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THE FINITE GROUP ACTION AND THE EQUIVARIANT DETERMINANT OF ELLIPTIC OPERATORS III

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Abstract

Let M be an even-dimensional closed oriented manifold and g a periodic automorphism of M of order p. In this paper, under the assumption that the fixed points of g^k $(1 \le k \le p-1)$ are isolated, a calculation formula is provided for the homomorphism $I_D : \mathbb{Z}_p \to \mathbb{R}/\mathbb{Z}$ defined in [6] for equivariant twisted signature operators D over M. The formula gives a new method to study the periodic automorphisms of oriented manifolds. As examples of the application of the formula, results about the existence of the cyclic group action for 2,4,6-dimensional closed oriented manifolds are obtained.

1. Introduction

Let M be a $2m$-dimensional closed oriented manifold and G a finite group acting on M. We assume that the action of G is effective and orientation-preserving. Let g be an element of G of order $p \ge 2$ and $\mathbb{Z}_p = \langle g \rangle$ the cyclic group generated by g. In this paper, we set the following assumption.

Assumption 1.1. *Some g^k $(1 \le k \le p-1)$ has a fixed point, and any fixed point of g^k is isolated for $1 \le k \le p-1$ if g^k has a fixed point.*

In [6] we introduce a group homomorphism I_D by using an elliptic operator D adapted to a geometric structure of a manifold, and in [7] we apply this homomorphism to the existence problem of finite group actions on almost complex manifolds under the assumption above.

Let D be a G-equivariant elliptic operator. Then a homomorphism I_D from G to $\mathbb{R}/\mathbb{Z}$ is defined by

$$I_D(g) = \frac{1}{2\pi\sqrt{-1}} \log \det(D, g) \in \mathbb{R}/\mathbb{Z}$$

for $g \in G$, where $\det(D, g)$ is defined by

$$\det(D, g) = \det(g|\ker D)/\det(g|\mathrm{coker}\, D) \in S^1 \subset \mathbb{C}^*$$

(see [6] Definition 2.1). Then as we see in [6] (3) the next equality holds

$$I_D(g) \equiv \frac{p-1}{2p}\mathrm{Ind}(D) - \frac{1}{p}\sum_{k=1}^{p-1} \frac{1}{1-\xi_p^{-k}}\mathrm{Ind}(D, g^k) \pmod{\mathbb{Z}} \tag{1}$$

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where Ind is the Atiyah-Singer index (see [3]) and ξ_p is the primitive p-th root of unity defined by $\xi_p = e^{2\pi\sqrt{-1}/p}$.

We can express the value $I_D(g)$ by the fixed point data of the g^k-action $(1 \leq k \leq p-1)$ by using the equality (1) and the fixed point formula of Atiyah-Segal-Singer [2], [3]. Since I_D is a homomorphism, the equalities $I_D(g^z) = zI_D(g)$, $I_D(gh) = I_D(g) + I_D(h)$ hold for any $g, h \in G$ and any integer z because $\mathbb{R}/\mathbb{Z}$ is an abelian group. These properties of I_D impose conditions on the fixed point data and a calculation formula to obtain the precise value of I_D can be used to examine the existence of a finite group action on M.

When M is a $2m$-dimensional closed almost complex manifold and the g-action preserves the almost complex structure of M, we give a formula to calculate $I_D(g)$ precisely for the $\otimes_{j=1}^m \left(\otimes^{\ell_j}\left(\wedge_{\mathbb{C}}^j TM\right)\right)$-valued Dolbeault operator D over M in [7] Theorem 2.2. Though the formula is useful for closed almost complex manifolds, we need a formula to calculate the precise value of $I_D(g)$ for equivariant elliptic operators D which is adapted for closed oriented manifolds, which do not necessarily have almost complex structure.

In [9] Zagier gives a formula which enables us to calculate the precise value of $I_D(g)$ for the equivariant non-twisted signature operator D under Assumption 1.1.

DEFINITION 1.2. For a real number x, a real number $((x))$ is defined by

$$((x)) := \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \text{ is not an integer} \\ 0 & \text{if } x \text{ is an integer} \end{cases}$$

where $[x]$ is the Gauss' symbol. Note that $((x+a)) = ((x))$ for an integer a and that $((-x)) = -((x))$ because $[-x] = -[x] - 1$ if x is not an integer.

Theorem 1.3 ([9] p.103)**.** *Let p be a natural number which is greater than* 1 *and $a_1, \cdots,$ a_n natural numbers which are prime to p. Then the following equality holds:*

$$\left(-\sqrt{-1}\right)^n \sum_{k=1}^{p-1} \cot\frac{\pi a_1 k}{p} \cdots \cot\frac{\pi a_n k}{p} = 2^n p \sum_{\substack{1 \leq k_1, \cdots, k_n < p \\ p \mid (a_1 k_1 + \cdots + a_n k_n)}} \left(\left(\frac{k_1}{p}\right)\right) \cdots \left(\left(\frac{k_n}{p}\right)\right).$$

As we see later (see (4)) the left hand side of the equality above is equal to the sum of indices for the equivariant non-twisted signature operator, and the equality implies that the sum is a rational number.

In this paper, generalizing the Zagier's formula, we give a formula to calculate the precise value of $I_D(g)$ for equivariant twisted signature operators D over $2m$-dimensional closed oriented manifolds to examine the existence of a finite group action on the manifolds.

2. Rotation angles of periodic automorphisms

Let M be a $2m$-dimensional closed oriented manifold and p a natural number which is greater than 1. Assume that the cyclic group $\mathbb{Z}_p = \langle g \rangle$ acts on M. Under Assumption 1.1, let Ω be the union of the fixed points of g^k for $1 \leq k \leq p-1$ and suppose that the image $\pi(\Omega)$ consists of b points $y_1, \cdots, y_b \in M/\mathbb{Z}_p$ where $\pi : M \longrightarrow M/\mathbb{Z}_p$ is the projection. In this paper, the $\mathbb{Z}_p$-action is called the $\mathbb{Z}_p$-action of isotropy orders $(p_1, \cdots, p_b)$ if the isotropy group at a point $q_i \in \pi^{-1}(y_i)$ $(1 \leq i \leq b)$ is the cyclic group of order p_i. Then for $1 \leq i \leq b$ the isotropy group at any points in $\pi^{-1}(y_i)$ is the cyclic group of order p_i generated by g^{r_i}

where $r_i = p/p_i$ and $\pi^{-1}(y_i)$ consists of r_i points $q_i,\ g\cdot q_i,\ \cdots,\ g^{r_i-1}\cdot q_i$.

REMARK 2.1. Let $e(M)$ be the Euler number of M. Then since $\mathbb{Z}_p$ acts freely on the punctured manifold $M_0 = M\setminus\{\cup_{i=1}^{b}\pi^{-1}(y_i)\}$, the next equality holds:

$$e(M) \equiv \sum_{i=1}^{b} r_i \pmod{p}. \tag{2}$$

Let $\xi_{p_i}^{\tau_{ij}},\ \xi_{p_i}^{-\tau_{ij}}$ $(1\le j\le m)$ be the eigenvalues of $g^{r_i}|T_{q_i}M$, and

$$\overrightarrow{u_{ij}} - \sqrt{-1}\,\overrightarrow{v_{ij}},\ \overrightarrow{u_{ij}} + \sqrt{-1}\,\overrightarrow{v_{ij}}\quad \left(\overrightarrow{u_{ij}},\ \overrightarrow{v_{ij}} \in T_{q_i}M \simeq \mathbb{R}^{2m}\right)$$

the corresponding eigenvectors such that

$$\omega_{q_i}\left(\overrightarrow{u_{i1}}\wedge\overrightarrow{v_{i1}}\wedge\cdots\wedge\overrightarrow{u_{im}}\wedge\overrightarrow{v_{im}}\right)$$

is positive with respect to a positive $2m$-form ω where τ_{ij} $(1\le j\le m)$ are natural numbers such that $1\le\tau_{ij}\le p_i-1$ (see ([1] p.473). Since the eigenvalues of $g^{r_i\kappa}|T_{q_i}M$ for $1\le\kappa\le p_i-1$ are $\xi_{p_i}^{\kappa\tau_{ij}},\ \xi_{p_i}^{-\kappa\tau_{ij}}$ $(1\le j\le m)$, it follows from Assumption 1.1 that τ_{ij} is prime to p_i for $1\le i\le b,\ 1\le j\le m$. In this paper the set $\{\tau_{ij}\}$ $(1\le j\le m)$ is called the rotation angle of g^{r_i} around the points in $\pi^{-1}(y_i)$ and the set $\{\tau_{ij}\}$ $(1\le i\le b,\ 1\le j\le m)$ is called the rotation angle of g. The rotation angle of g is also expressed as

$$((\tau_{11},\cdots,\tau_{1m}),\cdots,(\tau_{b1},\cdots,\tau_{bm}))\ .$$

Note that the equality

$$g^{r_i}\cdot\left(\overrightarrow{u_{ij}}-\sqrt{-1}\,\overrightarrow{v_{ij}}\right) = \xi_{p_i}^{\tau_{ij}}\left(\overrightarrow{u_{ij}}-\sqrt{-1}\,\overrightarrow{v_{ij}}\right)$$

implies the equality

$$\left(g^{r_i}\cdot\overrightarrow{u_{ij}},\ g^{r_i}\cdot\overrightarrow{v_{ij}}\right) = \left(\overrightarrow{u_{ij}},\ \overrightarrow{v_{ij}}\right)\begin{pmatrix}\cos\frac{2\pi\tau_{ij}}{p_i} & -\sin\frac{2\pi\tau_{ij}}{p_i}\\ \sin\frac{2\pi\tau_{ij}}{p_i} & \cos\frac{2\pi\tau_{ij}}{p_i}\end{pmatrix}$$

and that $\{\tau_{i1},\cdots,p_i-\tau_{ij},\cdots,p_i-\tau_{ik},\cdots,\tau_{im}\}$ $(j\ne k)$ is also the rotation angle of g^{r_i} when $m\ge 2$ because

$$\omega_{q_i}\left(\overrightarrow{u_{i1}}\wedge\overrightarrow{v_{i1}}\wedge\cdots\wedge\overrightarrow{v_{ij}}\wedge\overrightarrow{u_{ij}}\wedge\cdots\wedge\overrightarrow{v_{ik}}\wedge\overrightarrow{u_{ik}}\wedge\cdots\wedge\overrightarrow{u_{im}}\wedge\overrightarrow{v_{im}}\right)$$

is also positive. Hence if p_i is even and some τ_{ij} equals $p_i/2$, $\{\tau_{i1},\cdots,p_i-\tau_{ik},\cdots,\tau_{im}\}$ is also the rotation angle of g^{r_i} for $k\ne j$ and therefore we can assume that $1\le\tau_{ij}\le p_i/2$.

Corresponding to the irreducible representations of the rotation group, complex $\mathbb{Z}_p$-vector bundles $E^{\ell_1,\cdots,\ell_m}$ are defined for non-negative integers $\ell_1,\ \cdots,\ \ell_m$ as follows:

$$E^{\ell_1,\cdots,\ell_m} = \otimes_{j=1}^{m}\left(\otimes^{\ell_j}\left(\wedge_{\mathbf{C}}^{j}(TM\otimes\mathbb{C})\right)\right),$$

and the $E^{\ell_1,\cdots,\ell_m}$-valued signature operator $D_{\ell_1,\cdots,\ell_m}$ is defined. Note that $E^{0,\cdots,0}$ is a trivial line bundle and $D_{0,\cdots,0}$ is the equivariant non-twisted signature operator. Suppose that the rotation angle of g^{r_i} around the points q_i in $\pi^{-1}(y_i)$ $(1\le i\le b)$ is $\{\tau_{ij}\}$. Then the fixed point set of g^k $(1\le k\le p-1)$ exists if and only if k equals $r_i\kappa$ for $1\le i\le b,\ 1\le\kappa\le p_i-1$ and it follows from the signature theorem and the G-signature theorem that

$$\text{(3)}\qquad \mathrm{Ind}(D_{\ell_1,\cdots,\ell_m}) = 2^m \mathrm{Ch}(E^{\ell_1,\cdots,\ell_m})\widehat{\mathbf{L}}(M)[M] \quad \left(\widehat{\mathbf{L}}(M) = \prod_{j=1}^{m} \frac{x_j/2}{\tanh(x_j/2)}\right),$$

$$\text{(4)}\qquad \mathrm{Ind}(D_{\ell_1,\cdots,\ell_m}, g^{r_i\kappa}) = \sum_{i=1}^{b} r_i \mathrm{Ch}(E_{q_i}^{\ell_1\cdots\ell_m}, g^{r_i\kappa}) \prod_{j=1}^{m}\left(-\sqrt{-1}\cot\frac{\pi\kappa\tau_{ij}}{p_i}\right)$$

where $\sigma_j(x_1^2, \cdots, x_m^2)$ for the j-th elementary symmetric function σ_j is the j-th Pontrjagin class of M (see [3] (6.5), (6.6), [5] Theorem 13.9), $E_{q_i}^{\ell_1,\cdots,\ell_m}$ is the fiber of $E^{\ell_1,\cdots,\ell_m}$ at q_i and $\mathrm{Ch}(E_{q_i}^{\ell_1\cdots\ell_m}, g^{r_i\kappa})$ is the trace of the linear transformation $g^{r_i\kappa}$ of the fiber $E_{q_i}^{\ell_1\cdots\ell_m}$ (see [3] (6.12), (6.19), [4] Theorem 2.5), and hence we have

$$\begin{aligned}\text{(5)}\qquad & I_{D_{\ell_1,\cdots,\ell_m}}(g)\\ &\equiv \frac{p-1}{p}2^{m-1}\mathrm{Ch}(E^{\ell_1,\cdots,\ell_m})\widehat{\mathbf{L}}(M)[M]\\ &\quad - \frac{1}{p}\sum_{i=1}^{b} r_i \sum_{\kappa=1}^{p_i-1}\frac{1}{1-\xi_{p_i}^{-\kappa}}\mathrm{Ch}(E_{q_i}^{\ell_1\cdots\ell_m}, g^{r_i\kappa})\prod_{j=1}^{m}\left(-\sqrt{-1}\cot\frac{\pi\kappa\tau_{ij}}{p_i}\right) \pmod{\mathbb{Z}}.\end{aligned}$$

Remark 2.2. Since the Chern classes of $TM \otimes \mathbb{C}$ is expressed by the Pontrjagin classes of TM, the Chern character $\mathrm{Ch}(E^{\ell_1,\cdots,\ell_m})$ is expressed by the Pontrjagin classes of TM (see Proposition 2.6). Hence the equality (3) above implies that $\mathrm{Ind}(D_{\ell_1,\cdots,\ell_m}) = 0$ if m is odd.

