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DIFFERENTIAL EQUATIONS INVOLVING CUBIC THETA FUNCTIONS AND EISENSTEIN SERIES

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Abstract
In this paper, we derive systems of ordinary differential equations (ODEs) satisfied by modular forms of level three, which are level three versions of Ramanujan's system of ODEs satisfied by the classical Eisenstein series.

1. Introduction

Throughout this paper, let $\mathbb{N}_0$, and $\mathbb{N}$ denote the sets of nonnegative integers and positive integers. For positive integers j, k, and $n \in \mathbb{N}$, $d_{j,k}(n)$ denotes the number of positive divisors d of n such that $d \equiv j \bmod k$. Moreover, for k and $n \in \mathbb{N}$, $\sigma_k(n)$ is the sum of the k-th power of the positive divisors of n, and $d_{j,k}(n) = \sigma_k(n) = 0$ for $n \in \mathbb{Q} \setminus \mathbb{N}_0$. For each $n \in \mathbb{N}$, set

$$\left(\frac{n}{3}\right) = \begin{cases} +1, & \text{if } n \equiv 1 \bmod 3, \\ -1, & \text{if } n \equiv -1 \bmod 3, \\ 0, & \text{if } n \equiv 0 \bmod 3. \end{cases}$$

The *upper half plane* $\mathbb{H}^2$ is defined by $\mathbb{H}^2 = \{\tau \in \mathbb{C} \mid \Im\tau > 0\}$. Throughout this paper, set $q = \exp(2\pi i\tau)$ and define the *Dedekind eta function* by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) = q^{\frac{1}{24}} (q; q)_\infty.$$

The Eisenstein series E_2, E_4, and E_6 are defined by

$$E_2(q) = E_2(\tau) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n, \ \ E_4(q) = E_4(\tau) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

$$E_6(q) = E_4(\tau) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \ \ \sigma_k(n) = \sum_{d|n} d^k \ \text{for } k \in \mathbb{N}.$$

Many research efforts have been devoted to ordinary differential equations (ODEs) satisfied by modular forms. Classical examples include Ramanujan's coupled ODEs for Eisenstein series E_2, E_4, E_6; Pol and Rankin's fourth-order ODE satisfied by $\Delta = \eta^{24}(\tau)$; Jacobi's third-order ODE satisfied by the theta null functions $\vartheta_2, \vartheta_3, \vartheta_4$, which are defined by equation (2.1). The proofs of these respective ODEs can be found in the papers by Berndt [3,

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pp.92], Rankin [22], and Jacobi [12].

Halphen subsequently [10] rewrote Jacobi's ODE as a nonliear dynamical system:

$$X' + Y' = 2XY,\ Y' + Z' = 2YZ,\ Z' + X' = 2ZX.$$

In particular, ODEs of the quadratic type are known as Halphen-type systems.

Recently, Ohyama [18, 19] reconsidered Jacobi's ODE by taking into account Picard-Fuchs equations of elliptic modular surfaces, and following Jacobi's idea, derived a Halphen-type system satisfied by modular forms of level three. Following Ohyama, Mano [16] derived ODEs satisfied by modular forms of level five.

Ramanujan's system of ODEs is expressed as follows:

$$q\frac{dE_2}{dq} = \frac{(E_2)^2 - E_4}{12},\ q\frac{dE_4}{dq} = \frac{E_2E_4 - E_6}{3},\ q\frac{dE_6}{dq} = \frac{E_2E_6 - (E_4)^2}{2}, \tag{1.1}$$

which is equivalent to Chazy's third-order nonlinear ODE,

$$y''' = 2yy'' - 3(y')^2,\ y = \pi i E_2(\tau).$$

Ohyama [17] showed that Halphen's differential field is an extension of Ramanujan's differential field, whose Galois group is the symmetric group, S_3.

Ramamani [21] introduced

$$\mathcal{P} = 1 - 8\sum_{n=1}^{\infty}\frac{(-1)^n nq^n}{1-q^n},\ \tilde{\mathcal{P}} = 1 + 24\sum_{n=1}^{\infty}\frac{nq^n}{1+q^n},\ \mathcal{Q} = 1 + 16\sum_{n=1}^{\infty}\frac{(-1)^n n^3 q^n}{1-q^n},$$

and derived a system of ODEs satisfied by the $\mathcal{P}, \tilde{\mathcal{P}}, \mathcal{Q}$, modular forms of $\Gamma_0(2)$. Ablowitz [1] et al. showed that this system is equivalent to the following third-order non-linear ODE found by Bureau [5],

$$y''' = 2yy'' - (y')^2 + 2\frac{(y'' - yy')^2}{2y' - y^2},\ y = \mathcal{P}(\tau),$$

and that it is equivalent to a Halphen-type system. Maier [15] generalized these results to the Hecke group $\Gamma_0(N)$ $(N = 2, 3, 4)$.

In [11], Huber derived systems of ODEs satisfied by the cubic theta functions,

$$a(q) = \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+n^2},\ b(q) = \sum_{m,n\in\mathbb{Z}} \omega^{n-m} q^{m^2+mn+n^2},$$
$$c(q) = \sum_{m,n\in\mathbb{Z}} q^{(n+\frac{1}{3})^2+(n+\frac{1}{3})(m+\frac{1}{3})+(m+\frac{1}{3})^2},\ \omega = e^{\frac{2\pi i}{3}},\ |q| < 1.$$

The systems of ODEs are given by

$$q\frac{d}{dq}a = \frac{a\mathscr{P} - b^3}{3},\ q\frac{d}{dq}\mathscr{P} = \frac{\mathscr{P}^2 - ab^3}{3},\ q\frac{d}{dq}b^3 = \mathscr{P}b^3 - a^2b^3, \tag{1.2}$$

and

$$q\frac{d}{dq}a = \frac{c^3 - a\mathcal{P}}{3},\ q\frac{d}{dq}\mathcal{P} = \frac{ac^3 - \mathcal{P}^2}{3},\ q\frac{d}{dq}c^3 = c^3a^2 - \mathcal{P}c^3, \tag{1.3}$$

where

$$\mathscr{P}(q) = 1 - 6\sum_{n=1}^{\infty} \frac{\cos(2n\pi/3)nq^n}{1-q^n}, \ \mathcal{P}(q) = 9\sum_{n=1}^{\infty} \frac{n(q^n + q^{2n})}{1-q^{3n}}.$$

The aim of the research presented in this paper is to derive systems of ODEs satisfied by $a(q)$ by means of Farkas and Kra's theory of theta functions with rational characteristics.

Our main theorems are as follows.

