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LOCAL CONNECTEDNESS OF THE SPACE OF PUNCTURED TORUS GROUP

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Abstract

We will give a necessary condition for local connectedness of the space of Kleinian punctured torus group using Bromgerg's local coordinate system and provide a sufficient condition for local connectedness on a dense subset of the necessary condition. That is, the collection of the points where the boundary of the space of punctured torus group is not locally connected is a dense subset of the points satisfying the necessary condition.

1. Introduction

The space of Kleinian group that uniformizes a given compact 3-manifold M admits natural algebraic topology and is denoted by $AH(M)$. The interior of $AH(M)$ is well understood since late 1970's (See section 4.3 and 5.4 in [12] for more details). However, in the late 1990's it was discovered that the topology does not well-behaved at the boundary of the deformation space. Anderson-Canary [1] showed that the deformation space may contains components having intersecting closure and McMullen [13] showed that such phenomena can occur on the closure of a single component. Moreover Bromberg [3] showed that such phenomena, called now bumponomics, could be much complicated so that $AH(M)$ is not even locally connected. Recent works ([2],[10],[16]) show self-bumping phenomena in the Kleinian surface group, but local connectedness is not completely classified even in the simplest case, the space of Kleinian punctured torus group. In this paper, we give a necessary condition for which the space of Kleinian punctured torus group is not locally connected at given point σ and a sufficient condition in the sense of almost everywhere, that is the collection of points satisfying the sufficient condition is a dense subset of the collection of points satisfying the necessary condition, by using Bromberg's local coordinate system.

2. Preliminary

In this section we give some basic ingredients for our work.

2.1. Deformation space.

A Kleinian group is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$. Our interest lies on the space of Kleinian group that uniformizes a given compact 3-manifold M, defined as follows. Let G be a finitely generated group with n generators, and let $\mathcal{R}(G)$ be the space of representations of G in $\mathrm{PSL}(2,\mathbb{C})$. Then $\mathcal{R}(G)$ embeds in $(\mathrm{PSL}(2,\mathbb{C}))^n$ and this embedding gives $\mathcal{R}(G)$ a natural

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topology. Let $H(G)$ be the set of conjugacy classes of discrete, faithful representations of G in $\mathrm{PSL}(2,\mathbb{C})$. $H(G)$ admits a canonical topology called the algebraic topology by taking the induced topology on $H(G)$ as a subset of $\mathcal{R}(G)$. We denote $H(G)$ together with this topology as $AH(G)$.

If N is a compact, hyperbolizable 3-manifold possibly with boundary and G is isomorphic to $\pi_1(N)$, then we denote $\mathcal{R}(G)$ and $AH(G)$ as $\mathcal{R}(N)$ and $AH(N)$ respectively. Let $\mathcal{P}$ be a collection of disjoint, essential and homotopically distinct annuli and tori in ∂N. Then we denote $\mathcal{R}(N,\mathcal{P}) \subset \mathcal{R}(N)$, $AH(N,\mathcal{P}) \subset AH(N)$ for the collection of those representations σ for which $\sigma(\gamma)$ is parabolic if γ is freely homotopic into $\mathcal{P}$. A representation $\sigma \in AH(N,\mathcal{P})$ is minimally parabolic if $\sigma(\gamma)$ is parabolic if and only if γ is freely homotopic into $\mathcal{P}$. It is well known that the set of both minimally parabolic and geometrically finite representations in $AH(N,\mathcal{P})$, denoted by $MP(N,\mathcal{P})$, is the interior of $AH(N,\mathcal{P})$. In general, $MP(N,\mathcal{P})$ has many components. If the orientation for N is chosen, we will denote $MP_0(N,\mathcal{P})$ to be the component of $MP(N,\mathcal{P})$ that contains a representation with the orientation preserving marking map (See 2.2, 2.4 in [3] for more details).

2.2. Maskit slice.

We are interested in the most simplest case, the space of punctured torus group. Hereafter, we always assume that $N = \Sigma_{1,1} \times [-1,1]$, the trivial I-bundle of a compact torus with one boundary component and $\mathcal{P} = \partial\Sigma_{1,1} \times [-1,1]$. In this case, $\pi_1(N) = \pi_1(\Sigma_{1,1})$ is a free group of rank 2 and $AH(N,\mathcal{P})$ consists of the conjugacy classes of discrete, faithful representations $\sigma : \pi_1(N) \to \mathrm{PSL}(2,\mathbb{C})$ with added condition that it sends the commutator of the generators of $\pi_1(N)$ to a parabolic element. The space $AH(N,\mathcal{P})$ is a two-dimensional complex analytic space each of whose interior components is parametrized by the product of the Teichmüller space of once punctured torus (See 5.3 in [12]). The parametrization can be extended to $AH(N)$ by the following way. Let $\bar{\mathcal{T}}$ be the compactified Teichmüller space of once punctured torus by Thurston compactification, $\mathcal{AB}$ be the Alfors-Bers parametrization and let $\triangle$ be the diagonal in $\partial\bar{\mathcal{T}} \times \partial\bar{\mathcal{T}}$. The following theorem is due to Minsky [14] .

Theorem 2.1 (Minsky)**.** *There exists a continuous bijection*

$$\nu : \bar{\mathcal{T}} \times \bar{\mathcal{T}} - \triangle \to AH(N,\mathcal{P})$$

and the composition $\nu \circ \mathcal{AB}$ is the identity on $MP(N,\mathcal{P})$.

