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THE CHENG-YAU METRICS ON REGULAR CONVEX CONES AS HARMONIC IMMERSIONS INTO THE SYMMETRIC SPACE OF POSITIVE DEFINITE REAL SYMMETRIC MATRICES

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Abstract

A Riemannian metric g on a domain Ω in $\mathbb{R}^n$ defines a map F_g from (Ω, g) into the symmetric space of positive definite real symmetric $n\times n$ matrices $(\mathrm{Sym}^+(n), h)$, where h is the Cheng-Yau metric on $\mathrm{Sym}^+(n)$. We show that the map F_g is a harmonic immersion if Ω is a regular convex cone and g is the Cheng-Yau metric on Ω. We also prove that the map F_g is totally geodesic if Ω is a homogeneous self-dual regular convex cone and g is the Cheng-Yau metric on Ω.

Introduction

Let g be a Riemannian metric on a domain Ω in $\mathbb{R}^n$ with the standard coordinate $x = (x^1, \dots x^n)$ and $\mathrm{Sym}^+(n)$ the symmetric space of positive definite real symmetric $n\times n$ matrices. Then g defines a map $F_g : \Omega \to \mathrm{Sym}^+(n)$ which is given by $F_g(x) = [g_{ij}(x)]_{1\leq i,j\leq n}$. The goal of this paper is to show the following Main Theorem in Section 4.

Main Theorem. *Let g be the Cheng-Yau metric on a regular convex cone Ω in $\mathbb{R}^n$ and h the Cheng-Yau metric on $Sym^+(n)$. Then $F_g : (\Omega, g) \to (Sym^+(n), h)$ is a harmonic immersion. In particular, if Ω is a homogeneous self-dual regular convex cone, then F_g is totally geodesic.*

A *regular convex domain* is a convex domain in $\mathbb{R}^n$ which does not contain a full straight line. If a regular convex domain Ω satisfies that tx belongs to Ω for all x in Ω and all positive real number t, then Ω is said to be a *regular convex cone*. A regular convex cone Ω in $\mathbb{R}^n$ is called *homogeneous* if a subgroup of $\mathrm{GL}(\mathbb{R}^n)$ acts on Ω transitively. In addition, a regular convex cone Ω in $\mathbb{R}^n$ is said to be *self-dual* if there exists a inner product $\langle\cdot,\cdot\rangle$ of $\mathbb{R}^n$ such that the *dual cone* $\Omega^* = \{y \in \mathbb{R}^n \mid \langle x, y\rangle > 0 \text{ for all } x \in \bar{\Omega} \setminus \{0\}\}$ coincides with Ω. It is known that a homogeneous self-dual regular convex cone is a Riemannian symmetric space with respect to the Cheng-Yau metric (cf. Theorem 4.6 in [2]). A symmetric space $\mathrm{Sym}^+(n)$ is an example of homogeneous self-dual regular convex cones in the space of real symmetric $n\times n$ matrices $\mathrm{Sym}(n)$ which can be identified with $\mathbb{R}^{\frac{n(n+1)}{2}}$.

We denote by D the standard flat affine connection on $\mathbb{R}^n$. A Riemannian metric g on a domain Ω in $\mathbb{R}^n$ is said to be a *Hessian metric* if there exists a convex function $\varphi \in C^\infty(\Omega)$ such that

$$g = Dd\varphi,$$

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that is,

$$g_{ij} := g\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big) = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}.$$

There exists a special Hessian metric on a regular convex domain from the following.

Theorem 2.1 ([1])**.** *Let Ω be a regular convex domain in $\mathbb{R}^n$. Then there exists a unique convex solution $\varphi \in C^\infty(\Omega)$ of the* Monge-Ampére equation

$$\begin{cases} \det\Big[\dfrac{\partial^2\varphi}{\partial x^i \partial x^j}\Big]_{1\le i,j\le n} = e^{2\varphi} \\ \varphi(x) \to \infty \qquad\qquad (x \to \partial\Omega) \end{cases}$$

such that the Hessian metric $g = Dd\varphi$ is complete, which is called the Cheng-Yau metric.

We denote by ∇^g the Levi-Civita connection for a Riemannian metric g. The map $F_g : (\Omega, g) \to (\mathrm{Sym}^+(n), h)$ is said to be *harmonic* if the *tension field* $\mathrm{tr}_g(\nabla^{g,h} dF_g)$ equals to 0. In particular, F_g is called *totally geodesic* if $\nabla^{g,h} dF_g = 0$. To prove Main Theorem, the following theorem plays an important role.

Theorem 4.2. *If g is a Hessian metric, then we have*

$$(\nabla^{g,h} dF_g)_{ij}\Big(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\Big) = 2g\Big((\nabla^g_{\frac{\partial}{\partial x^i}} \gamma_g)\Big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\Big), \frac{\partial}{\partial x^l}\Big)$$

$$tr_g(\nabla^{g,h} dF_g)_{ij} = 2((\beta_g)_{ij} - (\alpha_g)_r (\gamma_g)^r{}_{ij}) = 2(\nabla^g \alpha_g)\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big).$$

In particular, $(\nabla^{g,h} dF_g)_{ij}\Big(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\Big)$ is symmetric with respect to i, j, k, l.

Here, $\gamma_g = \nabla^g - D$, $\alpha_g = \frac{1}{2} d \log\det F_g$ and $\beta_g = D\alpha_g$. The Cheng-Yau metric g on a regular convex cone Ω satisfies $\nabla^g \alpha_g = 0$ (cf. Proposition 2.6). Further, g satisfies $\nabla_g \gamma_g = 0$ if Ω is a homogebeous self-dual regular convex cone (cf. Proposition 2.7). Hence we can show Main Theorem.