Theorem 1.1. *For $q \in \mathbb{C}$ with $|q| < 1$, set*

$$P(q) = a(q) = \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+n^2}, \ Q(q) = E_2(q) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n, \tag{1.4}$$

$$R(q) = b^3(q) = \frac{(q;q)_\infty^9}{(q^3;q^3)_\infty^3} = 1 - 9\sum_{n=1}^{\infty} q^n \left(\sum_{d|n} d^2 \left(\frac{d}{3}\right)\right).$$

Then, we have

$$q\frac{d}{dq}P = \frac{3P^3 + PQ - 4R}{12}, \ q\frac{d}{dq}Q = \frac{-9P^4 + 8PR + Q^2}{12}, \ q\frac{d}{dq}R = \frac{-P^2R + QR}{4}. \tag{1.5}$$

Theorem 1.2. *For $q \in \mathbb{C}$ with $|q| < 1$, set*

$$P(q) = a(q) = \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+n^2}, \ Q(q) = E_2(q^3) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^{3n}, \tag{1.6}$$

$$R(q) = c^3(q) = 27q\frac{(q^3;q^3)_\infty^9}{(q;q)_\infty^3} = 27\sum_{n=1}^{\infty} q^n \left(\sum_{d|n} d^2 \left(\frac{n/d}{3}\right)\right).$$

Then, we have

$$q\frac{d}{dq}P = \frac{-3P^3 + 3PQ + 4R}{12}, \ q\frac{d}{dq}Q = \frac{-9P^4 + 8PR + 9Q^2}{36}, \ q\frac{d}{dq}R = \frac{P^2R + 3QR}{4}. \tag{1.7}$$

Section 2 reviews Farkas and Kra's theory of theta functions with rational characteristics. Section 3 treats some theta functional formulas. In particular, we prove equations (1.4) and (1.6). Section 4 describes preliminary results. Section 5 proves Theorem 1.1. In particular, our method recovers Ramanujan's system of ODEs (1.1). Section 6 proves Theorem 1.2. Section 7 expresses $a^2(q), a^3(q), a^4(q), a^5(q)$ and $a^6(q)$ in terms of modular forms and divisor functions. Section 8 shows Ramanujan's identity of $a(q)$, for selected cases that express $a(q)$ by Dedekind's eta functions. Section 9 derives more product-series identities.

Remark 1. We note the properties of the cubic theta functions;

$$a(q) = a(\tau) = 1 + 6\sum_{n=1}^{\infty}(d_{1,3}(n) - d_{2,3}(n))q^n, \tag{1.8}$$

$$b^3(q) = b^3(\tau) = \frac{(q;q)_\infty^9}{(q^3;q^3)_\infty^3}, \ c^3(q) = c^3(\tau) = 27q\frac{(q^3;q^3)_\infty^9}{(q;q)_\infty^3},$$

$$a^3(q) = b^3(q) + c^3(q), \quad q = \exp(2\pi i\tau).$$

For the proof, readers are referred to the books by Berndt [3, pp. 79], Dickson [7, pp. 68] and the paper by Borwein et al. [4].

Remark 2. Let us define the operators

$$\theta := q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}, \ \partial := \partial_k = 12\theta - kE_2(q), \ (k = 1, 2, 3, \ldots).$$

The properties of the operators θ and ∂ can be found in Lang's book [13, pp. 159-175]. The first ODE of Theorem 1.1 implies

$$\partial_1 a(q) = 3a^3(q) - 4b^3(q).$$

The first ODE of Theorem 1.2 was proved in Cooper [6, pp. 263].

Remark 3. Equation (1.2) and Theorem 1.1 implies that the following differential fields, $(\quad , qd/dq)$, are equal:

$$\begin{aligned}\mathbb{C}\langle a(q), \mathscr{P}(q), b^3(q)\rangle =&\mathbb{C}\langle a(q), \mathscr{P}(q)\rangle = \mathbb{C}\langle a(q), b^3(q)\rangle = \mathbb{C}\langle \mathscr{P}(q), b^3(q)\rangle \\ =&\mathbb{C}\langle a(q), E_2(q)\rangle = \mathbb{C}\langle a(q), E_2(q^3)\rangle = \mathbb{C}\langle a(q), E_2(q), b^3(q)\rangle.\end{aligned}$$

Moreover, Theorem 5.1 expresses $E_4(q)$ and $E_6(q)$ by $a(q)$ and $b^3(q)$, which implies that

$$\mathbb{C}(E_2(q), E_4(q), E_6(q)) \subset \mathbb{C}(a(q), E_2(q), b^3(q)).$$

2. Properties of the theta functions

2.1. Definitions. Following the work of Farkas and Kra [9], we introduce the *theta function with characteristics,* which is defined by

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\zeta,\tau) = \theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\zeta) := \sum_{n\in\mathbb{Z}} \exp\left(2\pi i\left[\frac{1}{2}\left(n+\frac{\epsilon}{2}\right)^2\tau + \left(n+\frac{\epsilon}{2}\right)\left(\zeta+\frac{\epsilon'}{2}\right)\right]\right),$$

where $\epsilon, \epsilon' \in \mathbb{R}$, $\zeta \in \mathbb{C}$, and $\tau \in \mathbb{H}^2$. The *theta constants* are given by

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix} := \theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0,\tau).$$

In particular, note that

$$\vartheta_2 = \theta\begin{bmatrix}1\\ 0\end{bmatrix}, \ \vartheta_3 = \theta\begin{bmatrix}0\\ 0\end{bmatrix}, \ \vartheta_4 = \theta\begin{bmatrix}0\\ 1\end{bmatrix}. \tag{2.1}$$

Furthermore, we denote the derivative coefficients of the theta function by

$$\theta'\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix} := \frac{\partial}{\partial\zeta}\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\zeta,\tau)\bigg|_{\zeta=0}, \ \theta''\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix} := \frac{\partial^2}{\partial\zeta^2}\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\zeta,\tau)\bigg|_{\zeta=0},$$

and

$$\theta'''\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix} := \frac{\partial^3}{\partial\zeta^3}\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\zeta,\tau)\bigg|_{\zeta=0}, \ \theta^{(n)}\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix} := \frac{\partial^n}{\partial\zeta^n}\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\zeta,\tau)\bigg|_{\zeta=0}, \ (n = 1, 2, 3, 4, \ldots).$$

In particular, Jacobi's derivative formula is given by

$$\theta'\begin{bmatrix}1\\ 1\end{bmatrix} = -\pi\theta\begin{bmatrix}0\\ 0\end{bmatrix}\theta\begin{bmatrix}1\\ 0\end{bmatrix}\theta\begin{bmatrix}0\\ 1\end{bmatrix}. \tag{2.2}$$

2.2. Basic properties. We first note that for $m, n \in \mathbb{Z}$,

$$\theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(\zeta + n + m\tau, \tau) = \exp(2\pi i)\left[\frac{n\epsilon - m\epsilon'}{2} - m\zeta - \frac{m^2\tau}{2}\right]\theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(\zeta, \tau), \tag{2.3}$$

and

$$\theta\begin{bmatrix} \epsilon + 2m \\ \epsilon' + 2n \end{bmatrix}(\zeta, \tau) = \exp(\pi i \epsilon n)\theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(\zeta, \tau). \tag{2.4}$$

