C}$ of $\mathscr{F} \cap C_c(E)$ is called a core if $\mathscr{C}$ is dense in $\mathscr{F}$ with $\mathscr{E}_1$-norm and dense in $C_c(E)$ with uniform norm, where $C_c(E)$ consists of all continuous functions in E that have compact support. $\mathscr{C}$ is called a special standard core if additionally $\mathscr{C}$ is a dense subalgebra of $C_c(E)$ and satisfies the following:

(1) For any $\varepsilon > 0$, there exists a function $\phi_\varepsilon(t)$ satisfying (2.1) such that $\phi_\varepsilon(u) \in \mathscr{C}$ whenever $u \in \mathscr{C}$.
(2) For any compact set K and relatively compact open set G with $K \subset G$, there exists $u \in \mathscr{C}$ such that $u \ge 0$, $u = 1$ on K and $u = 0$ on $E \setminus G$.

$(\mathscr{E}, \mathscr{F})$ is called regular if $(\mathscr{E}, \mathscr{F})$ possesses a core, and is called strongly local if $\mathscr{E}(u,v) = 0$ whenever $u, v \in \mathscr{F}$ have compact support and u is a constant on a neighbourhood of the support of v.

Now we focus on the regular and strongly local Dirichlet forms on $L^2(\mathbb{R}, m)$. Let $J := \langle a, b\rangle$ be an interval, where a or b may or may not be contained in J. Take a fixed point in the interior of J as follows

$$\text{(2.2)}\qquad e := \begin{cases} \dfrac{a+b}{2}, & |a| + |b| < \infty, \\ a+1, & a > -\infty, b = \infty, \\ b-1, & a = -\infty, b < \infty, \\ 0, & a = -\infty, b = \infty. \end{cases}$$

A function s is called a scale function on J if it is strictly increasing and continuous, with $\mathsf{s}(e) = 0$. Since s is increasing, we can define

$$\mathsf{s}(a) := \lim_{x \downarrow a} \mathsf{s}(x), \quad \mathsf{s}(b) := \lim_{x \uparrow b} \mathsf{s}(x).$$

Then s is called an adapted scale function in the sense that

($\mathbf{A}_R$) $a + \mathsf{s}(a) > -\infty$ if and only if $a \in J$;
($\mathbf{B}_R$) $b + \mathsf{s}(b) < \infty$ if and only if $b \in J$.

Moreover we introduce two conditions at the infinities:

(L_R) $a = -\infty$, $\mathsf{s}(-\infty) > -\infty$ and $m((-\infty, 0]) < \infty$;
(R_R) $b = \infty$, $\mathsf{s}(\infty) < \infty$ and $m([0, \infty)) < \infty$.

Then we state the following representation theorem of regular and strongly local Dirichlet form on $\mathbb{R}$. It is originally due to [7, Theorem 2.1].

Theorem 2.1. *$(\mathscr{E}, \mathscr{F})$ is a regular and strongly local Dirichlet form on $L^2(\mathbb{R}, m)$ if and only if there exists a set $\{(I_n, \mathsf{s}_n) : n \ge 1\}$, where I_n are a series of disjoint intervals and s_n are adapted scale functions on I_n, such that*

$$\mathscr{F} = \left\{u \in L^2(\mathbb{R}, m) : u|_{I_n} \in \mathscr{F}^{(\mathsf{s}_n)}, \sum_{n\geq 1} \mathscr{E}^{(\mathsf{s}_n)}(u|_{I_n}, u|_{I_n}) < \infty\right\},$$
$$\mathscr{E}(u, v) = \sum_{n\geq 1} \mathscr{E}^{(\mathsf{s}_n)}(u|_{I_n}, v|_{I_n}), \quad u, v \in \mathscr{F}, \tag{2.3}$$

where $(\mathscr{E}^{(\mathsf{s}_n)}, \mathscr{F}^{(\mathsf{s}_n)})$ *is given by*

$$\mathscr{F}^{(\mathsf{s}_n)} := \Big\{u \in L^2(I_n, m|_{I_n}) : u \ll \mathsf{s}_n,\ \frac{du}{d\mathsf{s}_n} \in L^2(I_n, d\mathsf{s}_n);$$
$$u(a) = 0 \text{ (resp. } u(b) = 0) \text{ whenever } (L_R) \text{ (resp. } (R_R))\Big\}, \tag{2.4}$$
$$\mathscr{E}^{(\mathsf{s}_n)}(u, v) := \frac{1}{2}\int_{I_n} \frac{du}{d\mathsf{s}_n}\frac{dv}{d\mathsf{s}_n} d\mathsf{s}_n, \quad u, v \in \mathscr{F}^{(\mathsf{s}_n)}.$$

