Cheng, Z., Mukherjee, S., Przytycki, J.H., Wang, X. and Yang, S.Y.
Osaka J. Math.
**56** (2019), 661–674

# ROOTED TREES WITH THE SAME PLUCKING POLYNOMIAL

Zhiyun CHENG, Sujoy MUKHERJEE, Józef H. PRZYTYCKI, Xiao WANG and Seung Yeop YANG

(Received March 28, 2018)

**Abstract**

In this paper we address the following question: When do two rooted trees have the same plucking polynomial? The solution provided in the present paper has an algebraic version (Theorem 2.5) and a geometric version (Theorem 1.2). Furthermore, we give a criterion for a sequence of non-negative integers to be realized as a rooted tree.

## 1. Introduction

Plane trees, sometimes also called ordered trees, are basic objects in combinatorics. It is well known that the number of unlabeled plane trees with $n$ edges and the number of plane trees with $n$ edges and $k$ leaves coincide with the $n$-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$ and the Narayana number $\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ respectively. In this paper, we are concerned with a new rooted tree polynomial $Q(T) \in \mathbb{Z}[q]$, called the *plucking polynomial*, which was recently introduced by the third author in [7].

Throughout this paper rooted trees are always drawn on the upper half plane with root lying on the bottom level. If $T$ consists of a single point then we set $Q(T) = 1$. If $|E(T)| \geq 1$, the plucking polynomial $Q(T)$ is defined recursively as follows

$$Q(T) = \sum_{v \in l(T)} q^{r(T,v)} Q(T - v).$$

Here $l(T)$ denotes the set of leaves of $T$, $r(T, v)$ is the number of edges of $T$ on the right side of the unique path connecting $v$ with the root (we assume the root is situated at the origin), and $T - v$ is the subtree of $T$ obtained by deleting $v$ from $T$. See Figure 1 for an example of $r(T, v)$. According to the definition, one can easily find that the rooted tree described in Figure 1 has plucking polynomial $[2]_q[3]_q[5]_q[6]_q$.

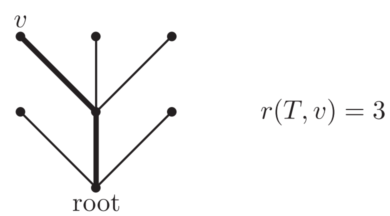


Fig. 1. An example of $r(T, v)$.

Note that with a given embedding of $T$, if we fix the direction from left to right then we

2010 Mathematics Subject Classification. 05C05, 05C31.

obtain an ordering on $V(T)$. It is easy to observe that $r(T,v)$ is nothing but the number of vertices which are "older" than $v$.

The definition of the plucking polynomial is motivated by the Kauffman bracket skein modules of 3-manifolds. For an oriented 3-manifold $M$, the *Kauffman bracket skein module* $\mathcal{S}(M)$ [6] is generated by all isotopy classes of framed links in $M$ and then one takes the quotient by

(1) skein relation: $[K] = A[K_\infty] + A^{-1}[K_0]$,
(2) framing relation: $[K \cup \bigcirc] = (-A^2 - A^{-2})[K]$.

Here $[\bigcirc]$ denotes the trivial framed knot and $K$, $K_\infty$, $K_0$ only differ in a small $D^3$, see Figure 2.

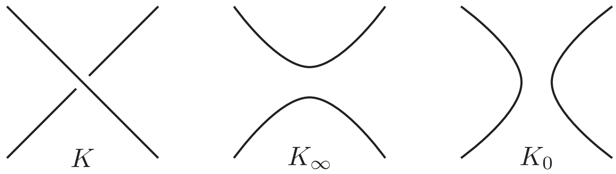


Fig. 2. Local diagrams in skein relation.

In [4], Dabkowski, Li and the third author studied $(m\times n)$-lattice crossing $L(m,n)$ in the relative Kauffman bracket skein module of $P\times I$, where $P$ denotes an $(m\times n)$ parallelogram with $(2m+2n)$ points on the boundary, see Figure 3.