The map ν is not a homeomorphism, but Minsky showed that the restriction of ν on some slices in $\bar{\mathcal{T}} \times \bar{\mathcal{T}} - \triangle$ is a homeomorphsim. If one choose a measured lamination in $\partial\bar{\mathcal{T}}$ whose leaves consist of the simple closed curve $\gamma_{p/q}$ where p/q is determined by the choice of presentation in $\pi_1(N) = \langle a, b\rangle$, then

Theorem 2.2 (Minsky)**.** *The restriction of ν to $(\bar{\mathcal{T}} - \{\gamma_{p/q}\}) \times \{\gamma_{p/q}\}$ is a homeomorphism onto its image.*

The image of $(\bar{\mathcal{T}} - \{\gamma_{p/q}\}) \times \{\gamma_{p/q}\}$ has a natural embedding in $\mathbb{C}$, which is known as the *Maskit slice*. Here we restrict to the case $\gamma = \gamma_{1/0}$, but it works for any p/q. For each $w \in \mathbb{C}$, define a representation $\sigma_w \in \mathcal{R}(N)$ by

$$\sigma_w(a) = \begin{pmatrix} iw & i \\ i & 0 \end{pmatrix}, \ \sigma_w(b) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Proposition 2.3 (Bromberg)**.** *The map*

$$w \to \sigma_w$$

is a homeomorphism from $\mathbb{C}$ to its image in $\mathcal{R}(N, \mathcal{P}')$ where $\mathcal{P}'$ is the union of $\partial\Sigma_{1.1} \times [-1, 1] \subset N$ and the annular neightborhood of the simple closed curve $\gamma_{1/0}$.

Now define the set $\mathcal{M}$ of $\mathbb{C}$ to be the preimage of $AH(N, \mathcal{P}')$ under the map $w \to \sigma_w$. It then consists of two components, each of which is a jordan domain canonically isomorphic to $\bar{\mathcal{T}} - \{\gamma_{1/0}\}$. Let $\mathcal{M}^+$ be the component that contains the preimage of $MP_0(N, \mathcal{P})$, and label the other component $\mathcal{M}^-$. Also fix the orientation for N such that $\mathcal{M}^+$ contains $w \in \mathbb{C}$ with $Im\, w > 0$. The set $\mathcal{M}^+$ is called the Maskit slice. There are some elementary facts about the set $\mathcal{M}$ due to Keen-Series [7].

Proposition 2.4 (Keen-Series)**.** (1) *If w is in $\mathcal{M}$, then $w + 2, -w$ and $\bar{w}$ are in $\mathcal{M}$.*

(2) *The set $\mathcal{M}$ does not intersect $\mathbb{R}$, and therefore $\mathcal{M}^+$ is contained in the upper half plane.*

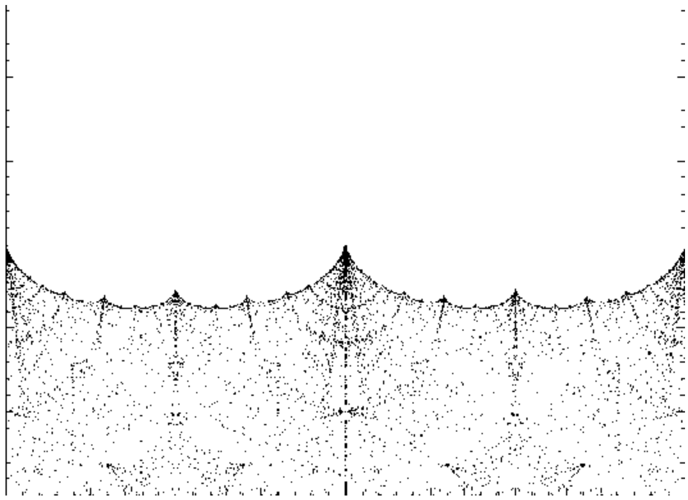

Fig. 1. The Maskit embedding in $\mathbb{C}$. Figure courtesy of David Wright.

2.3. Pleating rays.

Let $p/q \in \mathbb{Q}$. Let Ω_w be the unique invariant component of the region of discontinuity of $\sigma_w(\pi_1(\Sigma_{1,1}))$. The *$p/q$-pleating ray $\mathscr{P}_{p/q}$* is the set of points $w \in \mathcal{M}^+$ so that the component of the boundary of the convex core of $\mathbb{H}^3/\sigma_w(\pi_1(\Sigma_{1,1}))$ facing $\Omega_w/\sigma_w(\pi_1(\Sigma_{1,1}))$ is pleated along the simple closed geodesic labeled by p/q. Keen and Series [7] showed that those pleating rays, called rational pleating ray, give a partial foliation of $\mathcal{M}$ and can be extended to real pleating rays so that it gives a coordinate system of $\mathcal{M}$ called pleating coordinate. As we don't need a full strength of the pleating coordinate system, we just list some properties about rational pleating rays that we will use.

Proposition 2.5 (Keen-Series)**.** *For any $p/q \in \mathbb{Q}$, the pleating ray $\mathscr{P}_{p/q}$ is non-empty. Also, $\mathscr{P}_{p/q} \subset \{\mu \in \mathcal{M} : 2[p/q] \leq Re\, \mu \leq 2([p/q] + 1)\}$*

Proposition 2.6 (Keen-Series). *$\mathscr{P}_{p/q}$ meets $\partial\mathcal{M}$ only at a single point $\{w\}$ for which $\Omega_w/\sigma_w(\pi_1(N))$ is conformally equivalent to thrice punctured sphere $\Sigma - N(\gamma_{p/q})$. Moreover, $\mathscr{P}_{p/q}$ separates $\mathcal{M}$ into two connected pieces.*

2.4. Bromberg's local coordinate of $AH(N, \mathcal{P})$.

Bromberg provides a model space that is locally homeomorphic to $AH(N, \mathcal{P})$ near the Markit slice. In this subsection we will give a brief construction of the model space and some properties that we need. More details of the contents can be found in [3].