Remark 2.3. Assume that $p_i = 2$ for some i. Then since $\kappa = \tau_{ij} = 1$, we have

$$\prod_{j=1}^{m}\left(-\sqrt{-1}\cot\frac{\pi\kappa\tau_{ij}}{p_i}\right) = \prod_{j=1}^{m}\left(-\sqrt{-1}\cot\frac{\pi}{2}\right) = 0$$

in the equalities (4), (5).

Definition 2.4. Let p be a natural number which is greater than 1 and a, b integers where b is prime to p. Then using the Euler's function φ, we define integers $\langle a \rangle$, $\overline{b}$ as follows:

$$\langle a \rangle = a - p\left[\frac{a}{p}\right] \iff 0 \le \langle a \rangle \le p-1\,,\ \langle a \rangle \equiv a \mod p\,,$$

$$\overline{b} = \langle b^{\varphi(p)-1} \rangle \iff b\overline{b} \equiv 1 \mod p\,, \quad 1 \le \overline{b} \le p-1\,.$$

In this paper, we call $\overline{b}$ the mod.p-inverse of b. Note that $\langle a+p \rangle = \langle a \rangle$, $\overline{b+p} = \overline{b}$, $\overline{\overline{b}} \equiv b$ mod p because $(\overline{\overline{b}}-b)\overline{b} \equiv 0 \pmod{p}$ and that $\langle a_1 \rangle \cdot \langle a_2 \rangle \equiv \langle a_1a_2 \rangle$, $\overline{b_1}\cdot\overline{b_2} \equiv \overline{b_1b_2} \pmod{p}$ because $\left(\overline{b_1}\cdot\overline{b_2} - \overline{b_1b_2}\right)b_1b_2 \equiv 0 \pmod{p}$ where b_1, b_2 are prime to p. Note moreover that the equality $\langle a_1 \rangle + \langle a_2 \rangle \equiv \langle a_1 + a_2 \rangle \pmod{p}$ holds but the equality $\overline{b_1} + \overline{b_2} \equiv \overline{b_1 + b_2}$ $\pmod{p}$ does not hold.

Remark 2.5. Assume that a natural number z is prime to p. Then we have $(g^{kz})^{\overline{z}} = g^k$ for $1 \le k \le p-1$, and hence the fixed point set of g^{kz} coincides with that of g^k and the rotation angle of g^{r_iz} is equivalent to $\{z\tau_{ij}\}$ mod.p_i if the rotation angle of g^{r_i} is $\{\tau_{ij}\}$.

Proposition 2.6. *Suppose that the total Chern class of $TM \otimes \mathbb{C}$ equals $\prod_{j=1}^{m}(1 - x_j^2)$ and hence the j-th Pontrjagin class $p_j(TM)$ of M equals $\sigma_j(x_1^2, \cdots x_m^2)$ where σ_j is the j-th* *elementary symmetric function (see* [5] *p.228). Then we have*

$$\mathrm{Ch}(E^{\ell_1,\cdots,\ell_m}) = \prod_{j=1}^{m} \sigma_j(e^{x_1}, e^{-x_1}, \cdots, e^{x_m}, e^{-x_m})^{\ell_j}\,. \tag{6}$$

Proof. Since the total Chern class of $TM \otimes \mathbb{C}$ equals $\prod_{j=1}^{m}(1+x_j)(1-x_j)$, we have

$$\mathrm{Ch}(\wedge^j_{\mathbf{C}}(TM \otimes \mathbb{C})) = \sigma_j(e^{x_1}, e^{-x_1}, \cdots, e^{x_m}, e^{-x_m})\,.$$

Hence it follows that

$$\mathrm{Ch}(E^{\ell_1,\cdots,\ell_m}) = \prod_{j=1}^{m} \sigma_j(e^{x_1}, e^{-x_1}, \cdots, e^{x_m}, e^{-x_m})^{\ell_j}\,. \qquad \square$$

EXAMPLE 2.7. Assume that $m = 2$ and let p_1 be the first Pontrjagin number of M. Then we have

$$2^2\widehat{\mathbf{L}}(M) = 2^2 \prod_{j=1}^{2} \frac{x_j/2}{\tanh(x_j/2)} = 4 + \frac{1}{3}p_1(TM)\,,$$

$$\mathrm{Ch}(E^j) = \sigma_j(e^{x_1}, e^{-x_1}, e^{x_2}, e^{-x_2}) = \begin{cases} 4 + p_1(TM) & (j = 1, 3)\,, \\ 6 + 2p_1(TM) & (j = 2) \\ 1 & (j = 0, 4) \end{cases}\,,$$

where $E^j = \wedge^j_{\mathbf{C}}(TM \otimes \mathbb{C})$, and hence it follows from Proposition2.6 that

$$\mathrm{Ch}(\otimes^{\ell_j} E^j) = \left(\mathrm{Ch}(E^j)\right)^{\ell_j} = \begin{cases} 4^{\ell_j} + \ell_j 4^{\ell_j - 1} p_1(TM) & (j = 1, 3)\,, \\ 6^{\ell_j} + 2\ell_j 6^{\ell_j - 1} p_1(TM) & (j = 2) \\ 1 & (j = 0, 4) \end{cases}$$

$$\begin{aligned} \Longrightarrow\ \mathrm{Ch}(E^{\ell_1,\ell_2}) &= \left(\mathrm{Ch}(E^1)\right)^{\ell_1} \left(\mathrm{Ch}(E^2)\right)^{\ell_2} \\ &= 4^{\ell_1}6^{\ell_2} + 2 \cdot 4^{\ell_1 - 1}6^{\ell_2 - 1}\,(3\ell_1 + 4\ell_2)\,p_1(M) \\ \Longrightarrow\ \mathrm{Ind}D_{\ell_1,\ell_2} &= 2^2\mathrm{Ch}(E^{\ell_1,\ell_2})\widehat{\mathbf{L}}(M)[M] \\ &= 2 \cdot 4^{\ell_1}6^{\ell_2 - 1}(3\ell_1 + 4\ell_2 + 1)p_1 = 2^{2\ell_1 + \ell_2}3^{\ell_2 - 1}(3\ell_1 + 4\ell_2 + 1)p_1\,. \end{aligned}$$

DEFINITION 2.8. Let p be a natural number which is greater than 1. Then for any integer λ, a real number $\rho_p(\lambda)$ is defined by

$$\rho_p(\lambda) = \xi_p^{\lambda} + \xi_p^{-\lambda}\,.$$

Clearly ρ_p has the following properties:

$$\rho_p(-\lambda) = \rho_p(\lambda)\,,\ \ \rho_p(0) = 2\,,\ \ \rho_p(\lambda)\rho_p(\mu) = \rho_p(\lambda + \mu) + \rho_p(\lambda - \mu)\,,$$

which implies that

$$\rho_p(\lambda_1)\rho_p(\lambda_2)\cdots\rho_p(\lambda_j) = \rho_p(\lambda_1 \pm \lambda_2 \pm \cdots \pm \lambda_j) \tag{7}$$

for integers μ, λ_1, λ_2, $\cdots$, λ_j where

$$\rho_p(\lambda_1 \pm \lambda_2 \pm \cdots \pm \lambda_j) = \sum_{\varepsilon_i = \pm 1 (2 \le i \le j)} \rho_p(\lambda_1 + \varepsilon_2\lambda_2 + \cdots + \varepsilon_j\lambda_j)\,.$$

We set that the binomial coefficient $\binom{0}{0}$ and 0^0 are equal to 1 hereafter.

Proposition 2.9. *For* $1 \le i \le b$, $1 \le j \le m$, $1 \le k \le p_i - 1$ *set*

$$C_{ij}^k = \sum_{r=0}^{\left[\frac{j-1}{2}\right]} \binom{m-j+2r}{r} \sigma_{j-2r}(\rho_{p_i}(k\tau_{i1}), \cdots, \rho_{p_i}(k\tau_{im})) + \varepsilon_j \binom{m}{\frac{j}{2}}$$

where σ_{j-2s} *is the* $j-2s$*-th elementary symmetric function and*

$$\varepsilon_j = \frac{1+(-1)^j}{2} = \begin{cases} 1 & (j : even) \\ 0 & (j : odd) \end{cases} .$$

Then we have

$$\mathrm{Ch}(E_{q_i}^{\ell_1,\cdots,\ell_m}, g^{r_i k}) = \prod_{j=1}^m \left(C_{ij}^k\right)^{\ell_j} .$$

Proof. Since the eigenvalues of $g^{r_i k}|\left(T_{q_i}M \otimes \mathbb{C}\right)$ are $\lambda_1, \cdots, \lambda_{2m}$ where $\lambda_{2j-1} = \xi_{p_i}^{k\tau_{ij}}$, $\lambda_{2j} = \xi_{p_i}^{-k\tau_{ij}}$ $(1 \le j \le m)$, we have

$$\begin{aligned}
&\mathrm{Ch}(\wedge_{\mathbf{C}}^j \left(T_{q_i}M \otimes \mathbb{C}\right), g^{r_i k}) = \mathrm{Tr}\left(g^{r_i k} \left| \wedge_{\mathbf{C}}^j \left(T_{q_i}M \otimes \mathbb{C}\right)\right.\right) \\
&= \sum_{1 \le i_1 < \cdots < i_j \le 2m} \lambda_{i_1} \cdots \lambda_{i_j} = \sigma_j(\xi_{p_i}^{k\tau_{i1}}, \xi_{p_i}^{-k\tau_{i1}}, \cdots, \xi_{p_i}^{k\tau_{im}}, \xi_{p_i}^{-k\tau_{im}}) \\
&= \frac{1}{j!} \lim_{t \to 0} \left(\frac{d}{dt}\right)^j (1 + t\xi_{p_i}^{k\tau_{i1}})(1 + t\xi_{p_i}^{-k\tau_{i1}}) \cdots (1 + t\xi_{p_i}^{k\tau_{im}})(1 + t\xi_{p_i}^{-k\tau_{im}}) \\
&= \frac{1}{j!} \lim_{t \to 0} \left(\frac{d}{dt}\right)^j \prod_{k=1}^m (t^2 + \rho_{p_i}(k\tau_{ik})t + 1) \\
&= \sum_{n_1+\cdots+n_m=j} \prod_{k=1}^m \lim_{t \to 0} \frac{1}{n_k!} \left(\frac{d}{dt}\right)^{n_k} (t^2 + \rho_{p_i}(k\tau_{ik})t + 1)
\end{aligned}$$

$$(8) \qquad = \sum_{n_1+\cdots+n_m=j} \prod_{k=1}^m \chi(n_k)$$

where $n_1, \cdots, n_m$ are integers such that $0 \le n_1, \cdots, n_m \le 2$ and

$$\chi(n_k) = \begin{cases} \rho_{p_i}(k\tau_{ik}) & \text{if } n_k = 1 \\ 1 & \text{if } n_k = 0, 2 \end{cases} .$$

Let r, $j-2r$ be the numbers of 2's and 1's in $n_1, \cdots, n_m$. If $2r < j$ and $n_{s_1} = \cdots = n_{s_{j-2r}} = 1$, we have

$$(8) = \sum_{r=0}^{\left[\frac{j}{2}\right]} \sum_{1 \le s_1 < \cdots < s_{j-2r} \le m} \binom{m-j+2r}{r} \rho_{p_i}(k\tau_{is_1}) \cdots \rho_{p_i}(k\tau_{is_{j-2r}})$$

because the number of $(t_1, \cdots, t_r)$ such that $1 \le t_1 < \cdots < t_r \le m$ and that $n_{t_1} = \cdots = n_{t_r} = 2$ is $\binom{m-j+2r}{r}$. Hence we have

$$(8) = \sum_{r=0}^{\left[\frac{j-1}{2}\right]} \binom{m-j+2r}{r} \sigma_{j-2r}(\rho_{p_i}(k\tau_{i1}), \cdots, \rho_{p_i}(k\tau_{im})) + \varepsilon_j \binom{m}{\frac{j}{2}} = C_{ij}^k ,$$

because $\prod_{k=1}^m \chi(n_k) = 1$ if $2r = j$, $\left[\frac{j-1}{2}\right] = \left[\frac{j}{2}\right] - 1$ if j is even and $\left[\frac{j-1}{2}\right] = \left[\frac{j}{2}\right]$ if j is odd. Therefore we have

$$\mathrm{Ch}(E_{q_i}^{\ell_1 \cdots \ell_m}, g^{r_i k}) = \prod_{j=1}^m \left(C_{ij}^k\right)^{\ell_j} . \qquad \square$$

EXAMPLE 2.10. When $m = 1$, it follows from Proposition 2.9 that

$$\mathrm{Ch}(E_{q_i}^{\ell_1}, g^{r_i}) = \left(C_{i1}^1\right)^{\ell_1} = \left(\sigma_1(\rho_{p_i}(\tau_{i1}))\right)^{\ell_1} = \left(\rho_{p_i}(\tau_{i1})\right)^{\ell_1} .$$

