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ERROR ANALYSIS FOR APPROXIMATIONS TO ONE-DIMENSIONAL SDES VIA THE PERTURBATION METHOD

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Abstract

We study asymptotic error distributions associated with standard approximation scheme for one-dimensional stochastic differential equations driven by fractional Brownian motions. This problem was studied by, for instance, Gradinaru-Nourdin [6], Neuenkirch and Nourdin [14] and the second named author [13]. The aim of this paper is to extend their results to the case where the equations contain drift terms and simplify the proof of estimates of the remainder terms in [13]. To this end, we represent the approximation solution as the solution of the equation which is obtained by replacing the fractional Brownian path with a perturbed path. We obtain the asymptotic error distribution as a directional derivative of the solution by using this expression.

1. Introduction

For a one-dimensional fractional Brownian motion (fBm) B with the Hurst $1/3 < H < 1$, we consider a one-dimensional stochastic differential equation (SDE)

$$X_t = \xi + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,d^\circ B_s, \quad t \in [0,1], \tag{1.1}$$

where $\xi \in \mathbf{R}$ is a deterministic initial value and $d^\circ B$ stands for the symmetric integral in the sense of Russo-Vallois. We may write $X_t(\xi, B)$, $X_t(B)$ to indicate the dependence of the initial value and the driving path. We consider three schemes to approximate the solution to (1.1) and study asymptotic error distributions of them. We treat the Euler scheme, the Milstein type scheme and the Crank-Nicolson scheme as real-valued stochastic processes on the interval $[0,1]$.

There are several frameworks to treat SDEs driven by fBm. For multidimensional case, the Young integration theory and the rough path analysis are powerful tools [10, 11]. We can however deal with SDEs in dimension one more easily by using the theory of the symmetric integral [15]. The symmetric integral was proposed by Russo-Vallois [21] with a motivation to establish non-causal stochastic integration theory. Recently, Nourdin and his coauthors developed a theory of integration with respect to general integrators including fBm [15, 7] with a spirit of [21]. In the present article, we adopt the symmetric integral and give a meaning to (1.1).

The Euler scheme, the Milstein type scheme and the Crank-Nicolson scheme for SDEs driven by fBm are considered by many researchers. In the consideration of approximation

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schemes, they are interested in the sharp error bounds (convergence rates) and the limits of errors normalized by the convergence rates (asymptotic error distribution). In multidimensional case, Mishura-Shevchenko [12], Friz-Riedel [5] and Bayer et al. [1] obtain an almost sharp convergence rate of the Euler scheme and the Milstein type scheme, respectively. Hu-Liu-Nualart [8] consider asymptotic error distributions of the Euler scheme for SDEs driven by fBm with $1/2 < H < 1$. Liu-Tindel [9] treat the same problem in the case $1/3 < H < 1/2$. There are a lot of results on asymptotic error distributions of schemes for one-dimensional SDEs. For example, Neuenkirch-Nourdin [14] show the convergence of the normalized error of the Euler scheme for an SDE with a drift term driven by fBm with $1/2 < H < 1$. Gradinaru-Nourdin [6] deal with the Milstein type scheme for an SDE without a drift term, namely $b \equiv 0$ in (1.1), and prove that the normalized error of it converges to some random variable.

We next explain preceding results on the Crank-Nicolson scheme for one dimensional SDE. The first result on the error of it is obtained in [14]; the authors obtain an almost sharp convergence rate. In [6], the authors treat the error of the Crank-Nicolson scheme for an SDE without a drift term driven by a standard Brownian motion and obtain the convergence of the normalized error. The second named author in the present paper shows the convergence of the normalized error for fBm with $1/3 < H < 1/2$ in [13]. It is crucial to these studies that the solution is given by a function of B_t as $X_t(\xi, B) = \phi(\xi, B_t)$, where ϕ is a certain smooth increasing function depending only on σ. This is a Doss-Sussmann type representation of the solution. Let denote the approximation solution by $\bar{X}_t^{(m)}(\xi, B)$, where m is a positive integer. Let $B_t^{(m)}$ be the dyadic polygonal approximation of the fBm B such that $B^{(m)}_{\tau^m_k} = B_{\tau^m_k}$ for every $k = 0, \dots, 2^m$, where $\tau^m_k = k2^{-m}$. For the Wong-Zakai approximation, $\bar{X}_t^{(m)}(\xi, B) = \phi(\xi, B_t^{(m)})$ holds. Hence the analysis of the error $\bar{X}_t^{(m)} - X_t$ is almost similar to that of $B - B^{(m)}$ itself. Clearly, this simple relation does not hold any more for other approximation schemes such as the Euler, Milstein and Crank-Nicolson schemes. However, if the dispersion coefficient σ is strictly positive, there exists unique B-dependent random variable $h^{(m)}$ such that $\bar{X}^{(m)}_{\tau^m_k}(\xi, B) = \phi(\xi, B + h^{(m)})$ for all k. After obtaining this formula, it is clear that the analysis of $\{h^{(m)}\}$ is important to the study of the error $X_t(\xi, B) - \bar{X}_t^{(m)}(\xi, B)$. This is one of main ideas of the proof in [14, 13].

Even if the equations contain the drift terms, the Doss-Sussmann representation still holds and the solution mapping $B \mapsto X(\xi, B)$ is Lipschitz continuous in the uniform convergence topology in one dimensional cases. Further, under the nondegeneracy assumption of σ, we can show that there exists a unique piecewise linear $h^{(m)}$ such that $\bar{X}^{(m)}_{\tau^m_k}(\xi, B) = X_{\tau^m_k}(\xi, B + h^{(m)})$ $(0 \leq k \leq 2^m)$ hold. By this perturbation representation of the approximate solutions and the analysis of $h^{(m)}$, we can show the convergence of the normalized error distribution. Hence, the present paper is a natural extension of the preceding studies. We use central limit theorem for the Hermite variation process to see the asymptotic behavior of the normalized error similarly to [14, 13]. The proof that the remainder term is negligible in [13] was done by a long calculation. In this paper, we give simpler and shorter argument for estimates of remainder terms.

The organization of this paper is as follows. In Section 2, we explain three approximation schemes, that is, the Euler, Milstein and Crank-Nicolson scheme. We next state our main theorems which determine the asymptotic error distributions in Theorem 2.5, Theorem 2.6

and Theorem 2.7. The next two sections are preliminaries for the proofs of these theorems. In Section 3, we recall the definition of Russo-Vallois symmetric integral. We consider the solutions to SDEs driven by fractional Brownian motions with the Hurst parameter $1/3 < H \leq 1/2$. In this case, the solution has a Doss-Sussmann representation and the Russo-Vallois integral is the same as the symmetric Riemman-Stieltjes integral as Stratonovich integral. By using this, we obtain estimates of iterated integrals. Also we prepare lemmas for directional derivative of the solution with respect to the driving path. In Section 4, we collect necessary results for convergence of variation functionals. These are essential for the proof of our main theorems. We give the proof of these results in Appendixes B and C. In Section 5, we consider the Crank-Nicolson scheme and prove Theorem 2.7. For the reader's convenience, we give a skecth of the proof by using the perturbation path $h^{(m)}$ in Remark 5.4. The proof of other two theorems are essentially similar to that of this theorem. We give the sketch of the proof for other two schemes, the Euler scheme and Milstein type scheme in Section 6. In Appendix A, we prepare the Gaussian analysis and Malliavin calculus. In Appendixes B and C, we prove the results stated in Section 4.

Throughout this paper, we use the following notaion. For $m \in \mathbf{N}$, we denote by $\{\tau_k^m\}_{k=0}^{2^m}$ the m-th dyadic rationals, that is, $\tau_k^m = k2^{-m}$ for $k = 0, \dots, 2^m$. For $n \in \{0\} \cup \mathbf{N} \cup \{\infty\}$, $C^n(\mathbf{R}^d; \mathbf{R})$ denotes the set of all n-times continuously differentiable $\mathbf{R}$-valued functions defined on $\mathbf{R}^d$. For $n \in \{0\} \cup \mathbf{N} \cup \{\infty\}$, $C^n_{\mathrm{bdd}}(\mathbf{R}^d; \mathbf{R})$ (resp. $C^n_{\mathrm{poly}}(\mathbf{R}^d; \mathbf{R})$) stands for the set of all functions $f \in C^n(\mathbf{R}^d; \mathbf{R})$ which are bounded (resp. polynomial growth) together with all their derivatives. For $k, l \in \{0\} \cup \mathbf{N}$, $C^{k,l}(\mathbf{R}^2; \mathbf{R})$ denotes the set of all functions $f : \mathbf{R}^2 \to \mathbf{R}$ which is k-times (resp. l-times) continuously differentiable with respect to the first (resp. second) variable. We denote the set of right continuous paths on $\mathbf{R}^d$ whose left limit exist by $D([0, 1]; \mathbf{R}^d)$. For $\lambda \in (0, 1]$, $\mathscr{C}^\lambda([0, 1]; \mathbf{R})$ stands for the set of all λ-Hölder continuous functions from $[0, 1]$ to $\mathbf{R}$. The space $\mathscr{C}_0^\lambda([0, 1]; \mathbf{R})$ is the set of all functions $g \in \mathscr{C}^\lambda([0, 1]; \mathbf{R})$ starting from zero. For $g \in \mathscr{C}^\lambda([0, 1]; \mathbf{R})$ and $0 \leq t \leq 1$, we define the uniform norm by $\|g\|_{\infty,[0,t]} = \sup_{0 \leq s \leq t} |g_s|$. We simply write $\|g\|_\infty = \|g\|_{\infty,[0,1]}$. For fixed $0 < s < 1$, we define the shift operator θ_s by $(\theta_s g)(t) = g_{t+s} - g_s$ for $0 \leq t \leq 1 - s$.

2. Main results

We state our main result. For $b, \sigma \in C^\infty_{\mathrm{bdd}}(\mathbf{R}; \mathbf{R})$, we consider an SDE (1.1). Throughout this paper, we consider a solution X to (1.1) given by (3.3). We refer the meaning of SDEs driven by fBm to Section 3. To state our main results, we recall the definitions of three approximation schemes.

Definition 2.1 (The Euler scheme). For every $m \in \mathbf{N}$, the Euler scheme $\bar{X}^{(m)} : [0, 1] \to \mathbf{R}$ is defined by

$$\begin{cases} \bar{X}_0^{(m)} = \xi, \\ \bar{X}_t^{(m)} = \bar{X}^{(m)}_{\tau_{k-1}^m} + b(\bar{X}^{(m)}_{\tau_{k-1}^m})(t - \tau_{k-1}^m) + \sigma(\bar{X}^{(m)}_{\tau_{k-1}^m})(B_t - B_{\tau_{k-1}^m}) \quad \text{for } \tau_{k-1}^m < t \leq \tau_k^m. \end{cases}$$

Definition 2.2 (The Milstein type scheme). For every $m \in \mathbf{N}$, the Milstein type scheme $\bar{X}^{(m)} : [0, 1] \to \mathbf{R}$ is defined by

$$\begin{cases} \bar{X}_0^{(m)} = \xi, \\ \bar{X}_t^{(m)} = \bar{X}_{\tau_{k-1}^m}^{(m)} + b(\bar{X}_{\tau_{k-1}^m}^{(m)})(t - \tau_{k-1}^m) + \frac{1}{2} bb'(\bar{X}_{\tau_{k-1}^m}^{(m)})(t - \tau_{k-1}^m)^2 \\ \qquad + \frac{1}{2}[\sigma b' + \sigma' b](\bar{X}_{\tau_{k-1}^m}^{(m)})(t - \tau_{k-1}^m)(B_t - B_{\tau_{k-1}^m}) \\ \qquad + \sigma(\bar{X}_{\tau_{k-1}^m}^{(m)})(B_t - B_{\tau_{k-1}^m}) + \frac{1}{2}\sigma\sigma'(\bar{X}_{\tau_{k-1}^m}^{(m)})(B_t - B_{\tau_{k-1}^m})^2 \quad \text{for } \tau_{k-1}^m < t \le \tau_k^m. \end{cases}$$

Definition 2.3 (The Crank-Nicolson scheme). For every $m \in \mathbf{N}$, the Crank-Nicolson scheme $\bar{X}^{(m)} : [0,1] \to \mathbf{R}$ is defined by a solution to an equation

$$\begin{cases} \bar{X}_0^{(m)} = \xi, \\ \bar{X}_t^{(m)} = \bar{X}_{\tau_{k-1}^m}^{(m)} + \frac{1}{2}\left\{ b(\bar{X}_{\tau_{k-1}^m}^{(m)}) + b(\bar{X}_t^{(m)}) \right\}(t - \tau_{k-1}^m) \\ \qquad\qquad + \frac{1}{2}\left\{ \sigma(\bar{X}_{\tau_{k-1}^m}^{(m)}) + \sigma(\bar{X}_t^{(m)}) \right\}(B_t - B_{\tau_{k-1}^m}) \quad \text{for } \tau_{k-1}^m < t \le \tau_k^m. \end{cases}$$

Since the Crank-Nicolson scheme is an implicit scheme, we need to restrict the domain of it and assure an existence of a solution to the equation above. Roughly speaking, the existence of the solution is ensured for large m.

In order to state our main results concisely, we set $\mathsf{w} = \sigma b' - \sigma' b$ and

$$J_t = \exp\left(\int_0^t b'(X_u)\, du + \int_0^t \sigma'(X_u)\, d^\circ B_u \right). \tag{2.1}$$

We assume the following hypothesis in order to obtain an expression of the error of the scheme;

Hypothesis 2.4. $\inf \sigma > 0$.

The following are our main results.

Theorem 2.5 (The Euler scheme). *We consider the Euler scheme. Assume that Hypothesis* 2.4 *is satisfied. For* $1/2 < H < 1$*, we have*

$$\lim_{m \to \infty} 2^{m(2H-1)} \{ \bar{X}^{(m)} - X \} = \sigma(X) U + J \int_0^{\cdot} J_s^{-1} \mathsf{w}(X_s) U_s\, ds$$

in probability with respect to the uniform norm. Here U is defined by

$$U_t = \int_0^t f_2(X_u)\, du,$$

where $f_2 = -\sigma'/2$.

In this theorem, the limit is a continuous stochastic process indexed by the elements of the interval $[0,1]$. When we emphasize the time parameter t, we express the limit process as $\sigma(X_t)U_t + J_t \int_0^t J_s^{-1} \mathsf{w}(X_s) U_s\, ds$.

Theorem 2.6 (The Milstein type scheme). *Assume that Hypothesis* 2.4 *is satisfied. We consider Milsten type scheme. For* $1/3 < H < 1/2$ *(resp.* $H = 1/2$*), we have*

$$\lim_{m \to \infty} 2^{m(4H-1)} \{ \bar{X}^{(m)} - X \} = \sigma(X) U + J \int_0^{\cdot} J_s^{-1} \mathsf{w}(X_s) U_s\, ds$$

in probability (resp. weakly) with respect to the uniform norm. Here U is a stochastic process defined as follows; we set

$$\psi = -\frac{1}{4}\left[\frac{\sigma'(\sigma b' + \sigma' b) + \sigma(\sigma'' b + \sigma b'')}{\sigma}\right], \quad f_3 = -\frac{1}{3!}[(\sigma')^2 + \sigma\sigma''],$$
$$f_4^\dagger = \frac{1}{24}[\sigma^2\sigma''' + 6\sigma\sigma'\sigma'' + 3(\sigma')^3], \quad g_1 = \frac{\mathsf{w}}{\sigma}.$$

(1) *For $1/3 < H < 1/2$, we set*

$$U_t = 3\int_0^t f_4^\dagger(X_u)\,du.$$

(2) *For $H = 1/2$, we set*

$$U_t = \int_0^t \psi(X_u)\,du + \sqrt{6}\int_0^t f_3(X_u)\,dW_u + 3\int_0^t f_3(X_u)\circ dB_u + 3\int_0^t f_4^\dagger(X_u)\,du + \frac{1}{\sqrt{12}}\int_0^t g_1(X_u)\,d\tilde{W}_u,$$

where W and $\tilde{W}$ are standard Brownian motions and B, W and $\tilde{W}$ are independent. Also $dW_u, d\tilde{W}_u$ and $\circ\, dB_u$ stand for the Itô integral and the Stratonovich integral, respectively.

Theorem 2.7 (The Crank-Nicolson scheme)**.** *Assume that Hypothesis* 2.4 *is satisfied. For* $1/3 < H \leq 1/2$*, we have*

$$\lim_{m\to\infty} 2^{m(3H-1/2)}\{\bar{X}^{(m)} - X\} = \sigma(X)U + J\int_0^{\cdot} J_s^{-1}\mathsf{w}(X_s)U_s\,ds$$

weakly with respect to the uniform norm. Here U is a stochastic process defined as follows; we set

$$\psi = \frac{1}{4}[\sigma' b' + \sigma'' b], \qquad f_3 = \frac{1}{12}[(\sigma')^2 + \sigma\sigma''], \qquad g_1 = \frac{\mathsf{w}}{\sigma}.$$

(1) *For $1/3 < H < 1/2$, we set*

$$U_t = \sigma_{3,H}\int_0^t f_3(X_u)\,dW_u,$$

where $\sigma_{3,H}$ is a positive constant defined by (4.1) *and W is a standard Brownian motion independent of B.*

(2) *For $H = 1/2$, we set*

$$U_t = \int_0^t \psi(X_u)\,du + \sqrt{6}\int_0^t f_3(X_u)\,dW_u + 3\int_0^t f_3(X_u)\circ dB_u + \frac{1}{\sqrt{12}}\int_0^t g_1(X_u)\,d\tilde{W}_u,$$

where W and $\tilde{W}$ are standard Brownian motions and B, W and $\tilde{W}$ are independent.

We make remarks on our main results.

(1) We explain how we derive $f_i, g_1, \varphi_{\boldsymbol{i}}, \psi, f_4^\dagger$ ($i = 2, 3, 4, \boldsymbol{i} = 011, 101, 110$). Since Theorems 2.5, 2.6 and 2.7 are proved by the same method, we explain the case of the Crank-Nicolson scheme (Theorem 2.7) as an example. In the first step of our proof, we need to calculate one-step error $\hat{\kappa}_k$ of each approximation scheme as in (5.4). In

that calculation, the functions $\hat{f}_i, \hat{g}_1, \hat{\varphi}, \hat{\varphi}_i$, which are defined by σ and b, appear as the coefficients of the monomials of the increments of $\Delta B_k = B_{\tau_k^m} - B_{\tau_{k-1}^m}$ and $\Delta = 2^{-m}$ and iterated integrals of B_t and t (Lemma 5.6). We define the functions $f_i, g_1, \varphi, \varphi_i$ by using $\hat{f}_i, \hat{g}_1, \hat{\varphi}, \hat{\varphi}_i$ and express main part of the piecewise linear function $h^{(m)}$ in terms of $f_i, g_1, \varphi, \varphi_i$ (Lemma 5.7). Finally, we study asymptotic of $h^{(m)}$ and then define $\psi = \phi + (\varphi_{011} + \varphi_{110})/4$ (Lemma 5.10). In the case of the Euler and Milstein scheme, we show lemmas corresponding to Lemmas 5.6, 5.7 and 5.10. The function $f_4^\dagger$ in the Milstein scheme appears in studying in asymptotic of $h^{(m)}$.

(2) Theorem 2.5 is an extension of [14], but the proof is completely different and comparatively more simple.

(3) In [6], the authors consider higher order schemes for SDEs without drift terms. Theorem 2.6 coresponds to the second order scheme for an SDE containing a drift term.

(4) Theorem 2.7 is an extension of [6, 13]. To our knowledge, the convergence of the approximation solution itself is not unknown for $1/6 < H \leq 1/3$ ([16]). When $\sigma(x)^2$ is a quadratic function of x, Theorem 2.7 is proved in [14] for $1/6 < H < 1/2$. In the case where $H > 1/3$, the convergence of the approximation solution is a pathwise result, that is, the result holds for SDEs driven by Hölder continuous paths with Hölder exponent which is greater than $1/3$. However, the proof of [14] is due to a central limit theorem and it is not clear that this is also a pathwise result.

3. ODEs driven by Hölder continuous functions and SDEs

In this section, we define the symmetric integral in the sense of Russo-Vallois and discuss a unique existence and properties of a solution to an ordinary differential equation (ODE).

Let $1/3 < \lambda < 1$. For a λ-Hölder continuous function $g : [0, 1] \to \mathbf{R}$, we consider an ODE

$$x_t = \xi + \int_0^t b(x_u)\,du + \int_0^t \sigma(x_u)\,d^\circ g_u, \quad t \in [0, 1], \tag{3.1}$$

where $\xi \in \mathbf{R}$ and $d^\circ g$ denotes the symmetric integral. We shall also write $x_t(\xi, g)$, $x(\xi)$, or $x(g)$ for the solution x to emphasize dependence on the initial value ξ and/or the driver g. Since fBm with the Hurst $1/3 < H < 1$ is $(H - \epsilon)$-Hölder continuous with probability one, we can deal with SDE (1.1) in pathwise sense by using the theory of ODEs (3.1). We have $\lambda = H - \epsilon$ in mind. See Section 3.4.

We prepare notation. For $g \in \mathscr{C}^\lambda([0, 1]; \mathbf{R})$, we use the symbol C_g, which may change line by line, to denote a constant which has a bound

$$C_1 \left\{ 1 + \sup_{0 \leq s < t \leq 1} \frac{|g_t - g_s|}{(t - s)^\lambda} \right\}^{C_2}$$

for some constants C_1 and C_2, which may depend on the Hölder exponent λ but not on g.

3.1. Existence and uniqueness. We collect facts on the symmetric integral and a solution to an ODE (3.1). In what follows, we assume $1/3 < \lambda < 1$.

Definition 3.1. For continuous functions $f, g : [0,1] \to \mathbf{R}$, we define the symmetric integral in the sense of Russo-Vallois by

$$\int_0^t f_u \, d^\circ g_u = \lim_{\epsilon \downarrow 0} \int_0^t \frac{f_{(u+\epsilon)\wedge t} + f_u}{2} \cdot \frac{g_{(u+\epsilon)\wedge t} - g_u}{\epsilon} \, du$$

if the limit of the right-hand side exists.

Proposition 3.2 ([15, Theorem 4.1.7])**.** *Let $a \in \mathscr{C}^1([0,1];\mathbf{R})$ and $g \in \mathscr{C}_0^\lambda([0,1];\mathbf{R})$. Then, for any $f \in C^{1,3}(\mathbf{R}^2;\mathbf{R})$, $\int_0^t \partial_2 f(a_u, g_u) \, d^\circ g_u$ exists and it holds that*

$$f(a_t, g_t) = f(a_0, g_0) + \int_0^t \partial_1 f(a_u, g_u) \, da_u + \int_0^t \partial_2 f(a_u, g_u) \, d^\circ g_u.$$

Remark 3.3. Let $a \in \mathscr{C}^1([0,1];\mathbf{R})$ and $g \in \mathscr{C}_0^\lambda([0,1];\mathbf{R})$. Let $f \in C^{1,2}(\mathbf{R}^2;\mathbf{R}) \cap C^1(\mathbf{R}^2;\mathbf{R})$. Then, we can choose a primitive function $F \in C^{1,3}(\mathbf{R}^2;\mathbf{R}) \cap C^1(\mathbf{R}^2;\mathbf{R})$ with respect to the second variable, that is, $f(x,y) = \partial_2 F(x,y)$ for any $x, y \in \mathbf{R}$. Indeed, $F(x,y) = \int_0^y f(x,\eta) \, d\eta$ is a primitive function and the continuity of $\partial_1 f$ implies $\partial_1 F(x,y) = \int_0^y \partial_1 f(x,\eta) \, d\eta$. Hence, from Proposition 3.2, we see $\int_0^t f(a_u, g_u) \, d^\circ g_u$ exists and it holds that

$$\int_0^t f(a_u, g_u) \, d^\circ g_u = F(a_t, g_t) - F(a_0, g_0) - \int_0^t \partial_1 F(a_u, g_u) \, da_u.$$

The next proposition asserts that a symmetric integral is a limit of a modified Riemann sum.