Furthermore, it is easy to see that

$$\theta\begin{bmatrix} -\epsilon \\ -\epsilon' \end{bmatrix}(\zeta, \tau) = \theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(-\zeta, \tau) \text{ and } \theta'\begin{bmatrix} -\epsilon \\ -\epsilon' \end{bmatrix}(\zeta, \tau) = -\theta'\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(-\zeta, \tau).$$

For $m, n \in \mathbb{R}$, we see that

$$\begin{aligned} \theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}\left(\zeta + \frac{n + m\tau}{2}, \tau\right) \\ = \exp(2\pi i)\left[-\frac{m\zeta}{2} - \frac{m^2\tau}{8} - \frac{m(\epsilon' + n)}{4}\right]\theta\begin{bmatrix} \epsilon + m \\ \epsilon' + n \end{bmatrix}(\zeta, \tau). \end{aligned} \tag{2.5}$$

We note that $\theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(\zeta, \tau)$ has only one zero in the fundamental parallelogram, which is given by

$$\zeta = \frac{1 - \epsilon}{2}\tau + \frac{1 - \epsilon'}{2}.$$

2.3. Jacobi's triple product identity. All the theta functions have infinite product expansions, which are given by

$$\begin{aligned} \theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(\zeta, \tau) = \exp\left(\frac{\pi i \epsilon\epsilon'}{2}\right) x^{\frac{\epsilon^2}{4}} z^{\frac{\epsilon}{2}} \\ \times \prod_{n=1}^{\infty}(1 - x^{2n})(1 + e^{\pi i \epsilon'} x^{2n-1+\epsilon} z)(1 + e^{-\pi i \epsilon'} x^{2n-1-\epsilon}/z), \end{aligned} \tag{2.6}$$

where $x = \exp(\pi i \tau)$ and $z = \exp(2\pi i \zeta)$. Therefore, it follows from Jacobi's derivative formula (2.2) that

$$\theta'\begin{bmatrix} 1 \\ 1 \end{bmatrix}(0, \tau) = -2\pi q^{\frac{1}{8}} \prod_{n=1}^{\infty}(1 - q^n)^3, \;\; q = \exp(2\pi i \tau).$$

2.4. Spaces of N-th order θ-functions. Based on the results of Farkas and Kra [9], we define $\mathcal{F}_N\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$ to be the set of entire functions f that satisfy the two functional equations,

$$f(\zeta + 1) = \exp(\pi i \epsilon)\; f(\zeta),$$

and

$$f(\zeta + \tau) = \exp(-\pi i)[\epsilon' + 2N\zeta + N\tau]\; f(\zeta), \quad \zeta \in \mathbb{C}, \;\tau \in \mathbb{H}^2,$$

where N is a positive integer and $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} \in \mathbb{R}^2$. This set of functions is referred to as the space of *N-th order θ-functions with characteristic* $\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$. Note that

$$\dim \mathcal{F}_N \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} = N.$$

The proof of this space was reported by Farkas and Kra [9, pp.133].

2.5. The heat equation. The theta function satisfies the following heat equation:

$$\frac{\partial^2}{\partial \zeta^2}\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(\zeta, \tau) = 4\pi i \frac{\partial}{\partial \tau}\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(\zeta, \tau). \tag{2.7}$$

3. Some theta functional formulas

We introduce Weierstrass' $\wp$-function and σ-function:

$$\wp(z; \omega_1, \omega_2) = \wp(z) = \frac{1}{z^2} + \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n) \neq (0,0)}} \left(\frac{1}{(z - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right),$$

$$\sigma(z; \omega_1, \omega_2) = \sigma(z) = z \prod_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n) \neq (0,0)}} \left(1 - \frac{z}{m\omega_1 + n\omega_2} \right) \exp\left(\frac{z}{m\omega_1 + n\omega_2} + \frac{z^2}{2(m\omega_1 + n\omega_2)^2} \right),$$

where z, ω_1, ω_2 are complex numbers with $\omega_2/\omega_1 \notin \mathbb{R}$.

From Whittaker and Watson [24, pp. 437, 459], we recall the following formulas:

$$\frac{\sigma(4z)}{\sigma^4(z)} = -\wp'(z), \ \frac{\sigma(3z)}{\sigma^9(z)} = 3\wp(z)\wp'(z)^2 - \frac{1}{4}\wp''(z)^2, \tag{3.1}$$

and

$$\begin{aligned} \wp'(z)^2 &= 4\wp(z)^3 - g_2\wp(z) - g_3, \\ g_2(\omega_1, \omega_2) &= 60 \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n) \neq (0,0)}} \frac{1}{(m\omega_1 + n\omega_2)^4}, \ g_3(\omega_1, \omega_2) = 140 \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n) \neq (0,0)}} \frac{1}{(m\omega_1 + n\omega_2)^6}, \end{aligned}$$

which implies

$$\wp''(z) = 6\wp^2(z) - \frac{1}{2}g_2, \ \wp'''(z) = 12\wp(z)\wp'(z). \tag{3.2}$$

Moreover, from Farkas and Kra [9, pp. 124], we recall

$$\wp(z; 1, \tau) = \frac{1}{3} \frac{\theta''' \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}} - \frac{d^2}{dz^2} \log \theta \begin{bmatrix} 1 \\ 1 \end{bmatrix}(z, \tau),$$

$$\sigma(z;\omega_1,\omega_2)=\exp\left(\frac{\eta_1 z^2}{2\omega_1}\right)\frac{\omega_1}{\theta'\begin{bmatrix}1\\1\end{bmatrix}}\theta\begin{bmatrix}1\\1\end{bmatrix}\left(\frac{z}{\omega_1},\tau\right),$$

where $\omega_2/\omega_1=\tau\in\mathbb{H}^2$.