Next assume that $m = 2$ and that $(\ell_1, \ell_2) \neq (0, 0)$. Let σ_k the k-th elementary symmetric polynomial in $\rho_{p_i}(\tau_{i1})$, $\rho_{p_i}(\tau_{i2})$. Then it follows from Proposition 2.9 that

$$\begin{aligned}\mathrm{Ch}(E_{q_i}^{\ell_1,\ell_2}, g^{r_i}) &= \left(C_{i1}^1\right)^{\ell_1} \left(C_{i2}^1\right)^{\ell_2} = \sigma_1^{\ell_1} (\sigma_2 + 2)^{\ell_2} = \sum_{\kappa_2=0}^{\ell_2} \binom{\ell_2}{\kappa_2} 2^{\ell_2-\kappa_2} \sigma_1^{\ell_1} \sigma_2^{\kappa_2} \\ &= \sum_{\kappa_2=0}^{\ell_2} \binom{\ell_2}{\kappa_2} 2^{\ell_2-\kappa_2} \left(\rho_{p_i}(\tau_{i1}) + \rho_{p_i}(\tau_{i2})\right)^{\ell_1} \left(\rho_{p_i}(\tau_{i1})\rho_{p_i}(\tau_{i2})\right)^{\kappa_2} \\ &= \sum_{\mu_1=0}^{\ell_1} \sum_{\kappa_2=0}^{\ell_2} \binom{\ell_1}{\mu_1} \binom{\ell_2}{\kappa_2} 2^{\ell_2-\kappa_2} \rho_{p_i}(\tau_{i1})^{\mu_1+\kappa_2} \rho_{p_i}(\tau_{i2})^{\ell_1-\mu_1+\kappa_2} ,\end{aligned}$$

and using the property (7) we can express the value above as a summation of ρ_{p_i}'s. When $(\ell_1, \ell_2) = (0, 0)$, we have

$$\mathrm{Ch}(E_{q_i}^{0,0}, g^{r_i}) = \sigma_1^0 (\sigma_2 + 2)^0 = 1 \implies 2\mathrm{Ch}(E_{q_i}^{0,0}, g^{r_i}) = \rho(0) .$$

REMARK 2.11. Let $h(\ell_1, \cdots, \ell_m)$ be a non-negative integer and $\theta(\ell_1, \cdots, \ell_m)$ a natural number defined by

$$h(\ell_1, \cdots, \ell_m) = \prod_{j=1}^m \left(\varepsilon_j \binom{m}{\frac{j}{2}}\right)^{\ell_j} , \quad \theta(\ell_1, \cdots, \ell_m) = \begin{cases} 2 & (h(\ell_1, \cdots, \ell_m) \text{ is odd}) \\ 1 & (h(\ell_1, \cdots, \ell_m) \text{ is even}) \end{cases} .$$

It follows from Proposition 2.9 that $\mathrm{Ch}(E_{q_i}^{\ell_1, \cdots, \ell_m}, g^{r_i})$ is expressed as a summation of products of $\rho_{p_i}(\mu_i)$'s with integer coefficients and $h(\ell_1, \cdots, \ell_m)$ where μ_i's are non-negative integers. Hence $\theta(\ell_1, \cdots, \ell_m)\mathrm{Ch}(E_{q_i}^{\ell_1, \cdots, \ell_m}, g^{r_i})$ is expressed as a summation of $\rho_{p_i}(\mu)$'s with integer coefficients because $2 = \rho(0)$.

If $\theta\mathrm{Ch}(E_{q_i}^{\ell_1 \cdots \ell_m}, g^{r_i}) = \sum_s n_s \rho_{p_i}(\mu_{is})$ and hence $\theta\mathrm{Ch}(E_{q_i}^{\ell_1 \cdots \ell_m}, (g^{r_i})^{kz}) = \sum_s n_s \rho_{p_i}(kz\mu_{is})$ where $\theta = 1$ or 2, n_s's are natural numbers, z is a natural number which is prime to p and μ_{is}'s are non-negative integers, it follows from (3), (4) and (5) that

$$(9) \qquad \begin{aligned}\theta I_{D_{\ell_1, \cdots, \ell_m}}(g^z) &\equiv \frac{p-1}{p} 2^{m-1} \theta \mathrm{Ch}(E^{\ell_1, \cdots, \ell_m}) \widehat{\mathbf{L}}(M)[M] \\ &- \frac{1}{p} \sum_{i=1}^b r_i \sum_s n_s \sum_{\kappa=1}^{p_i-1} \frac{\rho_{p_i}(\kappa z \mu_{is})}{1 - \xi_{p_i}^{-\kappa}} \prod_{j=1}^m \left(-\sqrt{-1} \cot \frac{\pi \kappa z \tau_{ij}}{p_i}\right) \pmod{\mathbb{Z}} .\end{aligned}$$

So we can calculate $\theta I_{D_{\ell_1,\cdots,\ell_m}}(g^z)$ precisely if we can express the value

$$\sum_{k=1}^{p_i-1} \frac{\rho_{p_i}(kz\mu_{is})}{1-\xi_{p_i}^{-k}} \prod_{j=1}^{m} \left(-\sqrt{-1}\cot\frac{\pi kz\tau_{ij}}{p_i}\right)$$

as a rational number (see Theorem 3.6).

Proposition 2.12. *Let p be a natural number which is greater than* 1 *and $R(x)$ a rational function with real coefficients. Then $\sum_{k=1}^{p-1} R(\xi_p^k)$ is a real number.*

Proof. Since R is a rational function with real coefficients, we have

$$\overline{\sum_{k=1}^{p-1} R(\xi_p^k)} = \sum_{k=1}^{p-1} R(\overline{\xi_p^k}) = \sum_{k=1}^{p-1} R(\xi_p^{p-k}) = \sum_{p-k=1}^{p-1} R(\xi_p^{p-k}) = \sum_{k=1}^{p-1} R(\xi_p^k). \qquad \square$$

Proposition 2.13. *Let p be a natural number which is greater than* 1*, a an integer and b_j $(1 \le j \le 2\gamma)$ natural numbers which are prime to p. Then we have*

$$(-1)^{\gamma+1}\sum_{k=1}^{p-1} \xi_p^{-ka}\frac{\xi_p^k+1}{\xi_p^k-1}\prod_{j=1}^{2\gamma}\frac{\xi_p^{b_jk}+1}{\xi_p^{b_jk}-1} = \sum_{k=1}^{p-1}\sin\frac{2\pi ak}{p}\cot\frac{\pi k}{p}\prod_{j=1}^{2\gamma}\cot\frac{\pi b_jk}{p},$$

$$(-1)^{\gamma}\sum_{k=1}^{p-1} \xi_p^{-ka}\frac{\xi_p^k+1}{\xi_p^k-1}\prod_{j=1}^{2\gamma-1}\frac{\xi_p^{b_j\overline{b_{2\gamma}}k}+1}{\xi_p^{b_j\overline{b_{2\gamma}}k}-1} = \sum_{k=1}^{p-1}\cos\frac{2\pi ak}{p}\cot\frac{\pi k}{p}\prod_{j=1}^{2\gamma-1}\cot\frac{\pi b_j\overline{b_{2\gamma}}k}{p}.$$

Proof. It follows from Proposition 2.12 that the left-hand sides of the equalities above are real numbers. Hence it follows from the equalities below

$$(10)\qquad \frac{\xi_p^{k\tau}+1}{\xi_p^{k\tau}-1} = \frac{e^{2\pi ik\tau/p}+1}{e^{2\pi ik\tau/p}-1} = \frac{i^{-1}\frac{e^{\pi ik\tau/p}+e^{-\pi ik\tau/p}}{2}}{\frac{e^{\pi ik\tau/p}-e^{-\pi ik\tau/p}}{2i}} = -\sqrt{-1}\cot\frac{\pi k\tau}{p}$$

$$\frac{2}{1-\xi_p^{-k}} = \frac{2\xi_p^k}{\xi_p^k-1} = 1+\frac{\xi_p^k+1}{\xi_p^k-1} = 1-\sqrt{-1}\cot\frac{\pi k}{p},$$

that

$$\begin{aligned}
&(-1)^{\gamma+1}\sum_{k=1}^{p-1} \xi_p^{-ka}\frac{\xi_p^k+1}{\xi_p^k-1}\prod_{j=1}^{2\gamma}\frac{\xi_p^{b_jk}+1}{\xi_p^{b_jk}-1}\\
&= (-1)^{\gamma+1}\sum_{k=1}^{p-1}\left(\cos\frac{2\pi ak}{p}-\sqrt{-1}\sin\frac{2\pi ak}{p}\right)\left(-\sqrt{-1}\cot\frac{\pi k}{p}\right)\prod_{j=1}^{2\gamma}\left(-\sqrt{-1}\cot\frac{\pi b_jk}{p}\right)\\
&= \sum_{k=1}^{p-1}\sin\frac{2\pi ak}{p}\cot\frac{\pi k}{p}\prod_{j=1}^{2\gamma}\cot\frac{\pi b_jk}{p},
\end{aligned}$$

$$\begin{aligned}
&(-1)^{\gamma}\sum_{k=1}^{p-1} \xi_p^{-ka}\frac{\xi_p^k+1}{\xi_p^k-1}\prod_{j=1}^{2\gamma-1}\frac{\xi_p^{b_j\overline{b_{2\gamma}}k}+1}{\xi_p^{b_j\overline{b_{2\gamma}}k}-1}\\
&= (-1)^{\gamma}\sum_{k=1}^{p-1}\left(\cos\frac{2\pi ak}{p}-\sqrt{-1}\sin\frac{2\pi ak}{p}\right)\left(-\sqrt{-1}\cot\frac{\pi k}{p}\right)\prod_{j=1}^{2\gamma-1}\left(-\sqrt{-1}\cot\frac{\pi b_j\overline{b_{2\gamma}}k}{p}\right)
\end{aligned}$$

$$= \sum_{k=1}^{p-1} \cos\frac{2\pi ak}{p} \cot\frac{\pi k}{p} \prod_{j=1}^{2\gamma-1} \cot\frac{\pi b_j \overline{b_{2\gamma}} k}{p}. \qquad \square$$

DEFINITION 2.14. Let p be a natural number which is greater than 1 and $b_1, \cdots, b_{2n}$ natural numbers which are prime to p. Then a rational number $\zeta_j(p; b_1, \cdots, b_{2j})$ $(0 \le j \le n)$ is defined by

$$\begin{aligned}\zeta_j(p; b_1, \cdots, b_{2j}) &= (-1)^j 2^{2j} p \sum_{\substack{1\le k_1,\cdots,k_{2j}<p \\ p|(b_1k_1+\cdots+b_{2j}k_{2j})}} \left(\left(\frac{k_1}{p}\right)\right) \cdots \left(\left(\frac{k_{2j}}{p}\right)\right) \\ &= \frac{(-1)^j}{p^{2j-1}} \sum_{\substack{1\le k_1,\cdots,k_{2j}<p \\ p|(b_1k_1+\cdots+b_{2j}k_{2j})}} (2k_1 - p)\cdots(2k_{2j} - p)\end{aligned}$$

for $1 \le j \le n$ and $\zeta_0 = -1$. Set

$$c_k = \langle\, \overline{b_{2k-1}} b_{2k} \,\rangle \ (1 \le k \le n), \qquad d_k = \langle\, \overline{b_{2k}} b_{2k+1} \,\rangle \ (1 \le k \le n-1)$$

and let a be a non-negative integer. Then an integer $\eta_j(p; a, b_{2n-2j+1}, \cdots, b_{2n})$ $(0 \le j \le n)$ is defined by $\eta_0 = 1$, $\eta_j(p; 0, b_{2n-2j+1}, \cdots, b_{2n}) = 0$ $(1 \le j \le n)$ and if $a > 0$

$$\begin{aligned}&\eta_j(p; a, b_{2n-2j+1}, \cdots, b_{2n}) \\ &= \sum_{t_{n-1}=0}^{ac_n} \psi(a, c_n; t_{n-1}) \sum_{t_{n-2}=0}^{c_{n-1}d_{n-1}t_{n-1}} \psi(d_{n-1}t_{n-1}, c_{n-1}; t_{n-2}) \\ &\qquad \cdots \sum_{t_{n-j}=0}^{c_{n-j+1}d_{n-j+1}t_{n-j+1}} \psi(d_{n-j+1}t_{n-j+1}, c_{n-j+1}; t_{n-j})\end{aligned}$$

for $2 \le j \le n$ and

$$\eta_1(p; a, b_{2n-1}, b_{2n}) = \sum_{t_{n-1}=0}^{ac_n} \psi(a, c_n; t_{n-1})$$

where ψ is an integer defined by

$$\psi(a, c; t) = \begin{cases} -1 & (t = ac) \\ 4\left(\frac{t}{c} - a\right) & (0 < t < ac \text{ and } t \text{ is a multiple of } c) \\ 4\left(\left[\frac{t}{c}\right] - a\right) + 2 & (0 < t < ac \text{ and } t \text{ is not a multiple of } c) \\ -2a + 1 & (t = 0) \end{cases}.$$

REMARK 2.15. For $j > 0$ it follows from Theorem 1.3 that

$$\zeta_j(p; b_1, \cdots, b_{2j}) = \sum_{k=1}^{p-1} \cot\frac{\pi b_1 k}{p} \cdots \cot\frac{\pi b_{2j} k}{p}.$$

DEFINITION 2.16. For a non-negative integer n and narural numbers a, c, p, q, integers $\phi_n(a, c)$ and $\beta(n, p, q)$ are defined by

$$\phi_n(a,c) = \sum_{t=0}^{ac} \psi(a,c;t)t^n\,, \qquad \beta(n,p,q) = \sum_{k=p+1}^{q} k^n\,.$$

Proposition 2.17. *For narural numbers a, c, we have*

$$\phi_0(a,c) = -2a^2c\,, \tag{11}$$

$$\phi_n(a,c) = 4\sum_{j=1}^{a-1} j\beta(n, jc-1, (j+1)c-1) - 2(2a-1)\beta(n,0,ac-1) - c^n\{2\beta(n,0,a-1) + a^n\} \ (n \geq 1)\,. \tag{12}$$

Proof. Since there are $a-1$ t's $\{c, 2c, \cdots, (a-1)c\}$ which satisfy the conditions that $0 < t < ac$ and that t is a multiple of c, we have

$$\begin{aligned}
\phi_0(a,c) &= \sum_{t=0}^{ac} \psi(a,c;t) = -2a + 1 - 1 + \sum_{t=1}^{ac-1}\left\{4\left(\left[\frac{t}{c}\right] - a\right) + 2\right\} - 2(a-1) \\
&= 2ac(1-2a) + 4\sum_{t=1}^{ac-1}\left[\frac{t}{c}\right] = 2ac(1-2a) + 4\left(\sum_{j=1}^{a-1}\sum_{t=jc}^{(j+1)c-1} j\right) \\
&= 2ac(1-2a) + 4\left(\sum_{j=1}^{a-1} cj\right) = 2ac(1-2a) + 4c\frac{a(a-1)}{2} = -2a^2c\,.
\end{aligned}$$