In Section 1, we give a brief review of Hessian geometry. In Section 2, we discuss the Cheng-Yau metrics on regular convex cones. In Section 3, we give a brief summary of the Cheng-Yau metric on $\mathrm{Sym}^+(n)$. In Section 4, we consider the map $F_g : (\Omega, g) \to (\mathrm{Sym}^+(n), h)$ and prove Main Theorem. In Section 5, we conclude that the conditions in Main Theorem are crucial by giving examples.

1. Difference tensors and Koszul forms for Hessian metrics

In this section, we give a brief review of the properties of Difference tensors and Koszul forms for Hessian metrics. Note that we use Einstein's summation convention throughout this paper.

Definition 1.1. For a Riemannian metric g on a domain Ω in $\mathbb{R}^n$, we define the *difference tensor* γ_g by

$$\gamma_g(X, Y) = \nabla^g_X Y - D_X Y.$$

REMARK 1.2. The components ${(\gamma_g)^i}_{jk}$ of γ_g with respect to the standard coordinate coincide with the Christoffel symbols of ∇^g, that is,

$$ {(\gamma_g)^i}_{jk} = \frac{1}{2} g^{il} \Big(\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \Big). $$

Proposition 1.3. *Let g be a Hessian metric. Then we have*

$$ g(X, \gamma_g(Y,Z)) = \frac{1}{2}(D_X g)(Y,Z) = \frac{1}{2}(D_Y g)(X,Z) = g(Y, \gamma_g(X,Z)), $$

that is,

$$ (\gamma_g)_{ijk} = \frac{1}{2}\frac{\partial g_{jk}}{\partial x^i} = \frac{1}{2}\frac{\partial g_{ik}}{\partial x^j} = (\gamma_g)_{jik}, $$

$$ {(\gamma_g)^i}_{jk} = \frac{1}{2} g^{il} \frac{\partial g_{jk}}{\partial x^l} = \frac{1}{2} g^{il} \frac{\partial g_{lk}}{\partial x^j}, $$

where $(\gamma_g)_{ijk} = g_{il}{(\gamma_g)^l}_{jk}$.

Proof. By the definition of Hessian metrics we have

$$ \frac{\partial g_{lk}}{\partial x^j} = \frac{\partial g_{lj}}{\partial x^k} = \frac{\partial g_{jk}}{\partial x^l}. $$

Hence it follows from Remark 1.2 that

$$ {(\gamma_g)^i}_{jk} = \frac{1}{2} g^{il} \frac{\partial g_{jk}}{\partial x^l} = \frac{1}{2} g^{il} \frac{\partial g_{lk}}{\partial x^j}. $$

□

DEFINITION 1.4. For a Riemannian metric g on a domain Ω in $\mathbb{R}^n$, we define a closed 1-form α_g and a symmetric 2-form β_g by

$$ \alpha_g = \frac{1}{2} d \log \det[g_{ij}]_{1 \le i,j \le n}, \quad \beta_g = D\alpha_g. $$

The forms α_g and β_g are called *the first Koszul form* and *the second Koszul form*, respectively. If there exists a real number λ such that $\beta_g = g$, then g is called *Hesse-Einstein*.

REMARK 1.5. Let $(\alpha_g)_j = \alpha_g\Big(\frac{\partial}{\partial x^j}\Big)$ and $(\beta_g)_{ij} = \beta_g\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big)$. By the definition of Koszul forms we have

$$ (\alpha_g)_j = \frac{1}{2} g^{kl} \frac{\partial g_{kl}}{\partial x^j} = {(\gamma_g)^i}_{ij}, $$

$$ (\beta_g)_{ij} = \frac{1}{2} g^{kl} \Big(\frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} - \frac{\partial g_{kr}}{\partial x^i} g^{rs} \frac{\partial g_{sl}}{\partial x^j} \Big). $$

Proposition 1.6. *Let g be a Hessian metric and $(\alpha_g)^k = g^{kl}(\alpha_g)_l$. Then we have*

$$ (\alpha_g)^k = g^{ij}{(\gamma_g)^k}_{ij}. $$

Proof. From Remark 1.5 and Proposition 1.3, we obtain

$$\begin{aligned}(\alpha_g)^k &= g^{kl}(\alpha_g)_l \\ &= g^{kl}(\gamma_g)^j{}_{jl} \\ &= g^{kl}g^{ij}(\gamma_g)_{ijl} \\ &= g^{ij}g^{kl}(\gamma_g)_{lij} \\ &= g^{ij}(\gamma_g)^k{}_{ij}.\end{aligned}$$

□

Proposition 1.7 (c.f. Proposition 3.5 in [2]). *Let g be a Hessian metric on a domain Ω in $\mathbb{R}^n$ and g^T a Kähler metic on $T\Omega \cong \Omega \oplus \sqrt{-1}\mathbb{R}^n$ defined by*

$$g^T_{i\bar{j}}(x + \sqrt{-1}y) = g_{ij}(x).$$

Then the Ricci tensor R^T for g^T is expressed by

$$R^T_{i\bar{j}}(x + \sqrt{-1}y) = -\frac{1}{2}(\beta_g)_{ij}(x).$$

In particular, g^T is Kähler-Einstein if and only if g is Hesse-Einstein.

