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GRAPH INVARIANTS AND BETTI NUMBERS OF REAL TORIC MANIFOLDS

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Abstract

For a graph G, the graph cubeahedron $\square_G$ and the graph associahedron $\triangle_G$ are simple convex polytopes which admit (real) toric manifolds. In this paper, we introduce a graph invariant, called the b-number, and show that the b-numbers compute the Betti numbers of the real toric manifold $X^{\mathbb{R}}(\square_G)$ corresponding to $\square_G$. The b-number is a counterpart of the notion of a-number, introduced by S. Choi and the second named author, which computes the Betti numbers of the real toric manifold $X^{\mathbb{R}}(\triangle_G)$ corresponding to $\triangle_G$. We also study various relationships between a-numbers and b-numbers from the viewpoint of toric topology. Interestingly, for a forest G and its line graph $L(G)$, the real toric manifolds $X^{\mathbb{R}}(\triangle_G)$ and $X^{\mathbb{R}}(\square_{L(G)})$ have the same Betti numbers.

1. Introduction

Throughout this paper, we focus on simple convex polytopes constructed from a graph. We only consider a finite simple graph and use G or H for a generic symbol to denote a graph.

For a graph G, the *graph associahedron* of G, denoted by $\triangle_G$, is a simple convex polytope obtained from a product of simplices by truncating the faces corresponding to proper connected induced subgraphs of each component of G. See Section 2 for the precise construction. This polytope was first introduced by Carr and Devadoss in [4] whose motivation was the work of De Concini and Procesi, wonderful compactifications of hyperplane arrangements [12]. Graph associahedra have also appeared in a broad range of subjects such as the moduli space of curves [15, 13] and enumerative properties like h-vectors [22].

The h-vector of a simple polytope is a fundamental invariant of the polytope which encodes the number of faces of different dimensions. It is known that the h-vectors of graph associahedra give interesting integer sequences. For example, the h-vector of the graph associahedron $\triangle_{P_n}$ of a path P_n is given by the Narayana numbers: $h_i(\triangle_{P_n}) = N(n, i+1) = \frac{1}{n}\binom{n}{i+1}\binom{n}{i}$ for $i = 0, \dots, n-1$, see [22] for more examples.

A *graph cubeahedron* is a simple convex polytope introduced in [14], and it is deeply related to the moduli space of a bordered Riemann surface. The *graph cubeahedron* of G, denoted by $\square_G$, is defined to be a polytope obtained from a cube by truncating the faces corresponding to connected induced subgraphs. It was also shown in [14] that the graph cubeahedron $\square_{P_n}$ is combinatorially equivalent to the graph associahedron $\triangle_{P_{n+1}}$, and hence $h_i(\square_{P_n})$ is the Narayana number $N(n+1, i+1)$ for $i = 0, \dots, n$. But graph cubeahedra are

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not much known compared with graph associahedra.

On the other hand, there is a beautiful connection between the h-vector of a simple polytope and the Betti numbers of a toric variety in toric geometry. A compact non-singular toric variety (*toric manifold* for short) is over a simple polytope P if its quotient by the compact torus is homeomorphic to P as a manifold with corners. If a toric manifold X is over P, then the cohomology groups of X vanish in odd degrees and the $2i$-th Betti number of X is equal to $h_i(P)$, see [17, 11]. In fact, both graph associahedra and graph cubeahedra can admit toric manifolds over them. Hence the Betti numbers of toric manifolds associated with a path are Narayana numbers.

Unlike (complex) toric varieties, the real locus of a toric manifold, called a *real toric manifold*, is much less known for its cohomology. In coefficient $\mathbb{Z}_2$, the cohomology of a real toric manifold is very similar to the complex case according to [18]. For a toric manifold X and its real locus $X^{\mathbb{R}}$, the i-th $\mathbb{Z}_2$-Betti number of $X^{\mathbb{R}}$ is equal to the $2i$-th Betti number of X, and hence it is also determined by the h-vector of X/T.[1] But the Betti numbers of $X^{\mathbb{R}}$ are different from the h-vector of X/T in general. For example, both $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ are toric manifolds over the 2-cube $\square^2$ and have the same Betti numbers $\beta^0 = 1, \beta^2 = 2, \beta^4 = 1$ and $\beta^{odd} = 0$. The real loci of $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ are the 2-dimensional torus $\mathbb{T}$ and the Klein bottle $\mathbb{K}$, respectively. Note that $\mathbb{T}$ and $\mathbb{K}$ have the same $\mathbb{Z}_2$-Betti numbers: $\beta^0_{\mathbb{Z}_2} = 1$, $\beta^1_{\mathbb{Z}_2} = 2$, and $\beta^2_{\mathbb{Z}_2} = 1$. But their Betti numbers are different; $\beta^1(\mathbb{T}) = 2$ and $\beta^1(\mathbb{K}) = 1$.

Recently, the rational cohomology groups of real toric manifolds were studied in [9, 10, 26, 27], and the results allow us to compute the rational cohomology groups of real toric manifolds combinatorially. However, it is still difficult to compute their Betti numbers explicitly by these results in general. For some interesting families of real toric manifolds, their rational cohomology have been studied in [5, 8, 7, 20, 6]. In this paper, we study the integral cohomology groups of real toric manifolds arising from a graph in a combinatorial way.

A simple convex polytope of dimension n is called a *Delzant polytope* if the n primitive integral vectors (outward) normal to the facets meeting at each vertex form an integral basis of $\mathbb{Z}^n$. Delzant polytopes play an important role in toric geometry; each Delzant polytope P determines a toric manifold $X(P)$. Under the canonical Delzant realizations, both graph associahedra and graph cubeahedra become Delzant polytopes. For a graph G, we will denote by $X^{\mathbb{R}}(\triangle_G)$ and $X^{\mathbb{R}}(\square_G)$ the real loci of the toric manifolds $X(\triangle_G)$ and $X(\square_G)$, respectively.

The Betti numbers of the real toric manifold $X^{\mathbb{R}}(\triangle_G)$ are computed in [8], by a graph invariant called the *a-number*. We write $H \sqsubseteq G$ when H is an induced subgraph of a graph G, and $H \sqsubset G$ when H is a proper induced subgraph of G. We say that a graph G is *even* (respectively, *odd*) if every connected component of G has an even (respectively, odd) number of vertices. The a-number $a(G)$ of a graph G is an integer defined as[2]

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[1]Given a topological space X, the i-th *Betti number* of X, denoted by $\beta^i(X)$, is the free rank of the singular cohomology group $H^i(X;\mathbb{Z})$. For a field F the i-th *F-Betti number* of X, denoted by $\beta^i_F(X)$, is the dimension of $H^i(X;F)$ as a vector space over F. Note that $\beta^i(X) = \beta^i_{\mathbb{Q}}(X)$.

[2]The numbers $a(G)$ and $b(G)$ are originally defined in [8] in slightly different forms. For a connected graph G, $a(G)$ and $b(G)$ here correspond to $sa(G)$ and $(-1)^{|V(G)|}b(G)$ in [8] respectively. See Lemma 4.2.

$$a(G) = \begin{cases} 1 & \text{if } V(G) = \emptyset, \\ 0 & \text{if } G \text{ is not even,} \\ -\sum_{H:H\sqsubset G} a(H) & \text{otherwise,} \end{cases}$$

and the i-th Betti number of $X^{\mathbb{R}}(\triangle_G)$ is given as follows:

Theorem 1.1 ([8])**.** *Let G be a graph. For any integer $i \geq 0$, the i-th Betti number of $X^{\mathbb{R}}(\triangle_G)$ is*

$$\beta^i(X^{\mathbb{R}}(\triangle_G)) = \sum_{\substack{H \sqsubseteq G \\ |V(H)|=2i}} |a(H)|\,.$$

Although graph cubeahedra are very similar to graph associahedra in their constructions, real toric manifolds corresponding to those ploytopes $\triangle_G$ and $\square_G$ are quite different since their normal fans are not isomorphic in general. So, it is natural to ask if analogous theories hold for real toric manifolds corresponding to graph cubeahedra and how they can be stated. In this paper, we first focus on the Betti numbers of the real toric manifolds $X^{\mathbb{R}}(\square_G)$. Let us define the *b-number* of a graph G as follows.

$$b(G) = \begin{cases} 1 & \text{if } V(G) = \emptyset, \\ 0 & \text{if } G \text{ is not odd,} \\ -\sum_{H:H\sqsubset G} b(H) & \text{otherwise.} \end{cases}$$

We also obtain the b-number-analogue of Theorem 1.1 for $X^{\mathbb{R}}(\square_G)$ as follows.

Theorem 1.2. *Let G be a graph. For any integer $i \geq 0$, the i-th Betti number of $X^{\mathbb{R}}(\square_G)$ is*

$$\beta^i(X^{\mathbb{R}}(\square_G)) = \sum_{\substack{H \sqsubseteq G \\ |V(H)|+\kappa(H)=2i}} |b(H)|\,,$$

where $\kappa(H)$ is the number of connected components of H.

Recently, Manneville and Pilaud showed in [19] that for two connected graphs G and G', there exists a combinatorial equivalence between a graph associahedron $\triangle_G$ and a graph cubeahedron $\square_{G'}$ if and only if G is a tree with at most one vertex whose degree is greater than 2 and $G' = L(G)$, where $L(G)$ is the line graph of G. Hence $h_i(\triangle_G) = h_i(\square_{L(G)})$ for $i = 1, \ldots, |V(G)| - 1$, which implies that the toric manifolds $X(\triangle_G)$ and $X(\square_{L(G)})$ have the same Betti numbers. But $X(\triangle_G)$ and $X(\square_{L(G)})$ are not isomorphic as toric varieties even if G is the path P_4, see Section 3.

For a tree G, even though the polytopes $\triangle_G$ and $\square_{L(G)}$ are not combinatorially equivalent in general, we find an interesting relationship between the real toric manifolds $X^{\mathbb{R}}(\triangle_G)$ and $X^{\mathbb{R}}(\square_{L(G)})$ as follows.

Theorem 1.3. *For a forest G, the real toric manifolds $X^{\mathbb{R}}(\triangle_G)$ and $X^{\mathbb{R}}(\square_{L(G)})$ have the same Betti numbers, that is, for any integer $i \geq 0$,*

$$\beta^i(X^{\mathbb{R}}(\triangle_G)) = \beta^i(X^{\mathbb{R}}(\square_{L(G)})).$$

Note that the relation above does not hold in general, see Section 4. Meanwhile obtaining the theorem above, we also discuss additional various properties related to $a(G)$ and $b(G)$ such as Möbius inversion and Euler characteristics of $X^{\mathbb{R}}(\triangle_G)$ and $X^{\mathbb{R}}(\square_G)$.

This paper is organized as follows. In Section 2, we summarize the results of [8] about real toric manifolds over graph associahedra. In Section 3, we recall the definition of graph cubeahedra $\square_G$, introduce the toric manifold $X(\square_G)$, and briefly review the relationship between $X(\triangle_G)$ and $X(\square_G)$. In Section 4, we describe the Betti numbers of the real toric manifolds $X^{\mathbb{R}}(\square_G)$ in terms of the b-numbers of graphs (Theorem 1.2) and then study various relationships between the real toric manifolds $X^{\mathbb{R}}(\triangle_G)$ and $X^{\mathbb{R}}(\square_{L(G)})$ including Theorem 1.3. Section 5 is devoted to the proof of Theorem 1.2. Section 6 provides some interesting integer sequences arising from the Betti numbers of the real toric manifolds associated with some special graphs. In Section 7, we give some remarks on a graph associahedron of type B.