Moreover, the intervals $\{I_n : n \geq 1\}$ *and scale functions* $\{\mathsf{s}_n : n \geq 1\}$ *are uniquely determined, if the difference of order is ignored.*

Note that $m|_{I_n}$ and $u|_{I_n}$ above denote the restriction of measure m and function u to the interval I_n. The set $\{(I_n, \mathsf{s}_n) : n \geq 1\}$ is called the effective intervals of $(\mathscr{E}, \mathscr{F})$, and $\lambda_{\mathsf{s}} := \sum_{n\geq 1} d\mathsf{s}_n$ is called the scale measure. For a function $\mathfrak{f} \in \mathbf{S}(\mathbb{R})$, its induced measure $\lambda_{\mathfrak{f}}$ can be decomposed into the so-called 'effective part' $\lambda^{\mathrm{e}}_{\mathfrak{f}}$ and 'trivial part' $\lambda^{\mathrm{t}}_{\mathfrak{f}}$ with respect to $\{(I_n, \mathsf{s}_n) : n \geq 1\}$ in the following sense:

$$\lambda^{\mathrm{e}}_{\mathfrak{f}} := \lambda_{\mathfrak{f}}|_{\bigcup_{n\geq 1} I_n}, \quad \lambda^{\mathrm{t}}_{\mathfrak{f}} := \lambda_{\mathfrak{f}} - \lambda^{\mathrm{e}}_{\mathfrak{f}} = \lambda_{\mathfrak{f}}|_{(\bigcup_{n\geq 1} I_n)^c}.$$

We say that I_i and I_j are $\mathfrak{f}$-scale-connected with respect to the scale measure λ_{s}, if $\lambda^{\mathrm{t}}_{\mathfrak{f}}([e_i, e_j]) = 0$ and $\lambda_{\mathsf{s}}([e_i, e_j]) < \infty$, where e_i and e_j are two fixed points in I_i and I_j respectively. If any two of the intervals are not $\mathfrak{f}$-scale-connected, we say that $\{(I_n, \mathsf{s}_n) : n \geq 1\}$ is $\mathfrak{f}$-scale-isolated. With the above terminologies at hand, we are able to state the following criteria for $\mathscr{C}_{\mathfrak{f}}$ to be a special standard core of $(\mathscr{E}, \mathscr{F})$. We refer to [6, Lemma 5.1 and Corollary 5.7] for details.

Theorem 2.2. *$\mathscr{C}_{\mathfrak{f}}$ is a special standard core of $(\mathscr{E}, \mathscr{F})$ if and only if*

(1) *$\lambda^{\mathrm{e}}_{\mathfrak{f}}$ and λ_{s} are mutually absolutely continuous with each other, namely, $\lambda^{\mathrm{e}}_{\mathfrak{f}} \ll \lambda_{\mathsf{s}}$ and $\lambda_{\mathsf{s}} \ll \lambda^{\mathrm{e}}_{\mathfrak{f}}$;*

(2) *$\frac{d\lambda^{\mathrm{e}}_{\mathfrak{f}}}{d\lambda_{\mathsf{s}}} \in L^2_{loc}(\mathbb{R}, \lambda_{\mathsf{s}})$, i.e., for every $L > 0$*

$$\sum_{n\geq 1} \int_{I_n\cap(-L,L)} \left(\frac{d\lambda^{\mathrm{e}}_{\mathfrak{f}}}{d\lambda_{\mathsf{s}}}\right)^2 d\lambda_{\mathsf{s}} = \sum_{n\geq 1} \int_{I_n\cap(-L,L)} \left(\frac{d\lambda^{\mathrm{e}}_{\mathfrak{f}}}{d\lambda_{\mathsf{s}}}\right) d\lambda_{\mathfrak{f}} < \infty \tag{2.5}$$

(3) *$\{(I_n, \mathsf{s}_n) : n \geq 1\}$ is $\mathfrak{f}$-scale-isolated with respect to λ_{s}.*

3. Proof of Theorem 1.1 and 1.2

In this section we will prove Theorem 1.1 and 1.2. The proof of the sufficiency of Theorem 1.1 is similar to the counterpart of the proof of [4, Theorem 3.1.6], and for the proof of the necessity, we adopt a probabilistic method based on the effective intervals introduced in §2, which is originally due to [7].