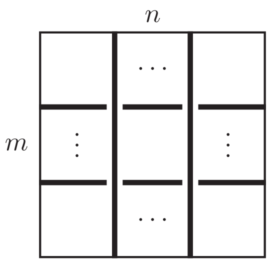


Fig. 3. $L(m,n)$.

Roughly speaking, in order to calculate $L(m,n)$ in the relative Kauffman bracket skein module of $P\times I$ one needs to smooth all $mn$ crossing points according to the skein relation $[K] = A[K_\infty] + A^{-1}[K_0]$ and then replace each trivial component with $(-A^2 - A^{-2})$. The result can be written in the form

$$[L(m,n)] = \sum_{C\in \mathrm{Cat}_{m,n}} r(C)C,$$

where $\mathrm{Cat}_{m,n}$ denotes all crossingless connections between the boundary points of $P$ and $r(C) \in \mathbb{Z}[A,A^{-1}]$. Note that not every element of $\mathrm{Cat}_{m,n}$ can be realized. It was proved in [4] that a Catalan state $C$ is realizable if and only if every vertical line cuts $C$ at most $m$ times and each horizontal line cuts $C$ at most $n$ times. For a Catalan state $C$ with no returns, the closed form formula for $r(C)$ was studied in [4]. After the paper [4] was finished, it was found that one can construct a rooted tree $T(C)$ for each Catalan state $C \in \mathrm{Cat}_{m,n}$, such that if $C$ has returns only on its ceiling then the coefficient $r(C)$ is given by

$$r(C) = A^{2|\mathbf{b}_M|-mn} Q(T(C))|_{q=A^{-4}}.$$

We refer the reader to [5] for the definition of $\mathbf{b}_M$ and the construction of $T(C)$.

Keeping in mind an important relation between the plucking polynomial $Q(T)$ and the coefficient $r(C)$ will develop further properties of $Q(T)$. Although the definition of $Q(T)$ seems to depend on a particular embedding of a rooted tree $T$, as it was shown in [7] (see Section 2), the polynomial $Q(T)$ does not. It is natural to ask the following two questions

(1) For a given polynomial $f(q) \in \mathbb{Z}[q]$, does there exist a rooted tree $T$ such that $Q(T) = f(q)$?

(2) When do two rooted trees have the same plucking polynomial?

The first question was answered in [1]. In this paper we will focus on the second question. Note that if the root of a rooted tree $T$ has only one child, i.e. there is only one edge incident with the root, then contracting this edge results in a new rooted tree $T'$. We will call this operation a *destabilization* and the inverse a *stabilization*. It is not hard to see that $Q(T) = Q(T')$, i.e. operations of stabilization and destabilization preserve the plucking polynomial. In particular, the plucking polynomial of any 1-ary rooted tree equals 1. We say a rooted tree is *reduced* if the root has more than one child.

At the end of [1], we introduced the *exchange move* for rooted trees, see Figure 4. More precisely, for a fixed embedded rooted tree $T$ and two vertices $v_1, v_2$, we consider two embedded circles $S_1, S_2$ such that $S_i \cap T = v_i$ $(i = 1, 2)$. We use $T_1$ and $T_2$ to denote the subtrees bounded by $S_1$ and $S_2$ respectively. In other words, $T_i$ is a rooted tree with root $v_i$ and it spans some children of $v_i$ and all of their descendants ($i = 1, 2$). If $|E(T_1)| = |E(T_2)|$ then we switch the positions of $T_1$ and $T_2$. We found in [1] that an exchange move preserves plucking polynomial. We also asked a question if any reduced rooted trees with the same plucking polynomial are related by a finite sequence of exchange moves. The answer to this question was positive for all examples given in [1].

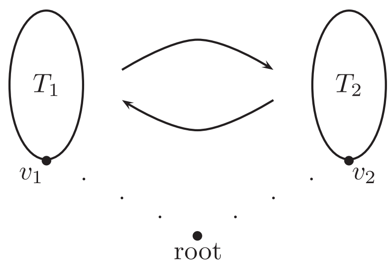


Fig.4. An exchange move.