2. The Cheng-Yau metrics on regular convex cones

In this section, we discuss the Cheng-Yau metrics which are examples of Hesse-Einstein metrics.

Theorem 2.1 ([1]). *Let Ω be a regular convex domain in $\mathbb{R}^n$. Then there exists a unique convex solution $\varphi \in C^\infty(\Omega)$ of the* Monge-Ampére equation

$$\begin{cases} \det\Big[\dfrac{\partial^2\varphi}{\partial x^i \partial x^j}\Big]_{1\le i,j\le n} = e^{2\varphi} \\ \varphi(x) \to \infty \qquad\qquad (x \to \partial\Omega) \end{cases}$$

such that the Hessian metric $g = Dd\varphi$ is complete, which is called the Cheng-Yau metric.

REMARK 2.2. The first and second Koszul forms for the Cheng-Yau metric $g = Dd\varphi$ are

$$\alpha_g = d\varphi, \quad \beta_g = g.$$

Proposition 2.3. *Let Ω be a regular convex domain in $\mathbb{R}^n$ and $g = Dd\varphi$ the Cheng-Yau metric on Ω. Assume that $a = [a^i{}_j]_{1\le i,j\le n} \in GL(n, \mathbb{R})$ satisfies $ax \in \Omega$ for all $x \in \Omega$. Then*

(1) $\varphi(ax) = \varphi(x) - \log|\det a|$.
(2) $a^* g = g$.

Proof. We define $\tilde{\varphi} \in C^\infty(\Omega)$ by

$$\tilde{\varphi}(x) = \varphi(ax) + \log|\det a|.$$

Then we obtain

$$\frac{\partial\tilde{\varphi}}{\partial x^j}(x) = a^l{}_j\frac{\partial\varphi}{\partial x^l}(ax),$$
$$\frac{\partial^2\tilde{\varphi}}{\partial x^i\partial x^j}(x) = a^k{}_i a^l{}_j\frac{\partial^2\varphi}{\partial x^k\partial x^l}(ax).$$

Hence we have

$$\begin{aligned}\det\Big[\frac{\partial^2\tilde{\varphi}}{\partial x^i\partial x^j}(x)\Big] &= (\det a)^2\det\Big[\frac{\partial^2\varphi}{\partial x^k\partial x^l}(ax)\Big]\\ &= (\det a)^2 e^{2\varphi(ax)}\\ &= e^{2(\varphi(ax)+\log|\det a|)}\\ &= e^{2\tilde{\varphi}(x)}.\end{aligned}$$

Therefore $\tilde{\varphi}$ is also the unique convex solution of the Monge-Ampére equation, that is,

$$\tilde{\varphi} = \varphi.$$

This implies the first assertion. The second assertion follows from

$$\frac{\partial^2\varphi}{\partial x^i\partial x^j}(x) = a^k{}_i a^l{}_j\frac{\partial^2\varphi}{\partial x^k\partial x^l}(ax).$$

□

Corollary 2.4. *Let Ω be a homogeneous regular convex domain in $\mathbb{R}^n$ and x_0 a fixed point in Ω. Assume that $a \in C^\infty(\Omega, GL(n,\mathbb{R}))$ satisfies $x = a(x)x_0$ for all $x \in \Omega$. Then the Cheng-Yau metric $g = Dd\varphi$ on Ω is expressed by*

$$g = -Dd\log|\det a(x)|.$$

Proof. It follows from Proposition 2.3 that

$$\varphi(x) = \varphi(a(x)x_0) = \varphi(x_0) - \log|\det a(x)|.$$

Hence we have

$$g = Dd\varphi = -Dd\log|\det a(x)|.$$

□

Proposition 2.5. *Let Ω be a regular convex cone in $\mathbb{R}^n$ and $g = Dd\varphi$ the Cheng-Yau metric on Ω. Then*

(1) $x^j\dfrac{\partial\varphi}{\partial x^j} = -n.$

(2) $grad\,\varphi := g^{ij}\dfrac{\partial\varphi}{\partial x^i}\dfrac{\partial}{\partial x^j} = -x^j\dfrac{\partial}{\partial x^j}.$

(3) $x^j(\gamma_g)_{ijk} = \dfrac{1}{2}x^j\dfrac{\partial g_{ik}}{\partial x^j} = -g_{ik}.$

Proof. Since Ω is a cone, $tx \in \Omega$ for all $x \in \Omega$ and all $t > 0$. Hence it follows from Proposition 2.3 that

$$\varphi(tx) = \varphi(x) - n\log t \quad \text{for } x \in \Omega \text{ and } t > 0.$$

Therefore

$$x^j \frac{\partial \varphi}{\partial x^j} = \frac{d}{dt}\Big|_{t=1} \varphi(tx) = -n.$$

Taking the derivative of both sides with respect to x^i, we have

$$\frac{\partial \varphi}{\partial x^i} + x^j g_{ij} = 0.$$

This implies (2). Further, taking the derivative of both sides respect to x^k, we obtain

$$g_{ki} + g_{ik} + x^j \frac{\partial g_{ij}}{\partial x^k} = 0,$$

that is,

$$x^j \frac{\partial g_{ik}}{\partial x^j} = -2 g_{ik}.$$

It follows from Proposition 1.3 that

$$x^j (\gamma_g)_{ijk} = \frac{1}{2} x^j \frac{\partial g_{ik}}{\partial x^j}.$$

□

Proposition 2.6. *Let $g = Dd\varphi$ be the Cheng-Yau metric on a regular convex cone Ω in $\mathbb{R}^n$. Then we have*

$$\nabla^g \alpha_g = \nabla^g d\varphi = 0.$$

Proof. It follows from Proposition 2.5 that

$$\begin{aligned}(\nabla^g d\varphi)\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big) &= \frac{\partial^2 \varphi}{\partial x^i \partial x^j} - (\gamma_g)^k{}_{ij} \frac{\partial \varphi}{\partial x^k} \\ &= g_{ij} - g^{kl} (\gamma_g)_{lij} \frac{\partial \varphi}{\partial x^k} \\ &= g_{ij} + x^l (\gamma_g)_{lij} \\ &= 0.\end{aligned}$$