Proof of Theorem 1.1. We begin with the proof of the necessity of Theorem 1.1. Suppose that (1.3) is closable on $L^2(\mathbb{R}, m)$. We denote the $\mathscr{E}_1$-closure of $\mathscr{D}[\mathscr{E}]$ by $\mathscr{F}$. Clearly,$(\mathscr{E}, \mathscr{F})$ is a regular and strongly local Dirichlet form with $\mathscr{C}_{\mathfrak{f}}$ as a special standard core. Denote the effective intervals of $(\mathscr{E}, \mathscr{F})$ by $\{(I_n, \mathsf{s}_n) : n \geq 1\}$. Since $\mathscr{C}_{\mathfrak{f}}$ is a special standard core of $(\mathscr{E}, \mathscr{F})$, it follows from Theorem 2.2 that $\lambda_{\mathfrak{f}}^{\mathrm{e}} \ll \lambda_{\mathsf{s}}$, $\lambda_{\mathsf{s}} \ll \lambda_{\mathfrak{f}}^{\mathrm{e}}$, $\frac{d\lambda_{\mathfrak{f}}^{\mathrm{e}}}{d\lambda_{\mathsf{s}}} \in L^2_{loc}(\mathbb{R}, \lambda_{\mathsf{s}})$ and $\{(I_n, \mathsf{s}_n) : n \geq 1\}$ is $\mathfrak{f}$-scale-isolated with respect to λ_{s}.

For a function $\varphi \in \mathscr{C}_{\mathfrak{f}}$, we know from (2.3), (2.4) and the fact $\lambda_{\mathfrak{f}}^{\mathrm{e}} \ll \lambda_{\mathsf{s}}$ that

$$\begin{aligned} 2\mathscr{E}(\varphi,\varphi) &= \sum_{n\geq 1}\int_{I_n}\left(\frac{d\varphi}{d\mathsf{s}_n}\right)^2 d\mathsf{s}_n = \sum_{n\geq 1}\int_{I_n}\left(\frac{d\varphi}{d\lambda_{\mathfrak{f}}^{\mathrm{e}}}\right)^2\left(\frac{d\lambda_{\mathfrak{f}}^{\mathrm{e}}}{d\lambda_{\mathsf{s}}}\right)^2 d\lambda_{\mathsf{s}} \\ &= \sum_{n\geq 1}\int_{I_n}\left(\frac{d\varphi}{d\lambda_{\mathfrak{f}}}\right)^2\left(\frac{d\lambda_{\mathfrak{f}}}{d\lambda_{\mathsf{s}}}\right) d\lambda_{\mathfrak{f}}. \end{aligned} \tag{3.1}$$

Define a measure μ on $\mathbb{R}$ as follows:

$$\mu|_{(\bigcup_{n\geq 1} I_n)^c} := 0, \quad \mu|_{I_n} := \left(\frac{d\lambda_{\mathfrak{f}}}{d\lambda_{\mathsf{s}}}\right)\Big|_{I_n} d\lambda_{\mathfrak{f}} = \left(\frac{d\lambda_{\mathfrak{f}}^{\mathrm{e}}}{d\lambda_{\mathsf{s}}}\right) d\lambda_{\mathfrak{f}}. \tag{3.2}$$

It is obvious that $\mu \ll \lambda_{\mathfrak{f}}$, and

$$\mathscr{E}(\varphi,\varphi) = \frac{1}{2}\int_{-\infty}^{+\infty}\left(\frac{d\varphi}{d\lambda_{\mathfrak{f}}}\right)^2 d\mu,$$

which implies $\mu = \nu$. Moreover, it follows form (2.5) that μ is a positive Radon measure. Denote the Radon-Nikodym derivative $\frac{d\mu}{d\lambda_{\mathfrak{f}}}$ by $a(x)$, and denote its regular set with respect to $\mathfrak{f}$ by $R_{\mathfrak{f}}(a)$ (see (1.4)). We now claim that

$$R_{\mathfrak{f}}(a) = \bigcup_{n\geq 1} \mathring{I}_n \tag{3.3}$$

where $\mathring{I}_n$ denotes the interior of I_n. On one hand, take $x \in \mathring{I}_n$, which means there exists an $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset \mathring{I}_n$. Since $\lambda_{\mathsf{s}} \ll \lambda_{\mathfrak{f}}^{\mathrm{e}}$, it follows that