2. Real toric manifolds over graph associahedra

In this section, we briefly summarize the result of [8], which studies real toric manifolds over graph associahedra. Recall that a real toric manifold is the real locus of a toric manifold and we refer the reader to [16] for more details of toric varieties.

For a graph G we denote by $\kappa(G)$ the number of connected components of a graph G, where a *connected component* (or a *component*) means a maximally connected subgraph of G. The *null graph* is the graph whose vertex set is empty, and it has no connected component. So $\kappa(G)$ for the null graph G is defined to be 0 by convention. We say that a graph G is *even* (respectively, *odd*) if every connected component of G has an even (respectively, odd) number of vertices. Note that the null graph is a unique graph that is both even and odd. A subgraph H of G is said to be *induced* if H includes all the edges between every pair of vertices in H if such edges exist in G. For $I \subseteq V(G)$, the subgraph induced by I is denoted by $G[I]$. For simplicity, we let $\mathcal{I}_G$ be the set of all $I \subseteq V(G)$ such that $G[I]$ is connected. Throughout this paper, we denote by $H \sqsubseteq G$ if H is either an induced subgraph of G or a null graph. When G is not the null graph, we denote by $H \sqsubset G$ if H is either a proper induced subgraph of G or a null graph. We denote a complete graph, a path, a cycle, and a star with n vertices by K_n, P_n, C_n, and $K_{1,n-1}$, respectively.

Construction of a graph associahedron. Let G be a connected graph with the vertex set $[n] := \{1, \dots, n\}$. Let us consider the standard simplex Δ^{n-1} whose facets are labeled by $1, \dots, n$. Then there is a one-to-one correspondence between the faces of Δ^{n-1} and the subsets of $[n]$. Hence each face of Δ^{n-1} can be labeled by a subset $I \subseteq [n]$. Then the graph associahedron, denoted by $\triangle_G$, is obtained from Δ^{n-1} by truncating the faces labeled by I for each proper connected subgraph $G[I]$ in increasing order of dimensions. If G has the connected components $G_1, \dots, G_\kappa$, then we define $\triangle_G = \triangle_{G_1} \times \cdots \times \triangle_{G_\kappa}$.

Lemma 2.1 ([7]). *Let P be a Delzant polytope and F a proper face of P. Then there is a canonical truncation of P along F such that the result is a Delzant polytope, say* $\mathrm{Cut}_F(P)$*, satisfying that the toric manifold* $X(\mathrm{Cut}_F(P))$ *is the blow-up of* $X(P)$ *along the submanifold* $X(F)$ *of* $X(P)$.

The lemma above assures that $\triangle_G$ is a well-defined Delzant polytope, and the toric manifold $X(\triangle_G)$ is an iterated blow-up of $X(\Delta^n) = \mathbb{C}P^n$. Note that if a face F of P is the intersection of the facets $F_1, \cdots, F_k$, then the normal vector of the new facet of $\mathrm{Cut}_F(P)$ arising from the truncation is the sum of the normal vectors of $F_1, \ldots, F_k$.

Remark 2.2. Note that, for disconnected graphs, the definitions of graph associahedra in [4] and [21] do not coincide, and we follow the definition in [21]; for a given graph G, the polytope constructed in [4] is combinatorially equivalent to $\triangle_G \times \Delta^{\kappa(G)-1}$.

Example 2.3. For a path P_3, the graph associahedron $\triangle_{P_3}$ is a pentagon, which is obtained from a triangle by truncating two vertices, see Figure 1. Since $X(\Delta^2)$ is the complex projective space $\mathbb{C}P^2$, the toric manifold $X(\triangle_{P_3})$ corresponds to blowing up of $\mathbb{C}P^2$ at two fixed points of the torus action, and hence $X(\triangle_{P_3}) = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \# \overline{\mathbb{C}P^2}$. Therefore, the real toric manifold $X^{\mathbb{R}}(\triangle_{P_3})$ is $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$.

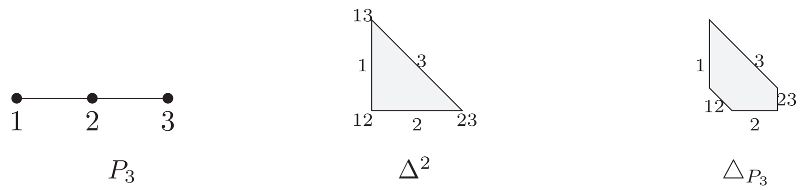


Fig. 1. A graph associahedron $\triangle_{P_3}$.

Note that the graph associahedra corresponding to a path P_n, a cycle C_n, a complete graph K_n, and a star $K_{1,n-1}$, are called an associahedron, a cyclohedron, a permutohedron, and a stellohedron, respectively. These polytopes are well-studied in various contexts such as [21] and [22].

The face structure of $\triangle_G$ can be described from the structure of the graph G.

Proposition 2.4 (Theorem 2.6 [4]). *For a connected graph G, there is a one-to-one correspondence between the facets of $\triangle_G$ and the proper connected induced subgraphs of G. We denote by F_I the facet corresponding to a proper connected induced subgraph $G[I]$. Furthermore, the facets $F_{I_1}, \ldots, F_{I_k}$ intersect if and only if $I_i \subseteq I_j$, $I_j \subseteq I_i$, or $I_i \cup I_j \notin \mathcal{I}_G$ for all $1 \leq i < j \leq k$.*

We can also write the (outward) primitive normal vector of each facet of $\triangle_G$ explicitly. When G is a connected graph, for each facet F_I corresponding to the proper connected induced subgraph $G[I]$, the primitive (outward) normal vector of F_I is

$$\begin{cases} -\sum_{i \in I} \mathbf{e}_i, & \text{if } n \notin I, \\ \sum_{j \notin I} \mathbf{e}_j, & \text{if } n \in I. \end{cases}$$

The Betti numbers of real toric manifolds associated with some interesting families of graphs are computed by using Theorem 1.1.

Corollary 2.5 ([8]). *Let G be a graph with $n+1$ vertices. For $1 \leq i \leq \lfloor \frac{n+1}{2} \rfloor$,*

$$\beta^i(X^{\mathbb{R}}(\triangle_G)) = \begin{cases} \binom{n+1}{2i} A_{2i} & \text{if } G = K_{n+1}, \\ \binom{n+1}{i} - \binom{n+1}{i-1} & \text{if } G = P_{n+1}, \\ \binom{n+1}{i} & \text{if } G = C_{n+1} \text{ and } 2i < n+1, \\ \frac{1}{2}\binom{2i}{i} & \text{if } G = C_{n+1} \text{ and } 2i = n+1, \\ \binom{n}{2i-1} A_{2i-1} & \text{if } G = K_{1,n}, \end{cases}$$

where A_k is the k-th Euler zigzag number given by

$$\sec x + \tan x = \sum_{k=0}^{\infty} A_k \frac{x^k}{k!}.$$

Refer to [23] for the formulae for $a(G)$ and $\beta^i(X^{\mathbb{R}}(\triangle_G))$ for a complete multipartite graph G.

We finish the section by noting flagness. A simple polytope is *flag* if any collection of pairwise intersecting facets has a nonempty intersection.

Proposition 2.6 (Corollary 7.2[22]). *For a graph G, the graph associahedron $\triangle_G$ is flag.*

3. Graph cubeahedra

In this section, we briefly review the construction of graph cubeahedra in [14] and the relationship between graph associahedra and graph cubeahedra. Set $[\overline{n}] = \{\overline{1}, \ldots, \overline{n}\}$.

Construction of a graph cubeahedron. Let us consider the standard cube $\square^n$ whose facets are labeled by $1, \ldots, n$ and $\overline{1}, \ldots, \overline{n}$, where the two facets labeled by i and $\overline{i}$ are parallel to each other. Then every face of $\square^n$ can be labeled by a subset I of $[n] \cup [\overline{n}]$ satisfying that $I \cap [n]$ and $\{i \in [n] \mid \overline{i} \in I\}$ are disjoint. Let G be a graph with the vertex set $[n]$. Recall that $\mathcal{I}_G$ is the set of all subsets I of $[n]$ such that $G[I]$ is connected. The *graph cubeahedron*, denoted by $\square_G$, is obtained from the standard cube $\square^n$ by truncating the faces labeled by $I \in \mathcal{I}_G$ in increasing order of dimensions. It follows from Lemma 2.1 that the graph cubeahedron $\square_G$ is also a Delzant polytope, and the toric manifold $X(\square_G)$ is an iterated blow-up of $X(\square^n) = (\mathbb{C}P^1)^n$.

Example 3.1. For paths P_2 and P_3, the graph cubeahedra $\square_{P_2}$ and $\square_{P_3}$ are illustrated in Figure 2. Note that the toric manifold corresponding to the standard cube $\square^2$ is $\mathbb{C}P^1 \times \mathbb{C}P^1$. Hence $X(\square_{P_2})$ corresponds to blow up of $\mathbb{C}P^1 \times \mathbb{C}P^1$ at one fixed point of the torus action. Thus $X(\square_{P_1}) = (\mathbb{C}P^1 \times \mathbb{C}P^1)\#\overline{\mathbb{C}P^2}$ and the real toric manifold $X^{\mathbb{R}}(\square_{P_2})$ is $(\mathbb{R}P^1 \times \mathbb{R}P^1)\#\mathbb{R}P^2$.

Let us describe the facets of the graph cubeahedron and their outward normal vectors. We label each facet of $\square_G$ by F_I, where $I \in \mathcal{I}_G$ or I is a singleton subset of $[\bar{n}]$. Then the primitive (outward) normal vector of the facet F_I is

$$\begin{cases} \sum_{i \in I} \mathbf{e}_i & \text{if } I \in \mathcal{I}_G, \\ -\mathbf{e}_i & \text{if } I = \{\bar{i}\} \text{ for some } i \in [n]. \end{cases} \tag{3.1}$$

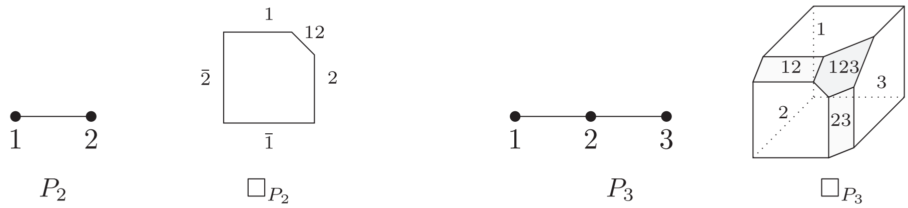


Fig. 2. Graph cubeahedra $\square_{P_2}$ and $\square_{P_3}$.

For a graph G consisting of the connected components $G_1, \ldots, G_\kappa$, one easily shows that $\square_G$ is combinatorially equivalent to $\square_{G_1} \times \cdots \times \square_{G_\kappa}$, where two Delzant polytopes are *equivalent* if their normal fans are isomorphic.

Now we describe the face poset of $\square_G$ as in the following, which was given in [14]. The flagness is implicitly stated in [14] in describing its face poset.