□

It is known that a homogeneous self-dual regular convex cone satisfies the following stronger condition than Proposition 2.6.

Proposition 2.7 (c.f. Proposition 4.12 in [2])**.** *Let g be the Cheng-Yau metric on a homogeneous self-dual regular convex cone Ω in $\mathbb{R}^n$. Then we have*

$$\nabla^g \gamma_g = 0.$$

3. The symmetric space of positive definite real symmetric matrices

In this section, we give a brief summary of the Cheng-Yau metric h on a regular convex cone $\mathrm{Sym}^+(n)$ in $\mathrm{Sym}(n)$ which is identified with $\mathbb{R}^{\frac{n(n+1)}{2}}$. Since $GL(n, \mathbb{R})$ acts on $\mathrm{Sym}^+(n)$ transitively by $A \cdot P := {}^t APA$ for $A \in GL(n, \mathbb{R})$ and $P \in \mathrm{Sym}^+(n)$, $\mathrm{Sym}^+(n)$ is homogeneous. Further, $\mathrm{Sym}^+(n)$ is self-dual with respect to an inner product $\langle P, Q \rangle := \mathrm{tr}\,(PQ)$ (c.f.

Example 4.1 in [2]).

We denote by $P = (P_{ij})_{1\le i\le j\le n}$ the standard coordinate of $\mathrm{Sym}^+(n)$. The Cheng-Yau metric h on $\mathrm{Sym}^+(n)$ is expressed by

$$h = -\frac{n+1}{2} Dd \log\det P.$$

Lemma 3.1. *For all $X_1, X_2, X_3 \in Sym(n)$ and $\sigma \in S_3$, we have*

$$tr\,(X_{\sigma(1)}X_{\sigma(2)}X_{\sigma(3)}) = tr\,(X_1X_2X_3).$$

Proof. We have

$$\begin{aligned}\mathrm{tr}\,(X_1X_2X_3) &= \mathrm{tr}\,{}^t(X_1X_2X_3)\\ &= \mathrm{tr}\,({}^tX_3{}^tX_2{}^tX_1)\\ &= \mathrm{tr}\,(X_3X_2X_1).\end{aligned}$$

The other equations follow from the commutativity of the trace. □

Proposition 3.2. *We identify $X = [X_{ij}]_{1\le i,j\le n} \in Sym(n)$ with $\sum_{i\le j} X_{ij}\Big(\frac{\partial}{\partial P_{ij}}\Big)_P \in T_P Sym^+(n)$. Then we have*

$$\begin{aligned} h_P(X,Y) &= \frac{n+1}{2}\,tr\,(P^{-1}XP^{-1}Y),\\ (Dh)_P(X,Y,Z) &= -(n+1)\,tr\,(P^{-1}XP^{-1}YP^{-1}Z),\\ (\gamma_h)_P(X,Y) &= -\frac{1}{2}(XP^{-1}Y + YP^{-1}X).\end{aligned}$$

Proof. For $X \in \mathrm{Sym}(n)$ we define a vector field $\tilde{X} = \sum_{i\le j} X_{ij}\frac{\partial}{\partial P_{ij}} \in \mathscr{X}(\mathrm{Sym}^+(n))$. Since $D_{\tilde{X}}\tilde{Y} = 0$ for all $X, Y \in \mathrm{Sym}(n)$, we have

$$\begin{aligned} h_P(X,Y) &= -\frac{n+1}{2}(D_{\tilde{Y}} d\log\det P)_P(X)\\ &= -\frac{n+1}{2}(\tilde{Y}\tilde{X}\log\det P)_P\\ &= -\frac{n+1}{2}(\tilde{Y}\,\mathrm{tr}\,(P^{-1}\tilde{X}))_P\\ &= \frac{n+1}{2}\,\mathrm{tr}\,(P^{-1}YP^{-1}X)\\ &= \frac{n+1}{2}\,\mathrm{tr}\,(P^{-1}XP^{-1}Y),\\ (Dh)_P(X,Y,Z) &= \frac{n+1}{2}(D_{\tilde{Z}}h)_P(X,Y)\\ &= \frac{n+1}{2}(\tilde{Z}h(\tilde{X},\tilde{Y}))_P\\ &= -\frac{n+1}{2}\,\mathrm{tr}\,(P^{-1}ZP^{-1}XP^{-1}Y + P^{-1}XP^{-1}ZP^{-1}Y).\end{aligned}$$