Therefore, we obtain the following theta functional formulas:

Theorem 3.1. *For every $z\in\mathbb{C}$, we have*

$$\frac{d^3}{dz^3}\log\theta\begin{bmatrix}1\\1\end{bmatrix}(z)=\theta'\begin{bmatrix}1\\1\end{bmatrix}^3\frac{\theta\begin{bmatrix}1\\1\end{bmatrix}(2z)}{\theta^4\begin{bmatrix}1\\1\end{bmatrix}(z)}, \tag{3.3}$$

$$\frac{\theta'\begin{bmatrix}1\\1\end{bmatrix}^8\theta\begin{bmatrix}1\\1\end{bmatrix}(3z)}{\theta^9\begin{bmatrix}1\\1\end{bmatrix}(z)}=3\left\{\frac{1}{3}\frac{\theta'''\begin{bmatrix}1\\1\end{bmatrix}}{\theta'\begin{bmatrix}1\\1\end{bmatrix}}-\frac{d^2}{dz^2}\log\theta\begin{bmatrix}1\\1\end{bmatrix}(z)\right\}\left\{\frac{\theta'\begin{bmatrix}1\\1\end{bmatrix}^3\theta\begin{bmatrix}1\\1\end{bmatrix}(2z)}{\theta^4\begin{bmatrix}1\\1\end{bmatrix}(z)}\right\}^2-\frac{1}{4}\left\{\frac{d^4}{dz^4}\log\theta\begin{bmatrix}1\\1\end{bmatrix}(z)\right\}^2 \tag{3.4}$$

and

$$\frac{d^5}{dz^5}\log\theta\begin{bmatrix}1\\1\end{bmatrix}(z)=12\left\{\frac{1}{3}\frac{\theta'''\begin{bmatrix}1\\1\end{bmatrix}}{\theta'\begin{bmatrix}1\\1\end{bmatrix}}-\frac{d^2}{dz^2}\log\theta\begin{bmatrix}1\\1\end{bmatrix}(z)\right\}\frac{\theta'\begin{bmatrix}1\\1\end{bmatrix}^3\theta\begin{bmatrix}1\\1\end{bmatrix}(2z)}{\theta^4\begin{bmatrix}1\\1\end{bmatrix}(z)}. \tag{3.5}$$

Corollary 3.2. *For every $\tau\in\mathbb{H}^2$, we have*

$$\frac{\eta^9(\tau)}{\eta^3(3\tau)}=1-9\sum_{n=1}^{\infty}q^n\left(\sum_{d|n}d^2\left(\frac{d}{3}\right)\right) \tag{3.6}$$

and

$$\frac{\eta^9(3\tau)}{\eta^3(\tau)}=\sum_{n=1}^{\infty}q^n\left(\sum_{d|n}d^2\left(\frac{n/d}{3}\right)\right), \tag{3.7}$$

where $q=\exp(2\pi i\tau)$.

Proof. The corollary can be proved by substituting $z=-1/3$ or $z=-\tau/3$ in equation (3.3) and applying Jacobi's triple product identity (2.6). □

4. Preliminary results

Proposition 4.1. *For every $\tau \in \mathbb{H}^2$, we have*

$$\frac{\theta'\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}}{\theta\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}} = -\frac{\pi}{\sqrt{3}}a(\tau), \text{ and } \frac{\theta'\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}}{\theta\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}} = \frac{\pi i}{3}a(\tau/3). \tag{4.1}$$

Proof. The proposition follows from Jacobi's triple product identity (2.6). □

Proposition 4.2. *For every $\tau \in \mathbb{H}^2$, we have*

$$3\frac{\theta''\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}}{\theta\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}} - \frac{\theta'''\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\theta'\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + 6\left\{\frac{\theta'\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}}{\theta\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}}\right\}^2 = 0, \tag{4.2}$$

$$3\frac{\theta''\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}}{\theta\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}} - \frac{\theta'''\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\theta'\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + 6\left\{\frac{\theta'\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}}{\theta\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}}\right\}^2 = 0, \tag{4.3}$$

$$3\frac{\theta''\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}}{\theta\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}} - \frac{\theta'''\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\theta'\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + 6\left\{\frac{\theta'\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}}{\theta\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}}\right\}^2 = 0, \tag{4.4}$$

and

$$3\frac{\theta''\begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \end{bmatrix}}{\theta\begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \end{bmatrix}} - \frac{\theta'''\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\theta'\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + 6\left\{\frac{\theta'\begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \end{bmatrix}}{\theta\begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \end{bmatrix}}\right\}^2 = 0. \tag{4.5}$$

Proof. Consider the following elliptic functions:

$$\varphi_1(z) = \frac{\theta^3\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}(z)}{\theta^3\begin{bmatrix} 1 \\ 1 \end{bmatrix}(z)}, \quad \varphi_2(z) = \frac{\theta^3\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}(z)}{\theta^3\begin{bmatrix} 1 \\ 1 \end{bmatrix}(z)}, \quad \varphi_3(z) = \frac{\theta^3\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}(z)}{\theta^3\begin{bmatrix} 1 \\ 1 \end{bmatrix}(z)}, \quad \varphi_4(z) = \frac{\theta^3\begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \end{bmatrix}(z)}{\theta^3\begin{bmatrix} 1 \\ 1 \end{bmatrix}(z)}.$$

In the fundamental parallelogram, the pole of $\varphi_j(z)$ $(j = 1, 2, 3, 4)$ is $z = 0$, which implies that $\mathrm{Res}(\varphi_j(z), 0) = 0$. Therefore, the proposition follows. □

Proposition 4.3. *For every $\tau \in \mathbb{H}^2$, we have*

$$\frac{\theta'''\begin{bmatrix}1\\1\end{bmatrix}}{\theta'\begin{bmatrix}1\\1\end{bmatrix}} = 4\pi i\frac{d}{d\tau}\log\theta'\begin{bmatrix}1\\1\end{bmatrix} = -\pi^2E_2(q),\ \ q = \exp(2\pi i\tau).$$

Proof. The proposition follows from Jacobi's triple product identity (2.6). □

5. Proof of Theorem 1.1

5.1. Notations. For every $\tau \in \mathbb{H}^2$, set $q = \exp(2\pi i\tau)$ and

$$X = \frac{\theta'\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}},\ \ Y = \frac{\theta'''\begin{bmatrix}1\\1\end{bmatrix}}{\theta'\begin{bmatrix}1\\1\end{bmatrix}},\ \ Z = \frac{\theta'\begin{bmatrix}1\\1\end{bmatrix}^3}{\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}^3}.$$

5.2. The ODE for $P(q)$. Proof. From equation (4.2), we have

$$\frac{\theta''\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}} = \frac{1}{3}Y - 2X^2. \tag{5.1}$$

By substituting $z = -1/3$ in equation (3.3), we obtain

$$\frac{\theta'''\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}} = -8X^3 + XY + Z. \tag{5.2}$$

Since

$$\begin{aligned}\frac{\theta'''\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}} &= 4\pi i\frac{d}{d\tau}\left\{\frac{\theta'\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}\right\} + \frac{\theta'\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}\cdot\frac{\theta''\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}\\ &= 4\pi iX' + \frac{1}{3}XY - 2X^3,\end{aligned}$$

it follows that

$$4\pi iX' = -6X^3 + \frac{2}{3}XY + Z, \tag{5.3}$$

where $' = d/d\tau$. Considering $d/d\tau = 2\pi iq\,d/dq$, we obtain the ODE for $P(q)$. □

5.3. The ODE for $Q(q)$. Proof. By substituting $z = -1/3$ in equation (3.4), we have

$$\pm 6XZ = \frac{\theta^{(4)}\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix}} - 10X^4 + 4X^2Y - 4XZ - \frac{1}{3}Y^2. \tag{5.4}$$