For $n \geq 1$ we have

$$\begin{aligned}
\phi_n(a,c) &= \sum_{t=0}^{ac} \psi(a,c;t)t^n \\
&= (-2a+1)\cdot 0^n + (-1)(ac)^n + \sum_{t=1}^{ac-1}\left\{4\left(\left[\frac{t}{c}\right] - a\right) + 2\right\}t^n - 2\sum_{j=1}^{a-1}(jc)^n \\
&= -a^nc^n + 4\sum_{t=1}^{ac-1} t^n\left[\frac{t}{c}\right] + (2-4a)\beta(n,0,ac-1) - 2c^n\beta(n,0,a-1) \\
&= 4\left(\sum_{j=1}^{a-1}\sum_{t=jc}^{(j+1)c-1} jt^n\right) - 2(2a-1)\beta(n,0,ac-1) - c^n\{2\beta(n,0,a-1) + a^n\} \\
&= 4\sum_{j=1}^{a-1} j\beta(n, jc-1, (j+1)c-1) - 2(2a-1)\beta(n,0,ac-1) \\
&\quad - c^n\{2\beta(n,0,a-1) + a^n\}\,. \qquad \square
\end{aligned}$$

EXAMPLE 2.18. Using the proposition above, we have

$$\eta_1(p; a, b_{2n-1}, b_{2n}) = \phi_0(a, c_n) = -2a^2c_n\,,$$

$$\phi_2(a,c)$$

$$= 4\sum_{j=1}^{a-1} j\beta(2, jc-1, (j+1)c-1) - 2(2a-1)\beta(2,0,ac-1) - c^2\left\{2\beta(2,0,a-1) + a^2\right\}$$

$$= 4\sum_{j=1}^{a-1} j \sum_{k=jc}^{(j+1)c-1} k^2 - 2(2a-1)\sum_{k=1}^{ac-1} k^2 - c^2\left\{2\sum_{k=1}^{a-1} k^2 + a^2\right\}$$
$$= -\frac{1}{3}a^2 c_n\left(1 + c_n^2\right) - \frac{1}{3}a^4 c_n^3,$$

and hence it follows that

$$\begin{aligned}
\eta_2 &= \eta_2(p; a, b_{2n-3}, \cdots, b_{2n}) \\
&= \sum_{t_{n-1}=0}^{ac_n} \psi(a, c_n; t_{n-1}) \sum_{t_{n-2}=0}^{c_{n-1}d_{n-1}t_{n-1}} \psi(d_{n-1}t_{n-1}, c_{n-1}; t_{n-2}) \\
&= \sum_{t_{n-1}=0}^{ac_n} \psi(a, c_n; t_{n-1})\phi_0(d_{n-1}t_{n-1}, c_{n-1}) \\
&= \sum_{t_{n-1}=0}^{ac_n} \psi(a, c_n; t_{n-1})\left(-2c_{n-1}d_{n-1}^2 t_{n-1}^2\right) = \left(-2c_{n-1}d_{n-1}^2\right)\phi_2(a, c_n) \\
&= \left(-2c_{n-1}d_{n-1}^2\right)\left(-\frac{1}{3}a^2 c_n\left(1 + c_n^2\right) - \frac{1}{3}a^4 c_n^3\right) = \frac{2}{3}a^2 c_{n-1}d_{n-1}^2 c_n(a^2c_n^2 + c_n^2 + 1)\,.
\end{aligned}$$

3. Main results

Theorem 3.1. *Let p be a natural number which is greater than 1, n a natural number, μ a non-negative integer and $b_1, \cdots, b_{2n}$ natural numbers which are prime to p. Then we have*

$$\sum_{k=1}^{p-1} \cos\frac{2\pi\mu k}{p} \prod_{j=1}^{2n} \cot\frac{\pi b_j k}{p} \equiv \sum_{k=0}^{n} \zeta_k(p; b_1, \cdots, b_{2k})\eta_{n-k}(p; a, b_{2k+1}, \cdots, b_{2n}) \pmod{p}$$

where $a = \mu\overline{b_{2n}}$.

Proof. When $\mu = 0$, as we see in Remark 2.15, the equality above holds. So we assume that $\mu > 0$ hereafter. Set

$$a_k = a_k(t_k) = \begin{cases} 0 & (k = 0) \\ d_k t_k (= \langle\, \overline{b_{2k}} b_{2k+1}\,\rangle t_k) & (1 \le k \le n-1) \\ a & (k = n) \end{cases}.$$

For $\gamma \ge 0$, set

$$F_\gamma(a, b_1, \cdots, b_{2\gamma}) = \begin{cases} (-1)^{\gamma+1}\displaystyle\sum_{k=1}^{p-1} \xi_p^{-ak}\frac{\xi_p^k + 1}{\xi_p^k - 1}\prod_{j=1}^{2\gamma}\frac{\xi_p^{b_j k} + 1}{\xi_p^{b_j k} - 1} & (\gamma \ge 1) \\ -\displaystyle\sum_{k=1}^{p-1} \xi_p^{-ak}\frac{\xi_p^k + 1}{\xi_p^k - 1} & (\gamma = 0) \end{cases},$$

$$G_\gamma(a, b_1, \cdots, b_{2\gamma}) = \begin{cases} (-1)^{\gamma}\displaystyle\sum_{k=1}^{p-1} \xi_p^{-ak}\frac{\xi_p^k + 1}{\xi_p^k - 1}\prod_{j=1}^{2\gamma-1}\frac{\xi_p^{b_j\overline{b_{2\gamma}}k} + 1}{\xi_p^{b_j\overline{b_{2\gamma}}k} - 1} & (\gamma \ge 1) \\ \displaystyle\sum_{k=1}^{p-1} \xi_p^{-ak} & (\gamma = 0) \end{cases},$$

$$f_\gamma(a,b_1,\cdots,b_{2\gamma}) = \begin{cases} \displaystyle\sum_{k=1}^{p-1} \sin\frac{2\pi ak}{p}\cot\frac{\pi k}{p}\prod_{j=1}^{2\gamma}\cot\frac{\pi b_j k}{p} & (\gamma \geq 1) \\ \displaystyle\sum_{k=1}^{p-1} \sin\frac{2\pi ak}{p}\cot\frac{\pi k}{p} & (\gamma = 0) \end{cases},$$

$$g_\gamma(a,b_1,\cdots,b_{2\gamma}) = \begin{cases} \displaystyle\sum_{k=1}^{p-1} \cos\frac{2\pi ak}{p}\cot\frac{\pi k}{p}\prod_{j=1}^{2\gamma-1}\cot\frac{\pi b_j\overline{b_{2\gamma}}k}{p} & (\gamma \geq 1) \\ \displaystyle\sum_{k=1}^{p-1} \cos\frac{2\pi ak}{p} & (\gamma = 0) \end{cases}.$$

Note that the values F_γ, G_γ, f_γ, g_γ are invariant under the substitution $a \to a+p$, $b_j \to b_j+p$, and that for any integer a we have

$$G_0(a) = g_0(a) = \begin{cases} p-1 & (\text{if } a \text{ is a multiple of } p) \\ -1 & (\text{if } a \text{ is not a multiple of } p) \end{cases} \equiv -1 \pmod{p}.$$

Note moreover that for a natural number λ which is prime to p, we have

$$g_\gamma(a,\lambda b_1,\cdots,\lambda b_{2\gamma}) = g_\gamma(a,b_1,\cdots,b_{2\gamma}) \tag{13}$$

because $\lambda b_j\overline{\lambda b_{2\gamma}} \equiv \lambda\overline{\lambda}b_j\overline{b_{2\gamma}} \equiv b_j\overline{b_{2\gamma}} \pmod{p}$. As we see in Proposition 2.13, F_γ, G_γ are real numbers and the following equalities hold for $\gamma \geq 1$:

$$F_\gamma(a,b_1,\cdots,b_{2\gamma}) = f_\gamma(a,b_1,\cdots,b_{2\gamma}),$$
$$G_\gamma(a,b_1,\cdots,b_{2\gamma}) = g_\gamma(a,b_1,\cdots,b_{2\gamma}).$$

Moreover it follows from Proposition 2.12 and (10) that

$$\begin{aligned} F_0(a) &= -\sum_{k=1}^{p-1}\xi_p^{-ak}\frac{\xi_p^k+1}{\xi_p^k-1} = -\sum_{k=1}^{p-1}\left(\cos\frac{2\pi ak}{p} - \sqrt{-1}\sin\frac{2\pi ak}{p}\right)\left(-\sqrt{-1}\cot\frac{\pi k}{p}\right) \\ &= \sum_{k=1}^{p-1}\sin\frac{2\pi ak}{p}\cot\frac{\pi k}{p} = f_0(a), \end{aligned}$$

and similarly that $G_0(a) = g_0(a)$.

Lemma 3.2. *For $0 \leq k \leq n$ the following equality holds:*

$$g_k(0,b_1,\cdots,b_{2k}) \equiv \zeta_k(p;b_1,\cdots,b_{2k}) \pmod{p}.$$

Proof. When $k=0$, we have

$$g_0(0) = \sum_{k=1}^{p-1}\cos 0 = p-1 \equiv \zeta_0 \pmod{p}.$$

For $1 \leq k \leq n$ since the map $\overline{b_{2k}}k \to k$ gives a bijection of $\mathbb{Z}_p$, we have

$$g_k(0,b_1,\cdots,b_{2k}) = \sum_{k=1}^{p-1}\cot\frac{\pi b_{2k}\overline{b_{2k}}k}{p}\prod_{j=1}^{2k-1}\cot\frac{\pi b_j\overline{b_{2k}}k}{p} = \sum_{k=1}^{p-1}\prod_{j=1}^{2k}\cot\frac{\pi b_j k}{p},$$

which is equal to $\zeta_k(p;b_1,\cdots,b_{2k})$ as we see in Remark 2.15. □

Lemma 3.3. *We have*

$$f_0(a) = \sum_{k=1}^{p-1} \sin\frac{2\pi ak}{p}\cot\frac{\pi k}{p} \equiv -2a \pmod{p}.$$

Proof. It follows from the Eisenstein's formula

$$\sum_{k=1}^{p-1} \sin\frac{2\pi ak}{p}\cot\frac{\pi k}{p} = -2p\left(\left(\frac{a}{p}\right)\right). \tag{14}$$

(see [9] p.103 (22)) that

$$f_0(a) = -2p\left(\left(\frac{a}{p}\right)\right) = -2p\left(\frac{a}{p} - \left[\frac{a}{p}\right] - \frac{1}{2}\right) \equiv -2a \pmod{p}. \qquad \square$$

The next Lemma is the key lemma to prove the theorem.

Lemma 3.4. *For $\gamma \geq 1$ we have*

$$g_\gamma(a, b_1, \cdots, b_{2\gamma}) = \sum_{t_{\gamma-1}=0}^{ac_\gamma} \psi(a, c_\gamma; t_{\gamma-1}) g_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1, \cdots, b_{2\gamma-2}) + g_\gamma(0, b_1, \cdots, b_{2\gamma}).$$

Proof. Since the map $b_{2\gamma}k \to k$ gives a bijection of $\mathbb{Z}_p$, for $\gamma \geq 1$ we have

$$\begin{aligned}
&F_\gamma(a+1, b_1, \cdots, b_{2\gamma}) - F_\gamma(a, b_1, \cdots, b_{2\gamma})\\
&= (-1)^{\gamma+1} \sum_{k=1}^{p-1} \xi_p^{-k(a+1)} \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma} \frac{\xi_p^{b_jk} + 1}{\xi_p^{b_jk} - 1}(1 - \xi_p^k)\\
&= (-1)^{\gamma} \sum_{k=1}^{p-1} \left(\xi_p^{-ak} + \xi_p^{-k(a+1)}\right) \prod_{j=1}^{2\gamma} \frac{\xi_p^{b_jk} + 1}{\xi_p^{b_jk} - 1}\\
&= (-1)^{\gamma} \sum_{k=1}^{p-1} \left(\xi_p^{-ka\overline{b_{2\gamma}}} + \xi_p^{-k(a+1)\overline{b_{2\gamma}}}\right) \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma-1} \frac{\xi_p^{b_j\overline{b_{2\gamma}}k} + 1}{\xi_p^{b_j\overline{b_{2\gamma}}k} - 1}\\
&= G_\gamma(a\overline{b_{2\gamma}}, b_1, \cdots, b_{2\gamma}) + G_\gamma((a+1)\overline{b_{2\gamma}}, b_1, \cdots, b_{2\gamma}),
\end{aligned}$$

and

$$F_0(a+1) - F_0(a) = (-1)^0 \sum_{k=1}^{p-1} \left(\xi_p^{-ka\overline{b_{2\gamma}}} + \xi_p^{-k(a+1)\overline{b_{2\gamma}}}\right) = G_0(a\overline{b_{2\gamma}}) + G_0((a+1)\overline{b_{2\gamma}}),$$

$$F_\gamma(0, b_1, \cdots, b_{2\gamma}) = f_\gamma(0, b_1, \cdots, b_{2\gamma}) = 0.$$

Hence it follows that

$$\begin{aligned}
&F_\gamma(a, b_1, \cdots, b_{2\gamma}) = F_\gamma(a, b_1, \cdots, b_{2\gamma}) - F_\gamma(0, b_1, \cdots, b_{2\gamma})\\
&= \sum_{t=0}^{a-1} \left\{F_\gamma(t+1, b_1, \cdots, b_{2\gamma}) - F_\gamma(t, b_1, \cdots, b_{2\gamma})\right\}\\
&= \sum_{t=0}^{a-1} \left\{G_\gamma(t\overline{b_{2\gamma}}, b_1, \cdots, b_{2\gamma}) + G_\gamma((t+1)\overline{b_{2\gamma}}, b_1, \cdots, b_{2\gamma})\right\}
\end{aligned}$$

$$= 2\sum_{t=0}^{a} G_\gamma(t\overline{b_{2\gamma}}, b_1, \cdots, b_{2\gamma}) - G_\gamma(a\overline{b_{2\gamma}}, b_1, \cdots, b_{2\gamma}) - G_\gamma(0, b_1, \cdots, b_{2\gamma})\,. \tag{15}$$