Following a seminar talk by the first author about plucking polynomial at Peking University in September 2016, Hao Zheng observed the following potential counterexample. Later we will show that this is really a counterexample, i.e. although the two rooted trees in Figure 5 have the same plucking polynomial $\frac{[8]_q[11]_q[12]_q[13]_q[14]_q[15]_q[16]_q[17]_q[18]_q[19]_q}{[2]_q^3[3]_q[5]_q[6]_q}$, none of them can be obtained from the other via finitely many exchange moves.

In order to answer the question (2) above, we introduce a more general version of an exchange move that we call a *permutation move*. Intuitively, a permutation move is an operation defined as follows. For a rooted tree, choose $n$ vertices $v_1, \cdots, v_n$ and the corresponding families of rooted subtrees $T_1, \cdots, T_m$ $(m \geq n)$ with roots $v_1, \cdots, v_n$. Then a

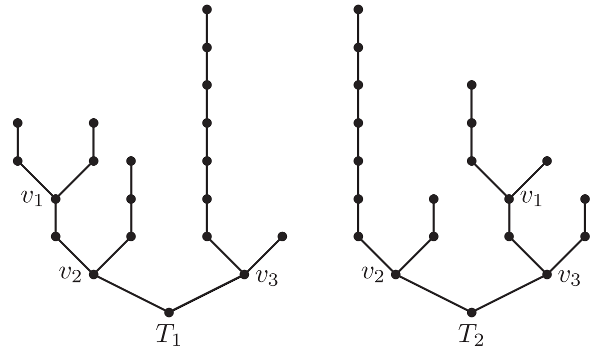


Fig. 5. Two rooted trees with the same plucking polynomial.

permutation move is simply a replacement of $T_1, \cdots, T_m$ such that the number of edges above each chosen vertex $v_i$ is preserved. The precise definition is as follows.

DEFINITION 1.1. Let us consider $n$ vertices $v_1, \cdots, v_n$ of a rooted tree $T$ and two sequences $\{\alpha_i\}_{0 \leq i \leq n}, \{\beta_i\}_{0 \leq i \leq n}$ which satisfy

$$0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \alpha_n = \beta_n > \beta_{n-1} > \cdots > \beta_1 > \beta_0 = 0.$$

For $v_1$, choose several children of it, say $w_1, \cdots, w_{\alpha_1}$, and draw embedded circles $S_i^1$ $(1 \leq i \leq \alpha_1)$ in the plane such that $S_i^1 \cap T = v_1$, $w_i$ is located in the interior of $S_i^1$ and other $w_j$ $(j \neq i)$ are located outside of $S_i^1$. We use $T_i$ to denote the subtree of $T$ bounded by $S_i^1$, see Figure 6 for an example of $S_1^1$. The other subtrees $T_i$ $(\alpha_1 + 1 \leq i \leq \alpha_n)$ can be defined in the same way. If for any $0 \leq i \leq n-1$ and some element $P \in \mathcal{S}_{\alpha_n}$, the symmetric group on the set $\{1, 2, \cdots, \alpha_n\}$, we have

$$\sum_{j=1}^{\alpha_{i+1}-\alpha_i} |E(T_{\alpha_i+j})| = \sum_{j=1}^{\beta_{i+1}-\beta_i} |E(T_{P(\beta_i+j)})|,$$

then as illustrated in Figure 6, for all $0 \leq i \leq n-1$ we replace $T_{\alpha_i+1} \vee T_{\alpha_i+2} \cdots \vee T_{\alpha_{i+1}}$ with $T_{P(\beta_i+1)} \vee T_{P(\beta_i+2)} \cdots \vee T_{P(\beta_{i+1})}$. We call this operation a *permutation move* on $T$.

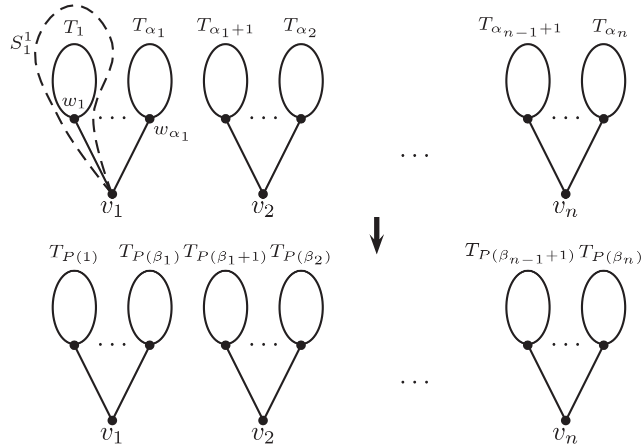


Fig. 6. A permutation move.