Proposition 3.4. *Let $a \in \mathscr{C}^1([0,1];\mathbf{R})$ and $g \in \mathscr{C}_0^\lambda([0,1];\mathbf{R})$. Let $0 = t_0 < \cdots < t_n = t$ be a partition of $[0,t]$. For any $f \in C^{1,2}(\mathbf{R}^2;\mathbf{R})$, we see that*

$$\sum_{k=1}^n \frac{f(a_{t_{k-1}}, g_{t_{k-1}}) + f(a_{t_k}, g_{t_k})}{2} (g_{t_k} - g_{t_{k-1}})$$

converges to $\int_0^t f(a_u, g_u) \, d^\circ g_u$ as $\max\{t_k - t_{k-1}; k = 1, \ldots, n\}$ tends to 0.

Proof. We use the formula in Remark 3.3. We have

$$\begin{aligned}
\int_s^t f(a_u, g_u) \, d^\circ g_u &= F(a_t, g_t) - F(a_s, g_s) - \int_s^t \partial_1 F(a_u, g_u) da_u \\
&= \left\{ F(a_t, g_t) - F(a_s, g_t) - \int_s^t \partial_1 F(a_u, g_u) \, da_u \right\} + \{F(a_s, g_t) - F(a_s, g_s)\} \\
&= \int_s^t \{\partial_1 F(a_u, g_t) - \partial_1 F(a_u, g_u)\} \, da_u \\
&\qquad + f(a_s, g_s)(g_t - g_s) + \partial_2 f(a_s, g_s) \frac{1}{2} (g_t - g_s)^2 \\
&\qquad + \partial_2^2 f(a_s, g_s + \theta(g_t - g_s)) \frac{(g_t - g_s)^3}{3!} \\
&= f(a_s, g_s)(g_t - g_s) + \partial_2 f(a_s, g_s) \frac{1}{2} (g_t - g_s)^2 + O(|t-s|^{1+\lambda}) + O(|t-s|^{3\lambda}),
\end{aligned}$$

where we used the Taylor formula and the Hölder continuity of g. On the other hand, by using the Taylor formula again, we have

$$\begin{aligned}\frac{f(a_s,g_s)+f(a_t,g_t)}{2}(g_t-g_s) &= f(a_s,g_s)(g_t-g_s)+\frac{1}{2}\{f(a_s,g_t)-f(a_s,g_s)\}(g_t-g_s)\\&\quad+\frac{1}{2}\{f(a_t,g_t)-f(a_s,g_t)\}(g_t-g_s)\\&= f(a_s,g_s)(g_t-g_s)+\frac{1}{2}\partial_2 f(a_s,g_s)(g_t-g_s)^2\\&\quad+\frac{1}{4}\partial_2^2 f(a_s,g_s+\theta(g_t-g_s))(g_t-g_s)^3\\&\quad+\frac{1}{2}\partial_1 f(a_s+\theta'(g_t-g_s),g_t)(a_t-a_s)(g_t-g_s).\end{aligned}$$

Therefore, we obtain

$$\int_s^t f(a_u,g_u)\,d^\circ g_u=\frac{f(a_t,g_t)+f(a_s,g_s)}{2}(g_t-g_s)+R(s,t),$$

where $|R(s,t)|\le C_g|t-s|^{(1+\lambda)\wedge(3\lambda)}$. By the additivity property of the integral, $\int_s^t f(a_u,g_u)\,d^\circ g_u + \int_t^v f(a_u,g_u)\,d^\circ g_u = \int_s^v f(a_u,g_u)\,d^\circ g_u$ $(s<t<v)$ and a limiting argument, we obtain the desired result. □

Next we consider properties of (3.1). Let us start our discussion with properties of the flow ϕ associated to σ, that is, ϕ is a unique solution ϕ to an ODE

$$\phi(\alpha,\beta)=\alpha+\int_0^\beta \sigma(\phi(\alpha,\eta))\,d\eta,\quad \beta\in\mathbf{R}. \tag{3.2}$$

Proposition 3.5 ([4, Lemma 2]). *Let $n\ge 1$. For any $\sigma\in C^n_{\mathrm{bdd}}(\mathbf{R};\mathbf{R})$ and an initial point $\alpha\in\mathbf{R}$, there exists a unique solution to* (3.2). *The unique solution ϕ satisfies the following:*

(1) $\phi\in C^{n,n+1}(\mathbf{R}^2;\mathbf{R})\cap C^n(\mathbf{R}^2;\mathbf{R})$,

(2) $\phi(\alpha,\beta)=\phi(\phi(\alpha,\beta'),\beta-\beta')$,

(3) $\partial_1\phi(\alpha,\beta)=\exp\left(\int_0^\beta \sigma'(\phi(\alpha,\eta))\,d\eta\right)$.

To state assertion about uniqueness of solutions to (3.1), we introduce a class $\mathfrak{C}$ of the solutions by

$$\mathfrak{C}=\left\{x\in\mathscr{C}^\lambda([0,1];\mathbf{R});\ \begin{array}{l}\text{there exist } f\in C^{1,3}(\mathbf{R}^2;\mathbf{R}) \text{ and } k\in\mathscr{C}^1([0,1];\mathbf{R})\\ \text{such that } x_t=f(k_t,g_t) \text{ for all } t\in[0,1]\end{array}\right\}.$$

Note that $\mathfrak{C}$ depends on $g\in\mathscr{C}_0^\lambda([0,1];\mathbf{R})$.

Proposition 3.6 ([15, Theorem 4.3.1], [18, Section 3]). *Let $g\in\mathscr{C}_0^\lambda([0,1];\mathbf{R})$. Assume that $b\in C^1_{\mathrm{bdd}}(\mathbf{R};\mathbf{R})$ and $\sigma\in C^2_{\mathrm{bdd}}(\mathbf{R};\mathbf{R})$. Then, a unique solution to* (3.1) *in the class $\mathfrak{C}$ exists and it is given by*

$$x_t=\phi(a_t,g_t), \tag{3.3}$$

where ϕ and $a\equiv a(\xi,g)$ are given by solutions to (3.2) *and*

$$a_t=\xi+\int_0^t f_{\sigma,b}(a_u,g_u)\,du,\quad t\in[0,1],$$

respectively. Here $f_{\sigma,b} = f_1 f_2$ *with*

$$f_1(x,y) = \exp\left(-\int_0^y \sigma'(\phi(x,\eta))\,d\eta\right), \qquad f_2(x,y) = b(\phi(x,y)).$$

Proof. It is easily shown that x given by (3.3) belongs to $\mathfrak{C}$ and satisfy (3.1). Indeed, Proposition 3.5 (1) implies $\phi \in C^{2,3}(\mathbf{R}^2;\mathbf{R}) \subset C^{1,3}(\mathbf{R}^2;\mathbf{R})$ and $a \in \mathscr{C}^1([0,1];\mathbf{R})$. From Proposition 3.2 and Proposition 3.5 (3), we see that x satisfies (3.1). To prove the uniqueness, we borrow results from [18, Section 3]. Let x be a solution in the class $\mathfrak{C}$ and given by $x = f(k,g)$ for $f \in C^{1,3}(\mathbf{R}^2;\mathbf{R})$ and $k \in \mathscr{C}^1([0,1];\mathbf{R})$. Since $\int_s^t x_u\,d^\circ g_u = \int_s^t f(k_u,g_u)\,d^\circ g_u$ is well-defined from Remark 3.3, set $A_{st} = \int_s^t x_u\,d^\circ g_u - \frac{1}{2}(x_t + s_s)(g_t - g_s)$. Then, we deduce that (x,A) is a solution to (3.1) in the sense of [18, Definition 3.1] from [18, Lemma 3.4 and Proposition 3.5]. Finally, [18, Corollary 3.7] implies $x_t = \phi(a_t, g_t)$. □

Proposition 3.7. *Let x be the solution to* (3.1) *given by* (3.3)*. For fixed* $0 < s < 1$*, we have* $x_{s+t}(\xi,g) = x_t(x_s(\xi,g),\theta_s g)$ *for any* $0 \le t \le 1 - s$*.*

Proof. We first prove $a_t\,(x_s(\xi,g),\theta_s g) = \tilde{a}_t := \phi(a_{s+t}(\xi,g),g_s)$. From Proposition 3.5, we see

$$\frac{1}{f_1(x,y')}\cdot f_1(x,y) = f_1(\phi(x,y'),y-y'), \qquad f_2(x,y) = f_2(\phi(x,y'),y-y').$$

Hence, it holds that

$$\begin{aligned}\frac{d}{dt}\tilde{a}_t &= \partial_1\phi(a_{s+t}(\xi,g),g_s)\frac{d}{dt}a_{s+t}(\xi,g)\\ &= \frac{1}{f_1(a_{s+t}(\xi,g),g_s)}\cdot f_1(a_{s+t}(\xi,g),g_{s+t})f_2(a_{s+t}(\xi,g),g_{s+t})\\ &= [f_1f_2](\phi(a_{s+t}(\xi,g),g_s),g_{s+t}-g_s)\\ &= f_{\sigma,b}(\tilde{a}_t,(\theta_s g)_t).\end{aligned}$$

By the definition of $\tilde{a}$ and Proposition 3.6, we have $\tilde{a}_0 = \phi(a_s(\xi,g),g_s) = x_s(\xi,g)$. It follows from the uniquness of a solution that $a_t\,(x_s(\xi,g),\theta_s g) = \tilde{a}_t$.

Combining Proposition 3.5 (2), Proposition 3.6 and this equality, we obtain

$$\begin{aligned}x_{s+t}(\xi,g) &= \phi(a_{s+t}(\xi,g),g_{s+t}) = \phi(\phi(a_{s+t}(\xi,g_s),g_{s+t}-g_s)\\ &= \phi(a_t(x_s(\xi,g),\theta_s g),(\theta_s g)_t) = x_t\,(x_s(\xi,g),\theta_s g)\,,\end{aligned}$$

which completes the proof. □

Remark 3.8. We assume the same assumption as in Proposition 3.6 and consider the solution x to (3.1) given by (3.3). In the proposition, we consider Hölder continuous paths. However it is easy to check that the mapping $g \mapsto x(g)$ can be extended to a continuous mapping on $C([0,1];\mathbf{R})$ with the uniform convergence norm $\|\cdot\|_\infty$. Further, by Remark 3.3, for any $f \in C^{1,2}(\mathbf{R}^2;\mathbf{R}) \cap C^1(\mathbf{R}^2;\mathbf{R})$, we have the continuity of the mapping in the uniform convergence topology :

$$C([0,1];\mathbf{R}) \ni g \mapsto \int_0^{\cdot} f(a_s(g),x_s(g))\,d^\circ g_s \in C([0,1];\mathbf{R}).$$

3.2. The Taylor expansion and its remainder estimates. For notational convenience, we set $g_t^0 = t$, $g_t^1 = g_t$ for $0 \le t \le 1$. Let x be the solution to (3.1) given by (3.3). Assume that $b \in C^1_{\mathrm{bdd}}(\mathbf{R};\mathbf{R})$ and $\sigma \in C^2_{\mathrm{bdd}}(\mathbf{R};\mathbf{R})$. For $0 \le s \le t \le 1$ and $f \in C^2_{\mathrm{bdd}}(\mathbf{R};\mathbf{R})$, we can define

$$I^0_{st} = \int_s^t f(x_u)\,dg^0_u, \qquad I^1_{st}(f) = \int_s^t f(x_u)\,d^\circ g^1_u.$$

Here, $I^0_{st}(f)$ is a usual Riemann integral. As for $I^1_{st}(f)$, the reasoning is as follows. By using functions ϕ and a given in Proposition 3.6, we have $f(x_u) = [f\circ\phi](a_u, g_u)$ and $f\circ\phi \in C^{1,2}(\mathbf{R}^2;\mathbf{R}) \cap C^1(\mathbf{R}^2;\mathbf{R})$. From Remark 3.3, we see $F(x,y) = \int_0^y f(x,\eta)\,d\eta$ belongs to $C^{1,3}(\mathbf{R}^2;\mathbf{R}) \cap C^1(\mathbf{R}^2;\mathbf{R})$ and it holds that

$$I^1_{st}(f) = \int_s^t [f\circ\phi](a_u,g_u)\,d^\circ g_u = F(a_t,g_t) - F(a_s,g_s) - \int_s^t \partial_1 F(a_u,g_u)\,da_u.$$

Hence we see $I^1_{st}(f)$ is well-defined. Further, for any $\alpha_1,\dots,\alpha_n \in \{0,1\}$, we can define the iterated integral

$$I^{\alpha_1\cdots\alpha_n}_{st}(f) = \int_s^t I^{\alpha_1\cdots\alpha_{n-1}}_{su}(f)\,d^\circ g^{\alpha_n}_u$$

inductively in the same way. For $f \equiv 1$, we set $g^{\alpha_1\cdots\alpha_n}_{st} = I^{\alpha_1\cdots\alpha_n}_{st}(f)$. We set $V_0 = b$, $V_1 = \sigma$ and define a vector field by $\mathcal{V}_\alpha f = V_\alpha f'$.

From Remark 3.3, we see the following estimate.

Lemma 3.9. *Assume that $b \in C^1_{\mathrm{bdd}}(\mathbf{R};\mathbf{R})$ and $\sigma \in C^2_{\mathrm{bdd}}(\mathbf{R};\mathbf{R})$. Let $f \in C^2_{\mathrm{bdd}}(\mathbf{R};\mathbf{R})$. Let $\alpha_1,\dots,\alpha_n \in \{0,1\}$ and set $r_i = \sharp\{k = 1,\dots,n; \alpha_k = i\}$. Then, there exists a constant $C = C_{f,g,\alpha_1,\dots,\alpha_n}$ which depends only on f, the Hölder constant of g and $\alpha_1,\dots,\alpha_n$ such that, for any $0 \le s < t \le 1$,*

$$|I^{\alpha_1\cdots\alpha_n}_{st}(f)| \le C(t-s)^{r_0 + r_1\lambda}.$$

We use the above Taylor expansion and the estimate of iterated integrals in the calculation below. Using Proposition 3.2, we can prove the following by induction on n;

Proposition 3.10. *Let $n \ge 0$. Assume that $b,\sigma \in C^{n+2}_{\mathrm{bdd}}(\mathbf{R};\mathbf{R})$. Then, for any $0 \le s < t \le 1$, we have*

$$\begin{aligned} x_t - x_s = &\sum_{k=1}^n \sum_{\alpha_1,\dots,\alpha_k\in\{0,1\}} [\mathcal{V}_{\alpha_1}\cdots\mathcal{V}_{\alpha_{k-1}}V_{\alpha_k}](x_s)\,g^{\alpha_1\cdots\alpha_k}_{st} \\ &+ \sum_{\alpha_1,\dots,\alpha_n,\alpha_{n+1}\in\{0,1\}} I^{\alpha_1\alpha_2\cdots\alpha_{n+1}}_{st}(\mathcal{V}_{\alpha_1}\mathcal{V}_{\alpha_2}\cdots\mathcal{V}_{\alpha_n}V_{\alpha_{n+1}}). \end{aligned}$$

We calculate each terms in Proposition 3.10. We first note that the p-th iterated integral $g^{\alpha\cdots\alpha}_{st}$ is equal to $(g^\alpha_t - g^\alpha_s)^p/p!$. This can be checked by a direct calculation.

Proposition 3.11. *Assume that $b,\sigma \in C^6_{\mathrm{bdd}}(\mathbf{R};\mathbf{R})$. Then, for any $0 \le s < t \le 1$, we have*

$$\begin{aligned} x_t - x_s = &\, b(x_s)(t-s) + \sigma(x_s)(g_t - g_s) + \frac{1}{2}[\sigma\sigma'](x_s)(g_t-g_s)^2 \\ &+ \frac{1}{3!}[\sigma(\sigma\sigma')'](x_s)(g_t-g_s)^3 + \frac{1}{4!}[\sigma(\sigma(\sigma\sigma')')'](x_s)(g_t-g_s)^4 \end{aligned}$$

$$+ [b\sigma'](x_s)(g_t - g_s)(t-s) + [\sigma b' - b\sigma'](x_s)g^{10}_{st} + \frac{1}{2}[b'b](x_s)(t-s)^2$$
$$+ [b\,(\sigma\sigma')'](x_s)g^{011}_{st} + [\sigma\,(b\sigma')'](x_s)g^{101}_{st} + [\sigma\,(\sigma b')'](x_s)g^{110}_{st} + r_{st},$$

where $|r_{st}| \le C_g(t-s)^{\min\{2+\lambda, 1+3\lambda, 5\lambda\}}$.

Proof. Set

$$J^k_{st} = \sum_{\alpha_1,\dots,\alpha_k\in\{0,1\}} [\mathcal{V}_{\alpha_1}\cdots\mathcal{V}_{\alpha_{k-1}}V_{\alpha_k}]\,(x_s)g^{\alpha_1\cdots\alpha_k}_{st},$$
$$\tilde{J}^k_{st} = \sum_{\alpha_1,\dots,\alpha_k\in\{0,1\}} I^{\alpha_1\cdots\alpha_k}_{st}\,(\mathcal{V}_{\alpha_1}\cdots\mathcal{V}_{\alpha_{k-1}}V_{\alpha_k})\,.$$

Then we see $x_t - x_s = J^1_{st} + \cdots + J^4_{st} + \tilde{J}^5_{st}$ and

$$J^1_{st} = b(x_s)g^0_{st} + \sigma(x_s)g^1_{st},$$
$$J^2_{st} = [bb'](x_s)g^{00}_{st} + [\sigma b'](x_s)g^{10}_{st} + [b\sigma'](x_s)g^{01}_{st} + [\sigma\sigma'](x_s)g^{11}_{st},$$
$$J^3_{st} = [\sigma(\sigma b')'](x_s)g^{110}_{st} + [\sigma(b\sigma')'](x_s)g^{101}_{st} + [b(\sigma\sigma')'](x_s)g^{011}_{st} + [\sigma(\sigma\sigma')'](x_s)g^{111}_{st} + r^{(3)}_{st},$$
$$J^4_{st} = [\sigma(\sigma(\sigma\sigma')')'](x_s)g^{1111}_{st} + r^{(4)}_{st},$$

where $r^{(3)}_{st}$ and $r^{(4)}_{st}$ satisfy $|r^{(3)}_{st}| \le C_g(t-s)^{2+\lambda}$ and $|r^{(4)}_{st}| \le C_g(t-s)^{1+3\lambda}$, respectively. In addition, we have $|\tilde{J}^5_{st}| \le C_g(t-s)^{5\lambda}$. Noting $[\sigma b'](x_s)g^{10}_{s,t} + [b\sigma'](x_s)g^{01}_{st} = [b\sigma'](x_s)(g_t - g_s)(t-s) + [\sigma b' - b\sigma'](x_s)g^{10}_{st}$, we complete the proof. □

3.3. Directional derivatives of solutions. In what follows, we assume that Hypothesis 2.4 is satisfied and find expressions of the solution $x \equiv x(g)$ to (3.1) given by (3.3) and its directional derivatives. We follow the approach employed in [3] in order to do so.

For $g \in \mathscr{C}^\lambda_0([0,1];\mathbf{R})$, we set

$$J_t(g) = \exp\left(\int_0^t b'(x_u(g))\,du + \int_0^t \sigma'(x_u(g))\,d^\circ g_u\right). \tag{3.4}$$

This is a deterministic version of (2.1). Note that $J_t(g)$ is expressed by

$$J_t(g) = \frac{\sigma(x_t(g))}{\sigma(x_0(g))}\exp\left(\int_0^t \left[\frac{\mathsf{W}}{\sigma}\right](x_u(g))\,du\right). \tag{3.5}$$

Indeed, we see

$$\log\sigma(x_t(g)) = \log(\sigma\circ\phi)(a_t(g), g_t)$$
$$= \log\sigma(x_0) + \int_0^t \left[\frac{\sigma' b}{\sigma}\right](x_u(g))\,du + \int_0^t \sigma'(x_u(g))\,d^\circ g_u$$

from Proposition 3.2. This implies

$$\sigma(x_t(g)) = \sigma(x_0)\exp\left(\int_0^t \left[\frac{\sigma' b}{\sigma}\right](x_u(g))\,du + \int_0^t \sigma'(x_u(g))\,d^\circ g_u\right).$$

Substituting the above to (3.5), we obtain (3.4).

Proposition 3.12. *Let* $b, \sigma \in C^{n+1}_{\mathrm{bdd}}(\mathbf{R};\mathbf{R})$ *for* $n \ge 1$*. Assume that Hypothesis* 2.4 *is satisfied. Then, the functional* $g \mapsto x_t(g)$ *is n-times Fréchet differentiable in* $\mathscr{C}^\lambda_0([0,1];\mathbf{R})$.

In particular, the derivatives satisfy the following;

(1) *For any* $h^1, \dots, h^\nu \in \mathscr{C}_0^\lambda([0,1];\mathbf{R})$, *we have*

$$|\nabla_{h^\nu} \cdots \nabla_{h^1} x_t(g)| \le C_\nu \|h^1\|_\infty \cdots \|h^\nu\|_\infty,$$

where C_ν *is a positive constant depending only on* b, σ *and* ν.

(2) *The first derivative* $\nabla_h x_t(g)$ *is expressed as*

$$\nabla_h x_t(g) = \sigma(x_t(g))h_t + \int_0^t J_t(g)(J_s(g))^{-1}\mathsf{w}(x_s(g))h_s\,ds.$$

(3) *If* h *is Lipschitz continuous, then* $\nabla_h x_t(g)$ *is expressed as*

$$\nabla_h x_t(g) = \int_0^t \dot{h}_s \sigma(x_s(g))J_t(g)(J_s(g))^{-1}\,ds = \sigma(x_t(g))\int_0^t \exp\left(\int_s^t \left[\frac{\mathsf{w}}{\sigma}\right](x_u(g))\,du\right)\dot{h}_s\,ds.$$

In order to prove Proposition 3.12, we set

$$F(x) = \int_0^x \frac{d\xi}{\sigma(\xi)}, \qquad G = F^{-1}, \qquad \tilde{b} = \left[\frac{b}{\sigma}\right]\circ G, \qquad y_0 = F(x_0).$$

We consider a solution y to an ODE

$$y_t = y_0 + \int_0^t \tilde{b}(y_u)\,du + g_t. \tag{3.6}$$

Then we obtain an expression of the solution x_t to (3.1) as follows;

Proposition 3.13. *Let* y *be a solution to* (3.6). *The solution* x *to* (3.1) *given by* (3.3) *is expressed by* $x = G(y)$.