Moreover since the map $b_{2\gamma-1}\overline{b_{2\gamma}}k \to k$ gives a bijection of $\mathbb{Z}_p$, for $\gamma \geq 1$ we have

$$\begin{aligned}
&G_\gamma(a+1, b_1, \cdots, b_{2\gamma}) - G_\gamma(a, b_1, \cdots, b_{2\gamma})\\
&= (-1)^\gamma \sum_{k=1}^{p-1} \xi_p^{-k(a+1)} \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma-1} \frac{\xi_p^{b_j\overline{b_{2\gamma}}k} + 1}{\xi_p^{b_j\overline{b_{2\gamma}}k} - 1}(1 - \xi_p^k)\\
&= (-1)^{\gamma+1} \sum_{k=1}^{p-1} \left(\xi_p^{-ka} + \xi_p^{-k(a+1)}\right) \frac{\xi_p^{b_{2\gamma-1}\overline{b_{2\gamma}}k} + 1}{\xi_p^{b_{2\gamma-1}\overline{b_{2\gamma}}k} - 1} \prod_{j=1}^{2\gamma-2} \frac{\xi_p^{b_j\overline{b_{2\gamma}}k} + 1}{\xi_p^{b_j\overline{b_{2\gamma}}k} - 1}\\
&= (-1)^{\gamma+1} \sum_{k=1}^{p-1} \left(\xi_p^{-ka\overline{b_{2\gamma-1}}b_{2\gamma}} + \xi_p^{-k(a+1)\overline{b_{2\gamma-1}}b_{2\gamma}}\right) \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma-2} \frac{\xi_p^{b_j\overline{b_{2\gamma-1}}k} + 1}{\xi_p^{b_j\overline{b_{2\gamma-1}}k} - 1}\\
&= -\Big\{F_{\gamma-1}(a\overline{b_{2\gamma-1}}b_{2\gamma}, b_1\overline{b_{2\gamma-1}}, \cdots, b_{2\gamma-2}\overline{b_{2\gamma-1}})\\
&\qquad + F_{\gamma-1}((a+1)\overline{b_{2\gamma-1}}b_{2\gamma}, b_1\overline{b_{2\gamma-1}}, \cdots, b_{2\gamma-2}\overline{b_{2\gamma-1}})\Big\}\\
&= -\left\{F_{\gamma-1}(a\overline{b_{2\gamma-1}}b_{2\gamma}, b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) + F_{\gamma-1}((a+1)\overline{b_{2\gamma-1}}b_{2\gamma}, b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma)\right\}\,,
\end{aligned}$$

where $b_j^\gamma = b_j\overline{b_{2\gamma-1}}$ $(1 \leq j \leq 2\gamma - 2)$ and hence it follows that

$$\begin{aligned}
&G_\gamma(a, b_1, \cdots, b_{2\gamma}) - G_\gamma(0, b_1, \cdots, b_{2\gamma})\\
&= \sum_{s=0}^{a-1} \left\{G_\gamma(s+1, b_1, \cdots, b_{2\gamma}) - G_\gamma(s, b_1, \cdots, b_{2\gamma})\right\}\\
&= -\sum_{s=0}^{a-1} \left\{F_{\gamma-1}(s\overline{b_{2\gamma-1}}b_{2\gamma}, b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) + F_{\gamma-1}((s+1)\overline{b_{2\gamma-1}}b_{2\gamma}, b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma)\right\}\\
&= -2\sum_{s=0}^{a} F_{\gamma-1}(s\overline{b_{2\gamma-1}}b_{2\gamma}, b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) + F_{\gamma-1}(a\overline{b_{2\gamma-1}}b_{2\gamma}, b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma)
\end{aligned}$$

because $F_{\gamma-1}(0, b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) = 0$. Therefore for $\gamma \geq 1$ it follows from (15) that

$$\begin{aligned}
&G_\gamma(a, b_1, \cdots, b_{2\gamma})\\
&= -2\sum_{s=0}^{a} F_{\gamma-1}(s\overline{b_{2\gamma-1}}b_{2\gamma}, b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma)\\
&\quad + F_{\gamma-1}(a\overline{b_{2\gamma-1}}b_{2\gamma}, b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) + G_\gamma(0, b_1, \cdots, b_{2\gamma})\\
&= -2\sum_{s_{\gamma-1}=0}^{a} F_{\gamma-1}(s_{\gamma-1}c_\gamma, b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) + F_{\gamma-1}(ac_\gamma, b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) \qquad (16)\\
&\quad + G_\gamma(0, b_1, \cdots, b_{2\gamma})\\
&\left(\text{Since } t_{\gamma-1}\overline{b_{2\gamma-2}^\gamma} \equiv a_{\gamma-1}(t_{\gamma-1})\,,\ s_{\gamma-1}c_\gamma\overline{b_{2\gamma-2}^\gamma} \equiv a_{\gamma-1}(s_{\gamma-1}c_\gamma) \pmod{p}\,.\right)
\end{aligned}$$

$$
\begin{aligned}
&= -2 \sum_{s_{\gamma-1}=0}^{a} \left\{ \begin{array}{l} 2 \sum_{t_{\gamma-1}=0}^{s_{\gamma-1}c_\gamma} G_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) \\ -G_{\gamma-1}(a_{\gamma-1}(s_{\gamma-1}c_\gamma), b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) - G_{\gamma-1}(0, b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) \end{array} \right\} \\
&\quad + 2 \sum_{t_{\gamma-1}=0}^{ac_\gamma} G_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) \\
&\quad - G_{\gamma-1}(a_{\gamma-1}(ac_\gamma), b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) - G_{\gamma-1}(0, b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) \\
&\quad + G_\gamma(0, b_1, \cdots, b_{2\gamma})
\end{aligned}
$$

$$
\begin{aligned}
&= -4 \sum_{s_{\gamma-1}=0}^{a} \sum_{t_{\gamma-1}=0}^{s_{\gamma-1}c_\gamma} G_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) \\
&\quad + 2 \sum_{s_{\gamma-1}=0}^{a} G_{\gamma-1}(a_{\gamma-1}(s_{\gamma-1}c_\gamma), b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) + 2 \sum_{s_{\gamma-1}=0}^{a} G_{\gamma-1}(0, b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) \\
&\quad + 2 \sum_{t_{\gamma-1}=0}^{ac_\gamma} G_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) - G_{\gamma-1}(a_{\gamma-1}(ac_\gamma), b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) \\
&\quad - G_{\gamma-1}(0, b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) + G_\gamma(0, b_1, \cdots, b_{2\gamma}) \\
&= \sum_{s_{\gamma-1}=0}^{a} \sum_{t_{\gamma-1}=0}^{s_{\gamma-1}c_\gamma} \left(\begin{array}{l} -4 + 2\delta(t_{\gamma-1}, s_{\gamma-1}c_\gamma) \\ +2\delta(t_{\gamma-1}, 0) + 2\delta(s_{\gamma-1}, a) \\ -\delta(s_{\gamma-1}, a)\delta(t_{\gamma-1}, s_{\gamma-1}c_\gamma) \\ -\delta(s_{\gamma-1}, a)\delta(t_{\gamma-1}, 0) \end{array} \right) G_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) \\
&\quad + g_\gamma(0, b_1, \cdots, b_{2\gamma}) \\
&= \sum_{t_{\gamma-1}=0}^{ac_\gamma} \Lambda(t_{\gamma-1}) G_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) + g_\gamma(0, b_1, \cdots, b_{2\gamma})
\end{aligned} \tag{17}
$$

where

$$
\Lambda(t) = \sum_{s_{\gamma-1}=N(t)}^{a} \left(\begin{array}{l} -4 + 2\delta(t, s_{\gamma-1}c_\gamma) \\ +2\delta(t, 0) + 2\delta(s_{\gamma-1}, a) \\ -\delta(s_{\gamma-1}, a)\delta(t, s_{\gamma-1}c_\gamma) \\ -\delta(s_{\gamma-1}, a)\delta(t, 0) \end{array} \right), \quad N(t) = \left\{ \begin{array}{ll} \dfrac{t}{c_\gamma} & \left(\text{if } c_\gamma | t\right) \\ \left[\dfrac{t}{c_\gamma}\right] + 1 & (\text{otherwise}) \end{array} \right. .
$$

Here we have

$$
\begin{aligned}
&\Lambda(ac_\gamma) = -4 + 2 + 0 + 2 - 1 - 0 = -1\,, \\
&\Lambda(t) = -4\left(a - \frac{t}{c_\gamma} + 1\right) + 2 + 0 + 2 - 0 - 0 = 4\left(\frac{t}{c_\gamma} - a\right) \\
&\qquad\qquad \text{if } 0 < t < ac_\gamma \text{ and } t \text{ is a multiple of } c_\gamma, \\
&\Lambda(t) = -4\left(a - \left[\frac{t}{c_\gamma}\right]\right) + 0 + 0 + 2 - 0 - 0 = 4\left(\left[\frac{t}{c_\gamma}\right] - a\right) + 2 \\
&\qquad\qquad \text{if } 0 < t < ac_\gamma \text{ and } t \text{ is not a multiple of } c_\gamma, \\
&\Lambda(0) = -4(a+1) + 2 + 2(a+1) + 2 - 0 - 1 = -2a + 1\,,
\end{aligned}
$$

which implies that $\Lambda(t_{\gamma-1})$ is equal to $\psi(a, c_\gamma; t_{\gamma-1})$. Therefore it follows from (13) that

$$\begin{aligned} g_\gamma(a, b_1, \cdots, b_{2\gamma}) &= G_\gamma(a, b_1, \cdots, b_{2\gamma}) \\ &= \sum_{t_{\gamma-1}=0}^{ac_\gamma} \Lambda(t_{\gamma-1}) G_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1^\gamma, \cdots, b_{2\gamma-2}^\gamma) + g_\gamma(0, b_1, \cdots, b_{2\gamma}) \\ &= \sum_{t_{\gamma-1}=0}^{c_\gamma a} \psi(a, c_\gamma; t_{\gamma-1}) g_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1, \cdots, b_{2\gamma-2}) + g_\gamma(0, b_1, \cdots, b_{2\gamma}) . \qquad \square \end{aligned}$$

For $2 \le \nu \le n-1$ set $\psi_\nu(t_{\nu-1}) = \psi(a_\nu(t_\nu), c_\nu; t_{\nu-1})$.

It follows from the result of Lemma 3.4 that

$$g_n(a, b_1, \cdots, b_{2n}) = \sum_{t_{n-1}=0}^{ac_n} \psi_n(t_{n-1}) g_{n-1}(a_{n-1}(t_{n-1}), b_1, \cdots, b_{2n-2}) + g_n(0, b_1, \cdots, b_{2n})$$

$$\begin{aligned} &= \sum_{t_{n-1}=0}^{ac_n} \psi_n(t_{n-1}) \left\{ \begin{array}{l} \displaystyle\sum_{t_{n-2}=0}^{a_{n-1}c_{n-1}} \psi_{n-1}(t_{n-2}) g_{n-2}(a_{n-2}(t_{n-2}), b_1, \cdots, b_{2n-4}) \\ + g_{n-1}(0, b_1, \cdots, b_{2n-2}) \end{array} \right\} \\ &\quad + g_n(0, b_1, \cdots, b_{2n}) \\ &= \sum_{t_{n-1}=0}^{ac_n} \psi_n(t_{n-1}) \sum_{t_{n-2}=0}^{a_{n-1}c_{n-1}} \psi_{n-1}(t_{n-2}) g_{n-2}(a_{n-2}(t_{n-2}), b_1, \cdots, b_{2n-4}) \\ &\quad + g_{n-1}(0, b_1, \cdots, b_{2n-2}) \sum_{t_{n-1}=0}^{ac_n} \psi_n(t_{n-1}) + g_n(0, b_1, \cdots, b_{2n}) \end{aligned}$$

$$\begin{aligned} (18) \quad &= \sum_{t_{n-1}=0}^{ac_n} \psi_n(t_{n-1}) \cdots \sum_{t_2=0}^{a_3c_3} \psi_3(t_2) \sum_{t_1=0}^{a_2c_2} \psi_2(t_1) g_1(a_1(t_1), b_1, b_2) \\ &\quad + \sum_{k=2}^{n-1} g_k(0, b_1, \cdots, b_{2k}) \sum_{t_{n-1}=0}^{ac_n} \psi_n(t_{n-1}) \cdots \sum_{t_k=0}^{a_{k+1}c_{k+1}} \psi_{k+1}(t_k) \\ &\quad + g_n(0, b_1, \cdots, b_{2n}) . \end{aligned}$$

Here it follows from (16), Example 2.18 and Lemma 3.3 that

$$\begin{aligned} g_1(a_1, b_1, b_2) &= -2 \sum_{s=0}^{a_1} f_0(sc_1) + f_0(a_1c_1) + g_1(0, b_1, b_2) \\ &\equiv -2 \sum_{s=0}^{a_1} (-2sc_1) - 2a_1c_1 + g_1(0, b_1, b_2) \\ &= 2a_1^2 c_1 + g_1(0, b_1, b_2) = -\phi_0(a_1, c_1) + g_1(0, b_1, b_2) \\ &\equiv g_0(0) \sum_{t_0=0}^{a_1c_1} \psi(a_1, c_1; t_0) + g_1(0, b_1, b_2) \pmod p , \end{aligned}$$

and hence we have

$$\begin{aligned} &\sum_{t_{n-1}=0}^{a_nc_n} \psi(a_n, c_n; t_{n-1}) \cdots \sum_{t_1=0}^{a_2c_2} \psi(a_2, c_2; t_1) g_1(a_1, b_1, b_2) \\ &\equiv g_0(0) \sum_{t_{n-1}=0}^{a_nc_n} \psi(a_n, c_n; t_{n-1}) \cdots \sum_{t_1=0}^{a_2c_2} \psi(a_2, c_2; t_1) \sum_{t_0=0}^{a_1c_1} \psi(a_1, c_1; t_0) \end{aligned}$$

$$+ g_1(0, b_1, b_2) \sum_{t_{n-1}=0}^{a_n c_n} \psi(a_n, c_n; t_{n-1}) \cdots \sum_{t_1=0}^{a_2 c_2} \psi(a_2, c_2; t_1)$$
$$\equiv g_0(0)\eta_n(p; a, b_1, \cdots, b_{2n}) + g_1(0, b_1, b_2)\eta_{n-1}(p; a, b_3, \cdots, b_{2n}) \pmod{p}.$$

Moreover we have

$$\sum_{k=2}^{n-1} g_k(0, b_1, \cdots, b_{2k}) \sum_{t_{n-1}=0}^{c_n a} \psi_n(t_{n-1}) \cdots \sum_{t_k=0}^{c_{k+1} a_{k+1}} \psi_{k+1}(t_k)$$
$$= \sum_{k=2}^{n-1} g_k(0, b_1, \cdots, b_{2k})\eta_{n-k}(p; a, b_{2k+1}, \cdots, b_{2n}),$$

and it follows from $\eta_0(a) = 1$ that

$$g_n(0, b_1, \cdots, b_{2n}) = g_n(0, b_1, \cdots, b_{2n})\eta_0(a).$$

Here since $k \to \overline{b_{2n}}k$ gives a bijection of $\mathbb{Z}_p$, it follows from Lemma 3.2 that

$$\sum_{k=1}^{p-1} \cos\frac{2\pi\mu k}{p} \prod_{j=1}^{2n} \cot\frac{\pi b_j k}{p} = \sum_{k=1}^{p-1} \cos\frac{2\pi a k}{p} \prod_{j=1}^{2n} \cot\frac{\pi b_j \overline{b_{2n}} k}{p} = g_n(a, b_1, \cdots, b_{2n})$$
$$\equiv \sum_{k=0}^{n} g_k(0, b_1, \cdots, b_{2k})\eta_{n-k}(p; a, b_{2k+1}, \cdots, b_{2n})$$
$$\equiv \sum_{k=0}^{n} \zeta_k(p; b_1, \cdots, b_{2k})\eta_{n-k}(p; a, b_{2k+1}, \cdots, b_{2n}) \pmod{p}.$$

This completes the proof of Theorem 3.1. □

Corollary 3.5. *Let p be a natural number which is greater than 1, μ a non-negative integer, q a natural number and $b_1, \cdots, b_{2q}$ natural numbers which are prime to p. Then*

$$p^{2q-1} \sum_{k=1}^{p-1} \cos\frac{2\pi\mu k}{p} \prod_{j=1}^{2q} \cot\frac{\pi b_j k}{p}$$

is an integer.