Clearly a permutation move preserves the plucking polynomial (see Proposition 2.4) and the exchange move described in Figure 4 is a special case of permutation move. The main result of this paper is as follows.

**Theorem 1.2.** *Let $T_I$ and $T_{II}$ be two rooted trees. Then $Q(T_I) = Q(T_{II})$ if and only if $T_I$ and $T_{II}$ are related by a finite number of stabilizations/destabilizations and one permutation move.*

The rest of this paper is organized as follows. Section 2 reviews some basic properties of plucking polynomial and then points out some other accessible calculation methods of the plucking polynomial. In section 3 we show that the trees shown in Figure 5 cannot be connected by a finite number of exchange moves, so indeed trees in Figure 5 provide a negative answer to our question given in [1]. Section 4 provides a proof of the main theorem. Finally we revisit the realization problem of the plucking polynomial and give an application of it, which may be of interest to experts in graph theory, algebraic combinatorics, or statistical mechanics.

## 2. Some properties of plucking polynomial

We first recall some standard notations in quantum calculus [3]. The $q$-analog of $n$, sometimes called the *q-bracket* or *q-number*, is defined to be $[n]_q = \frac{1-q^n}{1-q}$. Similarly, the *q-factorial* can be defined as $[n]_q! = \prod_{i=1}^{n}[i]_q$. Further, the *q-binomial coefficients* (also called *Gaussian binomial coefficients*) can be simply expressed as $\binom{m+n}{m,n}_q = \frac{[m+n]_q!}{[m]_q![n]_q!}$. In general, we define the *q-multinomial coefficient* as $\binom{n_1+\cdots+n_k}{n_1,\cdots,n_k}_q = \frac{[n_1+\cdots+n_k]_q!}{[n_1]_q!\cdots[n_k]_q!}$.

A crucial observation about the plucking polynomial in [7] can be described as follows.

**Lemma 2.1** ([7])**.** *Let $T_1, T_2$ be a pair of rooted trees on the upper half plane, and $T_1 \bigvee T_2$ the wedge product of $T_1$ and $T_2$ ($T_1$ on the left), then*

$$Q(T_1 \bigvee T_2) = \binom{|E(T_1)|+|E(T_2)|}{|E(T_1)|,|E(T_2)|}_q Q(T_1)Q(T_2).$$

Since $\binom{n_1+\cdots+n_k}{n_1,\cdots,n_k}_q = \binom{n_{k-1}+n_k}{n_{k-1},n_k}_q \binom{n_{k-2}+n_{k-1}+n_k}{n_{k-2},n_{k-1}+n_k}_q \cdots \binom{n_1+\cdots+n_k}{n_1,n_2+\cdots+n_k}_q$, repeating the lemma above $(k-1)$ times one can easily conclude the following result.

**Corollary 2.2** ([7])**.** *Let $T_1, \cdots, T_k$ be $k$ rooted trees on the upper half plane. Denote the wedge product of $T_1, \cdots, T_k$ by $\bigvee_{i=1}^{k} T_i$ ($T_i$ is on the left of $T_{i+1}$ ). Then*

$$Q(\bigvee_{i=1}^{k} T_i) = \binom{\sum_{i=1}^{k}|E(T_i)|}{|E(T_1)|,\cdots,|E(T_k)|}_q \prod_{i=1}^{k} Q(T_i).$$