Proposition 3.2 ([14]). *Let G be a graph with the vertex set $[n]$. Then two facets F_I and F_J of $\square_G$ intersect if and only if one of the following holds.*

(1) *Both I and J belong to $\mathcal{I}_G$ and they satisfy either $I \subseteq J$, $J \subseteq I$, or $I \cup J \notin \mathcal{I}_G$.*
(2) *Exactly one of I and J, say I, belongs to $\mathcal{I}_G$ and $J = \{\bar{j}\}$ for some $j \in [n] \setminus I$.*
(3) *Both I and J are singleton subsets of $[\bar{n}]$.*

Furthermore, the graph cubeahedron $\square_G$ is flag.

We can easily check that the map from $2^{[n]} \cup \{\{\bar{1}\}, \ldots, \{\bar{n}\}\}$ to $2^{[n+1]}$ defined by

$$I \mapsto \begin{cases} \{i\} & \text{for } I = \{\bar{i}\}, \\ [n+1] \setminus I & \text{for } I \subseteq [n], \end{cases}$$

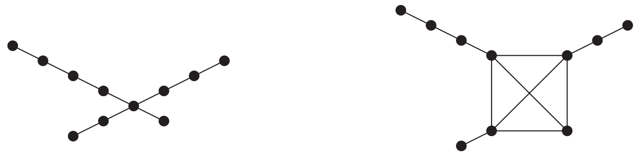
gives an isomorphism from the normal fan of $\square_{K_n}$ to the normal fan of $\triangle_{K_{1,n}}$, where K_n is a complete graph and $K_{1,n}$ is a star. We can also easily check that the map from the set of facets of $\triangle_{P_{n+1}}$ to that of $\square_{P_n}$ given by

$$I \mapsto \begin{cases} I & \text{if } I \subseteq [n], \\ \{\bar{j}\} & \text{if } n+1 \in I \text{ and } |I| = n+1-j, \end{cases}$$

gives an isomorphism from the face poset of $\triangle_{P_{n+1}}$ to that of $\square_{P_n}$. However, there is no isomorphism between the normal fan of $\triangle_{P_4}$ and that of $\square_{P_3}$ because $\triangle_{P_4}$ has a pair of square facets whose normal vectors are parallel but $\square_{P_3}$ has no such a pair of square facets.

The relationship above was noted in [19] between the two polytopes $\triangle_G$ and $\square_H$ when G is an octopus and H is a spider. An *octopus* is a tree with at most one vertex of degree more than two. A *spider* is a graph obtained from a complete graph K_n by attaching at most one path by one of its leaf to each vertex of K_n, see Figure 3. The *line graph* $L(G)$ of a graph G is the intersection graph of $E(G)$. In other words, the vertex set of $L(G)$ is $E(G)$ and two vertices e and e' of $L(G)$ are adjacent if and only if $e \cap e' \neq \emptyset$, that is, e and e' share an endpoint in G. Note that the line graph of an octopus is a spider. In Figure 3, the line graph of G is equal to H.

Proposition 3.3 ([19]). *For two connected graphs G and H, the polytopes $\triangle_G$ and $\square_H$ are combinatorially equivalent if and only if G is an octopus and H is a spider which is equal to the line graph of G. Furthermore, if G is a star $K_{1,n}$, the normal fan of $\triangle_G$ is isomorphic to the normal fan of $\square_{L(G)}$.*



An octopus G A spider H

Fig. 3. An octopus G and a spider H.

Thus, for an octopus G, the h-vector of $\triangle_G$ is equal to that of $\square_{L(G)}$. Hence the Betti numbers of the toric manifold $X(\triangle_G)$ are equal to those of the toric manifold $X(\square_{L(G)})$, and the $\mathbb{Z}_2$-Betti numbers of the real toric manifold $X^{\mathbb{R}}(\triangle_G)$ are equal to those of the real toric manifold $X^{\mathbb{R}}(\square_{L(G)})$.

REMARK 3.4. In [19], the authors define a family of complete simplicial fans, called *compatibility fans*, whose underlying simplicial complex is dual to the graph associahedron, and they also define a family of complete simplicial fans, called *design compatibility fans*, whose underlying simplicial complex is dual to the graph cubeahedron. The normal fan of $\triangle_G$ (respectively, $\square_G$) is a compatibility fan (respectively, a design compatibility fan) associated with G and the normal fan of $\triangle_G$ is not isomorphic to that of $\square_{L(G)}$ even if G is an octopus in general. But the normal fan of $\square_{L(G)}$ is isomorphic to some complete non-singular fan associated with $\triangle_G$, the dual compatibility fan, for an octopus G. See [19] for more details.

4. The a-number and the b-number of a graph and their relationships

In this section, we first study how the a-numbers and the b-numbers are related to each other and then describe the Betti numbers of the real toric manifold corresponding to a graph G in terms of the b-numbers. We also show that the real toric manifolds $X^{\mathbb{R}}(\triangle_G)$ and $X^{\mathbb{R}}(\square_{L(G)})$ have the same Betti numbers for a forest G.

Recall the two graph invariants $a(G)$ and $b(G)$ given in Section 1. For a graph G, $a(G)$ and $b(G)$, called the *a-number* and the *b-number* of G, respectively, are defined as follows.

$$a(G) = \begin{cases} 1 & \text{if } V(G) = \emptyset, \\ 0 & \text{if } G \text{ is not even}, \\ -\sum_{H:H\sqsubset G} a(H) & \text{otherwise}, \end{cases} \qquad b(G) = \begin{cases} 1 & \text{if } V(G) = \emptyset, \\ 0 & \text{if } G \text{ is not odd}, \\ -\sum_{H:H\sqsubset G} b(H) & \text{otherwise}. \end{cases}$$

EXAMPLE 4.1. Let us compute the a- and b-numbers of a path $G = P_4$, where $V(P_4) = \{1,2,3,4\}$ and $E(P_4) = \{\{1,2\},\{2,3\},\{3,4\}\}$. Then the a- and b-number of the induced subgraphs of P_4 are as follows:

$$\begin{aligned}
&a(G[1]) = a(G[2]) = a(G[3]) = a(G[4]) = a(P_1) = 0\\
&a(G[1,2]) = a(G[2,3]) = a(G[3,4]) = a(P_2) = -1\\
&a(G[1,3]) = a(G[1,4]) = a(G[2,4]) = a(P_1 \sqcup P_1) = 0\\
&a(G[1,2,3]) = a(G[2,3,4]) = a(P_3) = 0\\
&a(G[1,2,4]) = a(G[1,3,4]) = a(P_1 \sqcup P_2) = 0\\
&a(P_4) = -a(\emptyset) - 3a(P_2) = 2,
\end{aligned}$$

and

$$\begin{aligned}
&b(G[1]) = b(G[2]) = b(G[3]) = b(G[4]) = b(P_1) = -1\\
&b(G[1,2]) = b(G[2,3]) = b(G[3,4]) = b(P_2) = 0\\
&b(G[1,3]) = b(G[1,4]) = b(G[2,4]) = b(P_1 \sqcup P_1) = 1\\
&b(G[1,2,3]) = b(G[2,3,4]) = b(P_3) = -b(\emptyset) - 3b(P_1) - b(P_1 \sqcup P_1) = 1\\
&b(G[1,2,4]) = b(G[1,3,4]) = b(P_1 \sqcup P_2) = 0\\
&b(P_4) = 0,
\end{aligned}$$

where $\sqcup$ means "disjoint union".

One can observe that the invariants $a(G)$ and $b(G)$ are the Möbius invariants of some bounded posets as follows. A poset $\mathcal{P}$ is *bounded* if it has a unique maximum element, denoted by $\hat{1}$, and a unique minimum element, denoted by $\hat{0}$. For a finite bounded poset $\mathcal{P}$, the *Möbius invariant* of $\mathcal{P}$ is defined as $\mu(\mathcal{P}) = \mu_{\mathcal{P}}(\hat{0}, \hat{1})$.[3]

For a graph G, we define

$$\begin{aligned}
\mathcal{P}_G^{\text{even}} &= \{\emptyset \neq I \subsetneq V(G) \mid G[I] \text{ is even}\} \cup \{\hat{0}, \hat{1}\} \text{ and}\\
\mathcal{P}_G^{\text{odd}} &= \{\emptyset \neq I \subsetneq V(G) \mid G[I] \text{ is odd}\} \cup \{\hat{0}, \hat{1}\}.
\end{aligned}$$

Note that the definitions above are generalization of the poset $\mathcal{P}_G^{\text{even}}$ in [8, Lemma 4.7], where $|V(G)|$ is even. By the definitions of a- and b-numbers, $a(G) = \mu(\mathcal{P}_G^{\text{even}})$ when G is even, and $b(G) = \mu(\mathcal{P}_G^{\text{odd}})$ when G is odd. We can also check that the a- and b- numbers are multiplicative as follows.

Lemma 4.2. *Let G and H be two disjoint graphs. Then we have*

$$a(G \sqcup H) = a(G)a(H) \quad \textit{and} \quad b(G \sqcup H) = b(G)b(H).$$

Proof. If $G \sqcup H$ is not even, then both $a(G \sqcup H)$ and $a(G)a(H)$ are zero by definition. Similarly, if $G \sqcup H$ is not odd, then both $b(G \sqcup H)$ and $b(G)b(H)$ are zero by definition. If $G \sqcup H$ is even (respectively, odd), then we have an isomorphism $\mathcal{P}_{G\sqcup H}^{\text{even}} \cong \mathcal{P}_G^{\text{even}} \times \mathcal{P}_H^{\text{even}}$ (respectively, $\mathcal{P}_{G\sqcup H}^{\text{odd}} \cong \mathcal{P}_G^{\text{odd}} \times \mathcal{P}_H^{\text{odd}}$). Therefore, the proof is done by multiplicativity of Möbius invariants. □

A finite, pure simplicial complex K of dimension n is called *shellable* if there is an ordering $C_1, C_2, \ldots, C_t$ of the maximal simplices of K, called a *shelling*, such that $(\bigcup_{i=1}^{k-1} C_i) \cap C_k$ is pure of dimension $n-1$ for every $k = 2, 3, \ldots, t$. It is well-known in [24] that shellable complexes are Cohen-Macaulay and thus homotopy equivalent to a wedge of spheres of the same dimension. In [1], Björner presented a criterion for shellability of order complexes. If the order complex of a poset $\mathcal{P}$ is shellable, then we say $\mathcal{P}$ is shellable.

Note that [8, Proposition 4.9] says that for a connected even graph G, $\mathcal{P}_G^{\text{even}}$ is a pure

[3] The Möbius function μ can be defined inductively by the following relation: for a finite poset $\mathcal{P}$ and $s, t \in \mathcal{P}$,

$$\mu_{\mathcal{P}}(s,t) = \begin{cases} 1 & \text{if} \quad s = t \\ -\displaystyle\sum_{r\,:\, s \le r < t} \mu_{\mathcal{P}}(s,r) & \text{for} \quad s < t \\ 0 & \text{otherwise.} \end{cases}$$

shellable poset of length $\frac{|V(G)|}{2}$. But the proof of the local semimodularity of $\mathcal{P}_G^{\text{even}}$ still works even if G is disconnected or $|V(G)|$ is odd. For the length of the poset we need to be more careful. Recall that the length of a poset is defined to be

$$\ell(\mathcal{P}) := \max\{|\sigma| - 1 \mid \sigma \text{ is a chain of } \mathcal{P}\}.$$

If $G_1, \ldots, G_\kappa$ are the connected components of G, then every maximal chain of $\mathcal{P}_G^{\text{even}} \setminus \{\hat{0}, \hat{1}\}$ has $\sum_{i=1}^{\kappa} \lfloor \frac{|V(G_i)|}{2} \rfloor$ elements.[4] Let $\kappa^{\text{odd}}(G)$ be the number of odd connected components. Then $\sum_{i=1}^{\kappa} \lfloor \frac{|V(G_i)|}{2} \rfloor$ equals $\frac{|V(G)|-\kappa^{\text{odd}}(G)}{2}$, and hence $\mathcal{P}_G^{\text{even}}$ is of length $\frac{|V(G)|-\kappa^{\text{odd}}(G)}{2} + 1$.