Comparing the coefficients of the q-series, we obtain

$$-6XZ = \frac{\theta^{(4)}\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix}} - 10X^4 + 4X^2Y - 4XZ - \frac{1}{3}Y^2, \tag{5.5}$$

which implies that

$$\frac{\theta^{(4)}\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix}} = 10X^4 - 4X^2Y - 2XZ + \frac{1}{3}Y^2. \tag{5.6}$$

Since

$$\begin{aligned}\frac{\theta^{(4)}\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix}} &= 4\pi i\frac{d}{d\tau}\left\{\frac{\theta''\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix}}\right\} + \left\{\frac{\theta''\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix}}\right\}^2 \\ &= \frac{4}{3}\pi iY' + 28X^4 - 4X^2Y - 4XZ + \frac{1}{9}Y^2,\end{aligned}$$

it follows that

$$4\pi iY' = -54X^4 + 6XZ + \frac{2}{3}Y^2, \tag{5.7}$$

where $' = d/d\tau$. Considering $d/d\tau = 2\pi iq d/dq$, we obtain the ODE for $Q(q)$. □

5.4. The ODE for $R(q)$. Proof. By substituting $z = -1/3$ in equation (3.5), we have

$$\frac{\theta^{(5)}\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix}} = 106X^5 - \frac{80}{3}X^3Y - 14X^2Z + \frac{5}{3}XY^2 + \frac{10}{3}YZ, \tag{5.8}$$

and

$$a^2(q)b^3(q) = a^2(q)\frac{(q;q)_\infty^9}{(q^3;q^3)_\infty^3} = 1 + 3\sum_{n=1}^{\infty} q^n\left(\sum_{d|n} d^4\left(\frac{d}{3}\right)\right). \tag{5.9}$$

Since

$$\frac{\theta^{(5)}\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}} = 4\pi i\frac{d}{d\tau}\left\{\frac{\theta^{(3)}\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}\right\} + \frac{\theta^{(3)}\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}\cdot\frac{\theta''\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}$$
$$= 4\pi i Z' + 106X^5 - \frac{80}{3}X^3Y - 20X^2Z + \frac{5}{3}XY^2 + \frac{4}{3}YZ,$$

it follows that

$$4\pi i Z' = 6X^2Z + 2YZ, \tag{5.10}$$

where $'=d/d\tau$. Considering $d/d\tau = 2\pi i q d/dq$, we obtain the ODE for $R(q)$. □

5.5. Note on E_4 and E_6.

Theorem 5.1. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$E_4(q) = 9a^4(q) - 8a(q)b^3(q),\ \ E_6(q) = -27a^6(q) + 36a^3(q)b^3(q) - 8b^6(q). \tag{5.11}$$

Proof. For the proof, we use

$$\wp = \wp(z; 1, \tau) = \frac{1}{3}\frac{\theta'''\begin{bmatrix}1\\1\end{bmatrix}}{\theta'\begin{bmatrix}1\\1\end{bmatrix}} - \frac{d^2}{dz^2}\log\theta\begin{bmatrix}1\\1\end{bmatrix}(z).$$

We first note that

$$\wp'' = 6\wp^2 - g_2,\ \ g_2(1,\tau) = \frac{4\pi^4}{3}E_4(q).$$

Substituting $z = -1/3$, we have

$$g_2 = 108X^4 - 12XZ, \tag{5.12}$$

which implies

$$E_4(q) = 9P^4 - 8PR. \tag{5.13}$$

We next note that

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3,\ \ g_3(1,\tau) = \frac{8\pi^6}{27}E_6(q).$$

Substituting $z = -1/3$, we obtain

$$g_3 = -216X^6 + 36X^3Z - Z^2, \tag{5.14}$$

which implies

$$E_6(q) = -27P^6 + 36P^3R - 8R^2.$$

□

Theorem 5.2. *For every $\tau \in \mathbb{H}^2$, we have*

$$J(\tau) = J(q) = \frac{12^3 g_2^3}{g_2^3 - 27g_3^2}$$
$$= \frac{27a^3(q)\left(9a^3(q) - 8b^3(q)\right)^3}{b^9(q)\left(a^3(q) - b^3(q)\right)} = \frac{27a^3(q)(a^3(q) + 8c^3(q))^3}{b^9(q)c^3(q)}, \quad q = \exp(2\pi i\tau).$$

Proof. By equations (5.12) and (5.14), we have

$$J(\tau) = \frac{110592\left(9X^4 - XZ\right)^3}{Z^3\left(8X^3 - Z\right)},$$

which proves the theorem. The second formula follows from Ramanujan's cubic identity (1.8). □

5.6. Proof of Ramanujan's system of ODEs (1.1).

Theorem 5.3. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$q\frac{dE_2}{dq} = \frac{(E_2)^2 - E_4}{12}, \quad q\frac{dE_4}{dq} = \frac{E_2E_4 - E_6}{3}, \quad q\frac{dE_6}{dq} = \frac{E_2E_6 - (E_4)^2}{2}.$$

Proof. By Theorems 1.1 and 5.1, we first note that

$$q\frac{d}{dq}E_2 = q\frac{d}{dq}Q = \frac{Q^2 - (9P^2 - 8PR)}{12} = \frac{E_2^2 - E_4}{12}.$$

We next see that

$$q\frac{d}{dq}E_4 = q\frac{d}{dq}(9P^4 - 8PR) = 9P^6 + 3P^4Q - 12P^3R + \frac{8}{3}R^2 - \frac{8}{3}PQR$$
$$= \frac{E_2E_4 - E_6}{3}.$$

Finally, we obtain

$$q\frac{d}{dq}E_6 = q\frac{d}{dq}(-27P^6 + 36P^3R - 8R^2)$$
$$= -\frac{81}{2}P^8 - \frac{27}{2}P^6Q + 72P^5R + 18P^3QR - 32P^2R^2 - 4QR^2$$
$$= \frac{E_2E_6 - E_4^2}{2}.$$

□

6. Proof of Theorem 1.2

6.1. Notations. For every $\tau \in \mathbb{H}^2$, set $y = \exp(2\pi i\tau/3)$, $q = y^3$ and

$$X = \frac{\theta'\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}},\ Y = \frac{\theta'''\begin{bmatrix}1\\1\end{bmatrix}}{\theta'\begin{bmatrix}1\\1\end{bmatrix}},\ Z = -\frac{\theta'\begin{bmatrix}1\\1\end{bmatrix}^3}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}^3}.$$

6.2. The ODE for $P(q)$. Proof. From equation (4.3), we have

$$\frac{\theta''\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}} = \frac{1}{3}Y - 2X^2. \tag{6.1}$$

By substituting $z = -\tau/3$ in equation (3.3), we obtain

$$\frac{\theta'''\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}} = -8X^3 + XY + Z. \tag{6.2}$$