Proof. Due to Proposition 3.6, we see the assertion by showing $G(y) \in \mathfrak{C}$ and it satisfies (3.1). Note that the solution y is given by $y_t = \tilde{a}_t + g_t$, where $\tilde{a}$ is a solution to $\tilde{a}_t = y_0 + \int_0^t \tilde{b}(\tilde{a}_u + g_u)\,du$. Hence $G(y) \in \mathfrak{C}$. We prove that $G(y)$ satisfies (3.1). From Proposition 3.2, we see

$$\begin{aligned} G(y_t) - x_0 &= G(\tilde{a}_t + g_t) - G(\tilde{a}_0 + g_0) \\ &= \int_0^t G'(\tilde{a}_u + g_u)\,d\tilde{a}_u + \int_0^t G'(\tilde{a}_u + g_u)\,d^\circ g_u. \end{aligned}$$

The first term is equal to

$$\begin{aligned} \int_0^t \sigma(G(\tilde{a}_u + g_u))\tilde{b}(\tilde{a}_u + g_u)\,du &= \int_0^t \sigma(G(\tilde{a}_u + g_u))\left[\frac{b}{\sigma}\right](G(\tilde{a}_u + g_u))\,du \\ &= \int_0^t b(G(y_u))\,du \end{aligned}$$

and the second one is $\int_0^t \sigma(G(y_u))\,d^\circ g_u$. We see that $G(y)$ satisfies (3.1). The proof is completed. □

We see that the solution y_t to (3.6) with any coefficient $\tilde{b}$ and initial point y_0 is differentiable.

Proposition 3.14. *Assume that* $\tilde{b} \in C_{\mathrm{bdd}}^{n+1}(\mathbf{R};\mathbf{R})$ *for* $n \ge 1$. *The functional* $g \mapsto y_t(g)$ *is* n*-times Fréchet differentiable in* $\mathscr{C}_0^\lambda([0,1];\mathbf{R})$.

In particular, the derivatives satisfy the following;

(1) *For any* $h^1, \dots, h^\nu \in \mathscr{C}_0^\lambda([0,1]; \mathbf{R})$, *we have*

$$|\nabla_{h^\nu} \cdots \nabla_{h^1} y_t(g)| \le C_\nu \|h^1\|_\infty \cdots \|h^\nu\|_\infty$$

where C_ν *is a positive constant depending only on* $\tilde{b}$ *and* ν.

(2) *The first derivative* $\nabla_h y_t(g)$ *is expressed by*

$$\nabla_h y_t(g) = h_t + \int_0^t \exp\left(\int_s^t \tilde{b}'(y_u(g))\,du\right)\tilde{b}'(y_s(g))h_s\,ds.$$

For the sake of conciseness, we omit the proof of the above proposition and show Proposition 3.12.

Proof of Proposition 3.12. The differentiability and Assertion (1) follow from Propositions 3.13 and 3.14. Noting $\tilde{b}'(y_t(g)) = [\mathsf{w}/\sigma](x_t(g))$, we see that Assertion (2) is true. Assertion (3) follows from Assertion (2) and the integration by parts formula. □

3.4. SDEs driven by fBm. We consider the existence and properties of a solution to an SDE (1.1). Let us start our discussion with the definition of fBm;

Definition 3.15. A one-dimensional centered Gaussian process $B = \{B_t\}_{0\le t<\infty}$ starting from zero is called fractional Brownian motion (fBm) with the Hurst $0 < H < 1$ if its covariance is given by

$$\boldsymbol{E}[B_sB_t] = R(s,t) = \frac{1}{2}\left\{s^{2H} + t^{2H} - |t-s|^{2H}\right\}. \tag{3.7}$$

It is well known that fBm B has stationary increments in the sense of $\boldsymbol{E}[(B_t - B_s)(B_v - B_u)] = \boldsymbol{E}[(B_{t+a} - B_{s+a})(B_{v+a} - B_{u+a})]$ for any $0 \le s \le t \le u \le v < \infty$ and $0 \le a < \infty$ and that it has self-similarity, namely, for any $a > 0$, $\{a^{-H}B_{at}\}_{0\le t<\infty}$ is also fBm with the Hurst H. In addition, it has a modulus of continuity of trajectories; there exists a measurable subset Ω_0 of Ω such that $\boldsymbol{P}(\Omega_0) = 1$ and for any $0 < \epsilon < H$, there exists a nonnegative random variable G_ϵ such that $\boldsymbol{E}[G_\epsilon^p] < \infty$ for any $p \ge 1$ and

$$|B_t(\omega) - B_s(\omega)| \le G_\epsilon(\omega)|t-s|^{H-\epsilon} \tag{3.8}$$

for any $0 \le s, t < \infty$ and $\omega \in \Omega_0$.

Assume that $1/3 < H < 1$. From Proposition 3.6 and the Hölder continuity of fBm (3.8), we see existence of a unique solution to the SDE (1.1) in the pathwise sense. More precisely, since $B(\omega)$ for any $\omega \in \Omega_0$ is $(H-\epsilon)$-Hölder continuous, a solution X to (1.1) is give by (3.3) and it is unique in sense of Proposition 3.6. In the same way as x, we shall also write $X(\xi)$, $X(B)$, or $X(\xi, B)$ to emphasize dependence on the initial value ξ and/or the driver B.

Proposition 3.16. *Assume that* $b \in C^1_{\mathrm{bdd}}(\mathbf{R};\mathbf{R})$ *and* $\sigma \in C^2_{\mathrm{bdd}}(\mathbf{R};\mathbf{R})$. *Then there exists a unique solution* X *to* (1.1) *and the following are satisfied:*

(1) X *is adapted to the fBm filtration* $\{\mathcal{F}_t\}_{0\le t\le 1}$, *where* $\mathcal{F}_t = \sigma(B_u; 0 \le u \le t)$,

(2) $t \mapsto X_t$ *is* $(H-\epsilon)$*-Hölder continuous a.s. for every* $0 < \epsilon < H$,

(3) *for any* $r \ge 1$, *there exists a positive constant* C *such that*

$$\boldsymbol{E}[|X_t - X_s|^r]^{1/r} \le C(t-s)^H$$

for any $0 \le s < t \le 1$.

Proof. The first assertion follows from Proposition 3.6. We show the second and third assertion. We decompose $X_t - X_s$ into $\{\phi(a_t^B, B_t) - \phi(a_s^B, B_t)\} + \{\phi(a_s^B, B_t) - \phi(a_s^B, B_s)\}$. From Propositions 3.5 and 3.6, we have

$$|\phi(a_t^B, B_t) - \phi(a_s^B, B_t)| \leq e^{c_1|B_t|} \int_s^t c_2 e^{c_3|B_u|}\, du,$$
$$|\phi(a_s^B, B_t) - \phi(a_s^B, B_s)| \leq c_4|B_t - B_s|,$$

where c_1, c_2, c_3, c_4 are positive constants. The proof is completed. □

4. Convergence of variation functionals

Let $B = \{B_t\}_{0\leq t\leq 1}$ be an fBm with the Hurst $1/3 < H < 1$ and $X = \{X_t\}_{0\leq t\leq 1}$ the solution to (1.1) given by (3.3). We assume that $b, \sigma \in C^\infty_{\mathrm{bdd}}(\mathbf{R};\mathbf{R})$. For these processes, we define the weighted Hermite variations and the trapezoidal error variations. The purpose of this section is to present necessary results for asymptotics of the variations.

Let $f \in C^{2q}_{\mathrm{poly}}(\mathbf{R};\mathbf{R})$ for $q \geq 2$ and $g \in C^2_{\mathrm{poly}}(\mathbf{R};\mathbf{R})$. Let μ be a probability measure on $[0, 1]$. For every $0 \leq s < t \leq 1$ and continuous path $x : [0, 1] \to \mathbf{R}$, define

$$F_{st}(x) \equiv F^{f,\mu}_{st}(x) := \int_0^1 f(\theta x_t + (1-\theta)x_s)\,\mu(d\theta).$$

We define the weighted Hermite variations $U^{(m)}_q(t) \equiv U^{(m)}_{q,f,\mu}(t)$ by

$$U^{(m)}_q(t) = \sum_{k=1}^{\lfloor 2^m t\rfloor} F_{\tau^m_{k-1}\tau^m_k}(X) H_q(2^{mH} B_{\tau^m_{k-1}\tau^m_k})$$

and the trapezoidal error variations $\tilde{U}^{(m)}(t) \equiv \tilde{U}^{(m)}_g(t)$ by

$$\tilde{U}^{(m)}(t) = \sum_{k=1}^{\lfloor 2^m t\rfloor} g(X_{\tau^m_{k-1}}) \left(\frac{1}{2\cdot 2^m} B_{\tau^m_{k-1}\tau^m_k} - \int_{\tau^m_{k-1}}^{\tau^m_k} B_{\tau^m_{k-1}u}\, du \right).$$

Here, $B_{st} = B_t - B_s$ for $0 \leq s < t \leq 1$ and H_q is the q-th Hermite polynomial defined by

$$H_q(\xi) = (-1)^q e^{\xi^2/2} \frac{d^q}{d\xi^q} e^{-\xi^2/2}.$$

The first few Hermite polynomials are $H_1(\xi) = \xi$, $H_2(\xi) = \xi^2 - 1$, $H_3(\xi) = \xi^3 - 3\xi$, and $H_4(\xi) = \xi^4 - 6\xi^2 + 3$. We set $H_0(\xi) = 1$ by convention.

The following limit theorems are vital for our proof. These results are proved in Appendixes B and C.

Theorem 4.1. *Let $q \geq 2$ be even. We have*

$$\lim_{m\to\infty} 2^{m(qH-1)} \sum_{k=1}^{\lfloor 2^m \cdot\rfloor} F_{\tau^m_{k-1}\tau^m_k}(X)(B_{\tau^m_{k-1}\tau^m_k})^q = \boldsymbol{E}[Z^q] \int_0^{\cdot} f(X_s)\, ds$$

in probability with respect to the uniform norm. Here Z is a standard Gaussian random variable.

Theorem 4.2. *Let $q \geq 2$ and $1/2q < H < 1 - 1/2q$. We have*

$$\lim_{m\to\infty} \left(B, 2^{-m/2} U_q^{(m)}\right) = \left(B, \sigma_{q,H} \int_0^\cdot f(X_s)\, dW_s\right)$$

weakly in the Skorokhod topology, where $\sigma_{q,H}$ is a constant defined by (4.1) *and W is a standard Brownian motion independent of B.*

Theorem 4.3. *Let $q \geq 2$ and $H = 1/2$. We have*

$$\lim_{m\to\infty} \left(B, 2^{-m/2} U_q^{(m)}, 2^m \tilde{U}^{(m)}\right) = \left(B, \sqrt{q!} \int_0^\cdot f(X_s)\, dW_s, \frac{1}{\sqrt{12}} \int_0^\cdot g(X_s)\, d\tilde{W}_s\right)$$

weakly in the Skorokhod topology, where W and $\tilde{W}$ are standard Brownian motions and B, W and $\tilde{W}$ are independent.

Proposition 4.4. *If $0 < H < 1/2$ (resp. $1/2 \leq H < 1$), then the process $2^{mr}\tilde{U}^{(m)}$ for $0 < r < 2H$ (resp. $0 < r < 1$) converges to the process 0 in probability with respect to the uniform norm.*

In order to prove Theorems 4.2 and 4.3, we use a simplified version of them. Let $q \geq 2$. We set

$$V_q^{(m)}(t) = 2^{-m/2} \sum_{k=1}^{\lfloor 2^m t \rfloor} H_q(2^{mH} B_{\tau_{k-1}^m \tau_k^m})$$

and

$$\tilde{V}^{(m)}(t) = 2^{-m/2} \sum_{k=1}^{\lfloor 2^m t \rfloor} 2^{m(H+1)} \left(\frac{1}{2 \cdot 2^m} B_{\tau_{k-1}^m \tau_k^m} - \int_{\tau_{k-1}^m}^{\tau_k^m} B_{\tau_{k-1}^m u}\, du \right).$$

Then, we see $V_q^{(m)} = 2^{-m/2} U_{q,f,\mu}^{(m)}$ and $\tilde{V}^{(m)} = 2^{m(H+1/2)} \tilde{U}_g^{(m)}$ for $f = g \equiv 1$ and the following:

Proposition 4.5. *Assume $q \geq 2$ and $0 < H < 1 - 1/2q$. Then we have*

$$\lim_{m\to\infty} (B, V_q^{(m)}, \tilde{V}^{(m)}) = (B, \sigma_{q,H} W, \tilde{\sigma}_H \tilde{W})$$

weakly in the Skorokhod topology. Here W and $\tilde{W}$ are independent standard Brownian motions independent of B, and $\sigma_{q,H}$ and σ_H are positive constants given by

$$\sigma_{q,H}^2 = q! \left(1 + 2 \sum_{l=1}^{\infty} \rho_H(l)^q \right), \tag{4.1}$$

$$\tilde{\sigma}_H^2 = \frac{1}{4} \frac{1-H}{1+H} + 2 \sum_{l=1}^{\infty} \tilde{\rho}_H(l)$$

with

$$\rho_H(l) = \boldsymbol{E}[B_1(B_{l+1} - B_l)] = \frac{1}{2}(|l+1|^{2H} + |l-1|^{2H} - 2|l|^{2H}),$$

$$\tilde{\rho}_H(l) = \boldsymbol{E}\left[\left(\frac{1}{2} B_1 - \int_0^1 B_u\, du \right) \left(\frac{1}{2}(B_{l+1} - B_l) - \int_l^{l+1} (B_u - B_l)\, du \right) \right].$$

We close this section with making remarks on results above:

Remark 4.6. (1) In Appendix B, we show Proposition 4.5 by showing relative compactness (Lemma B.5) and convergence in the sense of finite-dimensional distributions (Lemma B.6). In the proof of Lemma B.6, we show independence of B, W and $\tilde{W}$ by using the multidimensional fourth moment theorem by Peccati and Tudor [20].

(2) In Appendix C, we show Theorems 4.1, 4.2 and 4.3 and Proposition 4.5. In order to prove Theorems 4.1, 4.2 and 4.3, we use good properties of the solution X: for example, the continuity of the solution map $B \mapsto X$, the continuity of the map $t \mapsto X_t$ and Malliavin differentiability of X_t. In addition, Proposition 4.5 is essential for Theorems 4.2 and 4.3. Since Proposition 4.5 is a consequence of the fourth moment theorem, these theorems are also consequences of it.

(3) Theorems 4.1 and 4.2 are slight extensions of [6, Theorem 2.1], [17, Theorem 1] and [13, Theorem 15]. In these references, the authors showed convergences of the weighted Hermite variations $U_q^{(m)}$ in which $F_{\tau_{k-1}^m \tau_k^m}(X)$ are replaced by $f(B_{\tau_{k-1}^m})$ or $F_{\tau_{k-1}^m \tau_k^m}(B)$, that is, they considered functionals which are expressed by fBm B explicitly. On the other hand, we consider functionals of the solution X to (1.1) in Theorems 4.1 and 4.2. Theorem 4.3 is an exention of Theorem 4.2 in the case $H = 1/2$.

(4) Since a standard Brownian motion has independent increments, we see $\rho_{1/2}(l) = 0$ and $\tilde{\rho}_{1/2}(l) = 0$ for $l \geq 1$. Hence we have $\sigma_{q,1/2} = \sqrt{q!}$ and $\sigma_{1/2} = 1/\sqrt{12}$.

5. The Crank-Nicolson scheme

In this section, we show Theorem 2.7. Below, we fix sufficiently small $0 < \epsilon < H$ and write $H^- = H - \epsilon$. For $m \in \mathbf{N}$, we may write $\Delta = 2^{-m}$, $\Delta B_k = B_{\tau_{k-1}^m \tau_k^m}$ $(1 \leq k \leq 2^m)$, $\Delta(\Delta B_k)^n = \Delta \cdot (\Delta B_k)^n$ $(n = 1, 2, \dots)$ and $\Delta(\Delta B_k) = \Delta(\Delta B_k)^1$. We use the notation $B_{st}^{\boldsymbol{i}}$ $(\boldsymbol{i} = 10, 01, 011, 101, 110)$ to denotes the iterated integral introduced in Section 3.2. We denote by $O(\Delta^p)$ the term which is less than or equal to $C\Delta^p$, where C does not depend on m and ξ.

5.1. Well-definedness of the Crank-Nicolson scheme. Since the Crank-Nicolson scheme is an implicit scheme, we need to define the set on which the scheme can be defined. Recall that $(\Omega, \mathcal{F}, \boldsymbol{P})$ denotes the canonical probability space which defines fBm $B(\omega)$ with the Hurst H and

$$\Omega_0 = \bigcap_{0<\epsilon<H} \{\omega \in \Omega; B(\omega) \in \mathscr{C}_0^{H-\epsilon}([0, 1]; \mathbf{R})\}.$$

For every $m \in \mathbf{N}$, we define

$$\Omega^{\mathrm{CN}(m)} = \Omega_0 \cap \left\{ \omega \in \Omega; \sup_{|t-s| \leq 2^{-m}} \frac{|B_t(\omega) - B_s(\omega)|}{(t-s)^{H-\epsilon}} \leq 1 \right\}.$$

Note that $\Omega^{\mathrm{CN}(m)} \subset \Omega^{\mathrm{CN}(m+1)}$ for any m and $\lim_{m\to\infty} P(\Omega^{\mathrm{CN}(m)}) = 1$ for the fBm with the Hurst H. We show that the Crank-Nicolson scheme is defined on $\Omega^{\mathrm{CN}(m)}$ for large m.

Proposition 5.1. *Suppose*

$$m > \max\left\{1 + \log_2(\sup|b'|), \frac{1 + \log_2(\sup|\sigma'|)}{H - \epsilon}\right\}. \tag{5.1}$$

Let $0 < s < t < 1$ *satisfy* $|t - s| \le 2^{-m}$. *Then for any* $\xi \in \mathbf{R}$ *and* $\omega \in \Omega^{\mathrm{CN}(m)}$, *there exists a unique* η_t *satisfying*

$$\eta_t = \xi + \frac{b(\xi) + b(\eta_t)}{2}(t - s) + \frac{\sigma(\xi) + \sigma(\eta_t)}{2}(B_t(\omega) - B_s(\omega)).$$

Proof. Set

$$F(\xi, \delta, \Delta; \eta) = \eta - \left[\xi + \frac{1}{2}\{b(\xi) + b(\eta)\}\delta + \frac{1}{2}\{\sigma(\xi) + \sigma(\eta)\}\Delta\right].$$

If $|\delta| < 1/(2\sup|b'|)$ and $|\Delta| < 1/(2\sup|\sigma'|)$, then $[\partial F/\partial\eta](\xi, \delta, \Delta; \eta) = 1 - \{(1/2)b'(\eta)\delta + (1/2)\sigma'(\eta)\Delta\}$ satisfies

$$\frac{\partial F}{\partial \eta}(\xi, \delta, \Delta; \eta) \ge 1 - \frac{1}{2}|b'(\eta)||\delta| - \frac{1}{2}|\sigma'(\eta)||\Delta| \ge \frac{1}{2},$$

which implies that $\eta \mapsto F(\xi, \delta, \Delta; \eta)$ is strictly increasing. Hence there exists a unique value $f(\xi, \delta, \Delta)$ such that $F(\xi, \delta, \Delta; f(\xi, \delta, \Delta)) = 0$ and $f(\xi, 0, 0) = \xi$.

Under the assumption on m and s, t, it holds that $t - s < 1/(2\sup|b'|)$ and $|B_t(\omega) - B_s(\omega)| < 1/(2\sup|\sigma'|)$ ($\omega \in \Omega^{\mathrm{CN}(m)}$). Hence η_t is uniquely defined as $\eta_t = f(\xi, t - s, B_t(\omega) - B_s(\omega))$. □

Remark 5.2. Clearly, the implicit function $f(\xi, \delta, \Delta)$ ($\xi \in \mathbf{R}, |\delta| < 1/(2\sup|b'|), |\Delta| < 1/(2\sup|\sigma'|)$) is a C^∞ function.

5.2. Proof of Theorem 2.7. The Crank-Nicolson approximation solution $\bar{X}^{(m)}$ can be defined on $\Omega^{\mathrm{CN}(m)}$ for m in (5.1). From now on, we assume m satisifes (5.1). For $\omega \notin \Omega^{\mathrm{CN}(m)}$, we always set $\bar{X}^{(m)}_t(\xi, B) \equiv \xi$.

To study the error $\bar{X}^{(m)} - X$, we prove that there exists a piecewise linear path h such that $X_{\tau^m_k}(\xi, B+h) = \bar{X}^{(m)}_{\tau^m_k}(\xi, B)$ for all $0 \le k \le 2^m$. Let h be a piecewise linear path defined on $[0, 1]$ with $h_0 = 0$ whose partition points are dyadic points $\{\tau^m_k\}_{k=0}^{2^m}$. Then h can be identified with the set of values at the partition points $\{h(\tau^k_k)\}_{k=1}^{2^m}$. We write $\kappa_k = h(\tau^m_k) - h(\tau^m_{k-1})$ $(1 \le k \le 2^m)$.

Lemma 5.3. *Let* $\omega \in \Omega_0$. *Then there exist unique* $\kappa_k \in \mathbf{R}$ $(1 \le k \le 2^m)$ *such that*

$$\bar{X}^{(m)}_{\tau^m_k}(\xi, B) = X_{\tau^m_k}(\xi, B + h), \qquad 1 \le k \le 2^m.$$

We denote the above h by $h^{(m)}$. Although κ_k depends on m similarly, we use the same notation κ_k for simplicity. $h^{(m)}(\omega)$ is defined for all $\omega \in \Omega_0$. Of course, the definition of $\bar{X}^{(m)}$ on $\Omega_0 \setminus \Omega^{\mathrm{CN}(m)}$ is essentially meaningless and the behavior of $h^{(m)}$ on $\Omega_0 \setminus \Omega^{\mathrm{CN}(m)}$ has nothing to do with the asymptotics of the error. Before proving the existence of $h^{(m)}$, we give a rough sketch how to prove Theorem 2.7 by using $h^{(m)}$.

Remark 5.4 (Rough sketch of the proof of Theorem 2.7). We decompose $h^{(m)}$ as $h^{(m)} = h^{(m)}_M + h^{(m)}_R$. Here, $h^{(m)}_M$ is the main term and we see

(5.2) $$\lim_{m\to\infty} 2^{m(3H-\frac{1}{2})} h_M^{(m)} = U \qquad \text{in law},$$

where U is a random variable. The term $h_R^{(m)}$ is the remainder term satisfying that for small $\delta > 0$,

(5.3) $$\lim_{m\to\infty} 2^{m(3H-\frac{1}{2}+\delta)} \|h_R^{(m)}\|_\infty = 0 \qquad \text{in probability}$$

By using the derivative of $X(\xi, B)$ with respect to B, we have $2^{m(3H-\frac{1}{2})}\{\bar{X}^{(m)}(\xi, B) - X(\xi, B)\} = I_1 + I_2 + I_3$, where

$$\begin{aligned} I_1 &= \nabla_{(2^m)^{3H-\frac{1}{2}} h_M^{(m)}} X(\xi, B), \\ I_2 &= (2^m)^{3H-\frac{1}{2}} \left\{ \bar{X}^{(m)}(\xi, B) - X(\xi, B + h_M^{(m)}) \right\}, \\ I_3 &= (2^m)^{3H-\frac{1}{2}} \left\{ X(\xi, B + h_M^{(m)}) - X(\xi, B) - \nabla_{h_M^{(m)}} X(\xi, B) \right\}. \end{aligned}$$

By the convergence $2^{m(3H-\frac{1}{2})} h_M^{(m)} \to U$ in law, we have $I_1 = \nabla_{2^{m(3H-\frac{1}{2})} h_M^{(m)}} X(\xi, B) \to \nabla_U X(\xi, B)$ in law. Since

$$I_2 \approx 2^{m(3H-\frac{1}{2})} \left\{ X(\xi, B + h^{(m)}) - X(\xi, B + h_M^{(m)}) \right\} \approx \nabla_{2^{m(3H-\frac{1}{2})} h_R^{(m)}} X(\xi, B),$$

the middle term converges to 0 in probability. For the third term, considering the second derivative, we have

$$I_3 \approx 2^{m(3H-\frac{1}{2})} \frac{1}{2} \nabla^2_{h_M^{(m)}} X(\xi, B).$$

Therefore this term also converges to 0 in probability because $h_M^{(m)}$ is of order $2^{-m(3H-\frac{1}{2})}$. In the following, $h_M^{(m)}$ and $h_R^{(m)}$ are piecewise linear paths corresponding to $\{\tilde{\kappa}_k\}$ and $\{R_k(\omega)\}$ in Lemma 5.7.