Proposition 4.3 ([8])**.** *For a graph G, $\mathcal{P}_G^{\text{even}}$ is a pure shellable poset of length* $\frac{|V(G)|-\kappa^{\text{odd}}(G)}{2}$ $+ 1$.

Hence the order complex of $\mathcal{P}_G^{\text{even}} \setminus \{\hat{0}, \hat{1}\}$ is homotopy equivalent to the wedge of $|\mu|$ copies of the spheres S^d, where $\mu = \mu(\mathcal{P}_G^{\text{even}})$ and $d = \frac{|V(G)|-\kappa^{\text{odd}}(G)}{2} - 1$. In fact, the a-number and the b-number determine each other as follows.

Theorem 4.4. *For a graph G, we have*

$$b(G) = (-1)^{|V(G)|} \sum_{H : H \sqsubseteq G} a(H) \tag{4.1}$$

and

$$a(G) = \sum_{H : H \sqsubseteq G} b(H). \tag{4.2}$$

Proof. Let us prove (4.1) first. If G is a connected even graph, then $b(G) = 0$ and $a(G) = -\sum_{H \sqsubset G} a(H)$ from their definitions, and hence (4.1) holds. The formula (4.1) trivially holds when $|V(G)| = 1$. Now we assume that G is a connected graph with $2k+1$ vertices for $k \geq 1$. Recall that every facet of $\triangle_G$ is labeled by I such that $G[I]$ is a proper connected induced subgraph and we write that facet by F_I. Then we have

$$\partial \triangle_G = \left(\bigcup_{|I|=\text{even}} F_I \right) \cup \left(\bigcup_{|I|=\text{odd}} F_I \right).$$

The former set $\bigcup_{|I|=\text{even}} F_I$ is homotopy equivalent to the order complex of $\mathcal{P}_G^{\text{even}} \setminus \{\hat{0}, \hat{1}\}$, and the latter set $\bigcup_{|I|=\text{odd}} F_I$ is homotopy equivalent to the order complex of $\mathcal{P}_G^{\text{odd}} \setminus \{\hat{0}, \hat{1}\}$. Actually, since $\triangle_G$ is a simple polytope, the dual of the set $\bigcup_{|I|=\text{even}} F_I$ (respectively, $\bigcup_{|I|=\text{odd}} F_I$) is a simplicial complex and it becomes the order complex of $\mathcal{P}_G^{\text{even}} \setminus \{\hat{0}, \hat{1}\}$ (respectively, $\mathcal{P}_G^{\text{odd}} \setminus \{\hat{0}, \hat{1}\}$) after suitable subdivisions, see Lemma 4.7 of [8].[5] Note that $\partial \triangle_G$ is homeomorphic to a sphere of dimension $2k-1$ and $\mathcal{P}_G^{\text{even}}$ is a shellable poset of length $k+1$ by Theorem 4.3. Hence we have $\mu(\mathcal{P}_G^{\text{odd}}) = \mu(\mathcal{P}_G^{\text{even}})$ from the Philip Hall theorem[6] and the Alexander duality[7] on the sphere $\partial \triangle_G$. Thus we obtain

[4]Note that for a given $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the maximal integer not greater than x.

[5]Note that this property holds only when G is connected.

[6]For a bounded poset $\mathcal{P}$, the reduced Euler characteristic of the order complex of $\mathcal{P} \setminus \{\hat{0}, \hat{1}\}$ is equal to the Möbius invariant $\mu(\mathcal{P})$.

[7]The Alexander duality says that if X is a compact, locally contractible subspace of a sphere S^n, then $\tilde{H}_q(X) \cong \tilde{H}^{n-q-1}(S^n \setminus X)$ for every q.

$$\begin{aligned} b(G) = \mu(\mathcal{P}_G^{\text{even}}) &= -\sum_{\substack{\hat{0} \le I < \hat{1} \\ G[I] \text{is even}}} \mu(\hat{0}, I) \\ &= -\sum_{\substack{\hat{0} \le I < \hat{1} \\ G[I] \text{is even}}} a(G[I]) \\ &= -\sum_{H \sqsubset G} a(H) \\ &= (-1)^{|V(G)|} \sum_{H \sqsubseteq G} a(H), \end{aligned}$$

where the last identity follows from the fact that $|V(G)|$ is odd.

When G has the connected components $G_1, \ldots, G_\kappa$, Lemma 4.2 implies

$$\prod_{i=1}^{\kappa} (-1)^{|V(G_i)|} \sum_{H : H \sqsubseteq G_i} a(H) = (-1)^{|V(G)|} \sum_{H : H \sqsubseteq G} a(H),$$

which proves (4.1).

Now let us show (4.2). Let $\mathcal{P}$ be the poset of all elements $H \sqsubseteq G$. Note that $\mathcal{P}$ is isomorphic to the Boolean algebra, the poset of all subsets of $[n]$. We apply the Möbius inversion formula[8] to (4.1). Then we immediately obtain that

$$a(G) = \sum_{H : H \sqsubseteq G} (-1)^{|V(H)|} b(H) \mu_{\mathcal{P}}(H, G).$$

Since $\mu_{\mathcal{P}}(H, G) = (-1)^{|V(G)| - |V(H)|}$ for every induced subgraph H of G, it holds that

$$a(G) = (-1)^{|V(G)|} \sum_{H : H \sqsubseteq G} b(H).$$

Note that $a(G) = 0$ whenever $|V(G)|$ is odd. Therefore, (4.2) holds. □

Remark 4.5. Note that for odd n, $\mathcal{P}_{K_n}^{\text{even}}$ and $\mathcal{P}_{K_n}^{\text{odd}}$ have the same number of elements, but for even n, the number of elements in $\mathcal{P}_{K_n}^{\text{even}}$ is smaller than that of $\mathcal{P}_{K_n}^{\text{odd}}$. Usually there are fewer even induced subgraphs than odd ones. One reason is that $G[I]$ can be odd even if $|I|$ is even. For example, P_4 and P_5 have three and seven proper even induced subgraphs, respectively. On the other hand, P_4 and P_5 have nine and eighteen proper odd subgraphs, respectively. Hence (4.1) is more efficient to compute the b-number than the definition.

In fact, the signs of $a(G)$ and $b(G)$ are completely determined by $|V(G)|$ and $\kappa(G)$.

Corollary 4.6. *For a graph G, the signs of $a(G)$ and $b(G)$ are determined as follows.*

(1) *If G is even, then $a(G) = (-1)^{\frac{|V(G)|}{2}} |a(G)|$.*
(2) *If G is odd, then $b(G) = (-1)^{\frac{|V(G)| + \kappa(G)}{2}} |b(G)|$.*

Proof. Note that (1) is already known in [8, Remark 2.2]. Let us prove (2). It is well-known that the Möbius function of $\mathcal{P}_G^{\text{even}}$ alternates in sign[9]. Let us write $\operatorname{sgn} x = x/|x|$ for a nonzero real number x. By (4.1), we have

[8] See Proposition 3.7.1 of [25].
[9] See Proposition 3.8.11 of [25].

$$\operatorname{sgn} b(G) = (-1)^{|V(G)|} \times (-1) \times \operatorname{sgn} \mu(\mathcal{P}_G^{\text{even}}). \tag{4.3}$$

For a maximal element H of $\mathcal{P}_G^{\text{even}} \setminus \{\hat{1}\}$, H is even and so by (1), we have $\operatorname{sgn} \mu_{\mathcal{P}_G^{\text{even}}}(\hat{0}, H) = (-1)^{\frac{|V(H)|}{2}}$. Thus the sign of $\mu(\mathcal{P}_G^{\text{even}})$ is equal to that of $(-1)^{\frac{|V(H)|}{2}+1}$. From the fact that $|V(H)| = |V(G)| - \kappa(G)$, (4.3) is equal to

$$(-1)^{|V(G)|+1+\frac{|V(H)|}{2}+1} = (-1)^{|V(G)|+\frac{|V(G)|-\kappa(G)}{2}} = (-1)^{\frac{3|V(G)|-\kappa(G)}{2}} = (-1)^{\frac{|V(G)|+\kappa(G)}{2}},$$

where the last equality is from the fact that $3|V(G)| - \kappa(G)$ and $|V(G)| + \kappa(G)$ have the same parity. $\square$

As the Betti numbers of $X^{\mathbb{R}}(\triangle_G)$ were formulated by the a-numbers, now we formulate the Betti numbers of $X^{\mathbb{R}}(\square_G)$ by the b-numbers. We restate the main theorem below, and its proof will be presented in Section 5.

Theorem 4.7 (Theorem 1.2)**.** *Let G be a graph. For any integer $i \geq 0$, the i-th Betti number of $X^{\mathbb{R}}(\square_G)$ is*

$$\beta^i(X^{\mathbb{R}}(\square_G)) = \sum_{\substack{H \sqsubseteq G \\ |V(H)|+\kappa(H)=2i}} |b(H)|\,.$$

The following implies that $a(G)$ and $b(G)$ are completely determined by the combinatorial structure of $\square_G$ and $\triangle_G$, respectively.

Corollary 4.8. *For a graph G, we have*

$$a(G) = \chi(X^{\mathbb{R}}(\square_G)) = \sum_i (-1)^i h_i(\square_G) \tag{4.4}$$

and

$$b(G) = (-1)^{|V(G)|}\chi(X^{\mathbb{R}}(\triangle_G)) = (-1)^{|V(G)|} \sum_i (-1)^i h_i(\triangle_G). \tag{4.5}$$

where $(h_0(P), h_1(P), \dots, h_n(P))$ is the h-vector of a simple polytope P.