Since $P \in \mathrm{Sym}^+(n)$, there exists $P^{\frac{1}{2}} \in \mathrm{Sym}^+(n)$ such that $(P^{\frac{1}{2}})^2 = P$. Hence it follows from Lemma 3.1 that

$$\begin{aligned}\operatorname{tr}(P^{-1}ZP^{-1}XP^{-1}Y) &= \operatorname{tr}(P^{-\frac{1}{2}}ZP^{-1}XP^{-1}P^{-1}YP^{-\frac{1}{2}})\\ &= \operatorname{tr}((P^{-\frac{1}{2}}ZP^{-\frac{1}{2}})(P^{-\frac{1}{2}}XP^{-\frac{1}{2}})(P^{-\frac{1}{2}}YP^{-\frac{1}{2}}))\\ &= \operatorname{tr}((P^{-\frac{1}{2}}XP^{-\frac{1}{2}})(P^{-\frac{1}{2}}YP^{-\frac{1}{2}})(P^{-\frac{1}{2}}ZP^{-\frac{1}{2}}))\\ &= \operatorname{tr}(P^{-1}XP^{-1}YP^{-1}Z).\end{aligned}$$

Therefore we have

$$(Dh)_P(X,Y,Z) = -(n+1)\operatorname{tr}(P^{-1}XP^{-1}YP^{-1}Z).$$

Moreover, we obtain

$$\begin{aligned}h_P\Big(-\frac{1}{2}(XP^{-1}Y+YP^{-1}X),Z\Big) &= -\frac{n+1}{4}\operatorname{tr}(P^{-1}XP^{-1}YP^{-1}Z+P^{-1}YP^{-1}XP^{-1}Z)\\ &= -\frac{n+1}{2}\operatorname{tr}(P^{-1}XP^{-1}YP^{-1}Z)\\ &= \frac{1}{2}(Dh)_P(X,Y,Z)\\ &= h_P((\gamma_h)_P(X,Y),Z),\end{aligned}$$

where the last equality follows from Proposition 1.3. Hence we have

$$(\gamma_h)_P(X,Y) = -\frac{1}{2}(XP^{-1}Y+YP^{-1}X).$$

□

4. Maps given by Riemannian metrics

In this section, we consider the map $F_g : (\Omega, g) \to (\mathrm{Sym}^+(n), h)$, where g is a Riemannian metric on a domain Ω in $\mathbb{R}^n$ and h is the Cheng-Yau metric on $\mathrm{Sym}^+(n)$. Since $T\mathrm{Sym}^+(n) \cong \mathrm{Sym}^+(n)\times\mathrm{Sym}(n)$, the space of all C^∞-sections of $F_g^*T\mathrm{Sym}^+(n)$ can be identified with $C^\infty(\Omega,\mathrm{Sym}(n))$. In particular, $(\nabla^{g,h}dF_g)\Big(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\Big)$ and $\mathrm{tr}_g(\nabla^{g,h}dF_g)$ belong to $C^\infty(\Omega,\mathrm{Sym}(n))$.

Proposition 4.1. *We have*

$$(\nabla^{g,h}dF_g)\Big(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\Big) = \frac{\partial^2F_g}{\partial x^k\partial x^l} - \frac{1}{2}\Big(\frac{\partial F_g}{\partial x^k}F_g^{-1}\frac{\partial F_g}{\partial x^l} + \frac{\partial F_g}{\partial x_l}F_g^{-1}\frac{\partial F_g}{\partial x_k}\Big) - (\gamma_g)^r{}_{kl}\frac{\partial F_g}{\partial x^r},$$

$$tr_g(\nabla^{g,h}dF_g)) = g^{kl}\Big(\frac{\partial^2F_g}{\partial x^k\partial x^l} - \frac{\partial F_g}{\partial x^k}F_g^{-1}\frac{\partial F_g}{\partial x^l} - (\gamma_g)^r{}_{kl}\frac{\partial F_g}{\partial x^r}\Big).$$

Proof. We obtain

$$\begin{aligned}(\nabla^{g,h}dF_g)\Big(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\Big) &= \nabla^h_{\frac{\partial}{\partial x^k}}dF_g\Big(\frac{\partial}{\partial x^l}\Big) - dF_g\Big(\nabla^g_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^l}\Big)\\ &= \frac{\partial^2F_g}{\partial x^k\partial x^l} + (\gamma_h)_{F_g}\Big(\frac{\partial F_g}{\partial x^k},\frac{\partial F_g}{\partial x_l}\Big) - dF_g\Big((\gamma_g)^r{}_{kl}\frac{\partial}{\partial x^r}\Big)\\ &= \frac{\partial^2F_g}{\partial x^k\partial x^l} - \frac{1}{2}\Big(\frac{\partial F_g}{\partial x^k}F_g^{-1}\frac{\partial F_g}{\partial x^l} + \frac{\partial F_g}{\partial x_l}F_g^{-1}\frac{\partial F_g}{\partial x_k}\Big) - (\gamma_g)^r{}_{kl}\frac{\partial F_g}{\partial x^r},\end{aligned}$$

where the last equality follows from Proposition 3.2 in the case of $X = \frac{\partial F_g}{\partial x^k}$, $Y = \frac{\partial F_g}{\partial x^l}$ and $P = F_g$. We also have

$$\begin{aligned} \mathrm{tr}_g(\nabla^{g,h} dF_g) &= g^{kl}(\nabla^{g,h} dF_g)\Big(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\Big) \\ &= g^{kl}\Big(\frac{\partial^2 F_g}{\partial x^k \partial x^l} - \frac{\partial F_g}{\partial x^k} F_g^{-1} \frac{\partial F_g}{\partial x^l} - (\gamma_g)^r{}_{kl} \frac{\partial F_g}{\partial x^r}\Big). \end{aligned}$$

□

We denote by $(\nabla^{g,h} dF_g)_{ij}\Big(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\Big)$ and $\mathrm{tr}_g(\nabla^{g,h} dF_g)_{ij}$ the (i, j)-components of $(\nabla^{g,h} dF_g)\Big(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\Big)$ and $\mathrm{tr}_g(\nabla^{g,h} dF_g)$, respectively.