Fix a simple closed curve $\gamma \in \partial N$ that is not homotopic into $\mathcal{P}$. let $\mathcal{P}'$ be the union of $\mathcal{P}$ and an annulus in ∂N whose core curve is homotopic to γ. Let b be a primitive element of $\pi_1(N)$ that is freely homotopic to γ. Let W be an open solid torus in the interior of N whose core curve is isotopic to γ. Let $\hat{N} = N \setminus W$ and let $\hat{\mathcal{P}}$ be the union of the preimage of $\mathcal{P}$ under the inclusion of $\hat{N}$ to N and the torus $T = \partial W$. Then $\pi_1(\hat{N})$ has presentation

$$\langle \pi_1(N), c \mid [b, c] \rangle$$

Now let σ_w be a representation of $\pi_1(N, \mathcal{P})$ so that

$$\sigma_w(a) = \begin{pmatrix} iw & i \\ i & 0 \end{pmatrix}, \ \sigma_w(b) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

The relation that b and c commute then gives one-dimension complex parameter extension of the representation σ_w to $\sigma_{w,z} : \pi_1(\hat{N}) \to \mathrm{PSL}(2, \mathbb{C})$ by

$$\sigma_{w,z}(c) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

Bromberg [3] showed that any representation of $\mathcal{R}(N, \mathcal{P}')$ is conjugate to a representation σ_w with the condition

$$\sigma_w(b) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and if two representations σ, σ' of $\mathcal{R}(N, \mathcal{P}')$ are conjugate and satisfy

$$\sigma(b) = \sigma'(b) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

then their one parameter extension by z are also conjugate. Hence for each $\sigma_w \in MP_0(N, \mathcal{P}')$ the map sending z to $\sigma_{w,z}$ is well-defined. Now define the set $\mathcal{A}_{\sigma_w} \subset \mathbb{C}$ by

$$\mathcal{A}_{\sigma_w} = \{z \in \mathbb{C} \mid \sigma_{w,z} \in AH(\hat{N}, \hat{P}) \text{ and } Im\, z > 0\}.$$

Define $\mathcal{A} \subset \mathcal{M}^+ \times \hat{\mathbb{C}}$ by

$$\mathcal{A} = \{(w, z) \mid z \in \mathcal{A}_{\sigma_w} \text{ or } z = \infty\}.$$

The following theorem due to Bromberg show that the set $\mathcal{A}$ can be used for a local model of $AH(N, \mathcal{P})$ at every point of $MP(N, \mathcal{P}') \subset AH(N, \mathcal{P})$

Theorem 2.7 (Bromberg). *For any $w \in \mathrm{int}(\mathcal{M})$, there exist a neighborhood U of (w, ∞) in $\mathcal{A}$, a neighborhood V of σ_w in $AH(N, \mathcal{P})$ and a homeomorphism $\Phi : U \to V$.*

For convenience, we outline Ito's summarization [6] about Bromberg's construction of the function Φ. See [3] for full details.

Given $w \in \mathrm{int}(\mathcal{M})$, choose a neighborhood U of (w, ∞) in $\mathcal{A}$ small enough so that the following holds. Let $(w, z) \in U$. If $z = \infty$, then $\Phi(w, z) = \sigma_w$. If $z \neq \infty$, the quotient manifold

$$\hat{M}_{w,z} = \mathbb{H}^3/\sigma_{w,z}(\pi_1(N))$$

has a rank-2 torus cusp whose monodromy group is generated by $\sigma_{w,z}(b)$ and $\sigma_{w,z}(c)$. Since we choose U small enough, by the filling theorem due to Hodgson, Kerckhoff and Bromberg (See Theorem 2.5 in [3]) $\hat{M}_{w,z}$ is c-fillable for every $(w, z) \in U$. More precisely, there is a complete hyperbolic manifold $M_{w,z}$ that is homeomorphic to the interior of N and an embedding $\phi_{w,z} : \hat{M}_{w,z} \to M_{w,z}$,called the $c - filling\ map$, which satisfsfy the following properties:

(1) the image of $\phi_{w,z}$ is equals to $M_{w,z}$ minus the geodesic representative of $(\phi_{w,z})_*(\sigma_{w,z}(b))$,
(2) $(\phi_{w,z})_*(\sigma_{w,z}(c))$ is trivial in $\pi_1(M_{w,z})$, and
(3) $\phi_{w,z}$ extends to a conformal map between the conformal boundaries of $\hat{M}_{w,z}$ and $M_{w,z}$.

$\Phi(w, z)$ is defined to be an element in $AH(N, \mathcal{P})$ associated to $M_{w,z}$ with the following marking. Since the restriction of $\sigma_{w,z}$ to $\pi_1(N)$ is equal to σ_w, the manifold $M_w = \mathbb{H}^3/\sigma_w(\pi_1(N))$ covers $\hat{M}_{w,z}$. Denote the covering map by $\Pi_{w,z} : M_w \to \hat{M}_{w,z}$. Let $f_w : N \to M_w$ be a homotopy equivalence which induces σ_w. Then $\sigma_{w,z}$ is defined to be a representation of $\pi_1(N)$ into $\mathrm{PSL}(2, \mathbb{C})$ induced from the marking $\phi_{w,z} \circ \Pi_{w,z} \circ f_w : N \to M_{w,z}$. This $\sigma_{w,z}$ is faithful, and hence is contained in $AH(N, \mathcal{P})$.

3. Main theorem

By Theorem 2.7, we may turn our attention to local topology of $\mathcal{A}$. Here is the key ingredient that describes the shape of neighborhood at a point $\sigma_w \in \mathcal{M}$.