Proof. It follows from the theorem above that

$$\sum_{k=1}^{p-1} \cos\frac{2\pi\mu k}{p} \prod_{j=1}^{2q} \cot\frac{\pi b_j k}{p} = \sum_{k=0}^{q} \zeta_k(p; b_1, \cdots, b_{2k})\, \eta_{q-k}(p; \mu\overline{b_{2q}}, b_{2k+1}, \cdots, b_{2q}) + p\nu$$

where ν is an integer. Here since $\eta_{q-k}(p; \mu\overline{b_{2q}}, b_{2k+1}, \cdots, b_{2q})$ and

$$p^{2q-1}\zeta_k(p; b_1, \cdots, b_{2k}) = (-1)^k p^{2q-2k} \sum_{\substack{1 \le k_1, \cdots, k_{2k} < p \\ p | (b_1 k_1 + \cdots + b_{2k} k_{2k})}} (2k_1 - p) \cdots (2k_{2k} - p)$$

are integers for $0 \le k \le q$, it follows that

$$p^{2q-1} \sum_{k=1}^{p-1} \cos\frac{2\pi\mu k}{p} \prod_{j=1}^{2q} \cot\frac{\pi b_j k}{p}$$

$$= p^{2q-1}\sum_{j=0}^{q}\eta_{q-j}(p;\mu\overline{b_{2q}},b_{2j+1},\cdots,b_{2q})\,\zeta_j(p;b_1,\cdots,b_{2j})+p^{2q}\nu$$

is an integer. □

Theorem 3.6. *Suppose that the total Chern class of $TM\otimes\mathbb{C}$ equals $\prod_{j=1}^{m}(1-x_j^2)$ and that $\theta\mathrm{Ch}(E_{q_i}^{\ell_1\cdots\ell_m},g^{r_i})=\sum_s n_s\rho_{p_i}(\mu_{is})$ where $\theta=\theta(\ell_1\cdots\ell_m)=1$ or 2 (see* Remark 2.11*), n_s's are natural numbers and μ_{is}'s are non-negative integers. Let a_{is}, b_{ij} $(1\le i\le b,\ 1\le j\le m)$, h_m be integers defined by*

$$a_{is}=\begin{cases}\mu_{is}\overline{\tau_{i\,2n}} & (\textit{if } m \textit{ is even})\\ z\mu_{is} & (\textit{if } m \textit{ is odd})\end{cases},\quad b_{ij}=\begin{cases}\bar{z} & (\textit{if } m \textit{ is odd and } j=2n)\\ \tau_{ij} & (\textit{otherwise})\end{cases},$$

$$h_m=\begin{cases}2^{m-1} & (\textit{if } m \textit{ is even})\\ 0 & (\textit{if } m \textit{ is odd})\end{cases}$$

and z a natural number which is prime to p. Then the next equality holds as an element of $\mathbb{R}/\mathbb{Z}$.

$$\theta I_{D_{\ell_1,\cdots,\ell_m}}(g^z)=\frac{p-1}{p}\theta h_m\prod_{j=1}^{m}\sigma_j(e^{x_1},e^{-x_1},\cdots,e^{x_m},e^{-x_m})^{\ell_j}\prod_{j=1}^{m}\frac{x_j/2}{\tanh(x_j/2)}[M]$$
$$+(-1)^{n+1}\frac{1}{p}\sum_{i=1}^{b}r_i\sum_s n_s\sum_{k=0}^{n}\zeta_k(p_i;b_{i1},\cdots,b_{i\,2k})\eta_{n-k}(p;a_{is},b_{i\,2k+1},\cdots,b_{i\,2n}).$$

Proof. It follows from (10) that

$$\sum_{k=1}^{p-1}\frac{\rho_{p_i}(kz\mu_{is})}{1-\xi_p^{-k}}\prod_{j=1}^{m}\left(-\sqrt{-1}\cot\frac{\pi kz\tau_{ij}}{p}\right)=\sum_{k=1}^{p-1}R(\xi_p^k)\tag{19}$$

where

$$R(\xi_p^k)=\frac{\xi_p^{kz\mu_{is}}+\xi_p^{-kz\mu_{is}}}{1-\xi_p^{-k}}\prod_{j=1}^{m}\frac{\xi_p^{kz\tau_{ij}}+1}{\xi_p^{kz\tau_{ij}}-1}.$$

Then Proposition 2.12 implies that the both sides of (19) are real numbers, and the map $kz\to k$ gives a bijection of $\mathbb{Z}_p$ because z is prime to p. Hence we have

$$\sum_{k=1}^{p-1}\frac{\rho_{p_i}(kz\mu_{is})}{1-\xi_p^{-k}}\prod_{j=1}^{m}\left(-\sqrt{-1}\cot\frac{\pi kz\tau_{ij}}{p}\right)$$
$$=\sum_{k=1}^{p-1}\frac{\xi_p^{kz\mu_{is}}+\xi_p^{-kz\mu_{is}}}{2}\frac{2}{1-\xi_p^{-k}}\prod_{j=1}^{m}\left(-\sqrt{-1}\cot\frac{\pi kz\tau_{ij}}{p}\right)$$
$$=\sum_{k=1}^{p-1}\cos\frac{2\pi kz\mu_{is}}{p}\left(1-\sqrt{-1}\cot\frac{\pi k}{p}\right)\prod_{j=1}^{m}\left(-\sqrt{-1}\cot\frac{\pi kz\tau_{ij}}{p}\right)$$

$$= \begin{cases} (-1)^n \sum_{k=1}^{p-1} \cos \frac{2\pi k z \mu_{is}}{p} \prod_{j=1}^{2n} \cot \frac{\pi k z \tau_{ij}}{p} \\ \qquad = (-1)^n \sum_{k=1}^{p-1} \cos \frac{2\pi k \mu_{is}}{p} \prod_{j=1}^{2n} \cot \frac{\pi k \tau_{ij}}{p} \quad (m = 2n) \\ (-1)^n \sum_{k=1}^{p-1} \cos \frac{2\pi k z \mu_{is}}{p} \cot \frac{\pi k}{p} \prod_{j=1}^{2n-1} \cot \frac{\pi k z \tau_{ij}}{p} \\ \qquad = (-1)^n \sum_{k=1}^{p-1} \cos \frac{2\pi k \mu_{is}}{p} \cot \frac{\pi k \bar{z}}{p} \prod_{j=1}^{2n-1} \cot \frac{\pi k \tau_{ij}}{p} \quad (m = 2n-1) \end{cases}$$

$$= (-1)^n \sum_{k=1}^{p-1} \cos \frac{2\pi \mu_{is} k}{p} \prod_{j=1}^{2n} \cot \frac{\pi b_j k}{p} .$$

Hence it follows from Theorem 3.1 that

$$\sum_{k=1}^{p-1} \frac{\rho_{p_i}(kz\mu_{is})}{1-\xi_p^{-k}} \prod_{j=1}^{m} \left(-\sqrt{-1} \cot \frac{\pi k z \tau_{ij}}{p} \right) \equiv (-1)^n \sum_{j=0}^{n} \zeta_j(p; b_1, \cdots, b_{2j})\, \eta_{n-j}(p; a_{is}, b_{2j+1}, \cdots, b_{2n}) . \tag{20}$$

Therefore it follows from (3), Proposition 2.6 and (9) that

$$\theta I_{D_{\ell_1,\cdots,\ell_m}}(g^z) = \frac{p-1}{p} \theta h_m \prod_{j=1}^{m} \sigma_j(e^{x_1}, e^{-x_1}, \cdots, e^{x_m}, e^{-x_m})^{\ell_j} \prod_{j=1}^{m} \frac{x_j/2}{\tanh(x_j/2)} [M]$$
$$+ (-1)^{n+1} \frac{1}{p} \sum_{i=1}^{b} r_i \sum_s n_s \sum_{k=0}^{n} \zeta_k(p_i; b_{i1}, \cdots, b_{i\,2k}) \eta_{n-k}(p; a_{is}, b_{i\,2k+1}, \cdots, b_{i\,2n}) .$$

This completes the proof of Theorem 3.6. □

Corollary 3.7. *Assume that the cyclic group $\mathbb{Z}_p = \langle g \rangle$ acts on $2m$-dimentional closed oriented manifold where m is even. Then for non-negative integers $\ell_1, \cdots, \ell_m$ we have $I_{D_{\ell_1,\cdots,\ell_m}}(g) = 0$ if p is odd and $2\theta I_{D_{\ell_1,\cdots,\ell_m}}(g) = 0$ if p is even.*

Proof. Since m is even, $\theta I_{D_{\ell_1,\cdots,\ell_m}}(g^z)$ does not depend on the natural number z which is prime to p because a_{is}, b_{ij} in the theorem above does not depend on z. Hence setting $z = p-1$ which is prime to p, we have

$$\begin{cases} p I_{D_{\ell_1,\cdots,\ell_m}}(g) = I_{D_{\ell_1,\cdots,\ell_m}}(g^p) = 0 , \\ \theta I_{D_{\ell_1,\cdots,\ell_m}}(g) = \theta I_{D_{\ell_1,\cdots,\ell_m}}(g^{p-1}) = \theta(p-1) I_{D_{\ell_1,\cdots,\ell_m}}(g) \end{cases}$$
$$\Longrightarrow \theta(2-p) I_{D_{\ell_1,\cdots,\ell_m}}(g) = 2\theta I_{D_{\ell_1,\cdots,\ell_m}}(g) = 0 ,$$

which imply that $I_{D_{\ell_1,\cdots,\ell_m}}(g) = 0$ if p is odd because 2θ is prime to p. □

4. Examples

Example 4.1. In this example we consider the case that $m = 1$, namely the case that M is a compact Riemann surface. Let p be an odd prime number and $\mathbb{Z}_p$ a cyclic group generated by g. Then we have $p_i = p$, in other words, $r_i = 1$ for $1 \leq i \leq b$. Let $t_1, \cdots, t_b$ be natural numbers such that $1 \leq t_1, \cdots, t_b \leq p-1$, z a natural number which is not a multiple of p, and set

$$f(t_1, \cdots, t_b, z) = \frac{1}{p} \sum_{i=1}^{b} (2zt_i + \zeta_1(p; t_i, \overline{z})) .$$

Then we have

$$\begin{aligned} f(t_1, \cdots, t_b, z) &\equiv \frac{1}{p} \sum_{i=1}^{b} \left(-2(zt_i)^2 \langle\, \overline{t_i}\,\overline{z}\, \rangle \cdot (-1) + 1 \cdot \zeta_1(p; t_i, \overline{z})\right) \pmod{\mathbb{Z}} \\ &= \frac{1}{p} \sum_{i=1}^{b} (\zeta_0\, \eta_1(p; zt_i, t_i, \overline{z}) + \zeta_1(p; t_i, \overline{z})\, \eta_0) . \end{aligned}$$

If $\{t_1, \cdots, t_b\}$ is the rotation angle of a $\mathbb{Z}_p$-action, namely if there exists an action of $\mathbb{Z}_p = \langle g \rangle$ on a compact Riemann surface such that the rotation angle of g is $\{t_1, \cdots, t_b\}$, it follows from Theorem 3.6 that $f(t_1, \cdots, t_b, z) = I_{D_1}(g^z) \in \mathbb{R}/\mathbb{Z}$ because $\mathrm{Ind}(D_1) = 0$ and $\mathrm{Ch}(E^1_{q_i}, g) = \rho_p(t_i)$ (see Example 2.10), and hence the following equality holds:

$$f(t_1, \cdots, t_b, z) - zf(t_1, \cdots, t_b, 1) \equiv 0 \pmod{\mathbb{Z}} . \tag{21}$$

Here as we see in the proof of Proposition 4.6 in [7], $\{t_1, \cdots, t_b\}$ is the rotation angle of a $\mathbb{Z}_p$-action if the equality

$$\sum_{i=1}^{b} \overline{t_i} \equiv 0 \pmod{p} \tag{22}$$

holds. In this example we show that the equality (22) follows from the equality (21), which implies that the equality (21), which corresponds to the equality $I_{D_1}(g^z) = zI_{D_1}(g)$, is the necessary and sufficient condition for $\{t_1, \cdots, t_b\}$ to be the rotation angle of a $\mathbb{Z}_p$-action.