Theorem 4.2. *If g is a Hessian metric, then we have*

$$\begin{aligned} (\nabla^{g,h} dF_g)_{ij}\Big(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\Big) &= 2g\Big((\nabla^g_{\frac{\partial}{\partial x^i}} \gamma_g)\Big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\Big), \frac{\partial}{\partial x^l}\Big) \\ tr_g(\nabla^{g,h} dF_g)_{ij} &= 2\left((\beta_g)_{ij} - (\alpha_g)^r (\gamma_g)_{rij}\right) = 2(\nabla^g \alpha_g)\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big). \end{aligned}$$

In particular, $(\nabla^{g,h} dF_g)_{ij}\Big(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\Big)$ *is symmetric with respect to* i, j, k, l.

Proof. From Proposition 4.1 and Proposition 1.3 we obtain

$$\begin{aligned} ((\nabla^{g,h} dF_g)_{ij}\Big(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\Big) &= \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} - \frac{1}{2}\Big(\frac{\partial g_{ir}}{\partial x^k} g^{rs} \frac{\partial g_{sj}}{\partial x^l} + \frac{\partial g_{ir}}{\partial x_l} g^{rs} \frac{\partial g_{sj}}{\partial x_k}\Big) - (\gamma_g)^r{}_{kl} \frac{\partial g_{ij}}{\partial x^r} \\ &= 2\frac{\partial (\gamma_g)_{jkl}}{\partial x^i} - 2g^{rs}\left((\gamma_g)_{rik}(\gamma_g)_{sjl} + (\gamma_g)_{ilr}(\gamma_g)_{jks}\right) - 2(\gamma_g)^r{}_{kl}(\gamma_g)_{rij} \\ &= 2\Big(\frac{\partial (\gamma_g)_{jkl}}{\partial x^i} - (\gamma_g)_{jrl}(\gamma_g)^r{}_{ik} - (\gamma_g)_{jkr}(\gamma_g)^r{}_{il} - (\gamma_g)_{rkl}(\gamma_g)^r{}_{ij}\Big) \\ &= 2g\Big((\nabla^g_{\frac{\partial}{\partial x^i}} \gamma_g)\Big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\Big), \frac{\partial}{\partial x^l}\Big). \end{aligned}$$

It follows from Proposition 1.6, Remark 1.5 and Proposition 1.3 that

$$\begin{aligned} \mathrm{tr}_g(\nabla^{g,h} dF_g)_{ij} &= g^{kl}\Big(\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} - \frac{\partial g_{ir}}{\partial x^k} g^{rs} \frac{\partial g_{sj}}{\partial x^l} - (\gamma_g)^r{}_{kl} \frac{\partial g_{ij}}{\partial x^r}\Big) \\ &= g^{kl}\Big(\frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} - \frac{\partial g_{kr}}{\partial x^i} g^{rs} \frac{\partial g_{sl}}{\partial x^j}\Big) - (\alpha_g)^r \frac{\partial g_{ij}}{\partial x^r} \\ &= 2\left((\beta_g)_{ij} - (\alpha_g)^r (\gamma_g)_{rij}\right) \\ &= 2(\nabla^g \alpha_g)\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big) \end{aligned}$$

□

Corollary 4.3. *If g is a Hessian metric on a domain* Ω *in* $\mathbb{R}^n$ *and the map* $F_g : (\Omega, g) \to (Sym^+(n), h)$ *is harmonic, then* $tr_g \beta_g$ *is nonnegative constant.*

Proof. Since $g^{ij}\,\mathrm{tr}_g(\nabla^{g,h}dF_g)_{ij} = 0$, it follows from Theorem 4.2 that

$$\begin{aligned} 0 &= g^{ij}((\beta_g)_{ij} - (\alpha_g)^r(\gamma_g)_{rij}) \\ &= \mathrm{tr}_g\beta_g - (\alpha_g)^r(\alpha_g)_r. \end{aligned}$$

Moreover, since $\nabla^g\alpha_g = 0$ by Theorem 4.2, $(\alpha_g)^r(\alpha_g)_r$ is nonnegative constant. □

Main Theorem. *Let g be the Cheng-Yau metric on a regular convex cone Ω in $\mathbb{R}^n$ and h the Cheng-Yau metric on $Sym^+(n)$. Then $F_g : (\Omega, g) \to (Sym^+(n), h)$ is a harmonic immersion. In particular, if Ω is a homogeneous self-dual regular convex cone, then F_g is totally geodesic.*

Proof. Let $a = a^k\Big(\dfrac{\partial}{\partial x^k}\Big)_x \in \mathrm{Ker}(dF_g)_x$, that is, $\dfrac{\partial g_{ij}}{\partial x^k}a^k = 0$ for all $1 \le i, j \le n$. It follows from Proposition 2.5 that

$$0 = x^i\frac{\partial g_{ij}}{\partial x^k}a^k = -2g_{jk}a^k$$

Hence $a = 0$. This implies that F_g is an immersion. It follows from Theorem 4.2 and Proposition 2.6 that $\mathrm{tr}_g(\nabla^{g,h}dF_g) = 2\nabla^g\alpha_g = 0$. If Ω is a homogeneous self-dual regular convex cone, we have $\nabla^{g,h}dF_g = 2\nabla^g\gamma_g = 0$ from Theorem 4.2 and Proposition 2.7. □

5. Examples of regular convex domains and non-self-dual homogeneous regular convex cones

The condition in Main Theorem that a regular convex domain is a cone is crucial to obtain a harmonic map. In fact, Example 5.1 shows that the Cheng-Yau metric does not give a harmonic map if the regular convex domain is not a cone. Another condition of Main Theorem that a homogeneous regular convex domain is self-dual is also necessary. Example 5.2 implies that the Cheng-Yau metric does not provide a totally geodesic map if a homogeneous regular convex cone is not self-dual.