Proposition 3.1 (Bromberg)**.** *Let (w, z) be a pair of complex numbers with* $\mathrm{Im}\, z \neq 0$ *and let s be the sign of* $\mathrm{Im}\, z$*. The representation $\sigma_{w,z}$ is in $AH(\hat{N}, \hat{P})$ if and only if there exists an integer n such that $w - snz \in \mathcal{M}^+$ and $w - s(n + 1)z \in \mathcal{M}^-$.*

Together with the construction of $\mathcal{A}$ and the definition of $\mathcal{M}$, this gives a numerical representation on $\mathcal{A}$ by

$$\mathcal{A} = \{(w, z) \in \mathcal{M}^+ \times \hat{\mathbb{C}} \mid z = \infty\ or\ z \in (\frac{w + \mathcal{M}^-}{n}) \cap (\frac{w + \mathcal{M}^+}{n + 1})\ for\ some\ n \in \mathbb{Z}\}.$$

We do not treat the case when $n = -1$ *or* 0 because in each of those two cases $z \in \pm w + \mathcal{M}^+$ hence it does not violate local connectedness. As Minsky showed that each component of $\mathcal{M}$ is homeomorphic to a closed half plane, our interest relies the shape of the set $(\frac{w+\mathcal{M}^-}{n}) \cap (\frac{w+\mathcal{M}^+}{n+1})$. From now on, we will denote it by $\mathcal{S}_{w,n}$ and call it the solution sets of w.

Lemma 3.2. *$\mathcal{A}$ is not locally connected at (w, ∞) if and only if there exists some integer n such that the solution set $\mathcal{S}_{w,n}$ contains a bounded component.*

To prove lemma 3.2, first we need the following lemma.

Lemma 3.3. *If $\mathcal{S}_{w,n} \neq \emptyset$, then $\mathcal{S}_{w,n}$ contains a bounded component if and only if $\frac{w+\partial\mathcal{M}^-}{n}$ is not conained in $\frac{w+\mathcal{M}^+}{n+1}$*

Proof. The only if case is clear. To prove the other direction, suppose $(\frac{w+\partial\mathcal{M}^-}{n}) \not\subset (\frac{w+\mathcal{M}^+}{n+1})$ while $\mathcal{S}_{w,n}$ is connected. Then there exists some point $z \in \partial\mathcal{M}^+$ so that $\frac{w-z}{n} \notin \frac{w+\mathcal{M}^+}{n+1}$. Since $\frac{w+\mathcal{M}^+}{n+1}$ is a closed subset of $\mathbb{C}$ and $\frac{w-z}{n}$ is not contained in the set, there exists $\epsilon > 0$ so that $N_\epsilon(\frac{w-z}{n}) \cap \frac{w+\mathcal{M}^+}{n+1} = \emptyset$.

Miyachi[15] showed that rational boundary points of the Maskit slice are simple zeros of the trace function of accidental parabolic, which is intersection point of associated pleating ray and the boundary of the Maskit slice. Since rational boundary points are dense in the boundary of the Maskit slice, we may assume that the point z is now a rational boundary point meeting a pleating ray $\mathscr{P}_{p/q}$. Then by proposition 2.6, the pleating ray $\mathscr{P}_{p/q}$ separates $\mathcal{M}^+$ into two connected components A, B and intersects $\partial\mathcal{M}^+$ only at z.

Since Im $\frac{w+\mathcal{M}^+}{n+1}$ is bounded below by Im $\frac{w+\frac{3}{2}i}{n+1}$ while $\mathscr{P}_{p/q}$ goes to ∞ in the direction of $2(p/q) + ti,\ t > 0$ (see 3.2, 5.1 and 5.3 of [7]), which implies $\frac{w-\mathscr{P}_{p/q}}{n}$ goes to ∞ in the direction of $\frac{(w-2(p/q))}{n} - ti,\ t > 0$, $\frac{w-\mathscr{P}_{p/q}}{n} \cap \mathcal{S}_{w,n}$ (possibly empty) consists of closed, bounded components. Since $\frac{w-\mathscr{P}_{p/q}}{n}$ meets $\frac{w+\partial\mathcal{M}^-}{n}$ only at $\frac{w-z}{n}$ which is not contained in $\mathcal{S}_{w,n}$, the end points of the component of $\frac{w-\mathscr{P}_{p/q}}{n} \cap \mathcal{S}_{w,n}$ must lie on $\frac{w+\partial\mathcal{M}^+}{n+1}$. Let α be a bounded component of $\frac{w-\mathscr{P}_{p/q}}{n} \cap \mathcal{S}_{w,n}$. As the end points of α lie on $\frac{w+\partial\mathcal{M}^+}{n+1}$, there is bounded subarc β of $\frac{w+\partial\mathcal{M}^+}{n+1}$ with the same end points. By proposition 2.5 and the bound on the real part of single period of the Maskit slice, the real parts of α and β are both bounded. This implies that the real parts of the region bounded by α and β are bounded and the bound does not depend on the choice of α and β.