Since

$$\begin{aligned}\frac{\theta'''\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}} &=4\pi i\frac{d}{d\tau}\left\{\frac{\theta'\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}\right\} + \frac{\theta'\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}\cdot\frac{\theta''\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}\\ &=4\pi iX' + \frac{1}{3}XY - 2X^3,\end{aligned}$$

it follows that

$$4\pi iX' = -6X^3 + \frac{2}{3}XY + Z, \tag{6.3}$$

where $' = d/d\tau$. Changing $\tau \to 3\tau$, we obtain the ODE for $P(q)$. □

6.3. The ODE for $Q(q)$. Proof. By substituting $z = -\tau/3$ in equation (3.4), we have

$$\pm 6XZ = \frac{\theta^{(4)}\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}} - 10X^4 + 4X^2Y - 4XZ - \frac{1}{3}Y^2. \tag{6.4}$$

Comparing the coefficients of the y-series, we obtain

$$-6XZ = \frac{\theta^{(4)}\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}} - 10X^4 + 4X^2Y - 4XZ - \frac{1}{3}Y^2, \tag{6.5}$$

which implies that

$$\frac{\theta^{(4)}\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}} = 10X^4 - 4X^2Y - 2XZ + \frac{1}{3}Y^2. \tag{6.6}$$

Since

$$\begin{aligned}\frac{\theta^{(4)}\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}} &= 4\pi i\frac{d}{d\tau}\left\{\frac{\theta''\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}\right\} + \left\{\frac{\theta''\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}\right\}^2\\ &= \frac{4}{3}\pi iY' + 28X^4 - 4X^2Y - 4XZ + \frac{1}{9}Y^2,\end{aligned}$$

it follows that

$$4\pi iY' = -54X^4 + 6XZ + \frac{2}{3}Y^2, \tag{6.7}$$

where $' = d/d\tau$. Changing $\tau \to 3\tau$, we obtain the ODE for $Q(q)$. □

6.4. The ODE for $R(q)$. Proof. By substituting $z = -\tau/3$ in equation (3.5), we have

$$\frac{\theta^{(5)}\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}} = 106X^5 - \frac{80}{3}X^3Y - 14X^2Z + \frac{5}{3}XY^2 + \frac{10}{3}YZ, \tag{6.8}$$

and

$$a^2(q)c^3(q) = a^2(q)\cdot 27q\frac{(q^3;q^3)_\infty^9}{(q;q)_\infty^3} = 27\sum_{n=1}^{\infty} q^n\left(\sum_{d|n} d^4\left(\frac{n/d}{3}\right)\right). \tag{6.9}$$

Since

$$\begin{aligned}\frac{\theta^{(5)}\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}} &= 4\pi i\frac{d}{d\tau}\left\{\frac{\theta^{(3)}\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}\right\} + \frac{\theta^{(3)}\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}\cdot\frac{\theta''\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}\\ &= 4\pi iZ' + 106X^5 - \frac{80}{3}X^3Y - 20X^2Z + \frac{5}{3}XY^2 + \frac{4}{3}YZ,\end{aligned}$$

it follows that

$$4\pi iZ' = 6X^2Z + 2YZ, \tag{6.10}$$

where $' = d/d\tau$. Changing $\tau \to 3\tau$, we obtain the ODE for $R(q)$. □

6.5. Note on E_4 and E_6.

Theorem 6.1. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$E_4(q^3) = a^4(q) - \frac{8}{9}a(q)c^3(q),\ E_6(q^3) = a^6(q) - \frac{4}{3}a^3(q)c^3(q) + \frac{8}{27}c^6(q). \tag{6.11}$$

Proof. For the proof, we use

$$\wp = \wp(z;1,\tau) = \frac{1}{3}\frac{\theta'''\begin{bmatrix}1\\1\end{bmatrix}}{\theta'\begin{bmatrix}1\\1\end{bmatrix}} - \frac{d^2}{dz^2}\log\theta\begin{bmatrix}1\\1\end{bmatrix}(z).$$

We first note that

$$\wp'' = 6\wp^2 - g_2,\ g_2(1,\tau) = \frac{4\pi^4}{3}E_4(q).$$

Substituting $z = -\tau/3$, we have

$$g_2 = 108X^4 - 12XZ, \tag{6.12}$$

which implies

$$E_4(q) = 9P(y)^4 - 8P(y)R(y). \tag{6.13}$$

We next note that

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3,\ g_3(1,\tau) = \frac{8\pi^6}{27}E_6(q).$$

Substituting $z = -\tau/3$, we obtain

$$g_3 = -216X^6 + 36X^3Z - Z^2, \tag{6.14}$$

which implies

$$E_6(q) = P(y)^6 - 36P(y)^3R(y) + 216R(y)^2.$$

Changing $\tau \to 3\tau$, we obtain the theorem. □

Theorem 6.2. *For every $\tau \in \mathbb{H}^2$, we have*

$$J(3\tau) = J(q^3) = \frac{27a^3(q)\left(9a^3(q) - 8c^3(q)\right)^3}{c^9(q)\left(a^3(q) - c^3(q)\right)} = \frac{27a^3(q)(a^3(q) + 8b^3(q))^3}{b^3(q)c^9(q)},\ q = \exp(2\pi i\tau).$$

Proof. By equations (6.12) and (6.14), we have

$$J(\tau) = \frac{12^3 g_2^3}{g_2^3 - 27g_3^2} = \frac{110592\left(9X^4 - XZ\right)^3}{Z^3\left(8X^3 - Z\right)},$$

which proves the theorem. The second formula follows from Ramanujan's cubic identity (1.8). □

Remark. The formulas of Theorems 5.1 and 6.1 were proved in Cooper [6, pp. 272].

7. Applications to number theory

Theorem 7.1 (Farkas and Kra [9, pp. 318])**.** *For every $\tau \in \mathbb{H}^2$, we have*

$$\frac{d}{d\tau}\log\frac{\eta(3\tau)}{\eta(\tau)}+\frac{1}{2\pi i}\left\{\frac{\theta'\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix}(0,\tau)}\right\}^2=0.$$

Proof. The heat equation (2.7) and equation (4.2) implies that

$$4\pi i\frac{d}{d\tau}\log\frac{\theta^3\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix}}{\theta'\begin{bmatrix}1\\ 1\end{bmatrix}}+6\left\{\frac{\theta'\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix}}\right\}^2=0.$$

The theorem follows from Jacobi's triple product identity (2.6). □

Theorem 7.2. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$a^2(q)=1+12\sum_{n=1}^{\infty}(\sigma_1(n)-3\sigma_1(n/3))q^n=\frac{1}{2}\left\{-E_2(q)+3E_2(q^3)\right\}.$$