Example 5.1. Let $\Omega = \{x = (x^1, x^2) \in \mathbb{R}^2 \mid x^2 - \frac{1}{2}(x^1)^2 > 0\}$. The regular convex domain Ω with the Cheng-Yau metric g is known as an example of Hessian manifolds of constant Hessian sectional curvature (c.f. Proposition 3.8 in [2]). The solution of the Monge-Ampére equation on Ω is

$$\varphi(x) = -\frac{3}{2}\log(x^2 - \frac{1}{2}(x^1)^2) + \log\frac{3}{2}.$$

We have

$$d\varphi = \frac{3}{2}\cdot\frac{1}{x^2 - (\frac{1}{2}(x^1)^2)}(x^1dx^1 - dx^2),$$

$$F_g = \frac{3}{2} \cdot \frac{1}{(x^2 - (\frac{1}{2}(x^1)^2))^2} \begin{bmatrix} x^2 + \frac{1}{2}(x^1)^2 & -x^1 \\ -x^1 & 1 \end{bmatrix},$$

$$F_g^{-1} = \frac{2}{3}(x^2 - \frac{1}{2}(x^1)^2) \begin{bmatrix} 1 & x^1 \\ x^1 & x^2 + \frac{1}{2}(x^1)^2 \end{bmatrix},$$

$$\operatorname{grad} \varphi = \begin{bmatrix} \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} \end{bmatrix} F_g^{-1} \begin{bmatrix} \frac{\partial \varphi}{\partial x^1} \\ \frac{\partial \varphi}{\partial x^2} \end{bmatrix} = -(x^2 - \frac{1}{2}(x^1)^2) \frac{\partial}{\partial x^2}.$$

Hence we obtain

$$\begin{aligned} \operatorname{tr}_g(\nabla^g \alpha_g) &= \operatorname{tr}_g \beta_g - (\alpha_g)^r (\alpha_g)_r \\ &= \operatorname{tr}_g g - d\varphi(\operatorname{grad}\varphi) \\ &= 2 - \frac{3}{2} \\ &= \frac{1}{2} \neq 0. \end{aligned}$$

Therefore $F_g : (\Omega, g) \to (\mathrm{Sym}^+(n), h)$ is not harmonic from Theorem 4.2.

EXAMPLE 5.2. We define a 5-dimensional vector space V and a regular convex cone Ω in V by

$$V = \left\{ v = \begin{bmatrix} v^1 & v^2 & v^4 \\ v^2 & v^3 & 0 \\ v^4 & 0 & v^5 \end{bmatrix} \in \mathrm{Sym}(3) \right\},$$

$$\Omega = \left\{ x = \begin{bmatrix} x^1 & x^2 & x^4 \\ x^2 & x^3 & 0 \\ x^4 & 0 & x^5 \end{bmatrix} \in V \;\middle|\; x' := \begin{bmatrix} x^1 & x^2 \\ x^2 & x^3 \end{bmatrix}, x'' := \begin{bmatrix} x^1 & x^4 \\ x^4 & x^5 \end{bmatrix} \in \mathrm{Sym}^+(2) \right\}.$$

The regular convex cone Ω is called the *Vinberg cone*, which is known as an example of non-self-dual homogeneous regular convex cones [3]. Let

$$G = \left\{ A = (A', A'') = \left(\begin{bmatrix} a & 0 \\ b_1 & c_1 \end{bmatrix}, \begin{bmatrix} a & 0 \\ b_2 & c_2 \end{bmatrix} \right) \in \mathrm{GL}(2, \mathbb{R})^2 \mid a, c_1, c_2 > 0 \right\}.$$

Then we can define a group representation $\rho : G \to \mathrm{GL}(V)$ by

$$(\rho(A)v)' = A'v'\,{}^tA', \quad (\rho(A)v)'' = A''v''\,{}^tA'' \quad \text{for } v \in V \text{ and } A = (A', A'') \in G.$$

We obtain $\rho(A)x \in \Omega$ for all $x \in \Omega$ and all $A \in G$, that is, $\rho(G)$ acts on Ω. We define $B : \Omega \to G$ by

$$B(x) = \left(\begin{bmatrix} \sqrt{x^1} & 0 \\ \frac{x^2}{\sqrt{x^1}} & \sqrt{\frac{x^1 x^3 - (x^2)^2}{x^1}} \end{bmatrix}, \begin{bmatrix} \sqrt{x^1} & 0 \\ \frac{x^4}{\sqrt{x^1}} & \sqrt{\frac{x^1 x^5 - (x^4)^2}{x^1}} \end{bmatrix} \right).$$

Then we have

$$\rho(B(x))\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x^1 & x^2 & x^4 \\ x^2 & x^3 & 0 \\ x^4 & 0 & x^5 \end{bmatrix} = x.$$