Here we assume that the equality (21) holds. Since

$$k_1 t_i + k_2 \overline{z} \equiv 0 \iff (k_1 t_i + k_2 \overline{z})z \equiv k_1 t_i z + k_2 \equiv 0 \iff k_2 \equiv -k_1 t_i z \pmod{p} ,$$

it follows that

$$\begin{aligned} \zeta_1(p; t_i, \overline{z}) &= -4p \sum_{\substack{1 \leq k_1, k_2 < p \\ p | (t_i k_1 + \overline{z} k_2)}} \left(\left(\frac{k_1}{p}\right)\right)\left(\left(\frac{k_2}{p}\right)\right) = -4p \sum_{k_1=1}^{p-1} \left(\left(\frac{k_1}{p}\right)\right)\left(\left(\frac{-k_1 t_i z}{p}\right)\right) \\ &= 4p \sum_{k=1}^{p-1} \left(\left(\frac{k}{p}\right)\right)\left(\left(\frac{k t_i z}{p}\right)\right) , \end{aligned}$$

and hence we have

$$f(t_1, \cdots, t_b, z) = \frac{1}{p} \sum_{i=1}^{b} \left(2zt_i + 4p \sum_{k=1}^{p-1} \left(\left(\frac{k}{p}\right)\right)\left(\left(\frac{k t_i z}{p}\right)\right)\right) .$$

Then since

$$2\left\{z\left(\left(\frac{kt_i}{p}\right)\right)-\left(\left(\frac{kt_iz}{p}\right)\right)\right\} = 2z\left(\frac{kt_i}{p}-\left[\frac{kt_i}{p}\right]-\frac{1}{2}\right)-2\left(\frac{kt_iz}{p}-\left[\frac{kt_iz}{p}\right]-\frac{1}{2}\right)$$
$$= 2\left[\frac{kt_iz}{p}\right]-2z\left[\frac{kt_i}{p}\right]+1-z$$

is an integer and the map $kt_i \to k$ gives a bijection of $\mathbb{Z}_p$, it follows that

$$zf(t_1,\cdots,t_b,1)-f(t_1,\cdots,t_b,z) = 4\sum_{i=1}^{b}\sum_{k=1}^{p-1}\left(\left(\frac{k}{p}\right)\right)\left\{z\left(\left(\frac{kt_i}{p}\right)\right)-\left(\left(\frac{kt_iz}{p}\right)\right)\right\}$$
$$= 2\sum_{i=1}^{b}\sum_{k=1}^{p-1}\left(\frac{2k}{p}-1\right)\left\{z\left(\left(\frac{kt_i}{p}\right)\right)-\left(\left(\frac{kt_iz}{p}\right)\right)\right\}$$
$$= \frac{4}{p}\sum_{i=1}^{b}\sum_{k=1}^{p-1}k\left\{z\left(\left(\frac{kt_i}{p}\right)\right)-\left(\left(\frac{kt_iz}{p}\right)\right)\right\}$$
$$= \frac{4}{p}\sum_{i=1}^{b}\sum_{k=1}^{p-1}k\overline{t_i}\left\{z\left(\left(\frac{k}{p}\right)\right)-\left(\left(\frac{kz}{p}\right)\right)\right\} = \frac{1}{p}T\times F(p,z)$$

where

$$T=\sum_{i=1}^{b}\overline{t_i}\,,\quad F(p,z)=4\sum_{k=1}^{p-1}k\left\{z\left(\left(\frac{k}{p}\right)\right)-\left(\left(\frac{kz}{p}\right)\right)\right\}.$$

Therefore it follows from the assumption that

$$(23)\qquad 2p\{2f(t_1,\cdots,t_b,1)-f(t_1,\cdots,t_b,2)\} = T\times 2F(p,2)\equiv 0 \pmod{p}.$$

Here we have

$$2F(p,2) = 8\sum_{k=1}^{p-1}k\left\{2\left(\left(\frac{k}{p}\right)\right)-\left(\left(\frac{2k}{p}\right)\right)\right\}$$
$$= 8\sum_{k=1}^{p-1}k\left\{\left(2\frac{k}{p}-0-1\right)-\left(\frac{2k}{p}-\left[\frac{2k}{p}\right]-\frac{1}{2}\right)\right\}$$
$$= 8\sum_{k=1}^{p-1}k\left[\frac{2k}{p}\right]-4\sum_{k=1}^{p-1}k = 8\sum_{k=1}^{p-1}k\left[\frac{2k}{p}\right]-2p(p-1)\equiv 8\sum_{k=1}^{p-1}k\left[\frac{2k}{p}\right] \pmod{p}$$
$$= 8\sum_{k=\frac{p+1}{2}}^{p-1}k = (3p-1)(p-1)\equiv 1 \pmod{p},$$

and hence it follows from (23) that $T\equiv 0 \pmod{p}$, namely the equality (22) holds.

EXAMPLE 4.2. In this example we consider the case that $m = 2$, namely the case that M is a 4-dimensional closed oriented manifold, which does not necessarily admit an almost complex structure (see Example 3.2 in [7]). In this example using Theorem 3.6, we show that a relationship between the rotation angle of a $\mathbb{Z}_p$-action on M and the signature of M exists (see (28),(29) below). Let g be an orientation-preserving periodic diffeomorphism of M of order p. Let $\sigma(M)$ be the signature of M and suppose that $\theta\mathrm{Ch}(E_{q_i}^{\ell_1,\ell_2}, g^{r_i}) = \sum_s n_s\rho_{p_i}(\mu_{is})$ ($\theta = 1$ or 2). Then since the first Pontrjagin number p_1 is equal to $3\sigma(M)$, it follows from Example 2.7, Example 2.18 and Theorem 3.6 that

$$\begin{aligned}\theta I_{D_{\ell_1,\ell_2}}(g) &= \frac{p-1}{2p}\,\theta\, 2^{2\ell_1+\ell_2}3^{\ell_2}(3\ell_1+4\ell_2+1)\sigma(M)\\ &\quad+\sum_{i=1}^{b}\frac{1}{p_i}\sum_s n_s\sum_{k=0}^{1}\zeta_k(p_i;\tau_{i1},\cdots,\tau_{i,2k})\eta_{1-k}(p;\mu_{is}\overline{\tau_{i2}},\tau_{i\,2k+1},\tau_{i2})\\ &=\frac{p-1}{2p}\,\theta\, 2^{2\ell_1+\ell_2}3^{\ell_2}(3\ell_1+4\ell_2+1)\sigma(M)\\ &\quad+\sum_{i=1}^{b}\frac{1}{p_i}\sum_s n_s\left\{(-1)\left(-2\,(\mu_{is}\overline{\tau_{i2}})^2\,\overline{\tau_{i1}}\tau_{i2}\right)+\zeta_1(p_i;\tau_{i1},\tau_{i2})\cdot 1\right\}\end{aligned}$$

$$\Longleftrightarrow$$

$$\begin{aligned}\text{(24)}\qquad p\theta I_{D_{\ell_1,\ell_2}}(g) &= \frac{p-1}{2}\,\theta\, 2^{2\ell_1+\ell_2}3^{\ell_2}(3\ell_1+4\ell_2+1)\sigma(M)\\ &\quad+\sum_{i=1}^{b}r_i\left(\sum_s 2n_s\mu_{is}^2\right)\overline{\tau_{i1}}\,\overline{\tau_{i2}}+\sum_{i=1}^{b}r_i\left(\sum_s n_s\right)\zeta_1(p_i;\tau_{i1},\tau_{i2}) \pmod{p}.\end{aligned}$$

Here as we see in Example 2.10 we have

$$\mathrm{Ch}(E_{q_i}^{\ell_1,\ell_2}, g) = \sigma_1^{\ell_1}(\sigma_2+2)^{\ell_2}$$

where σ_k is the k-th elementary symmetric polynomial in $\rho_{p_i}(\tau_{i1})$, $\rho_{p_i}(\tau_{i2})$. Hence we have

$$\begin{aligned}&\mathrm{Ch}(E_{q_i}^{0,0}, g) = 1 = \frac{1}{2}\rho_{p_i}(0) \iff 2\mathrm{Ch}(E_{q_i}^{0,0}, g) = \rho_{p_i}(0),\\ &\mathrm{Ch}(E_{q_i}^{1,0}, g) = \sigma_1 = \rho_{p_i}(\tau_{i1})+\rho_{p_i}(\tau_{i2}),\\ &\mathrm{Ch}(E_{q_i}^{0,1}, g) = \sigma_2+2 = \rho_{p_i}(\tau_{i1})\rho_{p_i}(\tau_{i2})+2 = \rho_{p_i}(\tau_{i1}+\tau_{i2})+\rho_{p_i}(\tau_{i1}-\tau_{i2})+\rho_{p_i}(0).\end{aligned}$$

Set $\lambda = 1$ if p is odd and $\lambda = 2$ if p is even. Then it follows from Corollary 3.7, (24) and the equalities above that

$$\text{(25)}\qquad 2\lambda p I_{D_{0,0}}(g) = \lambda(p-1)\sigma(M)+\lambda\sum_{i=1}^{b}r_i\zeta_1(p_i;\tau_{i1},\tau_{i2}) \equiv 0 \pmod{p},$$

$$\begin{aligned}p\lambda I_{D_{1,0}}(g) &= 8\lambda(p-1)\sigma(M)+2\lambda\sum_{i=1}^{b}r_i(\tau_{i1}^2+\tau_{i2}^2)\overline{\tau_{i1}\tau_{i2}}+2\lambda\sum_{i=1}^{b}r_i\zeta_1(p_i;\tau_{i1},\tau_{i2})\\ \text{(26)}\qquad &\equiv -8\lambda\sigma(M)+2\lambda\sum_{i=1}^{b}r_i(\tau_{i1}\overline{\tau_{i2}}+\overline{\tau_{i1}}\tau_{i2})+2\lambda\sum_{i=1}^{b}r_i\zeta_1(p_i;\tau_{i1},\tau_{i2}) \equiv 0 \pmod{p}.\end{aligned}$$

Since $(\tau_{i1}+\tau_{i2})^2+(\tau_{i1}-\tau_{i2})^2+0^2 = 2(\tau_{i1}^2+\tau_{i2}^2)$, the next equality also holds.

$$\begin{aligned}&p\lambda I_{D_{0,1}}(g)\\ \text{(27)}\qquad &\equiv -15\lambda\sigma(M)+4\lambda\sum_{i=1}^{b}r_i(\tau_{i1}\overline{\tau_{i2}}+\overline{\tau_{i1}}\tau_{i2})+3\lambda\sum_{i=1}^{b}r_i\zeta_1(p_i;\tau_{i1},\tau_{i2}) \equiv 0 \pmod{p}.\end{aligned}$$

Here we have

$$\begin{cases} 2\times(25)-(26) \equiv 6\lambda\sigma(M) - 2\lambda\sum_{i=1}^{b} r_i(\tau_{i1}\overline{\tau_{i2}} + \overline{\tau_{i1}}\tau_{i2}) \equiv 0 \pmod p \\ 3\times(25)-(27) \equiv 12\lambda\sigma(M) - 4\lambda\sum_{i=1}^{b} r_i(\tau_{i1}\overline{\tau_{i2}} + \overline{\tau_{i1}}\tau_{i2}) \equiv 0 \pmod p \end{cases}$$

$$\iff 2\lambda\sum_{i=1}^{b} r_i(\tau_{i1}\overline{\tau_{i2}} + \overline{\tau_{i1}}\tau_{i2}) \equiv 6\lambda\sigma(M) \pmod p\,,$$

and since 2 is prime to p if p is odd, it follows that

$$4\sum_{i=1}^{b} r_i(\tau_{i1}\overline{\tau_{i2}} + \overline{\tau_{i1}}\tau_{i2}) \equiv 12\sigma(M) \pmod p \quad \text{if } p \text{ is even}\,, \tag{28}$$

$$\sum_{i=1}^{b} r_i(\tau_{i1}\overline{\tau_{i2}} + \overline{\tau_{i1}}\tau_{i2}) \equiv 3\sigma(M) \pmod p \quad \text{if } p \text{ is odd}\,. \tag{29}$$

For example, we consider the case that $M = \mathbb{CP}^2$. Let p be a prime number which is greater than 3 and a, b natural numbers such that $1 < a < b < p$. Then

$$\mathbb{CP}^2 \ni q = [z_0 : z_1 : z_2] \to g\cdot q = [z_0 : \xi_p^a z_1 : \xi_p^b z_2] \in \mathbb{CP}^2$$

defines an action of the cyclic group $\mathbb{Z}_p = \langle g\rangle$. The fixed point set of g^k $(1 \le k \le p-1)$ consists of $q_1 = [1:0:0]$, $q_2 = [0:1:0]$, $q_3 = [0:0:1]$, and the rotation angle of g is

$$((\tau_{11},\tau_{12}),(\tau_{21},\tau_{22}),(\tau_{31},\tau_{32})) = ((a,b),(p-a,b-a),(p-b,p+a-b))\,.$$

Then $r_i = 1$ for $1 \le i \le 3$ and we have

$$\begin{aligned}\sum_{i=1}^{3} r_i(\tau_{i1}\overline{\tau_{i2}} + \overline{\tau_{i1}}\tau_{i2}) &\equiv a\overline{b} + \overline{a}b - a\overline{(b-a)} + \overline{-a}(b-a) - b\overline{(a-b)} - \overline{b}(a-b) \\ &\equiv a\overline{b} + \overline{a}b - a\overline{(b-a)} - \overline{a}b + 1 + b\overline{(b-a)} - \overline{b}a + 1 \\ &= (b-a)\overline{(b-a)} + 1 + 1 \equiv 3 = 3\sigma(\mathbb{CP}^2) \pmod p\,.\end{aligned}$$

On the other hand since $3 \equiv 3\sigma(M) \iff 3(\sigma(M)-1) \equiv 0 \pmod p$, M does not admit an $\mathbb{Z}_p$-action with the rotation angle above if $\sigma(M)-1$ is not a multiple of p.