We conclude this remark by making a comment on (5.2) and (5.3). The convergence (5.2) of the main term is shown by Theorem 4.2 and so on in Lemma 5.10. By using this result, we see the convergence (5.3) of the remainder in Lemma 5.7. We should mention that the method used in Lemma 5.7 makes estimate of the remainder simpler drastically than that of [13].

We now prove the existence of $h^{(m)}$. To this end, we need the bijectivity of the map $\kappa \mapsto X_t(\xi, B + \kappa\ell)$ which follows from the following lemma. Here $\ell_t = t$. This lemma is an immediate consequence of Proposition 3.12 (3).

Lemma 5.5. *There exist positive numbers C_1, C_2 which are independent of B, ξ, t such that*

$$C_1 t \le \frac{d}{d\kappa} X_t(\xi, B + \kappa\ell) \le C_2 t.$$

In particular, the mapping $\mathbf{R} \ni \kappa \mapsto X_t(\xi, B + \kappa\ell)$ is bijection on $\mathbf{R}$.

We prove Lemma 5.3. We write $\xi_k = \bar{X}^{(m)}_{\tau_k^m}(\xi, B)$.

Proof of Lemma 5.3. We prove this by an induction on k. Let $k = 1$. It suffices to prove the existence κ_1 satisfying $\xi_1 = X_{2^{-m}}(\xi, B + 2^m \kappa_1 \ell)$. Since $\kappa \mapsto X_{2^{-m}}(\xi, B + 2^m \kappa\ell)$ is a bijective mapping, κ_1 is uniquely determined. Suppose the equality holds upto k. Noting

$\xi_{k+1} = X_{\tau^m_{k+1}}(\xi, B+h)$ is equivalent to $\xi_{k+1} = X_{2^{-m}}(\xi_k, \theta_{\tau^m_k} B + 2^m \kappa_{k+1}\ell)$ and by applying Lemma 5.5, the proof is completed. □

In the rest of this subsection, we state some key lemmas (Lemmas 5.6, 5.7 and 5.10) for Theorem 2.7 and show the theorem. The key lemmas is shown in the next subsection. In these lemmas, we calculate κ_k and determine the main term of the error. By the definition, κ_k $(1 \le k \le 2^m)$ satisfies the equation

(5.4)

$$X_{2^{-m}}(\xi_{k-1}, \theta_{\tau^m_{k-1}} B + 2^m \kappa_k \ell) - X_{2^{-m}}(\xi_{k-1}, \theta_{\tau^m_{k-1}} B) = \{\xi_k - \xi_{k-1}\} - \{X_{2^{-m}}(\xi_{k-1}, \theta_{\tau^m_{k-1}} B) - \xi_{k-1}\}.$$

We set $\hat{\kappa}_k$ by the left-hand side of the above equality. The quantity $\hat{\kappa}_k$ is the 1-step error of the Crank-Nicolson scheme. We calculate $\hat{\kappa}_k$ and κ_k with small remainder terms. By this calculation and the Hölder continuity of B, we see that $\max_{1\le k\le 2^m} |\bar{X}^{(m)}_{\tau^m_k} - X_{\tau^m_k}|$ converges to 0 if $H > \frac{1}{3}$ (Lemma 5.6). This is a rough estimate. We improve it later by identifying the main term of the error (Lemma 5.7).

In order to express $\hat{\kappa}_k$, we introduce

$$\hat{f}_3 = \frac{1}{12}[\sigma^2\sigma'' + \sigma(\sigma')^2], \quad \hat{f}_4 = \frac{1}{24}[\sigma^3\sigma''' + 5\sigma^2\sigma'\sigma'' + 2\sigma(\sigma')^3], \quad \hat{g}_1 = \mathsf{w},$$
$$\hat{\varphi} = \frac{1}{4}\left[b(\sigma')^2 + \sigma^2 b''\right] + \frac{1}{2}[b\sigma\sigma'' + \sigma\sigma' b'],$$
$$\hat{\varphi}_{011} = -b(\sigma\sigma')', \quad \hat{\varphi}_{101} = -\sigma(b\sigma')', \quad \hat{\varphi}_{110} = -\sigma(\sigma b')'.$$

Here, we recall $\mathsf{w} = \sigma b' - \sigma' b$. We also see that the main term of κ_k is expressed by the following functions:

$$f_3 = \frac{1}{12}[\sigma\sigma'' + (\sigma')^2], \quad f_4 = \frac{1}{24}\sigma(\sigma\sigma''' + 3\sigma'\sigma''), \quad g_1 = \frac{\mathsf{w}}{\sigma},$$
$$\varphi = \frac{1}{4}\left[\frac{b(\sigma')^2}{\sigma} + \sigma b''\right] + \frac{1}{2}(b\sigma'' + \sigma' b'),$$
$$\varphi_{011} = -\frac{b(\sigma\sigma')'}{\sigma}, \quad \varphi_{101} = -(b\sigma')', \quad \varphi_{110} = -(\sigma b')'.$$

Note that $f_4 = (\hat{f}_4 - \sigma'\hat{f}_3)/\sigma$ and that $h = \hat{h}/\sigma$ for $h = f_3, g_1, \phi, \phi_{011}, \phi_{101}, \phi_{110}$. By a simple calculation, we have $f_4 = \sigma f_3'/2$. This identity is a key for the convergence of the main term of the error similarly to the case where $b \equiv 0$ ([14, 13]); see Lemma 5.10.

The expression of $\hat{\kappa}_k$ and the convergence of $\max_{1\le k\le 2^m} |\bar{X}^{(m)}_{\tau^m_k} - X_{\tau^m_k}|$ are obtained as follows:

Lemma 5.6. *For any $\omega \in \Omega^{\mathrm{CN}(m)}$, the following hold.*

(1) *We have*

$$\begin{aligned}\hat{\kappa}_k &= \hat{f}_3(\xi_{k-1})(\Delta B_k)^3 + \hat{f}_4(\xi_{k-1})(\Delta B_k)^4 + \hat{g}_1(\xi_{k-1})\left(\frac{\Delta}{2}\Delta B_k - B^{10}_{\tau^m_{k-1}\tau^m_k}\right)\\ &\quad + \hat{\varphi}(\xi_{k-1})\Delta(\Delta B_k)^2 + \hat{\varphi}_{011}(\xi_{k-1})B^{011}_{\tau^m_{k-1}\tau^m_k} + \hat{\varphi}_{101}(\xi_{k-1})B^{101}_{\tau^m_{k-1}\tau^m_k} + \hat{\varphi}_{110}(\xi_{k-1})B^{110}_{\tau^m_{k-1}\tau^m_k}\\ &\quad + O(\Delta^{5H^-}) + O(\Delta^{3H^-+1}) + O(\Delta^{H^-+2}).\end{aligned}$$

(2) *We have $\hat{\kappa}_k = O(\Delta^{3H^-})$, $\kappa_k = O(\Delta^{3H^-})$ and*

$$\max_{1\le k\le 2^m}|X_{\tau_k^m}(\xi,B)-\bar{X}^{(m)}_{\tau_k^m}(\xi,B)|=O(\Delta^{3H^- -1}).$$

In particular, the Crank-Nicolson approximation solution converges to the solution itself at the partition points uniformly if $H>\frac{1}{3}$.

(3) *We have*

$$\max_{0\le t\le 1}|\bar{X}^{(m)}_t(\xi,B)-X_t(\xi,B+h^{(m)})|=O(\Delta^{3H^-}).$$

The next lemma asserts that $\tilde{\kappa}_k$ is the main term of κ_k. As stated in Remark 5.4, in order to prove it, we use not only the Hölder regularity of B but also the convergence in law of the main term of $h^{(m)}$.

Lemma 5.7. *For* $1\le k\le 2^m$, *let*

$$\begin{aligned}\tilde{\kappa}_k&=f_3(X_{\tau_{k-1}^m})(\Delta B_k)^3+f_4(X_{\tau_{k-1}^m})(\Delta B_k)^4+g_1(X_{\tau_{k-1}^m})\left(\frac{\Delta}{2}\Delta B_k-B^{10}_{\tau_{k-1}^m\tau_k^m}\right)\\&\quad+\varphi(X_{\tau_{k-1}^m})\Delta(\Delta B_k)^2+\varphi_{011}(X_{\tau_{k-1}^m})B^{011}_{\tau_{k-1}^m\tau_k^m}+\varphi_{101}(X_{\tau_{k-1}^m})B^{101}_{\tau_{k-1}^m\tau_k^m}+\varphi_{110}(X_{\tau_{k-1}^m})B^{110}_{\tau_{k-1}^m\tau_k^m}\end{aligned}$$

and set $R_k(\omega)=\kappa_k-\tilde{\kappa}_k$. *Then there exists* $\delta>0$ *such that* $\lim_{m\to\infty}(2^m)^{3H-\frac{1}{2}+\delta}\max_{1\le k\le 2^m}|\sum_{i=1}^k R_i|=0$ *in probability.*

Remark 5.8. Although $\tilde{\kappa}_k$ and κ_k are defined on Ω_0, the definition of κ_k on $\Omega_0\setminus\Omega^{\mathrm{CN}(m)}$ is essentially meaningless. However, the statement of the convergence of R_k makes sense because $\lim_{m\to\infty}\boldsymbol{P}(\Omega^{\mathrm{CN}(m)})=1$.

The following processes are candidates of the main term of $h^{(m)}$:

$$\left\{\begin{aligned}\Phi_1(t)&=\sum_{k=1}^{\lfloor 2^m t\rfloor}\left\{f_3(X_{\tau_{k-1}^m})(\Delta B_k)^3+f_4(X_{\tau_{k-1}^m})(\Delta B_k)^4\right\},\\ \Phi_2(t)&=\sum_{k=1}^{\lfloor 2^m t\rfloor}g_1(X_{\tau_{k-1}^m})\left(\frac{\Delta}{2}\Delta B_k-B^{10}_{\tau_{k-1}^m\tau_k^m}\right),\\ \Phi_3(t)&=\sum_{k=1}^{\lfloor 2^m t\rfloor}\Big\{\varphi(X_{\tau_{k-1}^m})\Delta(\Delta B_k)^2+\varphi_{011}(X_{\tau_{k-1}^m})B^{011}_{\tau_{k-1}^m\tau_k^m}\\&\qquad+\varphi_{101}(X_{\tau_{k-1}^m})B^{101}_{\tau_{k-1}^m\tau_k^m}+\varphi_{110}(X_{\tau_{k-1}^m})B^{110}_{\tau_{k-1}^m\tau_k^m}\Big\},\\ \Phi_4(t)&=-\sum_{k=1}^{\lfloor 2^m t\rfloor}[g_1\sigma'](X_{\tau_{k-1}^m})\Delta B_k\left(\frac{\Delta}{2}\Delta B_k-B^{10}_{\tau_{k-1}^m\tau_k^m}\right).\end{aligned}\right. \tag{5.5}$$

Remark 5.9. The processes Φ_1, Φ_2 and Φ_3 are arising from the expression of $\tilde{\kappa}_k$. In order to prove Lemma 5.7, it is necessary to consider Φ_4 together.

By using Theorem 4.2, Theorem 4.3 and Proposition 4.4, we can show the next lemma, which gives us asymptotic of Φ_1, Φ_2, Φ_3 and Φ_4.

Lemma 5.10. *Let* W *and* $\tilde{W}$ *be standard Brownian motions. Assume that* B, W *and* $\tilde{W}$ *are independent. The next assertions hold.*

(1) *Let* $\frac{1}{3} < H < \frac{1}{2}$. *Then* $\left(B, (2^m)^{3H-\frac{1}{2}}(\Phi_1, \Phi_2, \Phi_3, \Phi_4)\right)$ *converges weakly to* $\left(B, \sigma_{3,H} \int_0^\cdot f_3(X_t)dW_t, 0, 0, 0\right)$ *in* $D([0,1]; \mathbf{R}^4)$ *with respect to the Skorokhod topology. Here,* $\sigma_{3,H}$ *is a constant defined by* (4.1).

(2) *Let* $H = \frac{1}{2}$. *Then* $(B, 2^m(\Phi_1, \Phi_2, \Phi_3, \Phi_4))$ *converges weakly to*

$$\left(B, \sqrt{6}\int_0^\cdot f_3(X_s)\,dW_s + 3\int_0^\cdot f_3(X_s)\circ dB_s, \frac{1}{\sqrt{12}}\int_0^\cdot g_1(X_s)d\tilde{W}_s,\right.$$
$$\left.\int_0^\cdot \varphi(X_s)\,ds + \frac{1}{4}\int_0^\cdot \{\varphi_{011}(X_s) + \varphi_{110}(X_s)\}\,ds, 0\right)$$

in $D([0,1]; \mathbf{R}^4)$ *with respect to the Skorokhod topology.*

We are in a position to show Theorem 2.7. Proofs of Lemmas 5.6, 5.7 and 5.10 are postponed in Section 5.3.

Proof of Theorem 2.7. We follow the idea in Remark 5.4. Let $h_M^{(m)}$ and $h_R^{(m)}$ be piecewise linear paths associated with $\{\tilde{\kappa}_k\}$ and $\{R_k\}$, respectively, in Lemma 5.7. By Lemma 5.10, we have the weak convergence in the Skorokhod topology in $D([0,1]; \mathbf{R}^2)$,

$$\left(B, (2^m)^{3H-\frac{1}{2}}(\Phi_1 + \Phi_2 + \Phi_3)\right) \to (B, U),$$

where U is the same process defined in Theorem 2.7. Since $h_M^{(m)}$ is a piecewise linear and $\Phi_1 + \Phi_2 + \Phi_3$ is step function, we have

$$\|h_M^{(m)} - (\Phi_1 + \Phi_2 + \Phi_3)\|_\infty = O(\Delta^{3H^-}) \qquad \omega \in \Omega^{\mathrm{CN}(m)}.$$

Hence $\lim_{m\to\infty}(2^m)^{3H-\frac{1}{2}}\|h_M^{(m)} - (\Phi_1 + \Phi_2 + \Phi_3)\|_\infty = 0$ in probability. Consequently, we have the weak convergence in the uniform convergence topology in $C([0,1]; \mathbf{R}^3)$:

$$\left(B, (2^m)^{3H-\frac{1}{2}} h_M^{(m)}\right) \to (B, U). \tag{5.6}$$

As stated in Remark 5.4, we have $(2^m)^{3H-\frac{1}{2}}\{\bar{X}^{(m)}(\xi, B) - X(\xi, B)\} = I_1 + I_2 + I_3$, where

$$\begin{aligned}
I_1 &= \nabla_{(2^m)^{3H-\frac{1}{2}} h_M^{(m)}} X(\xi, B),\\
I_2 &= (2^m)^{3H-\frac{1}{2}}\left\{\bar{X}^{(m)}(\xi, B) - X(\xi, B + h_M^{(m)})\right\},\\
I_3 &= (2^m)^{3H-\frac{1}{2}}\left\{X(\xi, B + h_M^{(m)}) - X(\xi, B) - \nabla_{h_M^{(m)}} X(\xi, B)\right\}.
\end{aligned}$$

We consider I_2 and I_3 first. By Taylor's theorem, we have

$$\begin{aligned}
|\bar{X}_t^{(m)}(\xi, B) - X_t(\xi, B + h_M^{(m)})| &\le |\bar{X}_t^{(m)}(\xi, B) - X_t(\xi, B + h^{(m)})|\\
&\quad + |X_t(\xi, B + h^{(m)}) - X_t(\xi, B + h_M^{(m)})|\\
&\le |\bar{X}_t^{(m)}(\xi, B) - X_t(\xi, B + h^{(m)})| + \left|\int_0^1 \nabla_{h_R^{(m)}} X_t(\xi, B + \theta h_R^{(m)})\,d\theta\right|.
\end{aligned}$$

By using Lemma 5.6 (3) and the boundedness of the derivative, we have

$$\|\bar{X}^{(m)}(\xi, B) - X(\xi, B + h_M^{(m)})\|_\infty \le C\{\Delta^{3H^-} + \|h_R^{(m)}\|_\infty\}.$$

Here C is a constant independent of m. Combining this and Lemma 5.7, we have $\|I_2\|_\infty$ converges to 0 in probability. Similarly, we have

$$\|I_3\|_\infty \le C(2^m)^{3H-\frac{1}{2}}\|h_M^{(m)}\|_\infty^2 \to 0 \qquad \text{in probability.}$$

We next consider the main term I_1. Let $J_t(g)$ be the continuous path defined by g in (3.4). By Remark 3.8, the mapping $g \mapsto J(g)$ is continuous on $C([0,1];\mathbf{R})$. From this, we have the continuity of the mapping

$$C([0,1];\mathbf{R}^2) \ni (g,z) \mapsto \sigma(x(g))z + J(g)\int_0^{\cdot} J_s^{-1}(g)\mathsf{w}(x_s(g))z_s\,ds \in C([0,1];\mathbf{R}).$$

Combining Proposition 3.12, (5.6) and the above, we complete the proof. □

5.3. Proof of key lemmas. In the rest of this section, we show Lemmas 5.6, 5.7 and 5.10. Lemma 5.6 follows from the next lemma immediately:

Lemma 5.11. *For any $\omega \in \Omega^{\mathrm{CN}(m)}$, the following hold.*

(1) *We have*

$$\begin{aligned}
\xi_k - \xi_{k-1} &= b(\xi_{k-1})\Delta + \sigma(\xi_{k-1})\Delta B_k + \frac{1}{2}[\sigma'\sigma](\xi_{k-1})(\Delta B_k)^2 \\
&+ \frac{1}{4}\left[\sigma(\sigma')^2 + \sigma^2\sigma''\right](\xi_{k-1})(\Delta B_k)^3 + \left[\frac{1}{12}\sigma'''\sigma^3 + \frac{3}{8}\sigma^2\sigma'\sigma'' + \frac{1}{8}\sigma(\sigma')^3\right](\xi_{k-1})(\Delta B_k)^4 \\
&+ \frac{1}{2}\left[\sigma' b + \sigma b'\right](\xi_{k-1})\Delta(\Delta B_k) + \frac{1}{4}\left[(b(\sigma')^2 + \sigma^2 b'') + 2(\sigma b\sigma'' + \sigma\sigma' b')\right](\xi_{k-1})\Delta(\Delta B_k)^2 \\
&+ \frac{1}{2}[bb'](\xi_{k-1})\Delta^2 + O(\Delta^{5H^-}) + O(\Delta^{3H^-+1}).
\end{aligned}$$

(2) *We have*

$$\begin{aligned}
&X_\Delta(\xi_{k-1}, \theta_{\tau^m_{k-1}}B) - \xi_{k-1} \\
&\quad = b(\xi_{k-1})\Delta + \sigma(\xi_{k-1})\Delta B_k + \frac{1}{2}\left[\sigma\sigma'\right](\xi_{k-1})(\Delta B_k)^2 + \frac{1}{3!}\left[\sigma(\sigma\sigma')'\right](\xi_{k-1})(\Delta B_k)^3 \\
&\qquad + \frac{1}{4!}\left[\sigma(\sigma(\sigma\sigma')')'\right](\xi_{k-1})(\Delta B_k)^4 + [b\sigma'](\xi_{k-1})\Delta(\Delta B_k) + [\sigma b' - b\sigma'](\xi_{k-1})B^{10}_{\tau^m_{k-1}\tau^m_k} \\
&\qquad + b(\sigma\sigma')'(\xi_{k-1})B^{011}_{\tau^m_{k-1}\tau^m_k} + \sigma(b\sigma')'(\xi_{k-1})B^{101}_{\tau^m_{k-1}\tau^m_k} + \sigma(\sigma b')'(\xi_{k-1})B^{110}_{\tau^m_{k-1}\tau^m_k} \\
&\qquad + \frac{1}{2}[b'b](\xi_{k-1})\Delta^2 + O(\Delta^{5H^-}) + O(\Delta^{3H^-+1}) + O(\Delta^{H^-+2}).
\end{aligned}$$

Proof. (1) ξ_k is determined by the equation

$$\xi_k = \xi_{k-1} + \frac{\sigma(\xi_{k-1}) + \sigma(\xi_k)}{2}\Delta B_k + \frac{b(\xi_{k-1}) + b(\xi_k)}{2}\Delta. \tag{5.7}$$

Since the implicit function is C^∞ as in Remark 5.2, there exist constants $a_{1,0}, \ldots, a_{4,0}$, $a_{0,1}$, $a_{1,1}$, $a_{2,1}$ and $a_{0,2}$ such that

$$\begin{aligned}
\xi_k - \xi_{k-1} &= \sum_{i=1}^{4} a_{i,0}(\Delta B_k)^i + a_{0,1}\Delta + a_{1,1}\Delta(\Delta B_k) + a_{2,1}\Delta(\Delta B_k)^2 + a_{0,2}\Delta^2 \\
&\quad + O(\Delta^{3H^-+1}) + O(\Delta^{5H^-}).
\end{aligned}$$

Putting this expansion of ξ_k into the equation (5.7) and compare the coefficients of the both sides of equation, we obtain the desired formula.

(2) This is a immediate consequence of Proposition 3.11. □

Proof of Lemma 5.6. (1) The assertion follows from Lemma 5.11 and the definition of $\hat{\kappa}_k$.
(2) The estimate $\hat{\kappa}_k = O(\Delta^{3H^-})$ follows from (1) and the Hölder continuity of B. It follows that $\kappa_k = O(\Delta^{3H^-})$ from the estimate of $\hat{\kappa}_k$ and Lemma 5.5. By combining $\kappa_k = O(\Delta^{3H^-})$ and the Lipschitz continuity of the mapping $B \mapsto X(B)$, we obtain the last assertion.
(3) Since Lemma 5.11 for $\Delta = t - \tau^m_{k-1}$ is still valid, for $\tau^m_{k-1} < t \leq \tau^m_k$, we have

$$\begin{aligned}\bar{X}^{(m)}_t(\xi, B) - X_t(\xi, B + h^{(m)}) &= \{\bar{X}^{(m)}_t(\xi, B) - \xi_{k-1}\} - \{X_{t-\tau^m_{k-1}}(\xi_{k-1}, \theta_{\tau^m_{k-1}}(B + h^{(m)})) - \xi_{k-1}\} \\ &= O(h^{(m)}_t - h^{(m)}_{\tau^m_{k-1}}).\end{aligned}$$

Noting $O(h^{(m)}_t - h^{(m)}_{\tau^m_{k-1}}) = O(\kappa_k) = O(\Delta^{3H^-})$, we see the assertion. □

Next we show Lemma 5.10. To prove this lemma, we use the following results concerning the Skorokhod topology.