Proof. The second formula in (4.5) is already known in [8, Remark 2.3]. The first identity in (4.4) is induced by the following chain of identities

$$\begin{aligned}
\chi(X^{\mathbb{R}}(\square_G)) &= \beta^0 - \beta^1 + \beta^2 - \cdots \text{ (by definition of the Euler characteristic)} \\
&= \sum_{H \sqsubseteq G} (-1)^{\frac{|V(H)|+\kappa(H)}{2}} |b(H)| \text{ (by Theorem 1.2)} \\
&= \sum_{H \sqsubseteq G} b(H) \text{ (by Corollary 4.6)} \\
&= a(G) \text{ (by Theorem 4.4).}
\end{aligned}$$

Recall that the $\mathbb{Z}_2$-Betti numbers of a real toric manifold are equal to the h-numbers of the corresponding polytope, that is, $\beta^i_{\mathbb{Z}_2}(X^{\mathbb{R}}(\square_G)) = h_i(\square_G)$. Consequently, the second identity in (4.4) follows from the fact that the Euler characteristic is independent of the choice of coefficient field. $\square$

Remark 4.9. Note that for an even-dimensional Delzant polytope P, the Euler characteristic of $X^{\mathbb{R}}(P)$ is equal to $(-1)^n\gamma_n(P)$, where $\gamma_n(P)$ is the n-th γ-number of a simple polytope P. See [2]. Hence it follows from Corollary 4.8 that

$$\begin{cases} a(G) = (-1)^n\gamma_n(\square_G) & \text{if } |V(G)| = 2n; \\ b(G) = (-1)^{n+1}\gamma_n(\triangle_G) & \text{if } |V(G)| = 2n+1. \end{cases}$$

From now on, we will see a significant application of our main result. Let us take a look into a result in [19] again. By Proposition 3.3, if G is an octopus and $L(G)$ is the corresponding spider, the line graph of G, then $h_i(\triangle_G) = h_i(\square_{L(G)})$ for each i. Hence the $\mathbb{Z}_2$-Betti numbers of the real toric manifolds $X^{\mathbb{R}}(\triangle_G)$ and $X^{\mathbb{R}}(\square_{L(G)})$ are the same. We can show that this phenomenon holds for the Betti numbers of $X^{\mathbb{R}}(\triangle_G)$ and $X^{\mathbb{R}}(\square_{L(G)})$ for every tree G and its line graph $L(G)$. We also note that the line graphs of trees form an important family of graphs in graph theory, called claw-free block graphs. In the rest of the section, we will prove the following by using Theorem 1.2, the main result.

Theorem 4.10 (Theorem 1.3)**.** *For a forest G, the real toric manifolds $X^{\mathbb{R}}(\triangle_G)$ and $X^{\mathbb{R}}(\square_{L(G)})$ have the same Betti numbers, that is, for any integer $i \geq 0$, we have*

$$\beta^i(X^{\mathbb{R}}(\triangle_G)) = \beta^i(X^{\mathbb{R}}(\square_{L(G)})).$$

Note that the theorem above does not hold in general. For example, the line graph of a cycle C_n is isomorphic to C_n itself, but the Betti numbers of $X^{\mathbb{R}}(\triangle_{C_n})$ are different from those of $X^{\mathbb{R}}(\square_{C_n})$, see Corollary 2.5 and Corollary 6.4. Before proving Theorem 1.3, we will see an interesting identity which shows a new relationship between the a- and b-numbers. The following lemma collects some simple observations. A *spanning subgraph H* of G is a subgraph of G such that $V(H) = V(G)$. For a graph G, let $\mathcal{S}(G)$ be the set of all spanning subgraphs of a graph G without isolated vertices.

Lemma 4.11. *For a forest G, the following hold.*

(1) *For each even subgraph H of G, $L(H)$ is an odd subgraph of $L(G)$.*
(2) *For each $H \in \mathcal{S}(G)$, $|V(L(H))| + \kappa(L(H)) = |V(G)|$.*

Proof. If H is an even subgraph of G, then each component of H has an odd number of edges and so each component of $L(H)$ has an odd number of vertices, which implies that $L(H)$ is odd. Thus (1) holds.

Take any $H \in \mathcal{S}(G)$. Since H is also a forest, we have $|E(H)| + \kappa(H) = |V(H)|$. Note that $|V(L(H))| = |E(H)|$ and $|V(H)| = |V(G)|$. Since H has no isolated vertex, $\kappa(L(H)) = \kappa(H)$. Thus (2) follows. □

The following proposition is for not only proving Theorem 1.3 but also providing a significant observation in a relationship between the a- and b-numbers.

Proposition 4.12. *For a forest G,*

$$a(G) = \sum_{H \in \mathcal{S}(G)} b(L(H)) \qquad \text{and} \qquad |a(G)| = \sum_{H \in \mathcal{S}(G)} |b(L(H))|.$$

Proof. Note that from the first equality, the second one follows immediately, since $a(G)$ and $b(L(G))$'s have the same sign by Corollary 4.6 and (2) of Lemma 4.11.

We first assume that $V(G)$ is an even forest and prove the first equality by induction on $|V(G)|$. It is clear for $|V(G)| = 2$. Now assume that the proposition is true for every even forest with at most $n-2$ vertices. Now take any even forest G with n vertices. By (1) of Lemma 4.11, $L(G)$ is an odd graph. Thus by the definition of the b-number,

$$b(L(G)) = -\sum_{L\colon L\sqsubset L(G)} b(L).$$

For each subgraph L of $L(G)$, we denote by H_L the minimal subgraph of G whose edges are the elements in $V(L)$, that is, $E(H_L) = V(L)$. Then

$$b(L(G)) = -\sum_{\substack{L\colon L\sqsubset L(G)\\ V(H_L)=V(G)}} b(L) - \sum_{\substack{L\colon L\sqsubset L(G)\\ V(H_L)\subsetneq V(G)}} b(L).$$

Hence we have

$$b(L(G)) + \sum_{\substack{L\colon L\sqsubset L(G)\\ V(H_L)=V(G)}} b(L) = -\sum_{\substack{L\colon L\sqsubset L(G)\\ V(H_L)\subsetneq V(G)}} b(L). \tag{4.6}$$

Then since for each $L \sqsubset L(G)$, any component of H_L is not an isolated vertex, the left-hand-side of (4.6) is equal to $\sum_{H\in\mathcal{S}(G)} b(L(H))$ by definition.

We will show that the right-hand-side of (4.6) is equal to $a(G)$. If H is even and $|V(H)| \leq n-2$, then it follows from the induction hypothesis that

$$a(H) = \sum_{\substack{L\colon L\sqsubseteq L(H)\\ H_L\in\mathcal{S}(H)}} b(L). \tag{4.7}$$

Even if H is not even, we still have the same identity (4.7). To see why, suppose that H is not even. Then $a(H) = 0$ and H has a component with an odd number of vertices. Since $H_L \in \mathcal{S}(H)$ and so H_L has no isolated vertex, H_L must have a component with an even number of edges. Thus $L(H_L) = L$ has a component with an even number of vertices, and so $b(L) = 0$. Therefore, the left and right hand sides of (4.7) are both equal to 0.

Thus the right-hand-side of (4.6) is equal to

$$-\sum_{H\colon H\sqsubset G}\ \sum_{\substack{L\colon L\sqsubseteq L(H)\\ H_L\in\mathcal{S}(H)}} b(L) = -\sum_{H\colon H\sqsubset G} a(H) = a(G)$$

where the first equality is from (4.7) and the last one is from the assumption that G is even. It proves the first equality.

If G is not even, then G must have an odd component and $a(G) = 0$ by definition. Furthermore, every graph in $\mathcal{S}(G)$ is not even. Hence the first identity holds even if G is not even. □

By Theorems 1.1 and 1.2, and Proposition 4.12, we can prove Theorem 1.3 as follows.

Proof of Theorem 1.3.

$$\begin{aligned}
\beta^i(X^{\mathbb{R}}(\triangle_G)) &= \sum_{\substack{H\colon H\sqsubseteq G\\ |V(H)|=2i}} |a(H)| && \text{(by Theorem 1.1)}\\
&= \sum_{\substack{H\colon H\sqsubseteq G\\ |V(H)|=2i}}\ \sum_{H'\in\mathcal{S}(H)} |b(L(H'))| && \text{(by Proposition 4.12)}
\end{aligned}$$

$$= \sum_{\substack{L\colon L \sqsubseteq L(G) \\ |V(L)|+\kappa(L)=2i}} |b(L)| \qquad \text{(by Lemma 4.11)}$$

$$= \beta^i(X^{\mathbb{R}}(\square_{L(G)})) \qquad \text{(by Theorem 1.2).}$$

□

5. Proof of Theorem 1.2

In this section, we first prepare some definitions and known results to prove our main theorem, and then give the proof of Theorem 1.2.

The cohomology of a real toric manifold. We present a result on the cohomology groups of a real toric manifold introduced in [3, 9, 10, 26, 27]. Let P be a Delzant polytope of dimension n and let $\mathcal{F}(P) = \{F_1, \ldots, F_m\}$ be the set of facets of P. Then the primitive outward normal vectors of P can be understood as a function ϕ from $\mathcal{F}(P)$ to $\mathbb{Z}^n$, and the composition map $\lambda\colon \mathcal{F}(P) \xrightarrow{\phi} \mathbb{Z}^n \xrightarrow{\text{mod } 2} \mathbb{Z}_2^n$ is called the (mod 2) *characteristic function* over P. Note that λ can be represented by a $\mathbb{Z}_2$-matrix Λ_P of size $n \times m$ as

$$\Lambda_P = \begin{pmatrix} \lambda(F_1) & \cdots & \lambda(F_m) \end{pmatrix},$$

where the i-th column of Λ_P is $\lambda(F_i) \in \mathbb{Z}_2^n$. For $\omega \in \mathbb{Z}_2^m$, we define P_ω to be the union of facets F_j such that the j-th entry of ω is nonzero. Then the following holds:

Theorem 5.1 ([26, 27])**.** *Let P be a Delzant polytope of dimension n. Then the i-th Betti number of the real toric manifold $X^{\mathbb{R}}(P)$ is given by*

$$\beta^i(X^{\mathbb{R}}(P)) = \sum_{S \subseteq [n]} \tilde{\beta}^{i-1}(P_{\omega_S}),$$

where ω_S is the sum of the k-th rows of Λ_P for all $k \in S$.

It is shown in [3] that the cohomology group of a real toric manifold $X^{\mathbb{R}}(P)$ is completely determined by the reduced cohomology groups of P_{ω_S}'s and the h-vector of P. In particular, if $\tilde{H}^*(P_{\omega_S})$ is torsion-free for every $S \subseteq [n]$, then the cohomology group of $X^{\mathbb{R}}(P)$ is

$$H^i(X^{\mathbb{R}}(P)) \cong \mathbb{Z}^{\beta^i} \oplus \mathbb{Z}_2^{h_i - \beta^i}, \tag{5.1}$$

where β^i is the i-th Betti number of $X^{\mathbb{R}}(P)$ and $(h_0, h_1, \ldots, h_n)$ is the h-vector of P.

The $\mathbb{Z}_2$-characteristic matrix of the real toric manifold $X^{\mathbb{R}}(\square_G)$. Let G be a graph with the vertex set $[n]$. Recall that $\mathcal{F}(\square_G) = \{F_I \mid I \in \mathcal{I}_G \text{ or } I \text{ is a singleton subset of } [\bar{n}]\}$. It follows from (3.1) that the (mod 2) characteristic function $\lambda\colon \mathcal{F}(\square_G) \to \mathbb{Z}_2^n$ is given by

$$\lambda(F_I) := \begin{cases} \sum_{i \in I} \mathbf{e}_i & \text{if } I \in \mathcal{I}_G, \\ \mathbf{e}_i & \text{if } I = \{\bar{i}\} \text{ for some } i \in [n]. \end{cases}$$

Let Λ_G be the $\mathbb{Z}_2$-characteristic matrix of $\square_G$. Then Λ_G is of size $n \times (|\mathcal{I}_G| + n)$.