The recursive definition of the plucking polynomial stated in the beginning of Section 1 is inconvenient to use to calculate the plucking polynomial. In general, one needs to pluck out the leaves of a rooted tree one by one and then calculate the plucking polynomial of the new rooted tree with fewer edges. Corollary 2.2 gives us a more convenient way to calculate the plucking polynomial of rooted trees. For a rooted tree $T$ and a fixed vertex $v$, we use the notation $d(v)$ to refer to the number of descendants of $v$. For example, if $v = r$, the root of $T$, then $d(r) = |V(T)| - 1 = |E(T)|$. Denote all children of $v$ by $v_1, \cdots, v_k$, we associate a *Boltzmann weight* $W(v)$ with $v$, which is defined by

$$W(v) = \binom{d(v)}{d(v_1)+1, \cdots, d(v_k)+1}_q.$$

Note that $d(v) = \sum_{i=1}^{k}(d(v_i)+1)$. The following state product formula of the plucking polynomial was proved in [7].

**Proposition 2.3** (State product formula [7])**.** $Q(T) = \prod_{v \in V(T)} W(v)$.

It follows immediately that plucking polynomial does not depend on a particular embedding of $T$, so the plucking polynomial is an invariant of rooted trees. On the other hand, since the plucking polynomial can be written as the product of some $q$-multinomial coefficients and each $q$-multinomial coefficient can be written as the product of some $q$-binomial coefficients, we conclude that plucking polynomial of rooted trees can be written as the product of some $q$-binomial coefficients. Based on this fact, in [1] we gave a complete answer to the first question mentioned in Section 1. To address the second question, we need to simplify further the formula for plucking polynomial.

**Proposition 2.4.** *Let $T$ be a rooted tree and $r$ the root of it, then* $Q(T) = \frac{[d(r)]_q!}{\prod_{v \in V(T)\setminus\{r\}} [d(v)+1]_q}$.

Proof. According to Proposition 2.3, the plucking polynomial $Q(T)$ has the form $\prod_{v \in V(T)} W(v)$. Let $v(\neq r)$ be a vertex of $T$ and $w$ its ancestor. Note that the numerator of $W(v)$ equals $[d(v)]_q!$, and the denominator of $W(w)$ has a factor $[d(v)+1]_q!$. After canceling $[d(v)]_q!$ for all non-root vertices the result follows. □

Proposition 2.4 motivates us to consider the multiset $D(T) = \{d(v) | v \in V(T)\}$[1]. According to Proposition 2.4, the plucking polynomial $Q(T)$ is determined by the set $D(T)$. The following proposition tells us that for reduced trees $Q(T)$ and $D(T)$ are essentially equivalent.

**Theorem 2.5.** *Assume $T_1$ and $T_2$ are two reduced rooted trees, then $Q(T_1) = Q(T_2)$ if and only if $D(T_1) = D(T_2)$.*

Proof. It suffices to prove that if $Q(T_1) = Q(T_2)$ then $D(T_1) = D(T_2)$. Assume $D(T_1) = \{a_1, \cdots, a_n\}$ and $D(T_2) = \{b_1, \cdots, b_m\}$, where $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_m$. Since $T_1$ and $T_2$ are both reduced, we observe that $d(r_1) = a_1 > a_2 + 1, d(r_2) = b_1 > b_2 + 1$ and $n = a_1 + 1, m = b_1 + 1$. Here $r_i$ denotes the root of $T_i$ $(i = 1, 2)$. It follows that

$$Q(T_1) = \frac{[a_1]_q!}{\prod_{i=2}^{n}[a_i+1]_q} = \frac{\prod_{i=1}^{a_1}(1-q^i)}{\prod_{i=2}^{n}(1-q^{a_i+1})} \text{ and } Q(T_2) = \frac{[b_1]_q!}{\prod_{i=2}^{m}[b_i+1]_q} = \frac{\prod_{i=1}^{b_1}(1-q^i)}{\prod_{i=2}^{m}(1-q^{b_i+1})}.$$

It is clear that $e^{\frac{2\pi i}{a_1}}$ is a root of $Q(T_1)$ with the minimal argument and $e^{\frac{2\pi i}{b_1}}$ is a root of $Q(T_2)$ with the minimal argument. Since $Q(T_1) = Q(T_2)$, we must have $a_1 = b_1$, therefore $n = a_1 - 1 = b_1 - 1 = m$.