Proposition 5.12. *The following hold.*

(1) *The mapping* $D([0, 1]; \mathbf{R}^d) \ni (x_i)^d_{i=1} \mapsto (\sum^d_{i=1} x_i) \in D([0, 1]; \mathbf{R})$ *is continuous.*
(2) *The mapping* $D([0, 1]; \mathbf{R}^d) \ni x \mapsto \sup_{0\leq t\leq 1} |x_t| \in \mathbf{R}$ *is continuous.*
(3) *We assume random variables in this statement are defined in the same probability space. Let* $\{X_n\}^\infty_{n=1}$ *and* $\{Y_n\}^\infty_{n=1}$ *be random variables with values in* $C([0, 1]; \mathbf{R}^{d_1})$ *and* $D([0, 1]; \mathbf{R}^{d_2})$*, respectively. Let* $\{Z_n\}^\infty_{n=1}$ *be random variables with values in* $D([0, 1]; \mathbf{R}^{d_3})$*. Let* $\varphi : C([0, 1]; \mathbf{R}^{d_1}) \to C([0, 1]; \mathbf{R}^{d_4})$ *be a continuous mapping. Suppose that* $(X_n, Y_n) \in D([0, 1]; \mathbf{R}^{d_1+d_2})$ *converges to* (X, Y) *in law with respect to the Skorokhod topology and* $\|Z_n\|_\infty \to 0$ *in probability. Then* $(X_n, Y_n, \varphi(X_n), Z_n)$ *converges in law in the Skorokhod topology to* $(X, Y, \varphi(X), 0) \in D([0, 1]; \mathbf{R}^{d_1+d_2+d_3+d_4})$.

Proof of Lemma 5.10. First, we consider Φ_1 and Φ_2. Recalling $f_4 = \sigma f'_3/2$, we have $f_3(X_{\tau^m_{k-1}}) + f_4(X_{\tau^m_{k-1}})\Delta B_k = \{f_3(X_{\tau^m_{k-1}}) + f_3(X_{\tau^m_k})\}/2 + O(\Delta^{2H^-}) + O(\Delta)$. Hence

$$\begin{aligned}(2^m)^{3H-\frac{1}{2}} &\left\{f_3(X_{\tau^m_{k-1}})(\Delta B_k)^3 + f_4(X_{\tau^m_{k-1}})(\Delta B_k)^4\right\} \\ &= (2^m)^{-1/2} \frac{f_3(X_{\tau^m_{k-1}}) + f_3(X_{\tau^m_k})}{2} H_3(2^{mH}\Delta B_k) \\ &\quad + (2^m)^{H-1/2} \frac{f_3(X_{\tau^m_{k-1}}) + f_3(X_{\tau^m_k})}{2} 3\Delta B_k + R_{m,k}(B)\end{aligned}$$

where $R_{m,k}(B) = O(\Delta^{5H^- -3H+\frac{1}{2}}) + O(\Delta^{3H^- -3H+\frac{3}{2}})$. Note that $\lim_{m\to\infty} \sum^{2^m}_{k=1} |R_{m,k}| = 0$ for any $\omega \in \bigcup_m \Omega^{\mathrm{CN}(m)}$. By Proposition 3.4, we have

$$\left\|\sum^{\lfloor 2^m \cdot\rfloor}_{k=1} \frac{f_3(X_{\tau^m_{k-1}}) + f_3(X_{\tau^m_k})}{2} \Delta B_k - \int^{\cdot}_0 f_3(X_s)\, d^\circ B_s\right\|_\infty \to 0, \qquad \omega \in \bigcup_m \Omega^{\mathrm{CN}(m)}.$$

By Remark 3.8, the mapping $B \mapsto \int^{\cdot}_0 f_3(X_s)\, d^\circ B_s$ is continuous in the uniform norm. By Theorem 4.2, Theorem 4.3, Proposition 4.4 and Proposition 5.12 (3),

$$\begin{aligned}&\left(B,(2^m)^{3H-\frac{1}{2}}(\Phi_1,\Phi_2)\right)\\&\to\begin{cases}\left(B,\sqrt{6}\int_0^{\cdot} f_3(X_s)\,dW_s+3\int_0^{\cdot} f_3(X_s)\,d^{\circ}B_s,\frac{1}{\sqrt{12}}\int_0^{\cdot} g_1(X_s)\,d\tilde{W}_s\right), & H=1/2,\\ \left(B,\sigma_{3,H}\int_0^{\cdot} f_3(X_s)\,dW_s,0\right), & 1/3<H<1/2\end{cases}\end{aligned}$$

weakly in the Skorokhod topology. Note that $\sigma_{3,\frac{1}{2}}=\sqrt{6}$. (See Remark 4.6.)

Next, we consider Φ_3. Suppose $1/3<H<1/2$. By Lemma 3.9, for any $\omega\in\Omega^{\mathrm{CN}(m)}$,

$$(2^m)^{3H-\frac{1}{2}}\sum_{k=1}^{2^m}\left(|\Delta(\Delta B_k)^2|+|B^{011}_{\tau^m_{k-1}\tau^m_k}|+|B^{101}_{\tau^m_{k-1}\tau^m_k}|+|B^{110}_{\tau^m_{k-1}\tau^m_k}|\right)=O(\Delta^{2H^--3H+\frac{1}{2}}).$$

Hence $\|\Phi_3\|_\infty$ converges to 0 in probability. We consider the case $H=\frac{1}{2}$. Then we have

$$\begin{aligned}B^{011}_{s,t}&=\int_s^t\left(\int_s^u(r-s)dB_r\right)dB_u+\frac{(t-s)^2}{4},\\B^{101}_{s,t}&=\int_s^t\left(\int_s^u(B_r-B_s)dr\right)dB_u,\\B^{110}_{s,t}&=\int_s^t\left(\int_s^u(B_r-B_s)dB_r\right)du+\frac{(t-s)^2}{4},\end{aligned}$$

where dB_r is the Itô integral. By the same reason as for Φ_3, we see that for almost all ω uniformly,

$$\lim_{m\to\infty}2^m\sum_{k=1}^{\lfloor 2^m\cdot\rfloor}\varphi_{\boldsymbol{i}}(X_{\tau^m_{k-1}})B^{\boldsymbol{i}}_{\tau^m_{k-1}\tau^m_k}=\begin{cases}\frac{1}{4}\int_0^{\cdot}\varphi_{\boldsymbol{i}}(X_s)\,ds, & \boldsymbol{i}=011,110,\\0, & \boldsymbol{i}=101.\end{cases}$$

By a similar calculation to the above, we have

$$\lim_{m\to\infty}2^m\sum_{k=1}^{\lfloor 2^m\cdot\rfloor}\varphi(X_{\tau^m_{k-1}})\Delta(\Delta B_k)^2=\int_0^{\cdot}\varphi(X_s)\,ds\quad\text{a.s. }\omega\text{ uniformly.}$$

Hence, we see that for almost all ω uniformly,

$$\lim_{m\to\infty}(2^m)^{3H-\frac{1}{2}}\Phi_3=\int_0^{\cdot}\varphi(X_s)\,ds+\frac{1}{4}\int_0^{\cdot}\{\varphi_{011}(X_s)+\varphi_{110}(X_s)\}\,ds.$$

Finally, we consider the term Φ_4. Suppose $1/3<H<1/2$. Then for any $\omega\in\Omega^{\mathrm{CN}(m)}$

$$(2^m)^{3H-\frac{1}{2}+\delta}\sum_{k=1}^{2^m}\left|\Delta B_k\left(\frac{\Delta}{2}\Delta B_k-B^{10}_{\tau^m_{k-1}\tau^m_k}\right)\right|=O(\Delta^{2H^--3H+\frac{1}{2}-\delta})=O\left(\Delta^{\frac{1}{2}-H-2\epsilon-\delta}\right).\tag{5.8}$$

Hence, if $\delta<\frac{1}{2}-H-2\epsilon$, $\lim_{m\to\infty}\|(2^m)^{3H-\frac{1}{2}+\delta}\Phi_4\|_\infty=0$ in probability. We consider the case where $H=\frac{1}{2}$. In this case, B_t is a standard Brownian motion and we have

$$\boldsymbol{E}\left[\Delta B_k\left(\frac{\Delta}{2}\Delta B_k-B^{10}_{\tau^m_{k-1}\tau^m_k}\right)\right]=0,\qquad\boldsymbol{E}\left[\left\{\Delta B_k\left(\frac{\Delta}{2}\Delta B_k-B^{10}_{\tau^m_{k-1}\tau^m_k}\right)\right\}^2\right]=\frac{\Delta^4}{3}.$$

Since $X_t(\xi,B)$ is $\sigma(\{B_u\mid 0\le u\le t\})$-adapted, by Doob's inequality, we have

$$\Delta^{-2}E\left[\sup_{0\le t\le 1}|\Phi_4(t)|^2\right]\le C\Delta.$$

This implies that for any $\delta<\frac{1}{2}$,

$$\lim_{m\to\infty}\Delta^{-1-\delta}\sup_{0\le t\le 1}|\Phi_4(t)|=0 \qquad \text{a.s. } \omega. \tag{5.9}$$

From the calculation above, Remark 3.8 and Proposition 5.12 (3), we see the conclusion. □

The next lemma is a corollary of Lemma 5.10 and Proposition 5.12, which is used in the proof of Lemma 5.7.

Lemma 5.13. *Set*

$$\Psi_{m,\delta}=(2^m)^{3H-\frac{1}{2}-\delta}\max_{0\le t\le 1}\left|\sum_{i=1}^{4}\Phi_i(t)\right|.$$

Then, for any $\delta>0$, $\lim_{m\to\infty}\Psi_{m,\delta}=0$ in probability.

Proof. From Proposition 5.12 (1) and (2), we see that $\sup_{0\le t\le 1}\left|\sum_i(2^m)^{3H-\frac{1}{2}}\Phi_i(t)\right|$ converges in law. Thus we obtain that $\lim_{m\to\infty}\Psi_{m,\delta}=0$ in probability. □

Next, we show Lemma 5.7. By using Lemmas 5.6 and 5.10, we obtain a representation of the main term of κ_k in terms of Δ, ΔB_k, $B^{\boldsymbol{i}}_{\tau^m_{k-1}\tau^m_k}$ and $X_{\tau^m_{k-1}}$. We divide this calculation into two steps. In the first step, we have the following. This estimate is a pathwise estimate. We use just Hölder continuity of the path of B.

Lemma 5.14. *Let $\omega\in\Omega^{\mathrm{CN}(m)}$. For k $(1\le k\le 2^m)$ and $x\in\mathbf{R}$, let*

$$\begin{aligned}
F_k(x,B)&=f_3(x)(\Delta B_k)^3+f_4(x)(\Delta B_k)^4+g_1(x)\left(\frac{\Delta}{2}\Delta B_k-B^{10}_{\tau^m_{k-1}\tau^m_k}\right)\\
&\quad+\varphi(x)\Delta(\Delta B_k)^2+\varphi_{011}(x)B^{011}_{\tau^m_{k-1}\tau^m_k}+\varphi_{101}(x)B^{101}_{\tau^m_{k-1}\tau^m_k}+\varphi_{110}(x)B^{110}_{\tau^m_{k-1}\tau^m_k},\\
G_k(x,B)&=-[g_1\sigma'](x)\Delta B_k\left(\frac{\Delta}{2}\Delta B_k-B^{10}_{\tau^m_{k-1}\tau^m_k}\right),\\
r_k&=\kappa_k-F_k(\xi_{k-1},B)-G_k(\xi_{k-1},B).
\end{aligned}$$

Then it holds that $r_k=O(\Delta^{3H^-+1})+O(\Delta^{5H^-})$.

Proof. By the Taylor formula, there exists $0<\rho<1$ such that

$$\begin{aligned}
\hat{\kappa}_k&=\xi_k-X_{2^{-m}}(\xi_{k-1},\theta_{\tau^m_{k-1}}B)\\
&=X_{2^{-m}}(\xi_{k-1},\theta_{\tau^m_{k-1}}B+2^m\kappa_k\ell)-X_{2^{-m}}(\xi_{k-1},\theta_{\tau^m_{k-1}}B)\\
&=\nabla_{2^m\kappa_k\ell}X_{2^{-m}}(\xi_{k-1},\theta_{\tau^m_{k-1}}B)+\frac{1}{2}\nabla^2_{2^m\kappa_k\ell}X_{2^{-m}}(\xi_{k-1},\theta_{\tau^m_{k-1}}B+\rho 2^m\kappa_k\ell).
\end{aligned}$$

Applying the estimate $\kappa_k=O(\Delta^{3H^-})$ and Proposition 3.12 (1), we see that the second term of the right-hand side is $O(\Delta^{6H^-})$. As for the first term, Proposition 3.12 (3), Lemma 5.6 (2) and Proposition 3.11 yield

$$\nabla_{2^m\kappa_k\ell}X_{2^{-m}}(\xi_{k-1},\theta_{\tau^m_{k-1}}B) = \sigma\left(X_\Delta(\xi_{k-1},\theta_{\tau^m_{k-1}}B)\right)\int_0^\Delta \exp\left(\int_s^\Delta \left[\frac{\mathsf{W}}{\sigma}\right]\left(X_u(\xi_{k-1},\theta_{\tau^m_{k-1}}B)\right)du\right)\frac{\kappa_k}{\Delta}\,ds$$
$$= \sigma(\xi_{k-1})\kappa_k + \left\{\sigma\left(X_\Delta(\xi_{k-1},\theta_{\tau^m_{k-1}}B)\right) - \sigma(\xi_{k-1})\right\}\kappa_k + O(\Delta^{3H^-+1})$$
$$= \{\sigma(\xi_{k-1}) + \sigma(\xi_{k-1})\sigma'(\xi_{k-1})\Delta B_k\}\kappa_k + O(\Delta^{5H^-}) + O(\Delta^{3H^-+1}).$$

Hence we see that $\hat{\kappa}_k$ and κ_k satisfy

$$\hat{\kappa}_k = \sigma(\xi_{k-1})\{1 + \sigma'(\xi_{k-1})\Delta B_k\}\kappa_k + O(\Delta^{3H^-+1}) + O(\Delta^{5H^-}).$$

Since $|\sigma'(\xi_{k-1})\Delta B_k| \le 1/2$ on $\Omega^{\mathrm{CN}(m)}$, we can solve this equation and using Lemma 5.6 (1),

$$\begin{aligned}\kappa_k &= \sigma(\xi_{k-1})^{-1}\{1 - \sigma'(\xi_{k-1})\Delta B_k\}\hat{\kappa}_k + O(\Delta^{3H^-+1}) + O(\Delta^{5H^-})\\ &= F_k(\xi_{k-1},B) + G_k(\xi_{k-1},B)\\ &\quad - [\sigma^{-1}\sigma'](\xi_{k-1})\Delta B_k\left\{\hat{\kappa}_k - \hat{f}_3(\xi_{k-1})(\Delta B_k)^3 - \hat{g}_1(\xi_{k-1})\left(\frac{\Delta}{2}\Delta B_k - B^{10}_{\tau^m_{k-1}\tau^m_k}\right)\right\}\\ &\quad + O(\Delta^{3H^-+1}) + O(\Delta^{5H^-}).\end{aligned}$$

Since $\hat{\kappa}_k - \hat{f}_3(\xi_{k-1})(\Delta B_k)^3 - \hat{g}_1(\xi_{k-1})\left(\frac{\Delta}{2}\Delta B_k - B^{10}_{\tau^m_{k-1}\tau^m_k}\right) = O(\Delta^{2H^-+1}) + O(\Delta^{4H^-})$, we complete the proof. □

Now, we are in a position to prove Lemma 5.7.

Proof of Lemma 5.7. Let $\epsilon_m = \max_{1\le k\le 2^m}|X_{\tau^m_k}(\xi,B) - \bar{X}^{(m)}_{\tau^m_k}(\xi,B)|$. We proved that $\lim_{m\to\infty}(2^m)^{3H^--1}\epsilon_m = 0$ for $\omega \in \bigcup_m \Omega^{\mathrm{CN}(m)}$. Our first task is to improve this estimate as $\lim_{m\to\infty}(2^m)^{3H-1/2-\delta}\epsilon_m = 0$ in probability for any $\delta > 0$ by using $\lim_{m\to\infty}\Psi_{m,\delta} = 0$ in probability (recall Lemma 5.13). To this end, let

$$\begin{aligned}\kappa_{k,1} &= F_k(X_{\tau^m_{k-1}},B) + G_k(X_{\tau^m_{k-1}},B),\\ \kappa_{k,2} &= F_k(\xi_{k-1},B) + G_k(\xi_{k-1},B) - \left(F_k(X_{\tau^m_{k-1}},B) + G_k(X_{\tau^m_{k-1}},B)\right),\end{aligned}$$

where F_k and G_k are the same functions as in Lemma 5.14. Then $\kappa_k = \kappa_{k,1} + \kappa_{k,2} + r_k$, $\tilde{\kappa}_k = F_k(X_{\tau^m_{k-1}},B)$ and $R_k = G_k(X_{\tau^m_{k-1}},B) + \kappa_{k,2} + r_k$ hold. Here, r_k is defined in Lemma 5.14. Let $h^{(m)}_i$ $(i = 1,2)$ be piecewise linear paths which are defined by $\{\kappa_{k,i}\}$. We define $h^{(m)}_r$ similarly by $\{r_k\}$. Note that $\|h^{(m)}_1\|_\infty = O(\Delta^{3H-1/2-\delta})\Psi_{m,\delta}$ holds. By the Lipschitz continuity of F_k and G_k with respect to x-variable, we have

$$\|h^{(m)}_2\|_\infty \le \sum_{k=1}^{2^m}|\kappa_{k,2}| \le K\epsilon_m, \qquad \omega \in \Omega^{\mathrm{CN}(m)}$$

where $K = O(\Delta^{3H^--1})$. By Lemma 5.14, we have

$$\|h^{(m)}_r\|_\infty \le \sum_{k=1}^{2^m}|r_k| = O(\Delta^{3H^-}) + O(\Delta^{5H^--1}), \qquad \omega \in \Omega^{\mathrm{CN}(m)}. \tag{5.10}$$

By the Lipschitz continuity of $B \mapsto X(\xi,B)$ in the uniform norm, we have

$$\epsilon_m = \max_{1\le k\le 2^m}|X_{\tau^m_k}(\xi,B) - X_{\tau^m_k}(\xi,B + h^{(m)}_1 + h^{(m)}_2 + h^{(m)}_r)| \tag{5.11}$$

$$\leq C \sum_{i=1}^{3} \|h_i^{(m)}\|_\infty = \tilde{K}\epsilon_m + \hat{K}, \qquad \omega \in \Omega^{\mathrm{CN}(m)},$$

where $\tilde{K} = CK = O(\Delta^{3H^- -1})$ and $\hat{K} = C(\|h_1^{(m)}\|_\infty + \|h_r^{(m)}\|_\infty)$. By applying the inequality (5.11), n-times and using the rough estimate $\epsilon_m = O(\Delta^{3H^- -1})$, we get

$$\epsilon_m \leq \tilde{K}^n O(\Delta^{3H^- -1}) + \hat{K}\left(1 + \sum_{j=1}^{n-1} \tilde{K}^j\right).$$

From this, we conclude that for $\omega \in \Omega^{\mathrm{CN}(m)}$, $\epsilon_m = \Psi_{m,\delta}(\omega) O(\Delta^{3H-1/2-\delta}) + O(\Delta^{3H^-}) + O(\Delta^{5H^- -1})$ holds for any $\delta > 0$. We now prove the estimate of the sum of R_k. Thanks for the the improved estimate of ϵ_m, we obtain for any $\delta > 0$

$$\sum_{k=0}^{2^m-1} |\kappa_{k,2}| = O(\Delta^{3H-1/2+3H^- -1-\delta})\Psi_{m,\delta}(\omega) + O(\Delta^{6H^- -1}) + O(\Delta^{8H^- -2}), \qquad \omega \in \Omega^{\mathrm{CN}(m)}.$$

We already proved the necessary estimates in (5.10), (5.8) and (5.9) for the sum of r_k and $G_k(X_{\tau_{k-1}^m}, B)$. Thus, we complete the proof. □

6. The Euler scheme and the Milstein scheme

In this section, we show Theorems 2.5 and 2.6, which are concerning with the Euler scheme and the Milstein scheme, respectively. Since the proofs are similar to one of Theorem 2.7, we omit the detail and give key lemmas. We denote by $\bar{X}^{(m)}$ the Euler scheme or the Milstein scheme and set $\xi_k = \bar{X}^{(m)}_{\tau_k^m}$.

Note that Lemma 5.3 holds for the Euler scheme and the Milstein scheme. We see Lemma 5.3 holds for the both of the schemes. We denote by $h^{(m)}$ the piecewise linear function which appears in Lemma 5.3 and we write $\kappa_k = h^{(m)}(\tau_k^m) - h^{(m)}(\tau_{k-1}^m)$ for every $1 \leq k \leq 2^m$. Because analysis of 1-step error $\hat{\kappa}_k = \{\xi_k - \xi_{k-1}\} - \{X_{2^{-m}}(\xi_{k-1}, \theta_{\tau_{k-1}^m} B) - \xi_{k-1}\}$ of the scheme and the main term $\tilde{\kappa}_k$ are essential in the proof, we state assertions on them, that is, we give counterparts of Lemmas 5.6, 5.7 and 5.10.

6.1. The Euler scheme. In this subsection, we assume $1/2 < H < 1$ and show Theorem 2.5. To state assertions, we set $\hat{f}_2 = -\sigma\sigma'/2$ and $f_2 = -\sigma'/2$. Then we see the following lemmas:

Lemma 6.1. *For any $\omega \in \Omega_0$, the following hold:*

(1) *We have $\hat{\kappa}_k = \hat{f}_2(\xi_{k-1})(\Delta B_k)^2 + O(\Delta^{H^- +1})$.*

(2) *We have $\hat{\kappa}_k = O(\Delta^{2H^-})$, $\kappa_k = O(\Delta^{2H^-})$ and*

$$\max_{1\leq k\leq 2^m} |X_{\tau_k^m}(\xi, B) - \bar{X}^{(m)}_{\tau_k^m}(\xi, B)| = O(\Delta^{2H^- -1}).$$

(3) *We have*

$$\max_{0\leq t\leq 1} |\bar{X}_t^{(m)}(\xi, B) - X_t(\xi, B + h^{(m)})| = O(\Delta^{2H^-}).$$

Lemma 6.2. *For $1 \leq k \leq 2^m$, let*

$$\tilde{\kappa}_k = f_2(X_{\tau_k^m})(\Delta B_k)^2$$

and set $R_k(\omega) = \kappa_k - \tilde{\kappa}_k$. *Then* $R_k = O(\Delta^{4H^- -1}) + O(\Delta^{H^- +1})$.

Lemma 6.3. *Let*

$$\Phi_1(t) = \sum_{k=1}^{\lfloor 2^m t \rfloor} f_2(X_{\tau_{k-1}^m})(\Delta B_k)^2.$$

Then, $\left(B, 2^{m(2H-1)}\Phi_1\right)$ *converges to* $\left(B, \int_0^{\cdot} f_2(X_u)\,du\right)$ *in* $D([0,1];\mathbf{R}^2)$ *with respect to the Skorokhod topology in probability.*

Here we make comments on proof of the lemmas above:

- Lemma 6.1 is seen by the similar way with Lemma 5.6.
- Lemma 6.2 follows from the equality $\hat{\kappa}_k = \sigma(\xi_{k-1})\kappa_k + O(\Delta^{3H^-})$ and Lemma 6.1 (note that we do not use Lemma 6.3).
- Lemma 6.3 is a direct consequence of Theorem 4.1.

Combining the lemmas, we obtain Theorem 2.5.