Simplicial complex $\mathbf{K}^{\text{odd}}_{\mathbf{G,S}}$ dual to $(\square_G)_{\omega_S}$. Let S be a subset of $[n]$, and let ω_S be the sum of the k-th rows of Λ_G for all $k \in S$. Then for each facet F_I, the I-entry of ω_S is nonzero if and only if either $I = \{\bar{i}\}$ for some $i \in S$ or $I \in \mathcal{I}_G$ such that $|I \cap S|$ is odd. Hence $(\square_G)_{\omega_S}$ is the union of facets F_I of $\square_G$, where the union is taken over all I satisfying that either $I = \{\bar{i}\}$ for some $i \in S$ or $I \in \mathcal{I}_G$ such that $|I \cap S|$ is odd. We let $K^{\mathrm{odd}}_{G,S}$ be the dual of $(\square_G)_{\omega_S}$. Then $K^{\mathrm{odd}}_{G,S}$ is a simplicial complex since $\square_G$ is a simple polytope. We write K^{odd}_{G} instead of $K^{\mathrm{odd}}_{G,[n]}$. Note that the set of vertices[10] of $K^{\mathrm{odd}}_{G,S}$ is

$$\{I \in \mathcal{I}_G \mid |I \cap S| \text{ is odd}\} \cup \{\bar{i} \mid i \in S\}.$$

Figure 4 shows examples for simplicial complexes $K^{\mathrm{odd}}_{G,S}$ and $K^{\mathrm{odd}}_{G[S]}$. Since $(\square_G)_{\omega_S}$ and its dual $K^{\mathrm{odd}}_{G,S}$ have the same homotopy type, by Theorem 5.1, we have

$$\beta^i(X^{\mathbb{R}}(\square_G)) = \sum_{S \subseteq [n]} \tilde{\beta}^{i-1}(K^{\mathrm{odd}}_{G,S}). \tag{5.2}$$

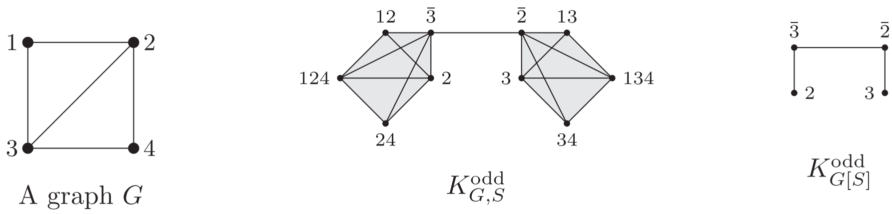


Fig.4. The simplicial complexes $K^{\mathrm{odd}}_{G,S}$ and $K^{\mathrm{odd}}_{G[S]}$ when $S = \{2, 3\}$.

Note that the two simplicial complexes $K^{\mathrm{odd}}_{G,S}$ and $K^{\mathrm{odd}}_{G[S]}$ in Figure 4 are homotopy equivalent and they are contractible. We will show that this phenomenon holds in general. More precisely, we will show that $K^{\mathrm{odd}}_{G,S}$ and $K^{\mathrm{odd}}_{G[S]}$ are homotopy equivalent for any $S \subseteq [n]$ (Lemma 5.3), and then $K^{\mathrm{odd}}_{G[S]}$ is contractible when $G[S]$ is a connected even graph (Lemma 5.4).

We mention one useful lemma to prove Lemmas 5.3 and 5.4.

Lemma 5.2 (Lemma 5.2 of [8])**.** *Let I be a vertex of a simplicial complex K and suppose that the link $\mathrm{Lk}_K I$ of I in K is contractible. Then K is homotopy equivalent to the complex $K \setminus \mathrm{St}_K I$, where $\mathrm{St}_K I$ is the star of I in K.*

We remark that for any two vertices I and J in $K := K^{\mathrm{odd}}_{G,S}$, it follows from Proposition 3.2 that $J \in \mathrm{Lk}_K I$ if and only if one of the following (a)~(d) is true: (a) $J \subsetneq I$, (b) $I \subsetneq J$, (c) $I \cup J \notin \mathcal{I}_G$, and (d) $J = \{\bar{j}\}$ for some $j \in [n] \setminus I$.

Lemma 5.3. *For any $S \subseteq [n]$, $K^{\mathrm{odd}}_{G,S}$ is homotopy equivalent to $K^{\mathrm{odd}}_{G[S]}$.*

Proof. For simplicity, we write $K := K^{\mathrm{odd}}_{G,S}$ and $K' := K^{\mathrm{odd}}_{G[S]}$. Let K^* be a minimal complex obtained by eliminating the stars of some vertices in $K \setminus K'$ without changing the homotopy type. We will show that $K^* = K'$.

Suppose that $K^* \supsetneq K'$. Let us take a vertex I in $K^* \setminus K'$ such that $|I \cap S|$ is minimal and $|I|$ is minimal.[11] Note that $I \in \mathcal{I}_G$ and $|I \cap S|$ is odd. Then $G[I \cap S]$ has a connected component I_1 with an odd number of vertices. Clearly, $I_1 \subseteq (I \cap S)$ and so $I_1 \subseteq S$. Thus $I_1 \in K'$ and $I_1 \neq I$. Note that if $J \in \mathrm{Lk}_{K^*} I$, then one of the following (a)~(d) is true : (a) $J \subsetneq I$, (b) $I \subsetneq J$,

[10]Note that a vertex is used for two meanings, one is for a graph and the other is for a simplicial complex.

[11]We first check the minimality of $|I \cap S|$ and then check the minimality of $|I|$.

(c) $I \cup J \notin \mathcal{I}_G$, and (d) $J = \{\bar{j}\}$ for some $j \in [n] \setminus I$. In the following, we will show that any vertex $J \in \mathrm{Lk}_{K^*} I$ with $J \neq I_1$ also belongs to $\mathrm{Lk}_{K^*} I_1$, that is, one of (a′)~(d′) is true: (a′) $J \subsetneq I_1$, (b′) $I_1 \subsetneq J$, (c′) $I_1 \cup J \notin \mathcal{I}_G$, and (d′) $J = \{\bar{j}\}$ for some $j \in [n] \setminus I_1$.

Suppose (a) $J \subsetneq I$. Then $J \cap S$ is a subset of $I \cap S$, and hence $|J \cap S| \leq |I \cap S|$ and $|J| < |I|$. Thus $J \in K'$ by the minimality conditions of I, that is, $J \subset S$. Then J is a subset of $I \cap S$ and so J is a connected graph contained in a connected component of $G[I \cap S]$, which implies that either (a′) $J \subsetneq I_1$ or (c′) $I_1 \cup J \notin \mathcal{I}_G$.

If (b) $I \subsetneq J$, then (b′) $I_1 \subsetneq J$ holds. If (c) $I \cup J \notin \mathcal{I}_G$, then (c′) $I_1 \cup J \notin \mathcal{I}_G$. If (d) $J = \{\bar{j}\}$ for some $j \in [n] \setminus I$, then (d′) $j \in [n] \setminus I_1$ holds.

Therefore, $\mathrm{Lk}_{K^*} I$ is the cone with the vertex I_1, and so $\mathrm{Lk}_{K^*} I$ is contractible and K^* is homotopy equivalent to $K^* \setminus \mathrm{St}_{K^*} I$ by Lemma 5.2. Then $K^* \setminus \mathrm{St}_{K^*} I$ is smaller than K^*, which contradicts the minimality of K^*. Therefore, $K^* = K'$. □

It follows from Lemma 5.3 that (5.2) is equivalent to the following:

$$\beta^i(X^{\mathbb{R}}(\square_G)) = \sum_{S \subseteq [n]} \tilde{\beta}^{i-1}(K^{\text{odd}}_{G[S]}). \tag{5.3}$$

Lemma 5.4. *If G is a connected even graph, then K^{odd}_G is contractible.*

Proof. For simplicity, let $K := K^{\text{odd}}_G$. Let K' be the induced subcomplex of K on the vertices $[\bar{n}]$, which is a simplex. Let K^* be a minimal complex obtained by eliminating the stars of some vertices $I \in \mathcal{I}_G$ such that $|I|$ is odd, without changing the homotopy type.

Suppose that $K' \subsetneq K^*$. Take a vertex I in $K^* \setminus K'$ such that $|I|$ is maximal. Since $|I|$ is odd and G is a connected even graph, there is a vertex $i \in [n] \setminus I$ of G such that $I \cup \{i\}$ induces a connected graph. Let $I_1 = \{\bar{i}\}$. Clearly, $I_1 \in K'$ and I_1 is in $\mathrm{Lk}_{K^*} I$. We will show that any vertex in $\mathrm{Lk}_{K^*} I$ is in the link of I_1. Take $J \in \mathrm{Lk}_{K^*} I$ with $J \neq I_1$. Then J satisfies either (a) $J = \{\bar{j}\}$ for some $j \in [n] \setminus I$ or (b) $J \in \mathcal{I}_G$ such that $|J|$ is odd. (a) If $J = \{\bar{j}\}$ for some $j \in [n] \setminus I$, then J is in the link of I_1 since $j \neq i$ and K' is a simplex. (b) Suppose that $J \in \mathcal{I}_G$ such that $|J|$ is odd. Then the maximality condition of I implies that $I \not\subset J$. Hence $J \subsetneq I$ or $G[I \cup J]$ is disconnected. If $J \subsetneq I$, then $i \notin J$. If $G[I \cup J]$ is disconnected, then any neighbor of a vertex in $G[I]$ does not belong to $G[J]$ and so $i \notin J$. So J is in $\mathrm{Lk}_{K^*} I_1$ by Proposition 3.2 (2). Therefore, $\mathrm{Lk}_{K^*} I$ is the cone with the vertex I_1. Hence $\mathrm{Lk}_{K^*} I$ is contractible and K^* is homotopy equivalent to $K^* \setminus \mathrm{St}_{K^*} I$ by Lemma 5.2. Then $K^* \setminus \mathrm{St}_{K^*} I$ is smaller than K^*, which contradicts the minimality of K^*. Therefore, $K^* = K'$. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. If $\kappa(G) = \kappa$ and $G_1, \ldots, G_\kappa$ are the connected components of G, then by definition

$$K^{\text{odd}}_G = K^{\text{odd}}_{G_1} * \cdots * K^{\text{odd}}_{G_\kappa} \simeq S^{\kappa-1} \wedge K^{\text{odd}}_{G_1} \wedge \cdots \wedge K^{\text{odd}}_{G_\kappa},$$

where $K_1 * K_2$ denotes the simplicial join of K_1 and K_2. Thus (5.3) is equivalent to

$$\beta^i(X^{\mathbb{R}}(\square_G)) = \sum_{S \subseteq [n]} \tilde{\beta}^{i-1}(K^{\text{odd}}_{G[S]}) = \sum_{S \subseteq [n]} \left(\sum_{\sum k_j = i - \kappa(G[S])} \prod_j \tilde{\beta}^{k_j}(K^{\text{odd}}_{G[S_j]}) \right), \tag{5.4}$$

where $G[S_j]$ means the j-th component of $G[S]$. If $G[S]$ is not odd, then $G[S]$ has a connected component with an even number of vertices, and then by Lemma 5.4, $\tilde{\beta}^{i-1}(K^{\text{odd}}_{G[S]}) = 0$. Thus (5.4) is equivalent to

$$\beta^i(X^{\mathbb{R}}(\Box_G)) \quad = \sum_{\substack{S\subseteq[n] \\ G[S] \text{ is odd}}} \tilde{\beta}^{i-1}(K^{\text{odd}}_{G[S]}).$$

For a graph H, let us denote by $(\partial\Box_H)^*$ the simplicial sphere which is the dual complex of $\partial\Box_H$. Then K^{odd}_H is an induced subcomplex of $(\partial\Box_H)^*$. We denote by K^{even}_H the induced subcomplex of $(\partial\Box_H)^*$ on the vertices not belonging to K^{odd}_H. Note that the vertices of K^{even}_H bijectively correspond to the connected even induced subgraphs of H. Since $(\partial\Box_{G[S]})^*$ is a sphere of dimension $|S|-1$, the Alexander duality implies

$$\beta^i(X^{\mathbb{R}}(\Box_G)) \quad = \sum_{\substack{S\subseteq[n] \\ G[S] \text{ is odd}}} \tilde{\beta}_{|S|-i-1}(K^{\text{even}}_{G[S]}). \tag{5.5}$$

When $|S|=1$, $K^{\text{even}}_{G[S]}$ is an empty simplicial complex. In this case we regard it as a sphere of dimension -1 for the formula above.