---

[1]Equivalently, instead of the set $D(T)$, one can also consider the generating function $\sum_i c_i x^i$, where $c_i$ denotes the multiplicity of $i$ in $D(T)$.

Since $\prod_{i=2}^{n}[a_i+1]_q = \prod_{i=2}^{n}[b_i+1]_q$, it follows that $a_i = b_i$, for all $2 \le i \le n$. $\square$

In this section, we only mentioned some basic properties of the plucking polynomial that are relevant to this paper. The reader interested in many other important properties of the polynomial is referred to [2], where the unimodality of its coefficients was proven, or to [7], where a connection between the plucking polynomial and homological algebra is discussed.

### 3. The exchange move is not sufficient

In this section we show that the two rooted trees depicted in Figure 5 cannot be connected by exchange moves, although they have the same plucking polynomial. In other words, exchange move is not sufficient to connect all pairs of rooted trees with the same plucking polynomial.

First we notice that one can only obtain finitely many rooted trees from the rooted tree $T_1$ in Figure 5 via exchange moves. By comparing them with $T_2$ one finds that $T_2$ is different from all of them.

Recall that in Section 2 we associate a Boltzmann weight $W(v)$ with each vertex $v$, which is defined by

$$W(v) = \binom{d(v)}{d(v_1)+1,\cdots,d(v_k)+1}_q.$$

Let $U(v)$ denote the unordered $k$-tuple $(d(v_1)+1,\cdots,d(v_k)+1)$, and $U(T)$ the multiset $\{U(v)\}_{v\in V(T)}$. Consider $v_1, v_2$ in Figure 4. If $U(v_1) = (a_1,\cdots,a_m,b_1,\cdots,b_n)$, $U(v_2) = (c_1,\cdots,c_s,d_1,\cdots,d_t)$, and $\sum_{i=1}^{n} b_i = \sum_{j=1}^{s} c_j$, then after the exchange move we have $U(v_1) = (a_1,\cdots,a_m,c_1,\cdots,c_s)$, $U(v_2) = (b_1,\cdots,b_n,d_1,\cdots,d_t)$ and all other tuples in $U(T)$ are preserved.

For the two rooted trees $T_1, T_2$ in Figure 5, we have

$$U(T_1) = \{(9,10),(3,6),(1,7),(2,2),(6),(5)\times 2,(4),(3),(2)\times 2,(1)\times 4,(0)\times 5\}$$

and

$$U(T_2) = \{(9,10),(2,7),(2,6),(1,3),(6),(5)\times 2,(4),(3),(2)\times 2,(1)\times 4,(0)\times 5\}.$$

A key observation is, although many rooted trees can be obtained from $T_1$ via exchange moves, most of them have the same set $U$ as $T_1$. The only exception is

$$U = \{(3,6,10),(9),(1,7),(2,2),(6),(5)\times 2,(4),(3),(2)\times 2,(1)\times 4,(0)\times 5\}.$$

It follows that $T_2$ cannot be obtained from $T_1$ by exchange moves.

One can directly show that there are essentially only four different rooted trees that can be obtained from $T_1$ by exchange moves (see Figure 7). Here we illustrate how one rooted tree can be obtained from another by exchange moves. Obviously $T_2$ is not one of them.

Before ending this section, we would like to remark that one can find some other pairs of "smaller" rooted trees with the same plucking polynomial but they cannot be connected by exchange moves. For example, the two rooted trees $T_3$ and $T_4$ described in Figure 8 have the same plucking polynomial. Similar as above one can check that they are not related by exchange moves. Note that $|E(T_3)| = |E(T_4)| = 18 < 19 = |E(T_1)| = |E(T_2)|$. These two rooted trees $T_3$ and $T_4$ can be regarded as a reduced version of Hao Zheng's $T_1, T_2$ described in Figure 5.