6.2. The Milstein scheme. In this subsection, we assume $1/3 < H \leq 1/2$ and Theorem 2.6. We set

$$\hat{f}_3 = -\frac{1}{3!}\sigma(\sigma\sigma')', \quad \hat{f}_4 = -\frac{1}{4!}\sigma(\sigma(\sigma\sigma')')',$$
$$f_3 = -\frac{1}{3!}(\sigma\sigma')', \qquad f_4 = -\frac{1}{4!}[\sigma^2\sigma''' - 3(\sigma')^3], \quad f_4^{\dagger} = \frac{1}{4!}[\sigma^2\sigma''' + 6\sigma\sigma'\sigma'' + 3(\sigma')^3].$$

Note that $f_4 = (\hat{f}_4 - \sigma'\hat{f}_3)/\sigma$ and $f_4^{\dagger} = f_4 - \sigma f_3'/2$. We set $\varphi = 0$ and use functions $\hat{g}_1$, g_1, $\hat{\varphi}_{\boldsymbol{i}}$, $\varphi_{\boldsymbol{i}}$ ($\boldsymbol{i} = 011, 101, 110$) introduced in Section 5.2. We define processes $\Phi_1, \ldots, \Phi_4$ by (5.5) with the functions above. Then we see the next lemmas:

Lemma 6.4. *For any* $\omega \in \Omega$, *the following hold.*

(1) *We have*

$$\begin{aligned}\hat{\kappa}_k &= \hat{f}_3(\xi_{k-1})(\Delta B_k)^3 + \hat{f}_4(\xi_{k-1})(\Delta B_k)^4 + \hat{g}_1(\xi_{k-1})\left(\frac{\Delta}{2}\Delta B_k - B^{10}_{\tau_{k-1}^m\tau_k^m}\right)\\ &\quad + \hat{\varphi}_{011}(\xi_{k-1})B^{011}_{\tau_{k-1}^m\tau_k^m} + \hat{\varphi}_{101}(\xi_{k-1})B^{101}_{\tau_{k-1}^m\tau_k^m} + \hat{\varphi}_{110}(\xi_{k-1})B^{110}_{\tau_{k-1}^m\tau_k^m}\\ &\quad + O(\Delta^{5H^-}) + O(\Delta^{3H^-+1}) + O(\Delta^{H^-+2}).\end{aligned}$$

(2) *We have* $\hat{\kappa}_k = O(\Delta^{3H^-})$, $\kappa_k = O(\Delta^{3H^-})$ *and*

$$\max_{1\leq k\leq 2^m} |X_{\tau_k^m}(\xi, B) - \bar{X}^{(m)}_{\tau_k^m}(\xi, B)| = O(\Delta^{3H^- -1}).$$

(3) *We have*

$$\max_{0\leq t\leq 1} |\bar{X}^{(m)}_t(\xi, B) - X_t(\xi, B + h^{(m)})| = O(\Delta^{3H^-}).$$

Lemma 6.5. *Let*

$$\begin{aligned}\tilde{\kappa}_k &= f_3(X_{\tau_{k-1}^m})(\Delta B_k)^3 + f_4(X_{\tau_{k-1}^m})(\Delta B_k)^4 + g_1(X_{\tau_{k-1}^m})\left(\frac{\Delta}{2}\Delta B_k - B^{10}_{\tau_{k-1}^m\tau_k^m}\right)\\ &\quad + \varphi_{011}(X_{\tau_{k-1}^m})B^{011}_{\tau_{k-1}^m\tau_k^m} + \varphi_{101}(X_{\tau_{k-1}^m})B^{101}_{\tau_{k-1}^m\tau_k^m} + \varphi_{110}(X_{\tau_{k-1}^m})B^{110}_{\tau_{k-1}^m\tau_k^m}\end{aligned}$$

and set $R_k(\omega) = \kappa_k - \tilde{\kappa}_k$. *Then there exists* $\delta > 0$ *such that* $\lim_{m\to\infty}(2^m)^{4H-1+\delta}\max_{1\leq k\leq 2^m}$

$|\sum_{i=1}^{k} R_i| = 0$ *in probability.*

Lemma 6.6. *The following hold:*

(1) *Let* $\frac{1}{3} < H < \frac{1}{2}$. *Then* $\Big(B, (2^m)^{4H-1}(\Phi_1, \Phi_2, \Phi_3, \Phi_4)\Big)$ *converges to* $\Big(B, 3\int_0^{\cdot} f_4^{\dagger}(X_s)\,ds,$ $0, 0, 0\Big)$ *in* $D([0,1];\mathbf{R}^4)$ *with respect to the Skorokhod topology in probability.*

(2) *Let* $H = \frac{1}{2}$. *Then* $(B, 2^m(\Phi_1, \Phi_2, \Phi_3, \Phi_4))$ *converges weakly to*

$$\Bigg(B, \sqrt{6}\int_0^{\cdot} f_3(X_s)\,dW_s + 3\int_0^{\cdot} f_3(X_s)\circ dB_s + 3\int_0^{\cdot} f_4^{\dagger}(X_s)\,ds,$$
$$\frac{1}{\sqrt{12}}\int_0^{\cdot} g_1(X_s)d\tilde{W}_s, \frac{1}{4}\int_0^{\cdot}\{\varphi_{011}(X_s) + \varphi_{110}(X_s)\}\,ds, 0\Bigg)$$

in $D([0,1];\mathbf{R}^4)$ *with respect to the Skorokhod topology.*

Note that in proof Lemma 6.6 we used the decomposition

$$\begin{aligned} f_3(X_{\tau^m_{k-1}}) + f_4(X_{\tau^m_{k-1}})\Delta B_k &= \left\{f_3(X_{\tau^m_{k-1}}) + \frac{1}{2}f_3'\sigma(X_{\tau^m_k})\Delta B_k\right\} + f_4^{\dagger}(X_{\tau^m_k})\Delta B_k \\ &= \frac{f_3(X_{\tau^m_{k-1}}) + f_3(X_{\tau^m_k})}{2} + O(\Delta^{2H^-}) + O(\Delta) + f_4^{\dagger}(X_{\tau^m_k})\Delta B_k \end{aligned}$$

and apply Theorems 4.1, 4.2 and 4.3.

A. Gaussian analysis and Malliavin calculus

We summarize basic results on Gaussian analysis and Malliavin calculus which we use to estimate some terms of error. For details, see [19].

Let $(\Omega, \mathcal{F}, \boldsymbol{P})$ be the canonical probability space for a one-dimensional centered continuous Gaussian process $X = \{X_t\}_{0\le t\le 1}$ with the covariance $\boldsymbol{E}[X_sX_t] = R(s,t)$, that is, Ω is the Banach space of continuous functions from $[0,1]$ to $\mathbf{R}$ starting at zero with the uniform norm $\|\cdot\|_\infty$, $\mathcal{F}$ the σ-field generated by the cylindrical subsets of Ω, and $\boldsymbol{P}$ a probability measure on Ω such that the canonical process $X(\omega) = \omega$, $\omega \in \Omega$, is the Gaussian process.

We construct an abstract Wiener space $(\Omega, \mathfrak{H}, \boldsymbol{P})$ and an isonormal Gaussian process $\{X(h)\}_{h\in\mathfrak{H}}$. The Hilbert space $\mathfrak{H}$ with the norm $\|\cdot\|_{\mathfrak{H}}$ and the inner product $\langle\cdot,*\rangle_{\mathfrak{H}}$ is defined by as follows; set $[\mathscr{R}\mathbf{1}_{[0,t)}](\cdot) = R(t,\cdot) = \boldsymbol{E}[X_tX_\cdot]$ and let $\mathfrak{H}_0$ be the linear span of functions $\mathscr{R}\mathbf{1}_{[0,t)}$ and $\mathfrak{H}$ the Hilbert space defined as the closure of $\mathfrak{H}_0$ with respect to the inner product $\langle\mathscr{R}\mathbf{1}_{[0,s)}, \mathscr{R}\mathbf{1}_{[0,t)}\rangle_{\mathfrak{H}} = \boldsymbol{E}[X_sX_t]$. We call the Hilbert space $\mathfrak{H}$ the Cameron-Martin subspace. Note the map $\mathfrak{H}_0 \ni \mathscr{R}\mathbf{1}_{[0,t)} \mapsto X(\mathbf{1}_{[0,t)}) \in \mathcal{L}^2(\Omega;\mathbf{R})$ is an isometry. Hence if $\{h_n\}_{n=1}^{\infty} \subset \mathfrak{H}_0$ converges to $h \in \mathfrak{H}$, then $\{X(h_n)\}_{n=1}^{\infty}$ converges to some element $X(h) \in \mathcal{L}^2(\Omega;\mathbf{R})$. Hence we obtain the isonormal Gaussian process $\{X(h)\}_{h\in\mathfrak{H}}$.

Next, we define the q-th Wiener integral I_q which is a map from the symmetric space $\mathfrak{H}^{\odot q}$ to the q-th Wiener chaos $\mathcal{H}_q$ for $q \in \mathbf{N}$.

In order to define $\mathfrak{H}^{\odot q}$, $\mathcal{H}_q$ and I_q, we denote by Λ the set of sequences $\lambda = (\lambda_1, \dots) \in (\mathbf{N}\cup\{0\})^\infty$ such that all the elements vanish except a finite number of them and set $\lambda! = \prod_{n=1}^{\infty}\lambda_n!$ for $\lambda \in \Lambda$. We take an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of $\mathfrak{H}$.

We denote by $\otimes$ the tensor product and by $\mathfrak{H}^{\otimes q}$ the tensor product space for $q \ge 2$. For $q = 0, 1$, we set $\mathfrak{H}^{\otimes 0} = \mathbf{R}$ and $\mathfrak{H}^{\otimes 1} = \mathfrak{H}$ by convention. We define the symmetrization $\tilde{h} \in \mathfrak{H}^{\otimes q}$ for $h \in \mathfrak{H}^{\otimes q}$ as follows: if h has the form of $h = h_1 \otimes \cdots \otimes h_q$ for $h_r \in \mathfrak{H}$, we set

$$(h_1 \otimes \cdots \otimes h_q)^{\sim} = \frac{1}{q!} \sum_{\sigma \in \mathfrak{S}_q} h_{\sigma(1)} \otimes \cdots \otimes h_{\sigma(q)},$$

where $\mathfrak{S}_q$ is the symmetric group on $\{1, \ldots, q\}$; we also define the symmetrization for general elements in $\mathfrak{H}^{\otimes q}$ by linearity. For notational simplicity, we set $h_1 \odot \cdots \odot h_q = (h_1 \otimes \cdots \otimes h_q)^{\sim}$. An element $h \in \mathfrak{H}^{\otimes q}$ is said to be symmetric if $\tilde{h} = h$. We denote by $\mathfrak{H}^{\odot q}$ the set of symmetric elements of $\mathfrak{H}^{\otimes q}$. The space $\mathfrak{H}^{\odot q}$ forms a Hilbert space with respect to the scaled norm $\sqrt{q!}\|\cdot\|_{\mathfrak{H}^{\otimes q}}$. For $\lambda \in \Lambda$, set

$$e^{\lambda} = \frac{1}{\sqrt{\lambda!}} e_1^{\odot \lambda_1} \odot e_2^{\odot \lambda_2} \odot \cdots .$$

Then, $\{e^{\lambda}; |\lambda| = q, \lambda \in \Lambda\}$ is an orthonormal basis of $\mathfrak{H}^{\odot q}$.

As we introduced in Section Section 4, H_q denotes the q-th Hermite polynomial. The q-th Wiener chaos $\mathcal{H}_q$ is defined as the closed subspace spanned by $\{H_q(X(h)); h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$ in $\mathcal{L}^2(\Omega; \mathbf{R})$. For $\lambda \in \Lambda$, set

$$H_{\lambda} = \frac{1}{\sqrt{\lambda!}} \prod_{n=1}^{\infty} H_{\lambda_n}(X(e_n)).$$

Then, $\{H_{\lambda}; |\lambda| = q, \lambda \in \Lambda\}$ is an orthonormal basis of $\mathcal{H}_q$.

The q-th Wiener integral I_q is defined by $I_q(e^{\lambda}) = H_{\lambda}$ and is extend by linearity. The mapping $I_q : \mathfrak{H}^{\odot q} \to \mathcal{H}_q$ provides a real linear isometry between $\mathfrak{H}^{\odot q}$ and $\mathcal{H}_q$.

Finally, we summarize results on Malliavin calculus. Let $\mathcal{S}$ be the totality of all smooth functionals which have the form of $F = f(X(h_1), \ldots, X(h_{\alpha}))$, where $h_{\beta} \in \mathfrak{H}$ and $f \in C^{\infty}_{\mathrm{poly}}(\mathbf{R}^{\alpha}; \mathbf{R})$. The Malliavin derivative DF of $F \in \mathcal{S}$ is an $\mathfrak{H}$-valued random variable and defined by

$$DF = \sum_{\beta=1}^{\alpha} \frac{\partial f}{\partial \xi_{\beta}}(X(h_1), \ldots, X(h_{\alpha}))h_{\beta}.$$

By the iteration, one can define n-th derivative $D^n F$, which is an $\mathfrak{H}^{\odot n}$-valued random variable, by

$$D^n F = \sum_{\beta_1, \ldots, \beta_n = 1}^{\alpha} \frac{\partial^n f}{\partial \xi_{\beta_1} \cdots \partial \xi_{\beta_n}}(X(h_1), \ldots, X(h_{\alpha}))h_{\beta_1} \otimes \cdots \otimes h_{\beta_n}.$$

As usual, for $n \in \mathbf{N}$ and $1 < p < \infty$, we define the Sobolev space $\mathcal{D}^{n,p}(\Omega; \mathbf{R})$ by the completion of $\mathcal{S}$ by the norm

$$\|F\|^p_{\mathcal{D}^{n,p}(\Omega;\mathbf{R})} = \sum_{k=0}^{n} \boldsymbol{E}[\|D^k F\|^p_{\mathfrak{H}^{\odot k}}].$$

We set $\mathcal{D}^{n,\infty-}(\Omega; \mathbf{R}) = \bigcap_{1<p<\infty} \mathcal{D}^{n,p}(\Omega; \mathbf{R})$.

Since the derivative operator D is a continuous operator from $\mathcal{D}^{1,2}(\Omega; \mathbf{R})$ to $\mathcal{L}^2(\Omega; \mathfrak{H})$, there exists its adjoint operator δ, which is called the divergence operator or the Skorokhod integral. Notice that the duality relationship

$$\boldsymbol{E}[F\delta(u)] = \boldsymbol{E}[\langle DF, u\rangle_{\mathfrak{H}}]$$

holds for any $F \in \mathcal{D}^{1,2}(\Omega; \mathbf{R})$ and u belonging to the domain of δ. By the iteration, we see that there exists an operator δ^n such that

$$\boldsymbol{E}[F\delta^n(u)] = \boldsymbol{E}[\langle D^n F, u\rangle_{\mathfrak{H}^{\otimes n}}] \tag{A.1}$$

for any $F \in \mathcal{D}^{n,2}(\Omega;\mathbf{R})$ and u belonging to the domain of δ^n. Notice that $h \in \mathfrak{H}^{\odot q}$ belongs to the domain of δ^q and $\delta^q(h) = I_q(h)$. From the Itô-Wiener expansion and the Stroock formula, we obtain the product formula:

$$I_p(h^{\odot p}) I_q(k^{\odot q}) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r}\binom{q}{r} (h,k)_{\mathfrak{H}}^r I_{p+q-2r}(h^{\odot p-r} \odot k^{\odot q-r}) \tag{A.2}$$

for every $h, k \in \mathfrak{H}$.

In what follows, we assume that fBm B is defined on the canonical probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$, that is, $B(\omega) = \omega$ for $\omega \in \Omega$ is fBm under the probability measure $\boldsymbol{P}$. In this setting, we can apply Gaussian analysis and Malliavin calculus to fBm. In particular, since $h \in \mathfrak{H}$ is given by $h_t = \boldsymbol{E}[ZB_t]$ for some square-integrable random variable Z, we see $|h_t - h_s| \leq \boldsymbol{E}[Z^2]^{1/2}\boldsymbol{E}[(B_t - B_s)^2]^{1/2} = \boldsymbol{E}[Z^2]^{1/2}(t-s)^H$, which implies $\mathfrak{H} \subset \mathscr{C}_0^H([0,1];\mathbf{R}) \subset \mathscr{C}_0^{H-\epsilon}([0,1];\mathbf{R})$. From Proposition 3.12 and the inclusion $\mathfrak{H} \subset \mathscr{C}_0^{H-\epsilon}([0,1];\mathbf{R})$, the functional $\omega \mapsto X_t(\omega)$ is Fréchet differentiable in $\mathfrak{H}$ and the derivative is integrable. Hence we see that X_t is Malliavin differentiable and have $\langle DX_t, h\rangle_{\mathfrak{H}} = \nabla_h X_t$ for any $h \in \mathfrak{H}$. More precisely, we obtain the following proposition.

Proposition A.1. *Let $b, \sigma \in C_{\mathrm{bdd}}^{n+1}(\mathbf{R};\mathbf{R})$ for $n \geq 1$. Assume that Hypothesis 2.4 is satisfied. Then $X_t \in \mathcal{D}^{n,\infty-}(\Omega;\mathbf{R})$ and*

$$|\langle D^\nu X_t, h^1 \odot \cdots \odot h^\nu\rangle_{\mathfrak{H}^{\odot\nu}}| \leq C_\nu \|h^1\|_\infty \cdots \|h^n\|_\infty,$$

for any $h^1, \ldots, h^\nu \in \mathfrak{H}$ and $1 \leq \nu \leq n$. Here C_ν is a positive constant depending only on b, σ and ν.

In what follows, we set

$$\delta_{st} = \mathscr{R}\mathbf{1}_{[s,t)}, \qquad \zeta_{st} = \mathscr{R}\left[\frac{1}{2}(t-s)\mathbf{1}_{[s,t)} - \int_s^t \mathbf{1}_{[s,v)}\, dv\right]$$

for $0 \leq s < t \leq 1$. Note

$$H_q(2^{mH} B_{\tau_{k-1}^m \tau_k^m}) = I_q((2^{mH}\delta_{\tau_{k-1}^m \tau_k^m})^{\odot q}) = 2^{mqH} I_q(\delta_{\tau_{k-1}^m \tau_k^m}^{\odot q}),$$

$$2^{m(H+1)}\left(\frac{1}{2\cdot 2^m} B_{\tau_{k-1}^m \tau_k^m} - \int_{\tau_{k-1}^m}^{\tau_k^m} B_{\tau_{k-1}^m u}\, du\right) = 2^{m(H+1)} I_1(\zeta_{\tau_{k-1}^m \tau_k^m}).$$

The functions δ_{st} and ζ_{st} are bounded functions as follows:

Proposition A.2. *For any $0 \leq s < t \leq 1$, we have*

$$\|\delta_{st}\|_\infty \leq \begin{cases} (t-s)^{2H}, & 0 < H < 1/2, \\ 2H(t-s), & 1/2 \leq H < 1, \end{cases}$$

$$\|\zeta_{st}\|_\infty \leq \begin{cases} \left(\dfrac{1}{2} + \dfrac{1}{2H+1}\right)(t-s)^{2H+1}, & 0 < H < 1/2, \\ 2H(t-s)^2, & 1/2 \leq H < 1. \end{cases}$$

Proof. Note

$$|\boldsymbol{E}[(B_t - B_s)B_u]| \le \begin{cases} (t-s)^{2H}, & 0 < H < 1/2, \\ 2H(t-s), & 1/2 \le H < 1, \end{cases}$$

for any $0 \le s < t \le 1$ and $0 \le u \le 1$. We can find this estimate in [17, Lemma 5,6]. The first assertion follows from this estimate and the identification $\delta_{st}(u) = [\mathscr{R}\mathbf{1}_{[s,t)}](u) = \boldsymbol{E}[(B_t - B_s)B_u]$. We see the second one from the expression

$$\zeta_{st}(u) = \frac{1}{2}(t-s)\boldsymbol{E}[(B_t - B_s)B_u] - \int_s^t \boldsymbol{E}[(B_v - B_s)B_u]\,dv.$$

The proof is completed. □

B. Proof of Proposition 4.5

In this section, we prove Proposition 4.5. The result of convergence of $(B, V_q^{(m)})$ can be found in [17]. Main contribution in this section is proof of convergence of $\tilde{V}^{(m)}$.

Throughout this section, we use the following notation:

$$a_{k,l} = \boldsymbol{E}\left[\left(\frac{1}{2}B_{k-1,k} - \int_{k-1}^{k} B_{k-1,u}\,du\right)\left(\frac{1}{2}B_{l-1,l} - \int_{l-1}^{l} B_{l-1,v}\,dv\right)\right],$$
$$a^{\dagger}_{k,l} = \boldsymbol{E}\left[B_{k-1,k}\left(\frac{1}{2}B_{l-1,l} - \int_{l-1}^{l} B_{l-1,u}\,du\right)\right]$$

for $k, l \ge 1$. It follows from the stationary increments of fBm that

(B.1) $$a_{k,l} = a_{1,l-k+1},$$

(B.2) $$a^{\dagger}_{k,l} = a^{\dagger}_{1,l-k+1}$$

for $1 \le k \le l$. For the same reason, we have

(B.3) $$a_{k,k} = a_{1,1} = \frac{1}{4}\frac{1-H}{1+H}.$$

B.1. Key estimates. Before starting to prove Proposition 4.5, we show the next three propositions:

Proposition B.1. *It holds that*

$$|a_{k,l}| \le C \begin{cases} |k-l|^{2H-4}, & |k-l| \ge 1, \\ 1, & |k-l| = 0 \end{cases}$$

for any k and l.

Proposition B.2. *It holds that*

$$|a^{\dagger}_{k,l}| \le C \begin{cases} |k-l|^{2H-3}, & |k-l| \ge 1, \\ 1, & |k-l| = 0, \end{cases}$$

for any $k, l \ge 1$.

Proposition B.3. *It holds that $a^{\dagger}_{k,l} + a^{\dagger}_{l,k} = 0$ for any $k, l \ge 1$.*

The following is a key lemma to prove Propositions B.1 and B.2:

Lemma B.4. *It holds that*

$$\boldsymbol{E}[(B_{x+k-1} - B_{s+k-1})(B_{y+l-1} - B_{t+l-1})] = \frac{1}{2}|k-l|^{2H}\left\{\binom{2H}{2}\frac{b_2(x,s,y,t)}{(k-l)^2} + \binom{2H}{3}\frac{b_3(x,s,y,t)}{(k-l)^3} + R(k-l;x,s,y,t)\right\}$$

for any $0 \le x, s, y, t \le 1$ *and* $k, l \in \mathbf{N}$ *with* $|k-l| \ge 2$. *Here*

$$b_2(x,s,y,t) = 2(xy - xt - sy + st),$$
$$b_3(x,s,y,t) = 3(x^2y - xy^2 - x^2t + xt^2 - s^2y + sy^2 + s^2t - st^2)$$

and R *satisfies* $|R(k-l;x,s,y,t)| \le C|k-l|^{-4}$ *for some positive constant* C.