Proof. The theorem follows from Proposition 4.1 and Theorem 7.1. □

Theorem 7.3. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$a^3(q)=1-9\sum_{n=1}^{\infty}q^n\left(\sum_{d|n}d^2\left(\frac{d}{3}\right)\right)+27\sum_{n=1}^{\infty}q^n\left(\sum_{d|n}d^2\left(\frac{n/d}{3}\right)\right).$$

Proof. From Theorems 1.1 and 1.2, we have

$$\left(12q\frac{d}{dq}-E_2(q)\right)a(q)=3a^3(q)-4\frac{(q;q)_\infty^9}{(q^3;q^3)_\infty^3},\tag{7.1}$$

and

$$\left(4q\frac{d}{dq}-E_2(q^3)\right)a(q)=-a^3(q)+36q\frac{(q^3;q^3)_\infty^9}{(q;q)_\infty^3}.\tag{7.2}$$

Eliminating $q(d/dq)a(q)$, we obtain

$$2(1+12\sum_{n=1}^{\infty}(\sigma_1(n)-3\sigma_1(n/3))q^n)a(q)=6a^3(q)-4\frac{(q;q)_\infty^9}{(q^3;q^3)_\infty^3}-108q\frac{(q^3;q^3)_\infty^9}{(q;q)_\infty^3}.$$

Therefore, it follows from Theorem 7.2 that

$$a^3(q)=\frac{(q;q)_\infty^9}{(q^3;q^3)_\infty^3}+27q\frac{(q^3;q^3)_\infty^9}{(q;q)_\infty^3}=b^3(q)+c^3(q),\tag{7.3}$$

which proves the theorem. □

Theorem 7.4. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$a^4(q) = 1 + 24\sum_{n=1}^{\infty}(\sigma_3(n) + 9\sigma_3(n/3))q^n.$$

Proof. By Theorems 5.1 and 6.1, we have

$$E_4(q) = 9a^4(q) - 8a(q)\frac{(q;q)_\infty^9}{(q^3;q^3)_\infty^3},\ \ E_4(q^3) = a^4(q) - 24a(q)\cdot q\frac{(q^3;q^3)_\infty^9}{(q;q)_\infty^3}.$$

Considering equation (7.3), we obtain

$$10a^4(q) = E_4(q) + 9E_4(q^3),$$

which proves the theorem. □

Theorem 7.5. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$a^5(q) = 1 + 3\sum_{n=1}^{\infty}q^n\left(\sum_{d|n}d^4\left(\frac{d}{3}\right)\right) + 27\sum_{n=1}^{\infty}q^n\left(\sum_{d|n}d^4\left(\frac{n/d}{3}\right)\right).$$

Proof. The theorem follows from equations (5.9), (6.9) and (7.3). □

Theorem 7.6. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$a^6(q) = 1 + \frac{252}{13}\sum_{n=1}^{\infty}(\sigma_5(n) - 27\sigma_5(n/3))q^n + \frac{216}{13}q(q;q)_\infty^6(q^3;q^3)_\infty^6.$$

Proof. By Theorems 5.1, 6.1 and equation (7.3), we have

$$E_6(q) - 27E_6(q^3) = -18a^6(q) - 8b^6(q) - 8c^6(q),$$

which implies

$$9a^6(q) + 4b^6(q) + 4c^6(q) = 13 + 252\sum_{n=1}^{\infty}(\sigma_5(n) - 27\sigma_5(n/3))q^n.$$

By equation (7.3), we obtain

$$13a^6(q) = 13 + 252\sum_{n=1}^{\infty}(\sigma_5(n) - 27\sigma_5(n/3))q^n + 8b^3(q)c^3(q),$$

which proves the theorem. □

Remark. Lomadze [14] proved Theorems 7.2, 7.3, 7.4, and 7.5 by means of specific Eisenstein series. He also treated

$$F_k = x_1^2 + x_1x_2 + x_2^2 + \cdots + x_{2k-1}^2 + x_{2k-1}x_{2k} + x_{2k}^2,\ (k = 1, 2, \ldots, 17).$$

Based on Ramanujan's theory of theta functions, Cooper [6, pp. 569] treats the case where $k = 1, 2, \ldots 8, 10, 12$.

For the elementary proof of Theorems 7.2 and 7.4, readers are referred to the book by Williams [23, pp. 224-227].

8. A selected example of Ramanujan's identity

8.1. Farkas and Kra's cubic identity.

Theorem 8.1 (Farkas and Kra [9, pp. 193])**.** *For every $\tau \in \mathbb{H}^2$, we have*

$$\theta^3\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} + \theta^3\begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \end{bmatrix} = \theta^3\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}, \tag{8.1}$$

and

$$\exp\left(\frac{\pi i}{3}\right)\theta^3\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} + \exp\left(\frac{2\pi i}{3}\right)\theta^3\begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \end{bmatrix} = \theta^3\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}. \tag{8.2}$$

Proof. Consider the following elliptic functions:

$$\varphi(z) = \frac{\theta^3\begin{bmatrix} 1 \\ 1 \end{bmatrix}(z)}{\theta\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}(z)\theta\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}(z)\theta\begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \end{bmatrix}(z)}, \quad \psi(z) = \frac{\theta^3\begin{bmatrix} 1 \\ 1 \end{bmatrix}(z)}{\theta\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}(z)\theta\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}(z)\theta\begin{bmatrix} \frac{5}{3} \\ \frac{1}{3} \end{bmatrix}(z)}.$$

We use $\varphi(z)$ to prove equation (8.1). Equation (8.2) can be obtained by using $\psi(z)$ in the same way.

Note that in the fundamental parallelogram, the poles of $\varphi(z)$ are $(\tau + 1)/3$, $\tau/3$, and $(\tau - 1)/3$. Direct computations yield

$$\operatorname{Res}\left(\varphi(z), \frac{\tau+1}{3}\right) = -\frac{\theta^3\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}}{\theta'\begin{bmatrix} 1 \\ 1 \end{bmatrix}\theta^2\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}}, \quad \operatorname{Res}\left(\varphi(z), \frac{\tau}{3}\right) = \frac{\theta^3\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}}{\theta'\begin{bmatrix} 1 \\ 1 \end{bmatrix}\theta^2\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}},$$

and

$$\operatorname{Res}\left(\varphi(z), \frac{\tau-1}{3}\right) = -\frac{\theta^3\begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \end{bmatrix}}{\theta'\begin{bmatrix} 1 \\ 1 \end{bmatrix}\theta^2\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}}.$$