For an odd graph H, K^{even}_H is homotopy equivalent to the order complex of the poset $\mathcal{P}^{\text{even}}_H \setminus \{\hat{0},\hat{1}\}$ (see Lemma 4.7 of [8]). The poset $\mathcal{P}^{\text{even}}_{G[S]} \setminus \{\hat{0},\hat{1}\}$ is pure and shellable by Theorem 4.3, and its length is $\frac{|S|-\kappa(G[S])}{2} - 1$. Hence

$$\tilde{\beta}_{|S|-i-1}(K^{\text{even}}_{G[S]}) = \left|\mu(\mathcal{P}^{\text{even}}_{G[S]})\right| = \begin{cases} \left|\sum_{H\sqsubseteq G[S]} a(H)\right| & \text{if } |S|-i-1 = \frac{|S|-\kappa(G[S])}{2} - 1; \\ 0 & \text{otherwise.} \end{cases}$$

Thus, together with (4.1), (5.5) is equivalent to the following

$$\beta^i(X^{\mathbb{R}}(\Box_G)) \quad = \sum_{\substack{S\subseteq[n] \\ G[S] \text{ is odd}}} |b(G[S])|, \tag{5.6}$$

where $|S|-i = \frac{|S|-\kappa(G[S])}{2}$, that is, $|S|+\kappa(G[S]) = 2i$. Note that (5.6) is equivalent to

$$\beta^i(X^{\mathbb{R}}(\Box_G)) = \sum_{\substack{H\sqsubseteq G \\ |V(H)|+\kappa(H)=2i}} |b(H)|,$$

which completes the proof of Theorem 1.2. □

REMARK 5.5. It follows from Theorem 4.3 and Lemmas 5.3 and 5.4 that $(\Box_G)_{\omega_S}$ (respectively, $(\triangle_G)_{\omega_S}$) is torsion-free for every $S \subseteq [n]$. Hence $H^*(X^{\mathbb{R}}(\Box_G))$ (respectively, $H^*(X^{\mathbb{R}}(\triangle_G))$) is completely determined by the b-numbers (respectively, a-numbers) of all $H \sqsubset G$ and the h-vector of $\Box_G$ (respectively, $\triangle_G$) from (5.1). Furthermore, for an octopus G, the real toric manifolds $X^{\mathbb{R}}(\triangle_G)$ and $X^{\mathbb{R}}(\Box_{L(G)})$ have the same cohomology groups.

6. Examples

In this section, we provide some interesting integer sequences arising from the b-number of a graph G and the Betti numbers of $X^{\mathbb{R}}(\Box_G)$ for some graph families such as paths, cycles, complete graphs, and stars.

Corollary 6.1 ([8])**.** *For an odd integer n, we have*

$$b(G) = \begin{cases} (-1)^{\frac{n+1}{2}} A_n & \text{if } G = K_n, \\ (-1)^{\frac{n+1}{2}} \mathrm{Cat}(\frac{n-1}{2}) & \text{if } G = P_n, \\ (-1)^{\frac{n-1}{2}} \binom{n-1}{(n-1)/2} & \text{if } G = C_n, \\ (-1)^{\frac{n+1}{2}} A_{n-1} & \text{if } G = K_{1,n-1}, \end{cases}$$

where A_k is the k-th Euler Zigzag number and $\mathrm{Cat}(n)$ *is the n-th Catalan number.*

For paths and complete graphs, their formulae for the Betti numbers are directly obtained by Corollary 2.5 and Proposition 3.3. From the equivalence of the normal fans of $\square_{P_n}$ and $\triangle_{P_{n+1}}$ in Proposition 3.3, a corollary follows from Corollary 1.4 of [8]. See the table on the left side of Table 1, and it makes up the Catalan's triangle.

Corollary 6.2. *For any integer $i \geq 0$, we have*

$$\beta^i(X^{\mathbb{R}}(\square_{P_n})) = \begin{cases} \binom{n+1}{i} - \binom{n+1}{i-1} & \text{if } 1 \leq i \leq \lfloor \frac{n+1}{2} \rfloor, \\ 0 & \text{otherwise}. \end{cases}$$

For a complete graph K_n with n vertices and a star $K_{1,n}$ with n leaves, recall that the normal fan of $\square_{K_n}$ is equivalent to that of $\triangle_{K_{1,n}}$ in Proposition 3.3. Hence from Corollary 2.5, we have the following.

Corollary 6.3 (Corollary 1.6 of [8])**.** *For any integer $i \geq 0$, we have*

$$\beta^i(X^{\mathbb{R}}(\square_{K_n})) = \binom{n}{2i-1} A_{2i-1}.$$

The graph cubeahedron corresponding to a cycle is called a halohedron in [14]. We are going to compute the Betti numbers of $X^{\mathbb{R}}(\square_{C_n})$. See the table on the right side of Table 1.

By a *word* we mean a finite sequence consisting of given alphabets. Recall that a *Dyck word* of length $2k$ is a grammatically correct expression consisting of k left parentheses '(' and k right parentheses ')'. It is well-known that the number of Dyck words of length $2k$ is

Table 1. The Betti numbers of $X^{\mathbb{R}}(\square_{P_n})$ and $X^{\mathbb{R}}(\square_{C_n})$.

$n \backslash i$	0	1	2	3	4	5
1	1	1				
2	1	2				
3	1	3	2			
4	1	4	5			
5	1	5	9	5		
6	1	6	14	14		
7	1	7	20	28	14	
8	1	8	27	48	42	
9	1	9	35	75	90	42

$\beta^i(X^{\mathbb{R}}(\square_{P_n}))$

$n \backslash i$	0	1	2	3	4	5
1	1	1				
2	1	2				
3	1	3	2			
4	1	4	6			
5	1	5	10	6		
6	1	6	15	20		
7	1	7	21	35	20	
8	1	8	28	56	70	
9	1	9	36	84	126	70

$\beta^i(X^{\mathbb{R}}(\square_{C_n}))$

the k-th Catalan number $\mathrm{Cat}(k)$.

Let $\pi\colon \mathbb{Z} \to \mathbb{Z}_n$ be the canonical quotient map. We say that a map $f\colon \mathbb{Z}_n \to \{(,),*\}$ is a *partial Dyck word* on $\mathbb{Z}_n$ if there are finitely many intervals $N_1, \ldots, N_k$ in $\mathbb{Z}$, $k \geq 1$, such that

(1) $\pi|_{N_i}$ is one-to-one for all i,
(2) $\{\pi(N_1), \ldots, \pi(N_k)\}$ is a partition of $\mathbb{Z}_n$, and
(3) $(f \circ \pi)|_{N_i}$ induces either a Dyck word or the word $*$ for each i.

For example,

$$()()***(())*, \qquad)**((())()(), \qquad **((())()()), \qquad)((((())))) $$

are partial Dyck words and $(()*())*((*))*$ is not. We also note that the second one and third one are distinguished as partial Dyck words, even though they are the same up to rotation. Moreover, we take the "finest" partition of $\mathbb{Z}_n$ in the sense that each N_i is as short as possible. That is, no Dyck word can be further divided to shorter ones. We say that a parenthesis is *outermost* if it is outermost in the Dyck word where it is contained. In the first example, we have three Dyck words $()$, $()$, and $(())$. The shaded ones in the following are outermost parentheses.

$$()()***(())*, \qquad)**((())()(), \qquad **((())()()), \qquad)((((())))) $$

Parentheses which are not outermost are called *inner*.

Theorem 6.4. *For any integer $i \geq 0$, we have*

$$\beta^i(X^{\mathbb{R}}(\square_{C_n})) = \begin{cases} \binom{n}{i} & \text{if } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, \\ \binom{n-1}{i-1} & \text{if } n \text{ is odd and } i = \frac{n+1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We apply Theorem 1.2 to a cycle C_n. If $i > n/2$, then the only possible nontrivial case is when $H = C_n$ is the whole graph and $i = (n+1)/2$, and $\beta^i(X^{\mathbb{R}}(\square_{C_n})) = |b(H)| = \binom{n-1}{(n-1)/2}$ by Corollary 6.1.

Now we suppose that $0 \leq i \leq n/2$. We identify $V(C_n)$ with $\mathbb{Z}_n$ so that $\{j, j+1\}$ is an edge of C_n for $j \in \mathbb{Z}_n$. For a partial Dyck word $f : \mathbb{Z}_n \to \{(,),*\}$, consider the set

$$I_f = \{j \in \mathbb{Z}_n \mid f(j) \text{ is a parenthesis which is either inner or left outermost}\}.$$

In other words, I_f excludes $*$ and right outermost parentheses. For simplicity, for each partial Dyck word f, let $H_f = C_n[I_f]$. Then H_f is odd and $|V(H_f)| + \kappa(H_f)$ is equal to the number of parentheses in f, that is, $2i$.

Let $\mathcal{D}_i$ be the set of all partial Dyck words having exactly i left parentheses. We define an equivalence relation $\sim$ on $\mathcal{D}_i$ by $f \sim g$ if and only if the inverse image of the outermost parentheses in f is equal to that in g. If a partial Dyck word f has q pairs of outermost parentheses and the j-th pair has $2k_j$ inner parentheses, then the size of the equivalence class $[f]$ is

$$|[f]| = \mathrm{Cat}(k_1) \times \cdots \times \mathrm{Cat}(k_q) = \prod_j |b(P_{2k_j+1})| = |b(H_f)|,$$

where the second equality is from $\mathrm{Cat}(k_j) = |b(P_{2k_j+1})|$ by Corollary 6.1, and the last equality follows from the fact that the j-th component of H_f is a path with $2k_j + 1$ vertices and from

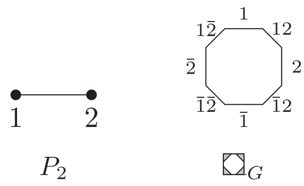


Fig. 5. A graph associahedron of type B

the definition of the b-number.

On the other hand, for an odd induced subgraph H of C_n such that $|V(H)| + \kappa(H) = 2i$, there is a partial Dyck word $f \in \mathcal{D}_i$ such that $H_f = H$. Thus, together with Theorem 1.2,

$$\beta^i(X^{\mathbb{R}}(\square_{C_n})) = \sum_{\substack{H \sqsubseteq G \\ |V(H)|+\kappa(H)=2i}} |b(H)| = \sum_{[f]\in\mathcal{D}_i/\sim} |b(H_f)| = \sum_{[f]\in\mathcal{D}_i/\sim} |[f]| = |\mathcal{D}_i|,$$

where $\mathcal{D}_i/\sim$ is the set of all equivalence classes.