$$\int_{x-\varepsilon}^{x+\varepsilon}\frac{1}{a(\xi)}d\lambda_{\mathfrak{f}}(\xi) = \int_{x-\varepsilon}^{x+\varepsilon}\left(\frac{d\lambda_{\mathsf{s}}}{d\lambda_{\mathfrak{f}}}\right)d\lambda_{\mathfrak{f}} = \mathsf{s}_n(x+\varepsilon) - \mathsf{s}_n(x-\varepsilon) < \infty, \tag{3.4}$$

which implies $x \in R_{\mathfrak{f}}(a)$ and $\bigcup_{n\geq 1}\mathring{I}_n \subset R_{\mathfrak{f}}(a)$. On the other hand, take $x \in \left(\bigcup_{n\geq 1}\mathring{I}_n\right)^c$ and suppose that $x \in R_{\mathfrak{f}}(a)$, i.e., there exists an $\varepsilon > 0$ such that

$$\int_{x-\varepsilon}^{x+\varepsilon}\frac{1}{a(\xi)}d\lambda_{\mathfrak{f}}(\xi) < \infty. \tag{3.5}$$

It follows from the definition of a and (3.2) that $a = 0$ $\lambda_{\mathfrak{f}}$-a.e. on $x \in \left(\bigcup_{n\geq 1}\mathring{I}_n\right)^c$. Combining with (3.5), we have

$$\lambda_{\mathfrak{f}}\left((x-\varepsilon, x+\varepsilon) \cap \left(\bigcup_{n\geq 1}\mathring{I}_n\right)^c\right) = 0. \tag{3.6}$$

We assert that (A) there exist two points $e_i, e_j \in (x-\varepsilon, x+\varepsilon)$, $e_i < e_j$, while e_i and e_j belong to different effective intervals I_i and I_j respectively. Indeed, if the above assertion does not hold true, we have the following cases:

Case 1: there exists an integer $k \geq 1$ such that $(x-\varepsilon, x+\varepsilon) \subset I_k$, but that contradicts the fact that $x \in \left(\bigcup_{n\geq 1} \mathring{I}_n\right)^c$;

Case 2: for every $n \geq 1$, $(x-\varepsilon, x+\varepsilon) \cap I_n = \emptyset$, which implies

$$\lambda_{\mathfrak{f}}\left((x-\varepsilon, x+\varepsilon) \cap \left(\bigcup_{n\geq 1} \mathring{I}_n\right)^c\right) = \lambda_{\mathfrak{f}}(x-\varepsilon, x+\varepsilon) > 0$$

since $\lambda_{\mathfrak{f}}$ is a Radon measure. That violates (3.6);

Case 3: there exists a unique $k \geq 1$ such that $(x-\varepsilon, x+\varepsilon) \cap I_k \neq \emptyset$ while $(x-\varepsilon, x+\varepsilon) \not\subset I_k$. That implies that $(x-\varepsilon, x+\varepsilon) \cap \left(\mathring{I}_k\right)^c$ contains an interval, which also violates (3.6).

In a word, $(x-\varepsilon, x+\varepsilon)$ must intersect with at least two effective intervals, which proves the assertion (A). Similar to (3.4), we have

$$\lambda_{\mathsf{s}}\left([e_i, e_j]\right) = \int_{e_i}^{e_j} \frac{1}{a(\xi)} d\lambda_{\mathfrak{f}}(\xi) < \infty$$

while

$$\lambda_{\mathfrak{f}}^{\mathsf{t}}\left([e_i, e_j]\right) \leq \lambda_{\mathfrak{f}}\left((x-\varepsilon, x+\varepsilon) \cap \left(\bigcup_{n\geq 1} \mathring{I}_n\right)^c\right) = 0$$

That implies that I_i and I_j are $\mathfrak{f}$-scale-connected, which contradicts the fact that $\{(I_n, \mathsf{s}_n) : n \geq 1\}$ is $\mathfrak{f}$-scale-isolated with respect to λ_{s}. In conclusion, if $x \in \left(\bigcup_{n\geq 1} \mathring{I}_n\right)^c$, then $x \notin R_{\mathfrak{f}}(a)$ so that (3.3) holds.

From the definition of a and (3.2) we know that a vanishes $\lambda_{\mathfrak{f}}$-a.e. on $\left(\bigcup_{n\geq 1} \mathring{I}_n\right)^c = S_{\mathfrak{f}}(a)$. That proves the necessity.