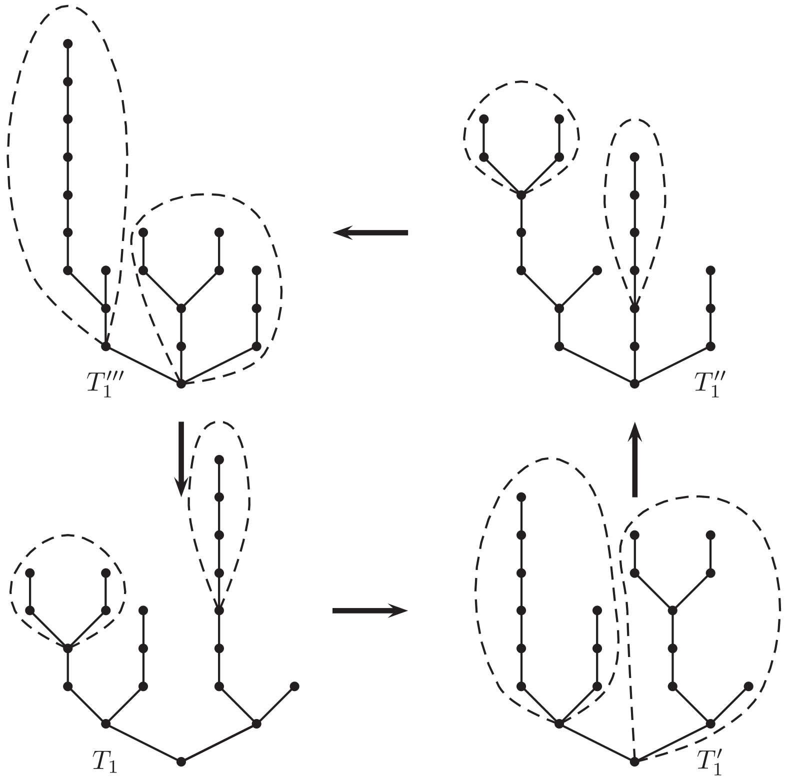


Fig. 7. Rooted trees obtained from $T_1$ by exchange moves.

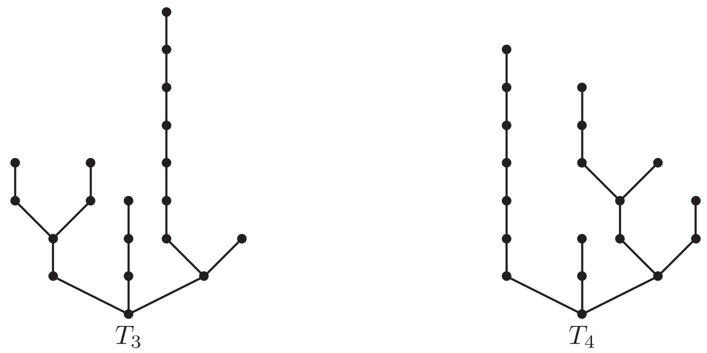


Fig. 8. Another counterexample with fewer edges.

## 4. The proof of Theorem 1.2

Now we give a proof of Theorem 1.2.

Proof. With some destabilization (if necessary) we may assume that $T_I, T_{II}$ are both reduced. By Theorem 2.5 we deduce that $D(T_I) = D(T_{II})$. In particular, $T_I$ and $T_{II}$ have the same number of edges. Now it is sufficient to prove that $T_I$ and $T_{II}$ are related by one permutation move.

The proof goes by induction on $|E(T_I)|\,(=|E(T_{II})|)$. When $|E(T_I)| = 1, 2, 3, 4$, there is no pair of distinct rooted trees with the same plucking polynomial. The first pair of rooted trees appear when $|E(T_I)| = |E(T_{II})| = 5$, see Figure 9. It is easy to see that the second tree can be obtained from the first tree by one exchange move, which interchanges the places of the subtrees bounded by dashed curves.

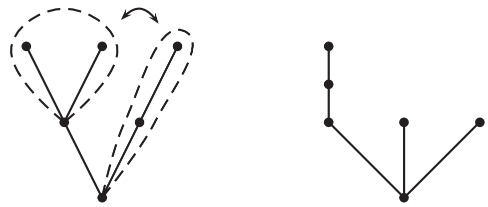

Fig. 9. Two 5-edge rooted trees with the same plucking polynomial.