Proof. From (3.7), we have

$$\begin{aligned}\boldsymbol{E}[(B_{x+k-1} - B_{s+k-1})(B_{y+l-1} - B_{t+l-1})] &= \frac{1}{2}\left\{-|x-y+k-l|^{2H} + |x-t+k-l|^{2H} + |s-y+k-l|^{2H} - |s-t+k-l|^{2H}\right\} \\ &= \frac{1}{2}|k-l|^{2H}\left\{-\left|1+\frac{x-y}{k-l}\right|^{2H} + \left|1+\frac{x-t}{k-l}\right|^{2H} + \left|1+\frac{s-y}{k-l}\right|^{2H} - \left|1+\frac{s-t}{k-l}\right|^{2H}\right\}.\end{aligned}$$

Applying the binomial theorem, we obtain

$$\boldsymbol{E}[(B_{x+k-1} - B_{s+k-1})(B_{y+l-1} - B_{t+l-1})] = \frac{1}{2}|k-l|^{2H}\left\{\sum_{\nu=0}^{3}\binom{2H}{\nu}a_\nu\left(\frac{x-y}{k-l},\frac{x-t}{k-l},\frac{s-y}{k-l},\frac{s-t}{k-l}\right) + R(k-l;x,s,y,t)\right\},$$

where $a_\nu(z_1,z_2,z_3,z_4) = -z_1^\nu + z_2^\nu + z_3^\nu - z_4^\nu$ and R is defined by

$$R(k-l;x,s,y,t) = \left\{-r_3\left(\frac{x-y}{k-l}\right) + r_3\left(\frac{x-t}{k-l}\right) + r_3\left(\frac{s-y}{k-l}\right) - r_3\left(\frac{s-t}{k-l}\right)\right\}$$

with the remainder term r_3. Note $|r_3(\xi)| \le C|\xi|^4$. Expanding the polynomials a_ν, we see

$$a_0\left(\frac{x-y}{k-l},\frac{x-t}{k-l},\frac{s-y}{k-l},\frac{s-t}{k-l}\right) = 0, \quad a_1\left(\frac{x-y}{k-l},\frac{x-t}{k-l},\frac{s-y}{k-l},\frac{s-t}{k-l}\right) = 0,$$
$$a_2\left(\frac{x-y}{k-l},\frac{x-t}{k-l},\frac{s-y}{k-l},\frac{s-t}{k-l}\right) = \frac{1}{(k-l)^2}\cdot b_2(x,s,y,t),$$
$$a_3\left(\frac{x-y}{k-l},\frac{x-t}{k-l},\frac{s-y}{k-l},\frac{s-t}{k-l}\right) = \frac{1}{(k-l)^3}\cdot b_3(x,s,y,t).$$

The proof is completed. □

Proof of Proposition B.1. The assertion for $|k-l| = 0, 1$ follows from the Hölder inequality and (B.3). We prove the assertion for $|k-l| \ge 2$. Note

$$\begin{aligned}\frac{1}{2}B_{k-1,k} - \int_{k-1}^{k} B_{k-1,u}\,du &= \frac{1}{2}(B_k - B_{k-1}) - \int_{k-1}^{k}(B_u - B_{k-1})\,du \\ &= \int_{k-1}^{k} du \int_{k-1}^{k} \mu_k(d\xi)\,(B_\xi - B_u)\end{aligned}$$

$$= \int_0^1 ds \int_0^1 \mu_1(dx)\,(B_{x+k-1} - B_{s+k-1}).$$

Here we set $\mu_k = (\delta_k + \delta_{k-1})/2$ by using the Dirac delta function δ_a. From this equality, we see

$$a_{k,l} = \int_0^1 ds \int_0^1 \mu_1(dx) \int_0^1 dt \int_0^1 \mu_1(dy)\, \boldsymbol{E}[(B_{x+k-1} - B_{s+k-1})(B_{y+l-1} - B_{t+l-1})].$$

Note that b_2 and b_3 in Lemma B.4 satisfy

$$\int_0^1 ds \int_0^1 \mu_1(dx) \int_0^1 dt \int_0^1 \mu_1(dy)\, b_\nu(x,s,y,t) = 0.$$

From Lemma B.4, we have

$$\begin{aligned}|a_{k,l}| &= \left| \int_0^1 ds \int_0^1 \mu_1(dx) \int_0^1 dt \int_0^1 \mu_1(dy)\, \frac{1}{2}|k-l|^{2H} R(k-l;x,s,y,t) \right| \\ &\le C|k-l|^{2H-4},\end{aligned}$$

which implies the conclusion for $|k-l| \ge 2$. The proof is completed. □

Proof of Proposition B.2. The assertion for $|k-l| = 0, 1$ follows from the Hölder inequality and (B.3). We prove the assertion for $|k-l| \ge 2$. We have

$$\begin{aligned}a^\dagger_{k,l} &= \boldsymbol{E}\left[(B_k - B_{k-1})\left(\frac{1}{2}(B_l - B_{l-1}) - \int_0^1 (B_{y+l-1} - B_{l-1})\,dy\right)\right] \\ &= \frac{1}{2}\boldsymbol{E}[(B_k - B_{k-1})(B_l - B_{l-1})] - \int_0^1 \boldsymbol{E}[(B_k - B_{k-1})(B_{y+l-1} - B_{l-1})]\,dy.\end{aligned}$$

From Lemma B.4, we have

$$\boldsymbol{E}[(B_k - B_{k-1})(B_l - B_{l-1})] = \frac{1}{2}|k-l|^{2H}\left\{\binom{2H}{2}\frac{2}{(k-l)^2} + R(k-l;1,0,1,0)\right\}$$

and

$$\begin{aligned}&\int_0^1 \boldsymbol{E}[(B_k - B_{k-1})(B_{y+l-1} - B_{l-1})]\,dy \\ &\qquad = \frac{1}{2}|k-l|^{2H}\left\{\binom{2H}{2}\frac{1}{(k-l)^2}\int_0^1 2y\,dy + \binom{2H}{3}\frac{1}{(k-l)^3}\int_0^1 3(y-y^2)\,dy \right.\\ &\qquad\qquad \left. + \int_0^1 R(k-l;1,0,y,0)\,dy\right\} \\ &\qquad = \frac{1}{2}|k-l|^{2H}\left\{\binom{2H}{2}\frac{1}{(k-l)^2} + \binom{2H}{3}\frac{1}{(k-l)^3}\frac{1}{2} + \int_0^1 R(k-l;1,0,y,0)\,dy\right\}\end{aligned}$$

From these equality, we have

$$\begin{aligned}a^\dagger_{k,l} &= \frac{1}{2}|k-l|^{2H}\left\{-\frac{1}{2}\binom{2H}{3}\frac{1}{(k-l)^3} + \frac{1}{2}R(k-l;1,0,1,0) - \int_0^1 R(k-l;1,0,y,0)\,dy\right\} \\ &= -\frac{1}{4}\binom{2H}{3}\frac{|k-l|^{2H}}{(k-l)^3} + \frac{1}{2}|k-l|^{2H}\left\{\frac{1}{2}R(k-l;1,0,1,0) - \int_0^1 R(k-l;1,0,y,0)\,dy\right\}.\end{aligned}$$

Recalling that R satisfies $|R(k-l;x,s,y,t)| \le C|k-l|^{-4}$ for some positive constant C, we obtain the conclusion. □

Proof of Proposition B.3. A direct computation yields

(B.4)
$$a^{\dagger}_{k,l} = \frac{1}{4}\left\{-|k-l+1|^{2H} + |k-l-1|^{2H}\right\} - \int_0^1 \frac{1}{2}\left\{-|k-l+1-s|^{2H} + |k-l-s|^{2H}\right\} ds$$

and

(B.5)
$$a^{\dagger}_{l,k} = \frac{1}{4}\left\{-|l-k+1|^{2H} + |l-k-1|^{2H}\right\} - \int_0^1 \frac{1}{2}\left\{-|k-l-t|^{2H} + |k-1+(1-t)|^{2H}\right\} dt.$$

The assertion follows from these two equalities.

We see (B.4) as follows:

$$\begin{aligned} a^{\dagger}_{k,l} &= \frac{1}{2}\boldsymbol{E}[(B_k - B_{k-1})(B_l - B_{l-1})] - \int_0^1 \boldsymbol{E}[(B_k - B_{k-1})(B_{s+l-1} - B_{l-1})]\, ds \\ &= \frac{1}{2}\frac{1}{2}\left\{|k-l+1|^{2H} + |k-l-1|^{2H} - 2|k-l|^{2H}\right\} \\ &\quad - \int_0^1 \frac{1}{2}\Big\{-|k-(s+l-1)|^{2H} + |k-(l-1)|^{2H} \\ &\qquad + |(k-1)-(s+l-1)|^{2H} - |(k-1)-(l-1)|^{2H}\Big\}\, ds. \end{aligned}$$

In order to prove (B.5), we exchange k and l in (B.4) and obtain

$$a^{\dagger}_{l,k} = \frac{1}{4}\left\{-|l-k+1|^{2H} + |l-k-1|^{2H}\right\} - \int_0^1 \frac{1}{2}\left\{-|l-k+1-s|^{2H} + |l-k-s|^{2H}\right\} ds.$$

From the integration by substitution $t = 1-s$, we see that the integral is equal to

$$\begin{aligned} &\int_1^0 \frac{1}{2}\left\{-|l-k+t|^{2H} + |l-k-(1-t)|^{2H}\right\}(-1)\, dt \\ &\qquad = \int_0^1 \frac{1}{2}\left\{-|k-l-t|^{2H} + |k-1+(1-t)|^{2H}\right\} dt. \end{aligned}$$

These two equalities imply (B.5). □

B.2. Relative compactness and convergence in fdds. We are ready to prove Proposition 4.5. We show relative compactness and convergence in the sense of finite-dimensional distributions (fdds).

Lemma B.5. *Under the assumption of Proposition* 4.5*, the sequence* $\{(B, V_q^{(m)}, \tilde{V}^{(m)})\}_{m=1}^{\infty}$ *is relative compact in the Skorokhod topology.*

Lemma B.6. *Under the assumption of Proposition* 4.5*, the sequence* $\{(B, V_q^{(m)}, \tilde{V}^{(m)})\}_{m=1}^{\infty}$ *converges in the sense of fdds. More precisely, we have, for* $0 \le s_1 < t_1 \le \cdots \le s_d < t_d \le 1$,

$$\begin{aligned} \lim_{m\to\infty} \Big(&B_{t_1} - B_{s_1}, V_q^{(m)}(t_1) - V_q^{(m)}(s_1), \tilde{V}^{(m)}(t_1) - \tilde{V}^{(m)}(s_1), \dots, \\ &B_{t_d} - B_{s_d}, V_q^{(m)}(t_d) - V_q^{(m)}(s_d), \tilde{V}^{(m)}(t_d) - \tilde{V}^{(m)}(s_d)\Big) \end{aligned}$$

$$= \Big(B_{t_1} - B_{s_1}, \sigma_H(W_{t_1} - W_{s_1}), \tilde{\sigma}_H(\tilde{W}_{t_1} - \tilde{W}_{s_1}), \dots, \\ B_{t_d} - B_{s_d}, \sigma_H(W_{t_d} - W_{s_d}), \tilde{\sigma}_H(\tilde{W}_{t_d} - \tilde{W}_{s_d})\Big)$$

weakly in $(\mathbf{R}^d)^3$*, where* W *and* $\tilde{W}$ *are standard Brownian motions and* B*,* W *and* $\tilde{W}$ *are independent.*

Before beginning our discussion, we note that, for any $0 \le s < t \le 1$ and $0 \le u < v \le 1$,

$$\text{(B.6)} \qquad \boldsymbol{E}[\{\tilde{V}^{(m)}(t) - \tilde{V}^{(m)}(s)\}\{\tilde{V}^{(m)}(v) - \tilde{V}^{(m)}(u)\}] = \frac{1}{2^m} \sum_{k=\lfloor 2^m s\rfloor+1}^{\lfloor 2^m t\rfloor} \sum_{l=\lfloor 2^m u\rfloor+1}^{\lfloor 2^m v\rfloor} a_{k,l}.$$

Applying (B.1) to (B.6), we see

$$\text{(B.7)} \quad \boldsymbol{E}[\{\tilde{V}^{(m)}(t) - \tilde{V}^{(m)}(s)\}^2] \\ = \frac{\lfloor 2^m t\rfloor - \lfloor 2^m s\rfloor}{2^m}\left(a_{1,1} + 2\sum_{j=1}^{\lfloor 2^m t\rfloor - \lfloor 2^m s\rfloor - 1} a_{1,j+1}\right) - \frac{2}{2^m}\sum_{j=1}^{\lfloor 2^m t\rfloor - \lfloor 2^m s\rfloor - 1} j a_{1,j+1}.$$

Proof of Lemma B.5. The assertion follows from

$$\boldsymbol{E}[\{V_q^{(m)}(t) - V_q^{(m)}(s)\}^4] \le C\left(\frac{\lfloor 2^m t\rfloor - \lfloor 2^m s\rfloor}{2^m}\right)^2,$$
$$\boldsymbol{E}[\{\tilde{V}^{(m)}(t) - \tilde{V}^{(m)}(s)\}^4] \le C\left(\frac{\lfloor 2^m t\rfloor - \lfloor 2^m s\rfloor}{2^m}\right)^2$$

for any $0 \le s < t \le 1$ and some constant C. The first estimate is proved in [17]. Combining (B.7) and Proposition B.1, we see

$$\boldsymbol{E}[\{\tilde{V}^{(m)}(t) - \tilde{V}^{(m)}(s)\}^2] \le C\frac{\lfloor 2^m t\rfloor - \lfloor 2^m s\rfloor}{2^m}.$$

Since $\tilde{V}^{(m)}(t) - \tilde{V}^{(m)}(s)$ is a Gaussian random variable, we have the second estimate. □

Proof of Lemma B.6. We show

$$\text{(B.8)} \qquad \lim_{m\to\infty} \boldsymbol{E}[\{V_q^{(m)}(t) - V_q^{(m)}(s)\}^4] = 3\sigma_H^4(t-s)^2,$$

$$\text{(B.9)} \qquad \lim_{m\to\infty} \boldsymbol{E}[\{V_q^{(m)}(t) - V_q^{(m)}(s)\}^2] = \sigma_H^2(t-s),$$

$$\text{(B.10)} \qquad \lim_{m\to\infty} \boldsymbol{E}[\{\tilde{V}^{(m)}(t) - \tilde{V}^{(m)}(s)\}^2] = \tilde{\sigma}_H^2(t-s),$$

$$\text{(B.11)} \qquad \lim_{m\to\infty} \boldsymbol{E}[\{B_t - B_s\}\{V_q^{(m)}(t) - V_q^{(m)}(s)\}] = 0,$$

$$\text{(B.12)} \qquad \lim_{m\to\infty} \boldsymbol{E}[\{V_q^{(m)}(t) - V_q^{(m)}(s)\}\{\tilde{V}^{(m)}(t) - \tilde{V}^{(m)}(s)\}] = 0,$$

$$\text{(B.13)} \qquad \lim_{m\to\infty} \boldsymbol{E}[\{B_t - B_s\}\{\tilde{V}^{(m)}(t) - \tilde{V}^{(m)}(s)\}] = 0$$

for $0 \le s < t \le 1$ and

$$\text{(B.14)} \qquad \lim_{m\to\infty} \boldsymbol{E}[\{V_q^{(m)}(t) - V_q^{(m)}(s)\}\{V_q^{(m)}(v) - V_q^{(m)}(u)\}] = 0,$$

$$\text{(B.15)} \qquad \lim_{m\to\infty} \boldsymbol{E}[\{\tilde{V}^{(m)}(t) - \tilde{V}^{(m)}(s)\}\{\tilde{V}^{(m)}(v) - \tilde{V}^{(m)}(u)\}] = 0,$$

$$\text{(B.16)} \qquad \lim_{m\to\infty} \boldsymbol{E}[\{B_t - B_s\}\{V_q^{(m)}(v) - V_q^{(m)}(u)\}] = 0,$$

$$\lim_{m\to\infty} \boldsymbol{E}[\{V_q^{(m)}(t) - V_q^{(m)}(s)\}\{\tilde{V}^{(m)}(v) - \tilde{V}^{(m)}(u)\}] = 0, \tag{B.17}$$

$$\lim_{m\to\infty} \boldsymbol{E}[\{B_t - B_s\}\{\tilde{V}^{(m)}(v) - \tilde{V}^{(m)}(u)\}] = 0 \tag{B.18}$$

for $0 \le s < t \le 1$ and $0 \le u < v \le 1$ with $(s,t) \cap (u,v) = \emptyset$. From these convergence and the fourth moment theorem in [20], we see the assertion.

The convergence (B.8), (B.9) and (B.14) are proved in [17].

We consider (B.10) and (B.15). Both convergence follows from (B.7) and Proposition B.1. In particular, (B.10) is a direct consequence from them. We show (B.15) for $s < t \le u < v$. From (B.6) and (B.1), we have

$$\begin{aligned}
&|\boldsymbol{E}[\{\tilde{V}^{(m)}(t) - \tilde{V}^{(m)}(s)\}\{\tilde{V}^{(m)}(v) - \tilde{V}^{(m)}(u)\}]| \\
&\qquad \le \frac{1}{2^m} \sum_{k=\lfloor 2^m s\rfloor+1}^{\lfloor 2^m t\rfloor} \sum_{l=\lfloor 2^m u\rfloor+1}^{\lfloor 2^m v\rfloor} |a_{1,l-k+1}| \le \frac{1}{2^m} \sum_{j=\lfloor 2^m u\rfloor+1-\lfloor 2^m t\rfloor}^{\lfloor 2^m v\rfloor-\lfloor 2^m s\rfloor-1} j|a_{1,j+1}| \le \frac{1}{2^m} \sum_{j=1}^{2^m} j|a_{1,j+1}|.
\end{aligned}$$

Combining this estimate and Proposition B.1, we obtain (B.15).

We study the equalities (B.11), (B.12), (B.16) and (B.17). Since $B_t - B_s$, $V_q^{(m)}(t) - V_q^{(m)}(s)$ and $\tilde{V}^{(m)}(t) - \tilde{V}^{(m)}(s)$ belongs to first, q-th, first Wiener chaos, the expectations in (B.11) and (B.12) are equal to 0. The same reason yields (B.16) and (B.17).

We prove (B.13) and (B.18). Set $B_t^{(m)} = B_{\lfloor 2^m t\rfloor/2^m} = \sum_{k=1}^{\lfloor 2^m t\rfloor} B_{\tau_{k-1}^m \tau_k^m}$. We decompose $\boldsymbol{E}[\{B_t - B_s\}\{\tilde{V}^{(m)}(v) - \tilde{V}^{(m)}(u)\}]$ into $I^{(m)} + \boldsymbol{E}[\{B_t^{(m)} - B_s^{(m)}\}\{\tilde{V}^{(m)}(v) - \tilde{V}^{(m)}(u)\}] + J^{(m)}$, where

$$\begin{aligned}
I^{(m)} &= \boldsymbol{E}[\{B_t - B_t^{(m)}\}\{\tilde{V}^{(m)}(v) - \tilde{V}^{(m)}(u)\}], \\
J^{(m)} &= \boldsymbol{E}[\{B_s^{(m)} - B_s\}\{\tilde{V}^{(m)}(v) - \tilde{V}^{(m)}(u)\}].
\end{aligned}$$

We can show convergence of $I^{(m)}$ and $J^{(m)}$ easily. In fact, we see

$$\begin{aligned}
|I^{(m)}| &\le \boldsymbol{E}[\{B_t - B_{\lfloor 2^m t\rfloor/2^m}\}^2]^{1/2} \boldsymbol{E}[\{\tilde{V}^{(m)}(v) - \tilde{V}^{(m)}(u)\}^2]^{1/2} \\
&\le \left(t - \frac{\lfloor 2^m t\rfloor}{2^m}\right)^H \left(C\frac{\lfloor 2^m u\rfloor - \lfloor 2^m v\rfloor}{2^m}\right)^{1/2}.
\end{aligned}$$

The same inequality holds for $J^{(m)}$. Hence we see the convergences.

We consider convergence of $\boldsymbol{E}[\{B_t^{(m)} - B_s^{(m)}\}\{\tilde{V}^{(m)}(v) - \tilde{V}^{(m)}(u)\}]$. Note

$$\boldsymbol{E}[\{B_t^{(m)} - B_s^{(m)}\}\{\tilde{V}^{(m)}(v) - \tilde{V}^{(m)}(u)\}] = 2^{-m(1/2+H)} \sum_{k=\lfloor 2^m s\rfloor+1}^{\lfloor 2^m t\rfloor} \sum_{l=\lfloor 2^m u\rfloor+1}^{\lfloor 2^m v\rfloor} a_{k,l}^{\dagger}.$$

In the case that $s = u$ and $t = v$, we see

$$\begin{aligned}
&\boldsymbol{E}[\{B_t^{(m)} - B_s^{(m)}\}\{\tilde{V}^{(m)}(t) - \tilde{V}^{(m)}(s)\}] \\
&\qquad = 2^{-m(1/2+H)} \sum_{\lfloor 2^m s\rfloor+1}^{\lfloor 2^m t\rfloor} a_{k,k}^{\dagger} + 2^{-m(1/2+H)} \sum_{\lfloor 2^m s\rfloor+1\le k<l\le\lfloor 2^m t\rfloor} (a_{k,l}^{\dagger} + a_{l,k}^{\dagger}) = 0.
\end{aligned}$$

In the last line, we used Proposition B.3. From this, we see (B.13).

In the case that $0 \le s < t \le u < v \le 1$, by noting (B.2), we have

$$|\boldsymbol{E}[\{B_t^{(m)} - B_s^{(m)}\}\{\tilde{V}^{(m)}(u) - \tilde{V}^{(m)}(v)\}]|$$
$$\leq 2^{-m(1/2+H)} \sum_{k=\lfloor 2^m s\rfloor+1}^{\lfloor 2^m t\rfloor} \sum_{l=\lfloor 2^m u\rfloor+1}^{\lfloor 2^m v\rfloor} |a_{1,l-k+1}^{\dagger}| \leq 2^{-m(1/2+H)} \sum_{j=\lfloor 2^m u\rfloor-\lfloor 2^m t\rfloor+1}^{\lfloor 2^m v\rfloor-\lfloor 2^m s\rfloor-1} j|a_{1,j+1}^{\dagger}|.$$

From Proposition B.2, we see

$$|\boldsymbol{E}[\{B_t^{(m)} - B_s^{(m)}\}\{\tilde{V}^{(m)}(u) - \tilde{V}^{(m)}(v)\}]| \leq C2^{-m(1/2+H)} \sum_{j=1}^{2^m} j \cdot j^{2H-3}.$$

In the case that $0 \leq u < v \leq s < t \leq 1$, we obtain the same inequality. We complete the proof of (B.18). □

C. Proof of convergence of variation functionals

C.1. Estimate on $\tilde{U}_m$. In this subsection, we prove Theorem 4.3. At the beginning, we give an estimate of $\boldsymbol{E}[|\tilde{U}^{(m)}(t) - \tilde{U}^{(m)}(s)|^2]$.

Proposition C.1. *There exists a positive constant C independent of m such that*

$$\left|\boldsymbol{E}[g(X_s)g(X_t)I_2(\zeta_{\tau_{k-1}^m\tau_k^m} \odot \zeta_{\tau_{l-1}^m\tau_l^m})]\right| \leq C \begin{cases} 2^{-m(4H+2)}, & 0 < H < 1/2, \\ 2^{-4m}, & 1/2 \leq H < 1 \end{cases}$$

for any $0 \leq s, t \leq 1$ and $1 \leq k, l \leq 2^m$.