For example let p be any prime number such that $p+3$ is a multiple of 4 and set $n = (p+3)/2$. Let M be the connected sum of n copies of $\mathbb{CP}^2$ and $n-2$ copies of $\overline{\mathbb{CP}^2}$ which is the underlying manifold of $\mathbb{CP}^2$ with the orientation reversed. Then the Euler number $e(M)$ of M is equal to $2n$ and the equality (2) is satisfied for $b = 3$. But since $\sigma(M)-1 = n-(n-2)-1 = 1$ is not a multiple of p, M does not admit an $\mathbb{Z}_p$-action with the rotation angle above. Note that M does not admit any almost complex structure because $e(M)+\sigma(M) = 2n+2$ is not a multiple of 4 (see [8] p.1625).

EXAMPLE 4.3. Let M be a 6-dimensional closed oriented manifold, which does not necessarily admit an almost complex structure (see Example 3.3 in [7]). In this example using Theorem 3.6, we show that M does not admit the $\mathbb{Z}_6$-action of isotropy orders

$$(p_1, p_2, p_3, \cdots, p_{\nu+2}) = (6, 3, 2, \cdots, 2)\,.$$

Here we assume that M admits a $\mathbb{Z}_6$-action of isotropy orders above. Then it follows from the equality (2) that the Euler number $e(M)$ of M satisfies the equality:

$$\sum_{i=1}^{\nu+2} r_i - e(M) = 1 + 2 + 3\nu - e(M) = 3\left(\nu + 1 - \frac{e(M)}{3}\right) \equiv 0 \pmod 6,$$

which implies that there does not exist such a $\mathbb{Z}_6$-action above unless the following condition is satisfied:

(30) $$e(M) \text{ is a multiple of 3 and } \nu + 1 - \frac{e(M)}{3} \text{ is an even number}.$$

So we assume that the condition above is satisfied.

Let g be a generator of $\mathbb{Z}_6$ and $\{\tau_{i1}, \tau_{i2}, \tau_{i3}\}$ the rotation angle of g^{r_i}. Note that $\tau_{1j} = 1$ or 5 and $\tau_{2j} = 1$ or 2 for $1 \le j \le 3$ and hence the mod.p_i-inverse of τ_{ij} is equal to τ_{ij} itself for $i = 1, 2$. Here it follows from Proposition 2.9 that

$$\mathrm{Ch}(E^{0,0,1}_{q_i}, g) = C^1_{i3} = \rho_{p_i}(\tau_{i1} \pm \tau_{i2} \pm \tau_{i3}) + 2\sum_{1\le j\le 3} \rho_{p_i}(\tau_{ij}) = \sum_s n_s \rho_{p_i}(\mu_{is})$$

where

$$(n_1, \mu_{i1}) = (1, \tau_{i1} + \tau_{i2} + \tau_{i3}), \quad (n_2, \mu_{i2}) = (1, \tau_{i1} + \tau_{i2} - \tau_{i3}),$$
$$(n_3, \mu_{i3}) = (1, \tau_{i1} - \tau_{i2} + \tau_{i3}), \quad (n_4, \mu_{i4}) = (1, \tau_{i1} - \tau_{i2} - \tau_{i3}),$$
$$(n_5, \mu_{i5}) = (2, \tau_{i1}), \quad (n_6, \mu_{i6}) = (2, \tau_{i2}), \quad (n_7, \mu_{i7}) = (2, \tau_{i3}).$$

Then since $\mathrm{Ind}(D_{0,0,1})$ vanishes as we see in Remark 2.2 and $\zeta_0 = -1$, $\eta_0 = 1$, it follows from Theorem 3.6 that

(31) $$I_{D_{0,0,1}}(g^z) = -\frac{1}{p}\sum_{i=1}^{\nu+2} r_i \sum_s n_s \mathrm{F}_s(i, z)$$

where

$$\mathrm{F}_s(i,z) = -\eta_2(p_i; z\mu_{is}, \tau_{i1}, \tau_{i2}, \tau_{i3}, \bar{z}) + \zeta_1(p_i; \tau_{i1}, \tau_{i2})\eta_1(p_i; z\mu_{is}, \tau_{i3}, \bar{z}) + \zeta_2(p_i; \tau_{11}, \tau_{12}, \tau_{13}, \bar{z}).$$

Here it follows from Example 2.18 for $n = 2$ that

$$\zeta_1 = \zeta_1(p_i; \tau_{i1}, \tau_{i2}) = -\frac{1}{p_i} \sum_{\substack{1\le k_1, k_2 < p_i \\ p_i \mid (\tau_{i1}k_1 + \tau_{i2}k_2)}} (2k_1 - p_i)(2k_2 - p_i),$$

$$\zeta_2 = \zeta_2(p_i; \tau_{i1}, \tau_{i2}, \tau_{i3}, z) = \frac{1}{p_i^3} \sum_{\substack{1\le k_1, k_2, k_3, k_4 < p_i \\ p_i \mid (\tau_{i1}k_1 + \tau_{i2}k_2 + \tau_{i3}k_3 + \bar{z}k_4)}} (2k_1 - p_i)(2k_2 - p_i)(2k_3 - p_i)(2k_4 - p_i),$$

$$\eta_1 = \eta_1(p_i; z\mu_{is}, \tau_{i3}, \bar{z}) = -2a_{is}^2 c_2 = -2(z\mu_{is})^2 \langle \overline{\tau_{i3}}\,\bar{z} \rangle,$$

$$\eta_2 = \eta_2(p_i; z\mu_{is}, \tau_{i1}, \tau_{i2}, \tau_{i3}, \bar{z}) = \frac{2}{3} a_{is}^2 c_1 d_1^2 c_2 \left(a_{is}^2 c_2^2 + c_2^2 + 1\right)$$
$$= \frac{2}{3}(z\mu_{is})^2 \langle \overline{\tau_{i1}}\tau_{i2} \rangle \langle \overline{\tau_{i2}}\tau_{i3} \rangle^2 \langle \overline{\tau_{i3}}\,\bar{z} \rangle \left\{(z\mu_{is})^2 \langle \overline{\tau_{i3}}\,\bar{z} \rangle^2 + \langle \overline{\tau_{i3}}\,\bar{z} \rangle^2 + 1\right\}.$$

When $i = 1$, $p_i = 6$, $z = 1$, $(\tau_{11}, \tau_{12}, \tau_{13}) = (1, 1, 1)$, we have

$$\mu_{11} = 1 + 1 + 1 = 3, \ \mu_{12} = 1 + 1 - 1 = 1, \ \mu_{13} = 1 - 1 + 1 = 1, \ \mu_{14} = 1 - 1 - 1 = -1,$$

$\mu_{15} = \mu_{16} = \mu_{17} = 1\,, \quad \overline{z} = 1\,,$

and direct computation shows that

$$\begin{aligned}
&\zeta_1(6;\tau_{11},\tau_{12}) = \zeta_1(6;1,1) = \frac{20}{3}\,,\\
&\zeta_2(6;\tau_{11},\tau_{12},\tau_{13},\overline{z}) = \zeta_2(6;1,1,1,1) = \frac{164}{9}\,,\\
&\eta_1(6;z\mu_{11},\tau_{13},\overline{z}) = \eta_1(6;3,1,1) = -18\,,\\
&\eta_1(6;z\mu_{14},\tau_{13},\overline{z}) = \eta_1(6;-1,1,1) = -2\,,\\
&\eta_1(6;z\mu_{1s},\tau_{13},\overline{z}) = \eta_1(6;1,1,1) = -2 \quad (s = 2,3,5,6,7)\,,\\
&\eta_2(6;z\mu_{11},\tau_{11},\tau_{12},\tau_{13},\overline{z}) = \eta_2(6;3,1,1,1,1) = 66\,,\\
&\eta_2(6;z\mu_{14},\tau_{11},\tau_{12},\tau_{13},\overline{z}) = \eta_2(6;-1,1,1,1,1) = 2\,,\\
&\eta_2(6;z\mu_{1s},\tau_{11},\tau_{12},\tau_{13},\overline{z}) = \eta_2(6;1,1,1,1,1) = 2 \quad (s = 2,3,5,6,7)\,,
\end{aligned}$$

and hence we have

$$\sum_s n_s \mathrm{F}_s(1,1) = \sum_{s=1}^{4}\left\{(-1)u_{1s} + \frac{20}{3}v_{1s} + \frac{164}{9}\right\} + 2\sum_{s=5}^{7}\left\{(-1)u_s + \frac{20}{3}v_s + \frac{164}{9}\right\} = -\frac{1276}{9}$$

where $(u_{11}, v_{11}) = (66, -18)$, $(u_{1s}, v_{1s}) = (2, -2)$ $(s = 2,3,4,5,6,7)$.

When $i = 2$, $p_i = 3$, $z = 1$, $(\tau_{21},\tau_{22},\tau_{23}) = (1,1,2)$, we have

$$\begin{aligned}
&\mu_{21} = 1+1+2 = 4\,,\ \mu_{22} = 1+1-2 = 0\,,\ \mu_{23} = 1-1+2 = 2\,,\ \mu_{24} = 1-1-2 = -2\,,\\
&\mu_{25} = \mu_{26} = 1\,,\ \mu_{27} = 2\,,
\end{aligned}$$

and direct computation shows that

$$\begin{aligned}
&\zeta_1(3;\tau_{21},\tau_{22}) = \zeta_1(3;1,1) = \frac{2}{3}\,,\\
&\zeta_2(3;\tau_{11},\tau_{12},\tau_{13},\overline{z}) = \zeta_2(3;1,1,2,1) = -\frac{2}{9}\,,\\
&\eta_1(3;z\mu_{21},\tau_{23},\overline{z}) = \eta_1(3;4,2,1) = -64\,,\\
&\eta_1(3;z\mu_{22},\tau_{23},\overline{z}) = \eta_1(3;0,2,1) = 0\,,\\
&\eta_1(3;z\mu_{2s},\tau_{13},\overline{z}) = \eta_1(3;2,2,1) = -16\ (s = 3,7)\,,\\
&\eta_1(3;z\mu_{24},\tau_{13},\overline{z}) = \eta_1(3;-2,2,1) = -16\,,\\
&\eta_1(3;z\mu_{2s},\tau_{13},\overline{z}) = \eta_1(3;1,2,1) = -4\ (s = 5,6)\,,\\
&\eta_2(3;z\mu_{21},\tau_{21},\tau_{22},\tau_{23},\overline{z}) = \eta_2(3;4,1,1,2,1) = 5888\,,\\
&\eta_2(3;z\mu_{22},\tau_{11},\tau_{12},\tau_{13},\overline{z}) = \eta_2(3;0,1,1,2,1) = 0\,,\\
&\eta_2(3;z\mu_{2s},\tau_{11},\tau_{12},\tau_{13},\overline{z}) = \eta_2(3;2,1,1,2,1) = 448\ (s = 3,7)\,,\\
&\eta_2(3;z\mu_{24},\tau_{11},\tau_{12},\tau_{13},\overline{z}) = \eta_2(3;-2,1,1,2,1) = 448\,,\\
&\eta_2(3;z\mu_{2s},\tau_{11},\tau_{12},\tau_{13},\overline{z}) = \eta_2(3;1,1,1,2,1) = 48\ (s = 5,6)\,,
\end{aligned}$$

and hence we have

$$\sum_s n_s \mathrm{F}_s(2,1) = \sum_{s=1}^{4}\left\{(-1)u_{2s} + \frac{2}{3}v_{2s} - \frac{2}{9}\right\} + 2\sum_{s=5}^{7}\left\{(-1)u_{2s} + \frac{2}{3}v_{2s} - \frac{2}{9}\right\} = -\frac{71732}{9}$$

where $(u_{21}, v_{21}) = (5888, -64)$, $(u_{22}, v_{22}) = (0,0)$, $(u_{2s}, v_{2s}) = (448, -16)$ $(s = 3,4,7)$, $(u_{2s}, v_{2s}) = (48, -4)$ $(s = 5, 6)$.

When $3 \leq i \leq \nu + 2$, $p_i = 2$ and it follows from Remark 2.3 and (20) that

$$\sum_s n_s \mathrm{F}_s(i, 1) = 0\,.$$

Hence it follows from (5) and (31) that

$$\begin{aligned} I_{D_{0,0,1}}(g) &= -\frac{1}{6}\sum_{i=1}^{\nu+2} r_i \sum_s n_s \mathrm{F}_s(i, z) \\ &= \frac{1}{6}\left(\frac{1276}{9} + 2\cdot\frac{71732}{9} + 0\right) = \frac{72370}{27} = \frac{10}{27} \in \mathbb{R}/\mathbb{Z}\,. \end{aligned}$$

Using the same method, we have

$$I_{D_{0,0,1}}(g^5) = \frac{17}{27} \in \mathbb{R}/\mathbb{Z}\,,$$

and therefore it follows that

$$I_{D_{0,0,1}}(g^5) - 5I_{D_{0,0,1}}(g) = \frac{7}{9} \in \mathbb{R}/\mathbb{Z}\,.$$

Namely the equality

$$I_{D_{0,0,1}}(g^5) = 5I_{D_{0,0,1}}(g)$$

does not hold for $(\tau_{11}, \tau_{12}, \tau_{13}; \tau_{21}, \tau_{22}, \tau_{23}) = (1,1,1;1,1,2)$.

The direct computation using the same argument shows that the equality

$$I_{D_{0,0,1}}(g^5) = 5I_{D_{0,0,1}}(g)$$

does not hold for $\tau_{1j} = 1,\ 5$ $(1 \leq j \leq 3)$ and $\tau_{2j} = 1,\ 2$ $(1 \leq j \leq 3)$. This is a contradiction and therefore M does not admit the $\mathbb{Z}_6$-action of isotropy orders above.

For example let N be a 4-dimensional closed oriented manifold, R a compact Riemann surface of genus r such that $r \equiv 1\,(3)$ and set $M = N \times R$. Then the Euler number of M is a multiple of 6 because the Euler number of R is $2(1 - r)$, and the condition (30) is satisfied for any odd natural number ν. Then it follows from the result above that M does not admit the $\mathbb{Z}_6$-action of isotropy orders $(p_1, p_2, p_3, \cdots, p_{\nu+2}) = (6, 3, 2, \cdots, 2)$.

Remark 4.4. In the example above, M is an almost complex manifold if N is an almost complex manifold. But the $\mathbb{Z}_6$-action does not necessarily preserve the almost complex structure and the result above can not be obtained from the methods in [7], where the action is assumed to preserve the almost complex structure.

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