Hence $\rho(G)$ acts on Ω transitively. Let $\tilde{B}(x) \in \mathrm{GL}(5, \mathbb{R})$ be the matrix representation of $\rho(B(x)) \in \mathrm{GL}(V)$ with respect to the standard basis

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Then we obtain

$$\tilde{B}(x) = \begin{bmatrix} x^1 & 0 & 0 & 0 & 0 \\ x^2 & \sqrt{x^1x^3 - (x^2)^2} & 0 & 0 & 0 \\ \frac{(x^2)^2}{x^1} & \frac{2x^2\sqrt{x^1x^3-(x^2)^2}}{x^1} & \frac{x^1x^3-(x^2)^2}{x^1} & 0 & 0 \\ x^4 & 0 & 0 & \sqrt{x^1x^5 - (x^4)^2} & 0 \\ \frac{(x^4)^2}{x^1} & 0 & 0 & \frac{2x^4\sqrt{x^1x^5-(x^4)^2}}{x^1} & \frac{x^1x^5-(x^4)^2}{x^1} \end{bmatrix}$$

Therefore it follows from Corollary 2.4 that the Cheng-Yau metric g on Ω is expressed by

$$\begin{aligned} g &= -Dd\log|\det \tilde{B}(x)| \\ &= -Dd\log\Big(\frac{(x^1x^3 - (x^2)^2)^{\frac{3}{2}}(x^1x^5 - (x^4)^2)^{\frac{3}{2}}}{(x^1)}\Big) \\ &= -Dd\Big(\frac{3}{2}\log(x^1x^3 - (x^2)^2) + \frac{3}{2}\log(x^1x^5 - (x^4)^2) - \log x^1\Big), \end{aligned}$$

that is,

$$F_g = \begin{bmatrix} \frac{3(x^3)^2}{2(x^1x^3-(x^2)^2)^2} + \frac{3(x^5)^2}{2(x^1x^5-(x^4)^2)^2} - \frac{1}{(x^1)^2} & \frac{-3x^2x^3}{(x^1x^3-(x^2)^2)^2} & \frac{3(x^2)^2}{2(x^1x^3-(x^2)^2)^2} & \frac{-3x^4x^5}{(x^1x^5-(x^4)^2)^2} & \frac{3(x^4)^2}{2(x^1x^5-(x^4)^2)^2} \\ \frac{-3x^2x^3}{(x^1x^3-(x^2)^2)^2} & \frac{3(x^1x^3+(x^2)^2)}{(x^1x^3-(x^2)^2)^2} & \frac{-3x^1x^2}{(x^1x^3-(x^2)^2)^2} & 0 & 0 \\ \frac{3(x^2)^2}{(x^1x^5-(x^4)^2)^2} & \frac{-3x^1x^2}{(x^1x^3-(x^2)^2)^2} & \frac{3(x^1)^2}{2(x^1x^3-(x^2)^2)^2} & 0 & 0 \\ \frac{-3x^4x^5}{(x^1x^5-(x^4)^2)^2} & 0 & 0 & \frac{3(x^1x^5+(x^4)^2)}{(x^1x^5-(x^4)^2)^2} & \frac{-3x^1x^4}{(x^1x^5-(x^4)^2)^2} \\ \frac{3(x^4)^2}{2(x^1x^5-(x^4)^2)^2} & 0 & 0 & \frac{-3x^1x^4}{(x^1x^5-(x^4)^2)^2} & \frac{3(x^1)^2}{2(x^1x^5-(x^4)^2)^2} \end{bmatrix}.$$

Let x_0 be the unit 3×3 matrix in Ω, that is, $x_0^1 = x_0^3 = x_0^5 = 1$ and $x_0^2 = x_0^4 = 0$. Since $(\gamma_g)_{22r} = \dfrac{1}{2}\dfrac{\partial g_{22}}{\partial x^r}$, we obtain

$$\begin{aligned} &(\gamma_g)_{221}(x_0) = (\gamma_g)_{223}(x_0) = -\frac{3}{2}, \\ &(\gamma_g)_{222}(x_0) = (\gamma_g)_{224}(x_0) = (\gamma_g)_{225}(x_0) = 0, \\ &\frac{\partial(\gamma_g)_{222}}{\partial x^2}(x_0) = 9. \end{aligned}$$

We also have

$$F_g(x_0) = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{2} \end{bmatrix}.$$

Hence

$$\begin{aligned} g_{x_0}\left((\nabla^g_{\frac{\partial}{\partial x^2}}\gamma_g)\left(\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^2}\right),\frac{\partial}{\partial x^2}\right) &= \frac{\partial(\gamma_g)_{222}}{\partial x^2}(x_0) - 3(\gamma_g)_{22r}(x_0)(\gamma_g)^r{}_{22}(x_0) \\ &= \frac{\partial(\gamma_g)_{222}}{\partial x^2}(x_0) - 3\Big(g^{11}(x_0)(\gamma_g)_{221}(x_0)^2 + g^{33}(\gamma_g)_{223}(x_0)^2\Big) \\ &= 9 - 3\Big(\frac{1}{2}\left(-\frac{3}{2}\right)^2 + \frac{2}{3}\Big(-\frac{3}{2}\Big)^2\Big) \\ &= \frac{9}{8} \neq 0. \end{aligned}$$

Therefore $F_g : (\Omega, g) \to (\mathrm{Sym}^+(n), h)$ is not totally geodesic from Theorem 4.2.

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