Proof. From the duality relationship (A.1), we have

$$\boldsymbol{E}[g(X_s)g(X_t)I_2(\zeta_{\tau_{k-1}^m\tau_k^m} \odot \zeta_{\tau_{l-1}^m\tau_l^m})] = \boldsymbol{E}\left[\left\langle D^2\{g(X_s)g(X_t)\}, \zeta_{\tau_{k-1}^m\tau_k^m} \odot \zeta_{\tau_{l-1}^m\tau_l^m}\right\rangle_{\mathfrak{H}^{\odot 2}}\right]$$

and the Leibniz rule implies

$$D^2\{g(X_s)g(X_t)\} = g'(X_s)g(X_t)D^2X_s + g''(X_s)g(X_t)(DX_s)^{\odot 2}$$
$$+ 2g'(X_s)g'(X_t)DX_s \odot DX_t + g(X_s)g''(X_t)(DX_t)^{\odot 2} + g(X_s)g'(X_t)D^2X_t.$$

From the Hölder inequality and Proposition A.1, we have

$$\boldsymbol{E}\left[g'(X_s)g(X_t)\left\langle D^2X_s, \zeta_{\tau_{k-1}^m\tau_k^m} \odot \zeta_{\tau_{l-1}^m\tau_l^m}\right\rangle_{\mathfrak{H}^{\odot 2}}\right] \leq \boldsymbol{E}[|g'(X_{\tau_{k-1}^m})g(X_{\tau_{l-1}^m})|^2]^{1/2} \cdot C\|\zeta_{\tau_{k-1}^m\tau_k^m}\|_\infty\|\zeta_{\tau_{l-1}^m\tau_l^m}\|_\infty$$
$$\leq C' \cdot \begin{cases} \left(\left(\dfrac{1}{2} + \dfrac{1}{2H+1}\right)^2 (2^{-m(2H+1)})^2, & 0 < H < 1/2, \\ (2H)^2(2^{-2m})^2, & 1/2 \leq H < 1. \end{cases}$$

In the last line, we used Proposition A.2 and the constant C and C' are independent of m. Since the other terms in the above also admit similar estimates, we see the assertion. □

Proposition C.2. *There exists a positive constant C independent of m such that*

$$\boldsymbol{E}[|\tilde{U}^{(m)}(t) - \tilde{U}^{(m)}(s)|^2] \leq C \cdot \frac{\lfloor 2^m t\rfloor - \lfloor 2^m s\rfloor}{2^m} \cdot \begin{cases} 2^{-4mH}, & 0 < H < 1/2, \\ 2^{-2m}, & 1/2 \leq H < 1 \end{cases}$$

for any $0 \leq s, t \leq 1$.

Proof. From the product formula (A.2), we have

$$|\tilde{U}^{(m)}(t) - \tilde{U}^{(m)}(s)|^2 = \sum_{k,l=\lfloor 2^m s\rfloor+1}^{\lfloor 2^m t\rfloor} g(X_{\tau^m_{k-1}})g(X_{\tau^m_{l-1}})I_1(\zeta_{\tau^m_{k-1}\tau^m_k})I_1(\zeta_{\tau^m_{l-1}\tau^m_l}) = S + T,$$

where

$$S = \sum_{k,l=\lfloor 2^m s\rfloor+1}^{\lfloor 2^m t\rfloor} g(X_{\tau^m_{k-1}})g(X_{\tau^m_{l-1}})\langle \zeta_{\tau^m_{k-1}\tau^m_k}, \zeta_{\tau^m_{l-1}\tau^m_l}\rangle_{\mathfrak{H}},$$

$$T = \sum_{k,l=\lfloor 2^m s\rfloor+1}^{\lfloor 2^m t\rfloor} g(X_{\tau^m_{k-1}})g(X_{\tau^m_{l-1}})I_2(\zeta_{\tau^m_{k-1}\tau^m_k} \odot \zeta_{\tau^m_{l-1}\tau^m_l}).$$

We estimate the expectations $\boldsymbol{E}[S]$ and $\boldsymbol{E}[T]$.

The expectation $|\boldsymbol{E}[S]|$ is estimated by

$$\begin{aligned}|\boldsymbol{E}[S]| &= \left|\sum_{k,l=\lfloor 2^m s\rfloor+1}^{\lfloor 2^m t\rfloor} \boldsymbol{E}[g(X_{\tau^m_{k-1}})g(X_{\tau^m_{l-1}})]\langle \zeta_{\tau^m_{k-1}\tau^m_k}, \zeta_{\tau^m_{l-1}\tau^m_l}\rangle_{\mathfrak{H}}\right| \\ &\leq \left(\sup_{0\leq t\leq 1} \boldsymbol{E}[|g(X_t)|^2]\right) \sum_{k,l=\lfloor 2^m s\rfloor+1}^{\lfloor 2^m t\rfloor} |\langle \zeta_{\tau^m_{k-1}\tau^m_k}, \zeta_{\tau^m_{l-1}\tau^m_l}\rangle_{\mathfrak{H}}|.\end{aligned}$$

Combining the self-similarity of fBm and Proposition B.1, we have

$$\begin{aligned}|\boldsymbol{E}[S]| &\leq \left(\sup_{0\leq t\leq 1} \boldsymbol{E}[|g(X_t)|^2]\right) \cdot 2^{-m(2H+2)} \cdot C(\lfloor 2^m t\rfloor - \lfloor 2^m s\rfloor) \\ &= C\left(\sup_{0\leq t\leq 1} \boldsymbol{E}[|g(X_t)|^2]\right) \frac{\lfloor 2^m t\rfloor - \lfloor 2^m s\rfloor}{2^m} \cdot 2^{-m(2H+1)}.\end{aligned}$$

We evaluate the expectation $\boldsymbol{E}[T]$. From Proposition C.1, we obtain

$$\begin{aligned}|\boldsymbol{E}[T]| &\leq (\lfloor 2^m t\rfloor - \lfloor 2^m s\rfloor)^2 \cdot C \begin{cases} 2^{-2m(4H+2)}, & 0 < H < 1/2, \\ 2^{-4m}, & 1/2 \leq H < 1, \end{cases} \\ &\leq C\frac{\lfloor 2^m t\rfloor - \lfloor 2^m s\rfloor}{2^m} \cdot \begin{cases} 2^{-4mH}, & 0 < H < 1/2, \\ 2^{-2m}, & 1/2 \leq H < 1. \end{cases}\end{aligned}$$

The proof is completed. □

Proof of Proposition 4.4. From Proposition C.2, we have

$$\begin{aligned}\boldsymbol{E}[|2^{mr}\tilde{U}^{(m)}(t) - 2^{mr}\tilde{U}^{(m)}(s)|^2] &= 2^{2mr}\boldsymbol{E}[|\tilde{U}^{(m)}(t) - \tilde{U}^{(m)}(s)|^2] \\ &\leq C\frac{\lfloor 2^m t\rfloor - \lfloor 2^m s\rfloor}{2^m} \cdot \begin{cases} 2^{-2m(2H-r)}, & 0 < H < 1/2, \\ 2^{-2m(1-r)}, & 1/2 \leq H < 1. \end{cases}\end{aligned}$$

This inequality implies convergence of in the sense of fdds and relative compactness. For relative compactness, see [2, Cororally 2.2]. The proof is completed. □

Proof of Theorem 4.3. The assertion follows from convergence of in the sense of fdds and relative compactness of $\{(B, 2^{-m/2}U_q^{(m)}, 2^m\tilde{U}^{(m)})\}_{m=1}^{\infty}$, that is, we obtain Theorem 4.3 from the following Lemmas C.3 and C.4. □

Lemma C.3. *Let* $0 \le t_1 < \cdots < t_d \le 1$. *Under the assumption of Theorem* 4.3, *we have*

$$\text{(C.1)} \quad \lim_{m\to\infty} \Big(B_{t_1}, 2^{-m/2}U_q^{(m)}(t_1), 2^m \tilde{U}^{(m)}(t_1), \dots, B_{t_d}, 2^{-m/2}U_q^{(m)}(t_d), 2^m \tilde{U}^{(m)}(t_d)\Big)$$
$$= \Big(B_{t_1}, \sqrt{q!}\int_0^{t_1} f(X_s)\,dW_s, \frac{1}{\sqrt{12}}\int_0^{t_1} g(X_s)\,d\tilde{W}_s, \dots,$$
$$B_{t_d}, \sqrt{q!}\int_0^{t_d} f(X_s)\,dW_s, \frac{1}{\sqrt{12}}\int_0^{t_d} g(X_s)\,d\tilde{W}_s,\Big)$$

weakly in $(\mathbf{R}^d)^3$, *where* W *and* $\tilde{W}$ *are standard Brownian motions and* B, W *and* $\tilde{W}$ *are independent.*

Lemma C.4. *Under the assumption of Theorem* 4.3, $\{(B, 2^{-m/2}U_q^{(m)}, 2^m\tilde{U}^{(m)})\}_{m=1}^\infty$ *is relative compact in the Skorokhod topology.*

Proof of Lemma C.3. We decompose $U_q^{(m)}(t)$ and $\tilde{U}^{(m)}(t)$ into $U_q^{(m,n)}(t) + R^{(m,n)}(t)$ and $\tilde{U}^{(m,n)}(t) + \tilde{R}^{(m,n)}(t)$ for $m \ge n$, respectively, where

$$U_q^{(m,n)}(t) = \sum_{k=1}^{\lfloor 2^m t\rfloor} f(X_{\eta_-^n(\tau_{k-1}^m)}) H_q(2^{mH} B_{\tau_{k-1}^m \tau_k^m}),$$
$$R^{(m,n)}(t) = \sum_{k=1}^{\lfloor 2^m t\rfloor} \Big\{F_{\tau_{k-1}^m \tau_k^m}(X) - f(X_{\eta_-^n(\tau_{k-1}^m)})\Big\} H_q(2^{mH} B_{\tau_{k-1}^m \tau_k^m}),$$
$$\tilde{U}^{(m,n)}(t) = \sum_{k=1}^{\lfloor 2^m t\rfloor} g(X_{\eta_-^n(\tau_{k-1}^m)}) I_1(\zeta_{\tau_{k-1}^m \tau_k^m}),$$
$$\tilde{R}^{(m,n)}(t) = \sum_{k=1}^{\lfloor 2^m t\rfloor} \Big\{g(X_{\tau_{k-1}^m}) - g(X_{\eta_-^n(\tau_{k-1}^m)})\Big\} I_1(\zeta_{\tau_{k-1}^m \tau_k^m}).$$

Here $\eta_-^n(t) = \sup\{\tau_k^n; \tau_k^n \le t, k = 0, \dots, 2^n - 1\}$. We prove

(1) The sequence $\{\{(B_{t_\alpha}, 2^{-m/2}U_q^{(m,n)}(t_\alpha), 2^m\tilde{U}^{(m,n)}(t_\alpha))_{\alpha=1}^d\}_{m=n}^\infty\}_{n=1}^\infty$ converges to the right-hand side of (C.1) as $m \to \infty$ and $n \to \infty$.

(2) $\lim_{n\to\infty} \limsup_{m\to\infty} \boldsymbol{E}[|2^{-m/2}R^{(m,n)}(t_\alpha)|^2] = 0$ for $\alpha = 1, \dots, d$,

(3) $\lim_{n\to\infty} \limsup_{m\to\infty} \boldsymbol{E}[|2^m\tilde{R}^{(m,n)}(t_\alpha)|^2] = 0$ for $\alpha = 1, \dots, d$.

Assertion (1) is a direct consequence of Proposition 4.5. To show Assertion (2), we use the product formula (A.2) and estimate the expectations. For detail, see [13, Lemmas 22 and 23].

In the rest of this proof we show Assertion (3) by using independent increments of the standard Brownian motion B. Set $\tilde{Y}_k^{(m,n)} = \{g(X_{\tau_{k-1}^m}) - g(X_{\eta_-^n(\tau_{k-1}^m)})\} I_1(\zeta_{\tau_{k-1}^m \tau_k^m})$ and $\mathcal{F}_t = \sigma(B_u; 0 \le u \le t)$. Then, for $k < l$, random variables $\tilde{Y}_k^{(m,n)}$ and $g(X_{\tau_{k-1}^m}) - g(X_{\eta_-^n(\tau_{k-1}^m)})$ are $\mathcal{F}_{\tau_{l-1}^m}$-measurable. In addition, $I_1(\zeta_{\tau_{l-1}^m \tau_l^m})$ is independent of $\mathcal{F}_{\tau_{l-1}^m}$. This implies $\boldsymbol{E}[I_1(\zeta_{\tau_{l-1}^m \tau_l^m})|\mathcal{F}_{\tau_{l-1}^m}] = \boldsymbol{E}[I_1(\zeta_{\tau_{l-1}^m \tau_l^m})] = 0$ a.s. Hence, we have

$$\boldsymbol{E}[\tilde{Y}_k^{(m,n)}\tilde{Y}_l^{(m,n)}|\mathcal{F}_{\tau_{l-1}^m}] = \tilde{Y}_k^{(m,n)}\{g(X_{\tau_{k-1}^m}) - g(X_{\eta_-^n(\tau_{k-1}^m)})\}\boldsymbol{E}[I_1(\zeta_{\tau_{l-1}^m \tau_l^m})|\mathcal{F}_{\tau_{l-1}^m}] = 0$$

a.s. for $k < l$. From this, we obtain $\boldsymbol{E}[\tilde{Y}_k^{(m,n)}\tilde{Y}_l^{(m,n)}] = 0$ for $k \ne l$. In addition, we have

$$\boldsymbol{E}[|\tilde{Y}_k^{(m,n)}|^2] \leq \boldsymbol{E}[\{g(X_{\tau_{k-1}^m}) - g(X_{\eta_-^n(\tau_{k-1}^m)})\}^4]^{1/2}\boldsymbol{E}[I_1(\zeta_{\tau_{k-1}^m\tau_k^m})^4]^{1/2} \leq C2^{-n}2^{-3m}$$

for some constant C. From these, we obtain

$$\boldsymbol{E}[|2^m\tilde{R}^{(m,n)}(t)|^2] = 2^{2m}\sum_{k=1}^{\lfloor 2^m t\rfloor}\boldsymbol{E}[|\tilde{Y}_k^{(m,n)}|^2] \leq 2^{2m}\cdot 2^m\cdot C2^{-n}2^{-3m} = C2^{-n},$$

which implies the third assertion.

The proof is completed. □

Proof of Lemma C.4. We can prove the assertion in the same way as [13, Proposition 18]. In the proof, we shall show that the processes satisfy some kind of moment condition for relative compactness. □

C.2. Weighted Hermite and power variations. In this subsection, we prove Theorems 4.1 and 4.2.

At the beginning, we give an estimate of $\boldsymbol{E}[|U_q^{(m)}(t) - U_q^{(m)}(s)|^2]$

Proposition C.5. *Let* μ *and* ν *be probability measure on* $[0,1]$ *and* $f, g \in C^{2q}_{\mathrm{poly}}(\mathbf{R};\mathbf{R})$. *Then there exists a constant C such that*

$$\left|\boldsymbol{E}\left[I_{2q-2r}(\delta_{\tau_{k-1}^m\tau_k^m}^{\odot q-r}\odot\delta_{\tau_{l-1}^m\tau_l^m}^{\odot q-r})F_{st}^{f,\mu}(X)F_{uv}^{g,\nu}(X)\right]\right| \leq C\begin{cases}2^{-4m(q-r)H}, & 0<H<1/2,\\ 2^{-2m(q-r)}, & 1/2\leq H<1,\end{cases}$$

for any $0\leq s<t\leq 1$, $0\leq u<v\leq 1$ *and* $1\leq k,l\leq 2^m$.

Proof. From the duality relationship (A.1) and the Leibniz rule, we see

$$\begin{aligned}
&\boldsymbol{E}\left[I_{2q-2r}(\delta_{\tau_{k-1}^m\tau_k^m}^{\odot q-r}\odot\delta_{\tau_{l-1}^m\tau_l^m}^{\odot q-r})F_{st}^{f,\mu}(X)F_{uv}^{g,\nu}(X)\right]\\
&\qquad=\boldsymbol{E}\left[\left\langle\delta_{\tau_{k-1}^m\tau_k^m}^{\odot q-r}\odot\delta_{\tau_{l-1}^m\tau_l^m}^{\odot q-r}, D^{2q-2r}\left\{F_{st}^{f,\mu}(X)F_{uv}^{g,\nu}(X)\right\}\right\rangle_{\mathfrak{H}^{\odot 2q-2r}}\right]\\
&\qquad=\sum_{a+b=2q-2r}\frac{(2q-2r)!}{a!b!}\boldsymbol{E}\left[\left\langle\delta_{\tau_{k-1}^m\tau_k^m}^{\odot q-r}\odot\delta_{\tau_{l-1}^m\tau_l^m}^{\odot q-r}, D^aF_{st}^{f,\mu}(X)\odot D^bF_{uv}^{g,\nu}(X)\right\rangle_{\mathfrak{H}^{\odot 2q-2r}}\right].
\end{aligned}$$

From Proposition A.2, we see that

$$\boldsymbol{E}[|\langle D^aF_{st}^{f,\mu}(X), h^1\odot\cdots\odot h^a\rangle_{\mathfrak{H}^{\odot a}}|^r]^{1/r} \leq C\|h^1\|_\infty\cdots\|h^a\|_\infty \leq C\begin{cases}2^{-2maH}, & 0<H<1/2,\\ (2H)^a2^{-ma}, & 1/2\leq H<1,\end{cases}$$

for $h^1,\ldots,h^a\in\{\delta_{\tau_{k-1}^m\tau_k^m},\delta_{\tau_{l-1}^m\tau_l^m}\}$. Combining them, the proof is completed. □

Proposition C.6. *Let* $q\geq 2$. *There exists a positive constant C such that*

$$\boldsymbol{E}[|U_q^{(m)}(t)-U_q^{(m)}(s)|^2] \leq C(\lfloor 2^mt\rfloor - \lfloor 2^ms\rfloor)\begin{cases}2^{m(1-2qH)}, & 0<H\leq 1/2q,\\ 1, & 1/2q<H<1-1/2q,\\ m, & H=1-1/2q,\\ 2^{m\{1-2q(1-H)\}}, & 1-1/2q<H<1,\end{cases}$$

for any $0\leq s<t\leq 1$.

Proof. We can prove this proposition in the same way as [13, Proposition 21] by using Proposition C.5 instead of [13, Proposition 19]. In more detail, we use (A.2) to rewrite $|U_q^{(m)}(t) - U_q^{(m)}(s)|^2$ by the Itô-Wiener integrals. Then we see that it is expressed by the summation of the integrand in Proposition C.5. From Proposition C.5, we see the conclusion. □

We prove Theorems 4.1 and 4.2.

Proof of Theorem 4.1. Recall the identity $\xi^q = \sum_{r=0}^{q} \binom{q}{r} \boldsymbol{E}[Z^{q-r}] H_r(\xi)$ for any $\xi \in \mathbf{R}$, where Z is a standard Gaussian random variable. Applying this identity, we see

$$\begin{aligned} 2^{m(qH-1)} \sum_{k=1}^{\lfloor 2^m \cdot \rfloor} F_{\tau_{k-1}^m \tau_k^m}(X)(B_{\tau_{k-1}^m \tau_k^m})^q &= 2^{-m} \sum_{k=1}^{\lfloor 2^m \cdot \rfloor} F_{\tau_{k-1}^m \tau_k^m}(X)(2^{mH} B_{\tau_{k-1}^m \tau_k^m})^q \\ &= 2^{-m} \sum_{r=0}^{q} \binom{q}{r} \boldsymbol{E}[Z^{q-r}] \sum_{k=1}^{\lfloor 2^m \cdot \rfloor} F_{\tau_{k-1}^m \tau_k^m}(X) H_r(2^{mH} B_{\tau_{k-1}^m \tau_k^m}) \\ &= \boldsymbol{E}[Z^q] \cdot 2^{-m} \sum_{k=1}^{\lfloor 2^m \cdot \rfloor} F_{\tau_{k-1}^m \tau_k^m}(X) + \sum_{r=2}^{q} \binom{q}{r} \boldsymbol{E}[Z^{q-r}] \cdot 2^{-m} U_r^{(m)}. \end{aligned}$$

We prove convergence of the first and second term in the following.

We consider the first term. Note

$$\begin{aligned} & 2^{-m} \sum_{k=1}^{\lfloor 2^m t \rfloor} F_{\tau_{k-1}^m \tau_k^m}(X) - \int_0^t f(X_s)\, ds \\ &\qquad = \sum_{k=1}^{\lfloor 2^m t \rfloor} \int_{\tau_{k-1}^m}^{\tau_k^m} F_{\tau_{k-1}^m \tau_k^m}(X)\, ds - \sum_{k=1}^{\lfloor 2^m t \rfloor} \int_{\tau_{k-1}^m}^{\tau_k^m} f(X_s)\, ds - \int_{\lfloor 2^m t \rfloor}^{t} f(X_s)\, ds \\ &\qquad = \sum_{k=1}^{\lfloor 2^m t \rfloor} \int_{\tau_{k-1}^m}^{\tau_k^m} \{F_{\tau_{k-1}^m \tau_k^m}(X) - f(X_s)\}\, ds - \int_{\lfloor 2^m t \rfloor}^{t} f(X_s)\, ds. \end{aligned}$$

Since X is $(H - \epsilon)$-Hölder continuous, we see that the absolute value of the above has an upper bound

$$\sum_{k=1}^{\lfloor 2^m t \rfloor} \int_{\tau_{k-1}^m}^{\tau_k^m} |F_{\tau_{k-1}^m \tau_k^m}(X) - f(X_s)|\, ds \le \sum_{k=1}^{\lfloor 2^m t \rfloor} \int_{\tau_{k-1}^m}^{\tau_k^m} C_X 2^{-m(H-\epsilon)}\, ds = C_X 2^{-m(H-\epsilon)},$$

where C_X is a random variable. Hence

$$\lim_{m\to\infty} 2^{-m} \sum_{k=1}^{\lfloor 2^m \cdot \rfloor} F_{\tau_{k-1}^m \tau_k^m}(X) = \int_0^{\cdot} f(X_s)\, ds$$

almost surely with respect to the uniform norm.

We prove convergence of the process $2^{-m} U_r^{(m)}$ to the process 0 for $r = 2, \dots, q$. It follows from Proposition C.6 that

$$E[|2^{-m}U_r^{(m)}(t) - 2^{-m}U_r^{(m)}(s)|^2] \leq C\frac{\lfloor 2^m t\rfloor - \lfloor 2^m s\rfloor}{2^m}\begin{cases} 2^{-2rmH}, & 0 < H \leq 1/2r, \\ 2^{-m}, & 1/2r < H < 1 - 1/2r, \\ m2^{-m}, & H = 1 - 1/2r, \\ 2^{-2rm(1-H)}, & 1 - 1/2r < H < 1, \end{cases}$$

$$\leq C\left(\frac{\lfloor 2^m t\rfloor - \lfloor 2^m s\rfloor}{2^m}\right)^{1+\kappa}\begin{cases} 2^{-\kappa m}, & 0 < H \leq 1/2r, \\ 2^{-\kappa m}, & 1/2r < H < 1 - 1/2r, \\ m2^{-\kappa m}, & H = 1 - 1/2r, \\ 2^{-\kappa m}, & 1 - 1/2r < H < 1, \end{cases}$$

where

$$\kappa = \begin{cases} rH, & 0 < H \leq 1/2r, \\ 1/2, & 1/2r < H < 1 - 1/2r, \\ 1/2, & H = 1 - 1/2r, \\ r(1-H), & 1 - 1/2r < H < 1. \end{cases}$$

This inequality implies convergence of $2^{-m}U_q^{(m)}$ to the zero process.

The proof is completed. □

Proof of Theorem 4.2. The assertion is proved in the same way as [13, Theorem 15] by using Proposition C.5 instead of [13, Proposition 19]. In this proof, we use Proposition 4.5. □

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