As for the sufficiency, we assume that $\nu \ll \lambda_{\mathfrak{f}}$ and the Radon-Nikodym derivative a vanishes $\lambda_{\mathfrak{f}}$-a.e. on $S_{\mathfrak{f}}(a)$. Suppose that $\{\phi_n\} \subset \mathscr{C}_{\mathfrak{f}}$ constitutes an $\mathscr{E}$-Cauchy sequence, i.e.,

$$\int_{\mathbb{R}} \left(\frac{d\phi_n}{d\lambda_{\mathfrak{f}}} - \frac{d\phi_m}{d\lambda_{\mathfrak{f}}}\right)^2 a \cdot d\lambda_{\mathfrak{f}} \to 0$$

as $n, m \to \infty$, and $\|\phi_n\|_{L^2(\mathbb{R},m)} \to 0$ as $n \to \infty$. Define $\psi_n := \dfrac{d\phi_n}{d\lambda_{\mathfrak{f}}}$. It is seen that $\{\psi_n\}$ is Cauchy in $L^2(\mathbb{R}, \nu)$, so that there exists a function ψ such that $\int_{\mathbb{R}} (\psi - \psi_n)^2 a \cdot d\lambda_{\mathfrak{f}} \to 0$. For any interval $[\alpha, \beta) \subset R_{\mathfrak{f}}(a)$,

$$\begin{aligned}\left(\int_\alpha^\beta \psi d\lambda_{\mathfrak{f}}\right)^2 &\leq 2\left(\int_\alpha^\beta (\psi - \psi_n) d\lambda_{\mathfrak{f}}\right)^2 + 2\left(\int_\alpha^\beta \psi_n d\lambda_{\mathfrak{f}}\right)^2 \\ &= 2\left(\phi_n(\beta) - \phi_n(\alpha)\right)^2 + 2\left(\int_\alpha^\beta (\psi - \psi_n) a \cdot \frac{1}{a} d\lambda_{\mathfrak{f}}\right)^2 := I_1 + I_2\end{aligned}$$

Since $\|\phi_n\|_{L^2(\mathbb{R},m)} \to 0$, $I_1 \to 0$ as $n \to \infty$ for m-a.e. α and β. We then apply Cauchy-Schwarz inequality to I_2:

$$I_2 \leq 2 \int_\alpha^\beta \frac{1}{a} d\lambda_{\mathfrak{f}} \int_\alpha^\beta (\psi - \psi_n)^2 a \cdot d\lambda_{\mathfrak{f}}.$$

It is seen that $\int_\alpha^\beta a^{-1} d\lambda_{\mathfrak{f}} < \infty$ from the fact that $[\alpha, \beta) \subset R_{\mathfrak{f}}(a)$, and it follows that $I_2 \to 0$ as $n \to \infty$. As a result, $\int_\alpha^\beta \psi d\lambda_{\mathfrak{f}} = 0$ for any $[\alpha, \beta) \subset R_{\mathfrak{f}}(a)$, which implies that $\psi = 0$ $\lambda_{\mathfrak{f}}$-a.e. on

$R_{\mathfrak{f}}(a)$. Since a vanishes $\lambda_{\mathfrak{f}}$-a.e. on $S_{\mathfrak{f}}(a)$, $\psi = 0$ $a \cdot d\lambda_{\mathfrak{f}}$-a.e. on $\mathbb{R}$. In conclusion,

$$\lim_{n\to\infty} \mathscr{E}(\phi_n, \phi_n) = \frac{1}{2} \int_{\mathbb{R}} \psi^2 a \cdot d\lambda_{\mathfrak{f}} = 0,$$

so that $(\mathscr{E}, \mathscr{D}[\mathscr{E}])$ is closable. □

From the proof of Theorem 1.1, we may get the proof of Theorem 1.2.