Now suppose z_1, z_2 be points in A and B respectively so that $\frac{w-z_1}{n}, \frac{w-z_2}{n}$ are both contained in $\mathcal{S}_{w,n}$. Let γ be a path in $\mathcal{S}_{w,n}$ joining $\frac{w-z_1}{n}$ and $\frac{w-z_2}{n}$. Since $\gamma \subset \mathcal{S}_{w,n} \subset \frac{w+\mathcal{M}^-}{n}$, γ must intersect $\frac{w-\mathscr{P}_{p/q}}{n} \cap \mathcal{S}_{w,n}$. Hence if $\frac{w-\mathscr{P}_{p/q}}{n} \cap \mathcal{S}_{w,n} = \emptyset$, it directly leads to a contradiction. Now suppose $\frac{w-\mathscr{P}_{p/q}}{n} \cap \mathcal{S}_{w,n} \neq \emptyset$ and let α be a component of $\frac{w-\mathscr{P}_{p/q}}{n} \cap \mathcal{S}_{w,n}$ that γ meets after starting from $\frac{w-z_1}{n}$ and take β for α as above. Since $\mathcal{S}_{w,n}$ is invariant under $z \to z+2$, the real part of z_1 can be chosen so that $\frac{w-z_1}{n}$ is not contained in the region bounded by α and β. Then γ cross α and enters the region, whose real part is uniformly bounded. Similar arguement about z_1 guarantees that z_2 can be chosen appropriately so that $\frac{w-z_2}{n}$ is not contained in the bounded region, hence γ must exit the region. Since $\gamma \subset \mathcal{S}_{w,n}$, γ cannot cross β and hence it cross α again. This implies that once γ cross some subarc of $\frac{w-\mathscr{P}_{p/q}}{n}$, it cross given subarc again and return to initial situation. By our choice of z_2, it implies γ cannot join $\frac{w-z_1}{n}$ and $\frac{w-z_2}{n}$, a contradiction. □

Proof of Lemma 3.2. Since $\mathcal{S}_{w,n}$ is the intersection of translated and scaled copies of $\mathcal{M}^+$ and $\mathcal{M}^-$ both of which are invariant under the translation action that sends z to $z + 2$, it is easy to see that either $\mathcal{S}_{w,n}$ is connected or contains a bounded component. Hence if there exists no such integer n, then $\mathcal{S}_{w,n}$ is connected for all $n \in \mathbb{Z}$, which gives $\mathcal{A}$ is locally connected at (w, ∞). This gives one direction.

Suppose $\mathcal{S}_{w,n}$ contains a bounded component for some integer n. Since $\mathcal{S}_{w,n}$ is invariant under the translation action $z \to z+2$, it implies that all the translated images of the bounded component are contained in $\mathcal{S}_{w,n}$, hence any neighborhood of ∞ contains infinitely many of them. To complete the proof, we must show that it happens in some neighborhood of (w, ∞) in $\mathcal{A}$,or equivalently show that there exists $\epsilon > 0$ such that $\bigcup_{w' \in N_\epsilon(w)} (\{w'\} \times \mathcal{S}_{w',n})$ contains a bounded component.

Note that if we translate w to $w + v$, the solution set $\mathcal{S}_{w,n} = (\frac{w+\mathcal{M}^-}{n}) \cap (\frac{w+\mathcal{M}^+}{n+1})$ goes to $\mathcal{S}_{w+v,n} = (\frac{w+v+\mathcal{M}^-}{n}) \cap (\frac{w+v+\mathcal{M}^+}{n+1})$, which shows that $\mathcal{S}_{w+v,n}$ is nothing but translate one half plane in the direction of $\frac{v}{n}$ and another in the direction of $\frac{v}{n+1}$,or equivalently translate one half plane in the direction of $\frac{v}{n(n+1)}$ relative to the other. Since $\mathcal{S}_{w,n}$ contains a bounded component, by lemma 3.3 there exists a rational boundary point $z \in \partial\mathcal{M}^+$ and $\epsilon_1 > 0$ small enough so that $N_{\epsilon_1}(\frac{w-z}{n}) \cap \frac{w+\mathcal{M}^+}{n+1} = \emptyset$. Above observation then shows that for any $w' \in N_{n(n+1)\epsilon_1}(w)$, $\frac{w'-z}{n} \notin \frac{w'+\mathcal{M}^+}{n+1}$. By Lemma 3.3, this implies $\mathcal{S}_{w',n}$ contains a bounded component.

It only remains to show that the product of those bounded components in small neighborhood of w is still bounded in $\mathcal{A}$. Our main idea is similar to the Bromberg's rectangular arguement in [3] with the fixed pleating ray $\frac{w-\mathscr{P}_{p/q}}{n}$ as the role of its vertical length, where $\mathscr{P}_{p/q}$ is the pleating ray associated to the chosen rational boundary point z in $\partial\mathcal{M}^+$.

Note that in the proof of Lemma 3.3, we observed that the intersection between $\frac{w-\mathscr{P}_{p/q}}{n}$ and $\mathcal{S}_{w,n}$ is contained in a compact, rectangular region $\{z \in \mathbb{C} \mid \text{Im } \frac{w+\frac{3}{2}i}{n+1} \le \text{Im } z \le \text{Im } \frac{w-\frac{3}{2}i}{n}, \text{ Re } (\frac{w-(2[p/q]+2)}{n} - 2) \le \text{ Re } z \le \text{ Re } (\frac{w-2[p/q]}{n} + 2)\}$. Thus in the remaining of the proof, Without lose of generality, we may consider only two period of $\mathcal{S}_{w,n}$ under the translation action $z \to z + 2$ to guarantee compactness.

CLAIM 1. The number of components of $\mathcal{S}_{w,n}$ that intersect $\frac{w-\mathscr{P}_{p/q}}{n}$ is finite.

Proof. Since the pleating ray $\frac{w-\mathscr{P}_{p/q}}{n}$ intersects $\frac{w+\partial\mathcal{M}^-}{n}$ only at $\frac{w-z}{n}$, $\frac{w-\mathscr{P}_{p/q}}{n} \setminus N_{\epsilon_1}(\frac{w-z}{n})$ has compact intersection with the rectangular region above and is disjoint with the closed subset $\frac{w+\partial\mathcal{M}^-}{n}$ of $\mathbb{C}$. Hence there exists $\epsilon_2 > 0$ such that ϵ_2 neighborhood of $\frac{w-\mathscr{P}_{p/q}}{n} \setminus N_{\epsilon_1}(\frac{w-z}{n})$ is disjoint with $\frac{w+\partial\mathcal{M}^-}{n}$. Since $N_{\epsilon_1}(\frac{w-z}{n}) \cap \frac{w+\mathcal{M}^+}{n+1} = \emptyset$, $\frac{w-\mathscr{P}_{p/q}}{n} \cap \mathcal{S}_{w,n} \subset \frac{w-\mathscr{P}_{p/q}}{n} \setminus N_{\epsilon_1}(\frac{w-z}{n})$ so we may replace the given pleating ray to its compact subarc whose ϵ_2 neighborhood misses $\frac{w+\partial\mathcal{M}^-}{n}$ while intersection with $\mathcal{S}_{w,n}$ is unchanged.