It remains to show that $|\mathcal{D}_i| = \binom{n}{i}$ for any integers i and n with $2i \leq n$. For a given $f \in \mathcal{D}_i$, one takes the set $\{i \in \mathbb{Z}_n \mid f(i) = (\}$. This set is distinguishable by f, and thus this gives an injective function from $\mathcal{D}_i$ to the set of all i-subsets of $\mathbb{Z}_n$. To show that this is surjective, take a subset $I \subseteq \mathbb{Z}_n$ such that $|I| = i$. Since $i \leq \frac{n}{2}$, there exists $j_1 \in I$ such that $j_1 + 1 \notin I$. Then one assigns $f(j_1) = ($ and $f(j_1 + 1) =)$, and then removes both ones to get a subset $I' = I \setminus \{j_1\}$ of $\mathbb{Z}_{n-2}$. Again, find $j_2 \in I'$ such that $j_2 + 1 \notin I'$ and then assign $f(j_2) = ($ and $f(j_2 + 1) =)$. In this way, we can assign i)'s inductively, and then we assign $*$ for remaining $n - 2i$ elements. □

For a star $K_{1,n}$, we have the following result.

Proposition 6.5. *For any integer $i \geq 0$, we have*

$$\beta^i(X^{\mathbb{R}}(\square_{K_{1,n}})) = \binom{n}{i} + \binom{n}{2i-2} A_{2i-2},$$

where $A_{-2} = 0$.

Proof. Note that by Corollary 6.1, $b(K_{1,2k}) = (-1)^{k+1}A_{2k}$. Each odd induced subgraph of $K_{1,n}$ is an edgeless graph induced from the leaves or a star $K_{1,2k}$ for $0 \leq k \leq n/2$. Hence an odd induced subgraph H such that $|V(H)| + \kappa(H) = 2i$ is either the edgeless graph with i isolated vertices or a subgraph isomorphic to $K_{1,2i-2}$. Hence, the proof is done. □

Remark 6.6. For a graph G with n vertices, one has $\beta^i(X^{\mathbb{R}}(\triangle_G)) = 0$ if $i > n/2$. For the graph cubeahedron $\square_G$, it can happen that $\beta^i(X^{\mathbb{R}}(\square_G)) > 0$ even though $i > \frac{n+1}{2}$ as in Proposition 6.5.

7. Remarks

A graph cubeahedron is obtained from the standard cube $\square^n$ by truncating the faces labeled by I for each $I \in \mathcal{I}_G$ in increasing order of dimensions. We introduce a slightly different polytope, which is also made from $\square^n$ by truncating the faces. Note that each facet

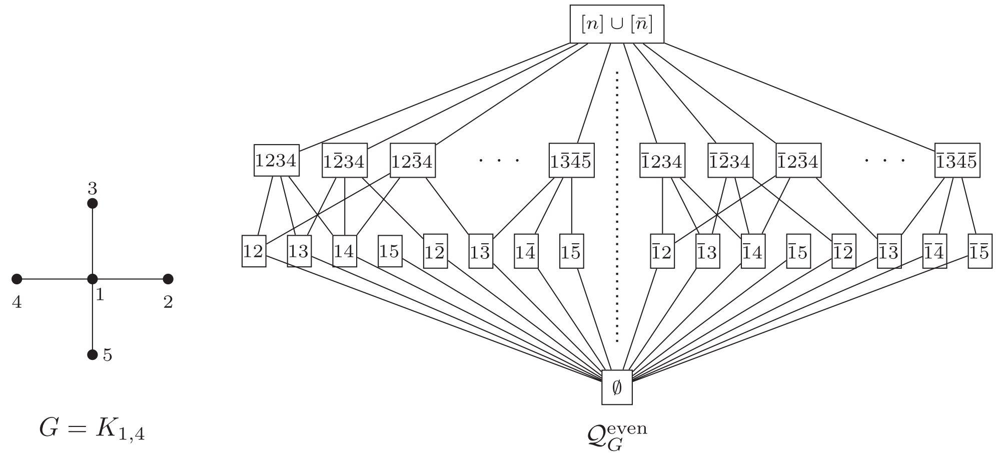


Fig. 6. When $G = K_{1,4}$, $\mathcal{Q}_G^{\text{even}}$ is not shellable.

of $\square^n$ corresponds to an element of $[n] \cup [\bar{n}]$. For $I \subseteq [n]$, we define $\mathcal{I}(I)$ as the set of subsets $\widetilde{I}$ of $[n] \cup [\bar{n}]$ satisfying two conditions

(1) i or $\bar{i}$ belongs to $\widetilde{I}$ if and only if $i \in I$, and
(2) $|\{i, \bar{i}\} \cap \widetilde{I}| = 1$ for each $i \in I$.

Let $\mathcal{F}(I)$ be the set of faces of $\square^n$ labelled by elements in $\mathcal{I}(I)$. Then $\mathcal{F}(I)$ is a disjoint union of faces of codimension $|I|$, and $\mathcal{F}(I)$ is of size $2^{|I|}$. Given a graph G, we denote by $\Box_G$ the polytope by truncating the faces labeled by $\widetilde{I}$ for every $\widetilde{I} \in \mathcal{I}(I)$ and $I \in \mathcal{I}_G$ in increasing order of dimensions. Note that for the complete graph K_n, $\Box_{K_n}$ is known as the type B permutohedron. So we call this polytope $\Box_G$ a *graph associahedron of type B*. Note that $\Box_G$ is also a Delzant polytope. For example, consider a complete graph K_2 with two vertices, and then Figure 5 shows a graph associahedron of type B.

The authors already checked that all statements of this paper corresponding to $\Box_G$ are well-established except shellability. More precisely, we can apply Theorem 5.1 and Lemma 5.3 to the case of $\Box_G$. Then the problem of computing the Betti numbers of a real toric manifold associated with $\Box_G$ is converted to studying a topology of the order complex of some special poset $\mathcal{Q}_G^{\text{even}}$, which is defined by

$$\mathcal{Q}_G^{\text{even}} = \{I \subset [n] \cup [\bar{n}] : I^+ \cap I^- = \emptyset,\ G[I^+ \cup I^-] \text{ is even}\} \cup \{\hat{0}, \hat{1}\},$$

where

$$I^+ = I \cap [n] \quad \text{and} \quad I^- = \{i \in [n] \mid \bar{i} \in I\}.$$

For a complete graph G, the Type B permutohedron $\Box_G$ is already studied in [6] and it was shown that $\mathcal{Q}_G^{\text{even}}$ is shellable. However, $\mathcal{Q}_G^{\text{even}}$ is not always shellable, and so it is not easy to compute the Betti numbers of the real toric manifold associated with $\Box_G$. Consider a star $K_{1,n}$ ($n \geq 4$) as in Figure 6. Then for any two elements I and J of $\mathcal{Q}_G^{\text{even}}$, if $1 \in I$ and $\bar{1} \in J$, then any maximal chain containing I and any maximal chain containing J do not intersect. Thus $\mathcal{Q}_G^{\text{even}}$ cannot be shellable.

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References

[1] A. Björner: *Shellable and Cohen-Macaulay partially ordered sets*, Trans. Amer. Math. Soc. **260** (1980), 159–183.

[2] V.M. Buchstaber and V.D. Volodin: *Combinatorial* 2*-truncated Cubes and Applications*. in Associahedra, Tamari Lattices and Related Structures, Progress in Mathematics **299**, Birkhäuser, Basel, 2012.

[3] L. Cai and S. Choi: *Integral cohomology groups of real toric manifolds and small covers*, preprint. arXiv:1604.06988.

[4] M. Carr and S.L. Devadoss: *Coxeter complexes and graph-associahedra*, Topology Appl. **153** (2006), 2155–2168.

[5] S. Choi, S. Kaji and H. Park: *The cohomology groups of real toric varieties associated to Weyl chambers of Type C and D*, to appear in P. Edinburgh Math. Soc. arXiv:1705.00275.

[6] S. Choi, B. Park and H. Park: *The Betti numbers of real toric varieties associated to Weyl chambers of type B*, Chin. Ann. Math. Ser. B. **38** (2017), 1213–1222.

[7] S. Choi, B. Park and S. Park: *Pseudograph and its associated real toric manifold*, J. Math. Soc. Japan **69** (2017), 693–714.

[8] S. Choi and H. Park: *A new graph invariant arises in toric topology*, J. Math. Soc. Japan **67** (2015), 699–720.

[9] S. Choi and H. Park: *On the cohomology and their torsion of real toric objects*, Forum Math. **29** (2017), 543–554.

[10] S. Choi and H. Park: *Multiplicative structure of the cohomology ring of real toric spaces*, preprint. arXiv:1711.04983.

[11] V.I. Danilov: *The geometry of toric varieties*, Uspekhi Mat. Nauk **33** (1978), 85–134.

[12] C. De Concini and C. Procesi: *Wonderful models of subspace arrangements*, Selecta Math. **1** (1995), 459–494.

[13] S. Devadoss: *Tessellations of moduli spaces and the mosaic operad*, Contemp. Math. **239** (1999), 91–114.

[14] S.L. Devadoss, T. Heath and W. Vipismakul: *Deformations of bordered surfaces and convex polytopes*, Notices of the AMS **58** (2011), 530–541.

[15] R. Ferreira da Rosa, D. Jensen and D. Ranganathan: *Toric graph associahedra and compactifications of* $M_{0,n}$, J. Algebraic Combin. **43** (2016), 139–151.

[16] W. Fulton: Introduction to Toric Varieties, Ann. of Math. Studies, **113**, Princeton Univ. Press, Princeton, N.J., 1993.

[17] J. Jurkiewicz: *Chow ring of projective nonsingular torus embedding*, Colloq. Math. **43** (1980), 261–270.

[18] J. Jurkiewicz: *Torus embeddings, polyhedra,* k^**-actions and homology*. Instytut Matematyczny Polskiej Akademi Nauk (Warszawa), 1985. available at http://eudml.org/doc/268485.

[19] T. Manneville and V. Pilaud: *Compatibility fans for graphical nested complexes*, J. Combin. Theo. Ser. A. **150** (2017), 36–107.

[20] B. Park and S. Park: *On shellability for a poset of even subgraphs of a graph*, preprint. arXiv:1705.06423.

[21] A. Postnikov: *Permutohedra, associahedra, and beyond*, Int. Math. Res. Not. IMRN (2009), 1026–1106.

[22] A. Postnikov, V. Reiner and L. Williams: *Faces of generalized permutohedra*, Doc. Math. **13** (2008), 207–273.

[23] S. Seo and H. Shin: *Signed a-polynomials of graphs and Poincaré polynomials of real toric manifolds*, Bull. Korean Math. Soc. **52** (2015), 467–481.

[24] R. Stanley: Combinatorics and Commutative Algebra, Second edition. Progress in Mathematics **41**, Birkhaüser Boston, Inc., Boston, MA, 1996.

[25] R. Stanley: Enumerative Combinatorics. Volume 1, Second edition, Cambridge Studies in Advanced Mathematics **49**. Cambridge University Press, Cambridge, 2012.

[26] A. Suciu and A. Trevisan: *Real toric varieties and abelian covers of generalized Davis-Januszkiewicz spaces*, preprint, 2012.

[27] A. Trevisan: *Generalized Davis-Januszkiewicz spaces and their applications in algebra and topology*, Ph.D. thesis, Vrije University Amsterdam, 2012.

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