Proof of Theorem 1.2. Assume that $(\mathscr{E}, \mathscr{F})$ is a regular and strongly local Dirichlet form on $L^2(\mathbb{R}, m)$ characterized by effective intervals $\{(I_n, \mathsf{s}_n) : n \geq 1\}$. From [6, Theorem 5.11] we know that there exists a function $\mathfrak{f} \in \mathbf{S}(\mathbb{R})$ such that $\mathscr{C}_{\mathfrak{f}}$ is a special standard core of $(\mathscr{E}, \mathscr{F})$. Then we can rewrite $\mathscr{E}$ as (3.1) and define the measure ν as (3.2). Clearly ν is absolutely continuous with respect to $\lambda_{\mathfrak{f}}$ and the Radon-Nikodym derivative a vanishes on $\lambda_{\mathfrak{f}}$-a.e. on its singular set $S_{\mathfrak{f}}(a)$ with respect to $\mathfrak{f}$, just as what we have proved in the proof of necessity of Theorem 1.1. □

REMARK 3.1. From the proof of [6, Theorem 5.11] we can see that the choice of $\mathfrak{f}$ is not unique for $\mathscr{C}_{\mathfrak{f}}$ to be a special standard core of a given regular and strongly local Dirichlet form $(\mathscr{E}, \mathscr{F})$. Hence the choice of Hamza pair $(\mathfrak{f}, a)$ for $(\mathscr{E}, \mathscr{F})$ is not unique. We will come back to this issue in the next section.

4. Recovering the effective intervals

In this section we start from a form of Hamza's type (1.3).It follows that $(\mathscr{E}, \mathscr{D}[\mathscr{E}])$ is closable, and denote its $\mathscr{E}_1$-closure by $(\mathscr{E}, \mathscr{F})$. Namely, $(\mathscr{E}, \mathscr{F})$ is generated by a Hamza pair $(\mathfrak{f}, a)$. Our goal is to characterize the effective intervals of $(\mathscr{E}, \mathscr{F})$ in terms of $(\mathfrak{f}, a)$.

From the definition of the regular set $R_{\mathfrak{f}}(a)$, we know that $R_{\mathfrak{f}}(a)$ is an open subset of $\mathbb{R}$, so that $R_{\mathfrak{f}}(a)$ can be written as a union of disjoint open intervals (Cf. [8, §1.3 Proposition 9]). Namely,

$$R_{\mathfrak{f}}(a) = \bigcup_{n\geq 1} (a_n, b_n). \tag{4.1}$$

For each n, fix a point $e_n \in (a_n, b_n)$ as (2.2) and define a function s_n on (a_n, b_n) by

$$\mathsf{s}_n(x) = \int_{e_n}^{x} \frac{1}{a(\xi)} d\lambda_{\mathfrak{f}}(\xi), \quad x \in (a_n, b_n). \tag{4.2}$$

Lemma 4.1. *s_n is continuous and strictly increasing on (a_n, b_n).*

Proof. From the definition of $R_{\mathfrak{f}}(a)$ we know that $\frac{1}{a} \in L^1_{loc}((a_n, b_n), \lambda_{\mathfrak{f}})$. The absolute continuity of the integral (Cf. [8, §18.3, Proposition 17]) suggests that for each $\varepsilon > 0$, there is a $\delta > 0$ such that for any measurable set E with $\lambda_{\mathfrak{f}}(E) < \delta$, we have $\int_E \frac{1}{a(\xi)} d\lambda_{\mathfrak{f}}(\xi) < \varepsilon$. For any $x \in (a_n, b_n)$, since $\mathfrak{f}$ is continuous, there exists a $\delta_0 > 0$ such that $(x-\delta_0, x+\delta_0) \subset (a_n, b_n)$ and $\lambda_{\mathfrak{f}}(x - \delta_0, x + \delta_0) < \delta$. Therefore, for any $y \in (x - \delta_0, x + \delta_0)$, we have

$$|\mathsf{s}_n(x) - \mathsf{s}_n(y)| = \left| \int_{y}^{x} \frac{1}{a(\xi)} d\lambda_{\mathfrak{f}}(\xi) \right| < \varepsilon.$$

The continuity of s_n follows. Clearly, s_n is increasing, and we argue that it is strictly increasing. Suppose that $\mathsf{s}_n(x) = \mathsf{s}_n(y)$ for some $x, y \in (a_n, b_n)$, $x < y$. Since $\mathfrak{f}$ is strictly increasing, it leads to $a(\xi) = \infty$ for $\lambda_{\mathfrak{f}}$-a.e. $\xi \in (x, y)$, which contradicts the fact that $\nu = a \cdot \lambda_{\mathfrak{f}}$ is a Radon measure. □

Since s_n is monotone on (a_n, b_n), we can extend it to its endpoints by

$$\mathsf{s}_n(a_n) := \lim_{x \downarrow a_n} \mathsf{s}_n(x), \quad \mathsf{s}_n(b_n) := \lim_{x \uparrow b_n} \mathsf{s}_n(x). \tag{4.3}$$

Set

$$I_n := \langle a_n, b_n \rangle \tag{4.4}$$

where $a_n \in I_n$ if $a_n + \mathsf{s}_n(a_n) > -\infty$ and $b_n \in I_n$ if $b_n + \mathsf{s}_n(b_n) < \infty$. We assert that $\{I_n : n \geq 1\}$ still remain disjoint by the next lemma.