Suppose the number of components of $\mathcal{S}_{w,n}$ that intersect $\frac{w-\mathscr{P}_{p/q}}{n}$ is infinite. Take some infinite sequence $\{C_i\}$ where C_i is the intersection of $\frac{w+\partial\mathcal{M}^+}{n+1}$ and some component of $\mathcal{S}_{w,n}$ intersecting $\frac{w-\mathscr{P}_{p/q}}{n}$. Since $\mathcal{M}^+$ is jordan domain, we may consider a homeomorphism $f : \mathbb{R} \to \partial\mathcal{M}^+$ and the inverse image of $\{C_i\}$ under the mapping f. It consists of closed subarcs in bounded region of $\mathbb{R}$, hence there exists subsequence that converges to some point. The image of such limit point under f is contained in the pleating ray, as the pleating ray is closed and $\{C_i \cap \frac{w-\mathscr{P}_{p/q}}{n}\}$ converges to the same limit point of $\{C_i\}$. But since each C_i is the boundary of some connected component of $\mathcal{S}_{w,n}$, its boundary points ∂C_i must lie in $\frac{w+\partial\mathcal{M}^-}{n}$, which lies outside of ϵ_2-neighborhood of the pleating ray. This implies the limit point of $\{C_i\}$ can not be contained in $\frac{w-\mathscr{P}_{p/q}}{n}$, a contradiction. □

CLAIM 2. There exists $\epsilon, \epsilon' > 0$ such that for any $w' \in N_{n(n+1)\epsilon}(w)$, $\mathcal{S}_{w',n} \subset N_{\epsilon'}(\mathcal{S}_{w,n})$

Proof. By claim 1, we may denote $\mathcal{S}_{w,n}$ as $\mathcal{S}_{w,n} = D \cup (\cup_{i=1}^n D_i)$ where D_i is the component of $\mathcal{S}_{w,n}$ that intersects the pleating ray and D is the union of remaining components in two period of $\mathcal{S}_{w,n}$. As D and D_i are pairwise disjoint compact subsets of $\mathbb{C}$, there exists $\epsilon_3 > 0$ small enough so that $\epsilon_3 < \epsilon_1$ and $N_{3\epsilon_3}(D)$, $N_{3\epsilon_3}(D_i)$ are pairwise disjoint. We further assume that ϵ_3 is so small so that $N_{3\epsilon_3}(D)$ is disjoint with the pleating ray.

Let $t = \frac{w+\partial\mathcal{M}^-}{n} \setminus N_{\epsilon_3}(\mathcal{S}_{w,n})$ and $s = \frac{w+\partial\mathcal{M}^+}{n+1} \setminus N_{\epsilon_3}(\mathcal{S}_{w,n})$. Then both t and s consist of finite subarcs of $\frac{w+\partial\mathcal{M}^-}{n}$ and $\frac{w+\partial\mathcal{M}^+}{n+1}$ respectively, otherwise there exists two infinite sequence $\{t_j\}$, $\{d_j\}$ of $\frac{w+\partial\mathcal{M}^-}{n}$ with $t_j \subset t, d_j \subset D$ for each $j \in \mathbb{Z}$ ($\{s_j\}, \{d'_j\}$ for $s_j \subset s, d'_j \subset D$ respectively) whose inverse images under the homeomorphism $f : \mathbb{R} \to \frac{w+\partial\mathcal{M}^-}{n}$ ($f : \mathbb{R} \to \frac{w+\partial\mathcal{M}^+}{n+1}$ respectively) appears alternately in a compact subset of $\mathbb{R}$, hence converges to the same limit point which implies all but finite of $\{t_j\}$ lie in $N_{\epsilon_3}(\mathcal{S}_{w,n})$, contradiction. Since each t and s consists of finite compact sets, there exists $\epsilon_4 > 0$ such that ϵ_4 neighborhood of $t \cup s$ are disjoint componentwise.

Now let $\epsilon = min(\epsilon_3, \epsilon_4)$ and $\epsilon' = 3\epsilon_3$. Suppose $w' \in N_{n(n+1)\epsilon}(w)$, or equivalently $w' = w + n(n+1)\delta e^{i\theta}$ for some $\delta < \epsilon$. Then for any $x' \in \mathcal{S}_{w',n}$, after taking translation $z \to z - (n+1)\delta e^{i\theta}$ to fixing one copy of the Maskit slice $\frac{w+\mathcal{M}^-}{n}$, $x' \in \frac{w+\mathcal{M}^+}{n+1} + \delta e^{i\theta}$. This gives $x = x' - \delta e^{i\theta} \in \frac{w+\mathcal{M}^+}{n+1}$.