From the residue theorem, it follows that

$$\operatorname{Res}\left(\varphi(z), \frac{\tau+1}{3}\right) + \operatorname{Res}\left(\varphi(z), \frac{\tau}{3}\right) + \operatorname{Res}\left(\varphi(z), \frac{\tau-1}{3}\right) = 0,$$

which implies equation (8.1). □

8.2. Ramanujan's identity.

Theorem 8.2 (Ramanujan [2, pp. 346])**.** *For every $\tau \in \mathbb{H}^2$, we have*

$$a(q) = \frac{\eta^3(\tau/3) + 3\eta^3(3\tau)}{\eta(\tau)}.$$

Proof. From the results obtained by Farkas [8], we recall the following identity:

$$\frac{6\theta'\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}(0,\tau)}{\zeta_6\theta^3\begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\end{bmatrix}(0,\tau)+\theta^3\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}(0,\tau)+\zeta_6^5\theta^3\begin{bmatrix}\frac{1}{3}\\\frac{5}{3}\end{bmatrix}(0,\tau)} \tag{8.3}$$

$$=\frac{2\pi i q^{\frac{1}{12}}}{\prod_{n=0}^{\infty}(1-q^{3n+1})(1-q^{3n+2})}=2\pi i\frac{e^{\frac{\pi i}{6}}}{\sqrt{3}}\frac{\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}(0,\tau)}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}(0,3\tau)},$$

where $q=\exp(2\pi i\tau)$ and $\zeta_6=\exp(2\pi i/6)$.

Theorem 8.1 yields that

$$\theta^3\begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\end{bmatrix}=-\exp\left(\frac{2\pi i}{3}\right)\theta^3\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}+\theta^3\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix},$$

and

$$\theta^3\begin{bmatrix}\frac{1}{3}\\\frac{5}{3}\end{bmatrix}=\exp\left(\frac{\pi i}{3}\right)\theta^3\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}-\theta^3\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix},$$

which imply

$$\frac{\theta'\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}=\frac{\pi\exp(\frac{2\pi i}{3})}{3\sqrt{3}}\times\frac{3\theta^3\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}+\sqrt{3}i\theta^3\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}(0,3\tau)}.$$

Therefore, the theorem follows from Proposition 4.1 and Jacobi's triple product identity (2.6). □

9. Additional product-series identities

9.1. Selected theta functional formulas.

Proposition 9.1. *For every $(z,\tau)\in\mathbb{C}\times\mathbb{H}^2$, we have*

$$\theta^2\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}(z)\theta\begin{bmatrix}1\\\frac{5}{3}\end{bmatrix}(z)+\exp\left(\frac{\pi i}{3}\right)\theta^2\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}(z)\theta\begin{bmatrix}\frac{5}{3}\\1\end{bmatrix}(z) \\ -\theta\begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\end{bmatrix}\theta\begin{bmatrix}\frac{1}{3}\\\frac{5}{3}\end{bmatrix}\theta^2\begin{bmatrix}1\\1\end{bmatrix}(z)=0, \tag{9.1}$$

$$\theta^2\begin{bmatrix}\frac{1}{3}\\\frac{5}{3}\end{bmatrix}\theta\begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\end{bmatrix}(z)\theta\begin{bmatrix}\frac{5}{3}\\\frac{5}{3}\end{bmatrix}(z)+\theta^2\begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\end{bmatrix}\theta\begin{bmatrix}\frac{1}{3}\\\frac{5}{3}\end{bmatrix}(z)\theta\begin{bmatrix}\frac{5}{3}\\\frac{1}{3}\end{bmatrix}(z) \\ -\exp\left(\frac{2\pi i}{3}\right)\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}\theta^2\begin{bmatrix}1\\1\end{bmatrix}(z)=0. \tag{9.2}$$

Proof. We prove equation (9.1). Equation (9.2) can be proved in the same way. We first note that $\dim \mathcal{F}_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 2$, and

$$\theta \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}(z,\tau)\theta \begin{bmatrix} 1 \\ \frac{5}{3} \end{bmatrix}(z,\tau),\ \theta \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}(z,\tau)\theta \begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix}(z,\tau),\ \theta^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}(z,\tau) \in \mathcal{F}_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, there exist some complex numbers, x_1, x_2, and x_3, not all of which are zero, such that

$$x_1\theta \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}(z,\tau)\theta \begin{bmatrix} 1 \\ \frac{5}{3} \end{bmatrix}(z,\tau) + x_2\theta \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}(z,\tau)\theta \begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix}(z,\tau) + x_3\theta^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}(z,\tau) = 0.$$

Note that in the fundamental parallelogram, the zero of $\theta \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}(z)$, $\theta \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}(z)$, or $\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix}(z)$ is $z = 1/3$, $\tau/3$ or 0. Substituting $z = 1/3, \tau/3$, and 0, we have

$$\begin{aligned}
& & x_2 \exp\left(-\frac{\pi i}{3}\right)\theta \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}\theta \begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \end{bmatrix} + x_3\theta^2 \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} &= 0,\\
& x_1\theta \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}\theta \begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \end{bmatrix} & + x_3\theta^2 \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} &= 0,\\
& -x_1\theta^2 \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} & + x_2 \exp\left(-\frac{\pi i}{3}\right)\theta^2 \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} &= 0.
\end{aligned}$$

Solving this system of equations, we have

$$(x_1, x_2, x_3) = \alpha \left(\theta^2 \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}, \exp\left(\frac{\pi i}{3}\right)\theta^2 \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}, -\theta \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}\theta \begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \end{bmatrix}\right) \quad \text{for some } \alpha \in \mathbb{C} \setminus \{0\},$$

which proves the proposition. □

9.2. Product-series identities.

Theorem 9.2. *For every $\tau \in \mathbb{H}^2$, we have*

$$\frac{\eta^{10}(3\tau)}{\eta^3(\tau)\eta^3(9\tau)} = 1 + 3\sum_{n=1}^{\infty}(\sigma_1(n) - 9\sigma_1(n/9))q^n, \tag{9.3}$$

and

$$\frac{\eta^3(\tau)\eta^3(9\tau)}{\eta^2(3\tau)} = \sum_{n=0}^{\infty}\sigma_1(3n+1)q^{3n+1} - \sum_{n=0}^{\infty}\sigma_1(3n+2)q^{3n+2}. \tag{9.4}$$

where $q = \exp(2\pi i\tau)$.

Proof. By equations (4.2), (4.3), and (9.1), we derive equation (9.3). Equation (9.4) can be proved in the same way.

Comparing the coefficients of the term z^2 in equation (9.1), we have

$$\left\{\theta'\begin{bmatrix}1\\1\end{bmatrix}\right\}^2 \frac{\theta\begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\end{bmatrix}\theta\begin{bmatrix}\frac{1}{3}\\\frac{5}{3}\end{bmatrix}}{\theta^2\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}\theta^2\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}} = \frac{3}{2}\left(\frac{\theta''\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}} - \frac{\theta''\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}\right)$$

$$= 6\pi i\frac{d}{d\tau}\log\frac{\theta\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}}{\theta\begin{bmatrix}1\\\frac{1}{3}\end{bmatrix}}.$$

Therefore, equation (9.3) can be obtained by Jacobi's triple product identity (2.6). □

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