Lemma 4.2. *$\{I_n : n \geq 1\}$ are mutually disjoint.*

Proof. Suppose there exist two intervals $I_i = \langle a_i, b_i]$ and $I_j = [a_j, b_j\rangle$, and $b_i = a_j$. By the definition of I_i and I_j, $\mathsf{s}_i(b_i) < \infty$ and $\mathsf{s}_j(a_j) > -\infty$. It follows from (4.2) and (4.3) that

$$\int_{e_i}^{e_j} \frac{1}{a(\xi)} d\lambda_{\mathfrak{f}}(\xi) = \int_{e_i}^{b_i} \frac{1}{a(\xi)} d\lambda_{\mathfrak{f}}(\xi) - \int_{e_j}^{a_j} \frac{1}{a(\xi)} d\lambda_{\mathfrak{f}}(\xi) = \mathsf{s}_i(b_i) - \mathsf{s}_j(a_j) < \infty.$$

Hence $a_j (= b_i) \in R_{\mathfrak{f}}(a)$, which contradicts (4.1). □

Thus we obtain a set $\{(I_n, \mathsf{s}_n) : n \geq 1\}$, where I_n are a series of disjoint intervals and s_n are adapted scale functions on I_n, and it is by definition a class of effective intervals. The next theorem shows that the Dirichlet form corresponds to $\{(I_n, \mathsf{s}_n) : n \geq 1\}$ coincides the $\mathscr{E}_1$-closure of $(\mathscr{E}, \mathscr{D}[\mathscr{E}])$.

Theorem 4.3. *Let $(\mathscr{E}, \mathscr{F})$ be the Dirichlet form generated by a Hamza pair $(\mathfrak{f}, a)$.The interval I_n and scale function s_n on I_n are defined by* (4.4) *and* (4.2) *respectively for each $n \geq 1$. Then $(\mathscr{E}, \mathscr{F})$ is a regular and strongly local Dirichlet form characterized by the effective intervals $\{(I_n, \mathsf{s}_n) : n \geq 1\}$.*

Proof. Denote the Dirichlet form corresponding to $\{(I_n, \mathsf{s}_n) : n \geq 1\}$ by $(\bar{\mathscr{E}}, \bar{\mathscr{F}})$, and we argue that $(\mathscr{E}, \mathscr{F}) = (\bar{\mathscr{E}}, \bar{\mathscr{F}})$. Firstly we prove that $\mathscr{C}_{\mathfrak{f}}$ is a special standard core of $(\bar{\mathscr{E}}, \bar{\mathscr{F}})$ by checking the conditions in Theorem 2.2. Indeed, the fact that $\lambda_{\mathsf{s}} \ll \lambda_{\mathfrak{f}}^{\mathrm{e}}$ is clear from the definition and (4.2). Clearly $\frac{1}{a} > 0$ $\lambda_{\mathfrak{f}}$-a.e. on I_n, so that $d\lambda_{\mathfrak{f}}^{\mathrm{e}} = a \cdot d\lambda_{\mathsf{s}}$ on I_n. It follows that $\lambda_{\mathfrak{f}}^{\mathrm{e}} \ll \lambda_{\mathsf{s}}$ and $a = \dfrac{d\lambda_{\mathfrak{f}}^{\mathrm{e}}}{d\lambda_{\mathsf{s}}} \in L^2_{loc}(\mathbb{R}, \lambda_{\mathsf{s}})$, due to the fact that

$$\sum_{n\geq 1} \int_{I_n \cap (-L,L)} a^2 \cdot d\lambda_{\mathsf{s}} = \sum_{n \geq 1} \int_{I_n \cap (-L,L)} a \cdot d\lambda_{\mathfrak{f}} = \nu(-L, L) < \infty.$$