CASE 1. $x \in N_{2\epsilon_3}(\mathcal{S}_{w,n})$

In this case, since $d(x, x') = \delta < \epsilon \le \epsilon_3$, $x' \in N_{3\epsilon_3}(\mathcal{S}_{w,n}) = N_{\epsilon'}(\mathcal{S}_{w,n})$

CASE 2. $x \notin N_{2\epsilon_3}(\mathcal{S}_{w,n})$

Since $x \in \frac{w+\mathcal{M}^+}{n+1}$, it must be the case $x \notin \frac{w+\mathcal{M}^-}{n}$. On the other hand, since $x' = x + \delta e^{i\theta} \in \frac{w+\mathcal{M}^-}{n}$, we have $N_\epsilon(x) \cap \frac{w+\mathcal{M}^-}{n} \neq \emptyset$. It implies $N_\epsilon(x) \cap \frac{w+\partial\mathcal{M}^-}{n} \neq \emptyset$ which gives $N_\epsilon(x) \cap t \neq \emptyset$, otherwise by definition of t, $N_\epsilon(x) \cap \frac{w+\partial\mathcal{M}^-}{n} \subset N_{\epsilon_3}(\mathcal{S}_{w,n})$ which tells that x is within $\epsilon+\epsilon_3 \le 2\epsilon_3$ of $\mathcal{S}_{w,n}$. It leads to a contradiction. Also $N_\epsilon(x) \cap \frac{w+\partial\mathcal{M}^+}{n+1} \neq \emptyset$, since if not, then $x \in \frac{w+\mathcal{M}^+}{n+1}$ implies $N_\epsilon(x) \subset \frac{w+\mathcal{M}^+}{n+1}$, but then $x' + \delta e^{i\theta} \in N_\epsilon(x') \subset \frac{w+\mathcal{M}^+}{n+1} + \delta e^{i\theta}$, or equivalently $x' \in \frac{w+\mathcal{M}^+}{n+1}$, which implies that $x' \in \mathcal{S}_{w,n}$. It implies that $N_{\epsilon_3}(x) \cap \mathcal{S}_{w,n} \neq \emptyset$, which is a contradiction. The similar areguement in the case t then guarantees that $N_\epsilon(x) \cap s \neq \emptyset$. Since both t and s intersects $N_\epsilon(x)$, their ϵ_4 neighborhood intersect at x, contradicts to our choice of ϵ_4. □

We claim that $U = \bigcup_{w' \in N_{n(n+1)\epsilon}(w)} \{w'\} \times \mathcal{S}_{w',n}$ contains bounded component, which implies that, since second coefficients are invariant under the translation action $z \to z + 2$, any open subneighborhood of (w, ∞) in U contains infinitely many components, which completes the proof.

Suppose, for a contrary, that U is connected. As the second coefficient is invariant under the translation action, for convenience we may assume z_1, z_2 are points in the proof of lemma 3.3 satisfying $(w, z_1), (w, z_2) \in U$. Any path γ joining (w, z_1) and (w, z_2) in U must cross $\frac{w-\mathcal{P}_z}{n}$ in some slice $\mathcal{S}_{w',n}$. Since $\mathcal{S}_{w',n} \subset N_{\epsilon'}(\mathcal{S}_{w,n})$ whose intersection with the pleating ray is admitted only at $\{N_{\epsilon'}(D_i)\}$, the path γ cross the pleating ray in the subarc $\frac{w+\mathcal{P}_z}{n} \cap N_{\epsilon'}(D_i)$ for some i. Since $N_{\epsilon'}(D)$, $N_{\epsilon'}(D_i)$ are pairwise disjoint and whenever $w' \in N_{n(n+1)\epsilon}(w)$ the solution set $\mathcal{S}_{w',n}$ is contained in $N_{\epsilon'}(D) \cup \{\cup_i N_{\epsilon'}(D_i)\}$, γ cannot get across the boundary of the closure of $N_{\epsilon'}(D_i)$. Uniform bound on the real part of $N_{\epsilon'}(D_i)$ is also clear. Then by similar arguement in the proof of Lemma 3.3 with replacing $N_{\epsilon'}(D_i)$ in place of the component of $\mathcal{S}_{w,n}$, we lead a contradiction. □

By Lemma 3.2 and 3.3, we have seen that whether $\mathcal{A}$ is not locally connected or not at some point (w, ∞) in the Maskit slice depends only on the shape of the solution set $\mathcal{S}_{w,n}$. It is an easy observation that the condition $(\frac{w+\partial\mathcal{M}^-}{n}) \not\subset (\frac{w+\mathcal{M}^+}{n+1})$ is satisfied only when $(\frac{w+\partial\mathcal{M}^-}{n}) \cap (\frac{w+\partial\mathcal{M}^+}{n+1}) \neq \emptyset$. This condition can be solved in the following way.

Suppose $(\frac{w+\partial\mathcal{M}^-}{n}) \cap (\frac{w+\partial\mathcal{M}^+}{n+1}) \neq \emptyset$ and let $z \in (\frac{w+\partial\mathcal{M}^-}{n}) \cap (\frac{w+\partial\mathcal{M}^+}{n+1})$. This gives $z = \frac{w-u}{n} = \frac{w+v}{n+1}$ for some $u, v \in \partial\mathcal{M}^+$. If one solve it, one obtains $w = (n+1)u + nv$ and $z = u + v$. Let $B_n = \{w \in \mathcal{M} \mid w = (n+1)u + nv \; for\ some\ u, v \in \partial\mathcal{M}^+\} = (n+1)\partial\mathcal{M}^+ + n\partial\mathcal{M}^+$ and let $B = \bigcup_{n\in\mathbb{Z}-\{-1,0\}} B_n$. We now state our main theorem.