To check that $\{(I_n, \mathsf{s}_n) : n \geq 1\}$ is $\mathfrak{f}$-scale-isolated with respect to λ_{s}, suppose that there exist I_i and I_j which are $\mathfrak{f}$-scale-connected, $i < j$, i.e., $\lambda_{\mathfrak{f}}^{\mathrm{t}}([e_i, e_j]) = 0$ and $\lambda_{\mathsf{s}}([e_i, e_j]) < \infty$. From (4.1) we know that there exists a point $x \in (e_i, e_j) \cap S_{\mathfrak{f}}(a)$. Take $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset [e_i, e_j]$, and we have

$$\int_{x-\varepsilon}^{x+\varepsilon} \frac{1}{a(\xi)} d\lambda_{\mathfrak{f}}(\xi) = \int_{x-\varepsilon}^{x+\varepsilon} \frac{1}{a(\xi)} d\lambda_{\mathfrak{f}}^{\mathrm{e}}(\xi) + \int_{x-\varepsilon}^{x+\varepsilon} \frac{1}{a(\xi)} d\lambda_{\mathfrak{f}}^{\mathrm{t}}(\xi).$$

The second term vanishes since $\lambda_{\mathfrak{f}}^{\mathrm{t}}([e_i,e_j]) = 0$, and the first term equals $\lambda_{\mathsf{s}}(x-\varepsilon, x+\varepsilon) \le \lambda_{\mathsf{s}}([e_i,e_j]) < \infty$. In other words, $\int_{x-\varepsilon}^{x+\varepsilon} \frac{1}{a(\xi)} d\lambda_{\mathfrak{f}}(\xi) < \infty$, which contradicts the fact that $x \in S_{\mathfrak{f}}(a)$.

From Theorem 2.2 we can conclude that $\mathscr{C}_{\mathfrak{f}}$ is a special standard core of $(\bar{\mathscr{E}}, \bar{\mathscr{F}})$. Moreover, for any $\varphi \in \mathscr{C}_{\mathfrak{f}}$, similar to (3.1),

$$\begin{aligned} 2\bar{\mathscr{E}}(\varphi,\varphi) &= \sum_{n\ge1}\int_{I_n}\left(\frac{d\varphi}{d\mathsf{s}_n}\right)^2 d\mathsf{s}_n = \sum_{n\ge1}\int_{I_n}\left(\frac{d\varphi}{d\lambda_{\mathfrak{f}}}\right)^2\left(\frac{d\lambda_{\mathfrak{f}}}{d\lambda_{\mathsf{s}}}\right)d\lambda_{\mathfrak{f}} \\ &= \sum_{n\ge1}\int_{I_n}\left(\frac{d\varphi}{d\lambda_{\mathfrak{f}}}\right)^2 a\cdot d\lambda_{\mathfrak{f}} = \sum_{n\ge1}\int_{I_n}\left(\frac{d\varphi}{d\lambda_{\mathfrak{f}}}\right)^2 d\nu \\ &= 2\mathscr{E}(\varphi,\varphi). \end{aligned}$$

Thus $\mathscr{E} = \bar{\mathscr{E}}$ on $\mathscr{C}_{\mathfrak{f}} \times \mathscr{C}_{\mathfrak{f}}$ and thus $\mathscr{F} = \bar{\mathscr{F}}$. That finishes the proof. □

Now we come back to Remark 3.1 below the proof of Theorem 1.2. We shall answer under what conditions, two different Hamza pairs $(\mathfrak{f}_1, a_1)$ and $(\mathfrak{f}_2, a_2)$ generate the same Dirichlet form.

Corollary 4.4. *Two Hamza pairs $(\mathfrak{f}_1, a_1)$ and $(\mathfrak{f}_2, a_2)$ generate the same Dirichlet form if and only if*

(1) $R_{\mathfrak{f}_1}(a_1) = R_{\mathfrak{f}_2}(a_2)$;

(2) *Write $R_{\mathfrak{f}_1}(a_1) = \bigcup_{n\ge1}(a_n, b_n)$, then for every $n \ge 1$,*

$$\int_{e_n}^x \frac{1}{a_1(\xi)} d\lambda_{\mathfrak{f}_1}(\xi) = \int_{e_n}^x \frac{1}{a_2(\xi)} d\lambda_{\mathfrak{f}_2}(\xi), \quad x \in (a_n, b_n).$$

Proof. The proof follows from the unique characterization of regular and strongly local Dirichlet forms in the form of effective intervals (Cf. Theorem 2.1), and Theorem 4.3. □

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Wenjie Sun
Shanghai Center for Mathematical Sciences
Fudan University
Shanghai 200433
China
e-mail: wjsun14@fudan.edu.cn

Jiangang Ying
School of Mathematical Sciences
Fudan University
Shanghai 200433
China
e-mail: jgying@fudan.edu.cn