Theorem 3.4 (Main theorem)**.** *If $AH(N, \mathcal{P})$ is not locally connected at σ, then $\sigma = \sigma_w$ up to conjugate in $PSL(2, \mathbb{C})$ for some $w \in B$. The converse holds in the sense of almost everywhere. That is, B contains the set of the representations at which $AH(N)$ is not locally connected as a dense subset.*

Proof. By Theorem 2.7, $AH(N, \mathcal{P})$ is not locally connected at $\sigma_w \in AH(N, \mathcal{P}')$ if and only if $\mathcal{A}$ is not locally connected at (w, ∞). Lemma 3.2 and the previous arguement shows that it happens only when $w \in B$. The following theorem due to Ma[10] then gives those are the only possible choices of local disconnectedness, since $AH(N, \mathcal{P})$ consists of two components and hence local disconnectedness implies phenomenon of self-bumping. This gives one half of the proof.

Theorem 3.5 (Ma)**.** *If $\sigma \in AH(\Sigma)$ is a self-bumping point, then σ is geometrically finite and has exactly one accidental parabolic class.*

To show the remaining part, suppose $w \in B$. We will show that for any neighborhood of w, it contains a point w' such that $\mathcal{A}$ is not locally connected at (w', ∞). Now fix a neighborhood $N_\varepsilon(w) \subset \mathcal{M}^+$. Since $w \in B$, we can write w as $w = (n+1)u + nv$ for some $n \in \mathbb{Z}$ and for some $u, v \in \partial\mathcal{M}^+$. Moreover $z = u + v \in \mathcal{S}_{w,n}$ corresponds to the point of $\mathcal{S}_{w,n}$ where $\frac{w+\partial\mathcal{M}^-}{n}$ and $\frac{w+\partial\mathcal{M}^+}{n+1}$ intersect. Since $\mathcal{M}^+$ is closed in $\mathbb{C}$ and z lies on the boundary of its scaled and translated copy, there exists some point $z + \frac{\delta}{n(n+1)} \in N_{\frac{\varepsilon}{n(n+1)}}(z)$ such that $z + \frac{\delta}{n(n+1)} \notin \frac{w+\mathcal{M}^+}{n+1}$. Hence for $w' = w + \delta \in N_\varepsilon(w)$, the previous observation gives $\frac{w'+\partial\mathcal{M}^-}{n} \not\subset \frac{w+\mathcal{M}^+}{n+1}$ since $z + \frac{\delta}{n} \in \frac{w+\partial\mathcal{M}^-}{n} + \frac{\delta}{n} = \frac{w'+\partial\mathcal{M}^-}{n}$ but $z + \frac{\delta}{n} = z + \frac{\delta}{n(n+1)} + \frac{\delta}{n+1} \notin \frac{w+\mathcal{M}^+}{n+1} + \frac{\delta}{n+1} = \frac{w'+\mathcal{M}^+}{n+1}$. Thus by lemma 3.2 and 3.3, $\mathcal{A}$ is not locally connected at (w', ∞). This completes the proof. □

4. A geometric question related to numerical data

It is quite natural to ask that the local connectedness is a geometric property. That is, one can check whether $AH(N)$ is locally connected or not at the given representation σ by using only geometric condition. In this section we will give an observation that there is some geometric condition that locally disconnected points enjoy but is too week to convince the question is supposed to be true.

Note that each component B_n of B is an annulus, as it is a continuous image of $\mathbb{R}^2$ and is homotopic to a scaled and translated copy of the jordan curve $\partial\mathcal{M}^+$. One can see that each B_n has two disjoint boundaries one of which consists of the points whose solution set has an empty interior, and the other consists of the points that $\mathcal{A}$ is locally connected. since the set of points for which $\mathcal{A}$ is not locally connected is the dense subset of B, the boundary of B_n associated to the locally connected points must be contained in the boundary of the set of local connectedness. It tells that those jordan curves take the role of barometa for local connectedness of $\mathcal{A}$. Hence we can replace the question to the following one; do those jordan curves come from geometric data? Unfortunately, it is hard to parametrize the points on the boundary component $\partial^+ B_n$ explicitly.

Now we turn to geometric observation on the solution sets. By the construction in [3], the pair $(w, z) \in \mathcal{A}$ with $z \neq \infty$ is mapped to a representation $\sigma_{w,z} \in AH(N)$ which uniformizes a quasi-fuchsian group determined by two conformal marking, which are associated to the conformal boundary of $\sigma_{w-nz}, \sigma_{w-(n+1)z} \in AH(N, \mathcal{P}')$ that are not pair of pants, respectively. If one takes the translation action $z \to z + 2$ on the second coordinate of (w, z) for m times, it effects on the conformal boundary of σ_{w-nz} by Dehn twist along γ_∞ nm times and on the conformal boundary of $\sigma_{w-(n+1)z}$ by $(n+1)m$ times. By Ito's result[5] it corresponds to the sequence that converges exotically near σ_w.

Note that if $w \in B_n$ and hence two boundary curves of $\mathcal{S}_{w,n}$ intersect, then for any $z \in \mathcal{S}_{w,n}$, either $w - nz \in \mathcal{M}^-$ or $w - (n+1)z \in \mathcal{M}^+$ close to the boundary of the Maskit slice in uniform sense. If we take the Maskit slice to the upper half plane by Matthews biholomorphic identification of teichmuller space of punctured torus [11], it gives uniform upper bound on the imaginary part, which encodes length condition on γ. It implies that there exists some weak geometric condition for $w \in B_n$; if $w \in B_n$, then $\sigma_w \in AH(N)$ is an exotic limit point of the sequence of quasi-fuchsian group $qf(\tau^{nm}X, \tau^{(n+1)m}Y)$ where τ is the Dehn twist along simple closed curve γ, $X, Y \in \mathcal{T}$ and either in X or Y, the length of γ is uniformly bounded below. This gives a weak geometric condition for local disconnectedness. Note that such condition appears in local connectedness of the linear slices, see [6],[9].

Question 4.1. Is there a geometric property that determines local connectedness of $\mathcal{A}$?

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