Now we assume that any two reduced rooted trees can be transformed into each other by one permutation move if they have $(k-1)$ edges and the same plucking polynomial. It is sufficient to prove that the statement is still correct for the case $|E(T_I)| = |E(T_{II})| = k$. Suppose that $T_I, T_{II}$ are two reduced rooted trees with the same plucking polynomial and $|E(T_I)| = |E(T_{II})| = k$. We need to prove that $T_I$ and $T_{II}$ differ by one permutation move.

Denote the roots of $T_I, T_{II}$ by $r_1, r_2$ respectively. Let us assume that $D(T_I) = D(T_{II}) = \{a_1, a_2, \cdots, a_{k+1}\}$, where $a_1 \geq a_2 \geq \cdots \geq a_{k+1}$. It is clear that $a_1 = d(r_1) = d(r_2) = k$. Choose vertices $u_1, u_2$ in $T_I, T_{II}$ respectively such that $d(u_1) = d(u_2) = a_2$. Then $u_i$ must be a child of $r_i$ $(i = 1, 2)$. We use $e_i$ to denote the edge between $r_i$ and $u_i$ $(i = 1, 2)$, and by taking edge contractions on $e_1$ and $e_2$ we will obtain two new rooted trees $T'_I$ and $T'_{II}$. Note that for these two new rooted trees we have $D(T'_I) = D(T'_{II}) = \{a_1 - 1, a_3, \cdots, a_{k+1}\}$, and therefore $T'_I$ and $T'_{II}$ have the same plucking polynomial. By the induction assumption, it follows that $T'_{II}$ can be obtained from $T'_I$ by making one permutation move. Since there is a one-to-one correspondence between $E(T_I)\backslash\{e_1\}$ and $E(T'_I)$, $V(T_I)\backslash\{u_1\}$ and $V(T'_I)$, we will use the same notation to denote an edge (vertex) in $T_I$ and its corresponding edge (vertex) in $T'_I$. In particular, the roots of $T'_I$ and $T'_{II}$ are still denoted by $r_1$ and $r_2$.

If $r_1$ coincides with some element in $\{v_1, \cdots, v_n\}$ (see Figure 6), without loss of generality, let us assume that $v_1 = r_1$. In $T'_I$, denote the children of $v_1$ involved in the permutation move by $w_1, \cdots, w_{\alpha_1}$. Then in $T_I$, some elements of $\{w_1, \cdots, w_{\alpha_1}\}$ are adjacent to $u_1$ and others are adjacent to $r_1$. Without loss of generality, we assume $w_1, \cdots, w_\gamma$ are adjacent to $u_1$ and $w_{\gamma+1}, \cdots, w_{\alpha_1}$ are adjacent to $r_1$, where $0 \leq \gamma \leq \alpha_1$. As before, for each $i = 1, 2, \cdots, \alpha_1$ we still use $T_i$ to denote the subtree involved in the permutation move which contains the vertex $w_i$. After the permutation move, the places of $\bigvee_{i=1}^{\alpha_1} T_i$ will be occupied by $\bigvee_{i=1}^{\beta_1} T_{P(i)}$, where $P$ is the corresponding permutation of $\{1, \cdots, \alpha_n\}$ (see Figure 6). Similarly, in the original rooted tree $T_{II}$, some of $\{T_{P(1)}, \cdots, T_{P(\beta_1)}\}$ are attached to $u_2$ and others are attached to $r_2$. Let us assume $\bigvee_{i=1}^{\delta} T_{P(i)}$ are attached to $u_2$ and $\bigvee_{i=\delta+1}^{\beta_1} T_{P(i)}$ are attached to $r_2$.

Now let us consider the remaining children (and their descendants) of $r_1$, which form a rooted subtree of $T'_I$. This rooted subtree can be regarded as the wedge sum of four rooted subtrees, say $T_{A_1} \bigvee T_{B_1} \bigvee T_{C_1} \bigvee T_{D_1}$. Without loss of generality, we suppose that in $T_I$ the subtree $T_{A_1} \bigvee T_{C_1}$ is attached to $u_1$ and $T_{B_1} \bigvee T_{D_1}$ is attached to $r_1$. Since $T'_I$ and $T'_{II}$ differ by one permutation move, and the rest children of $r_1$ are not involved in

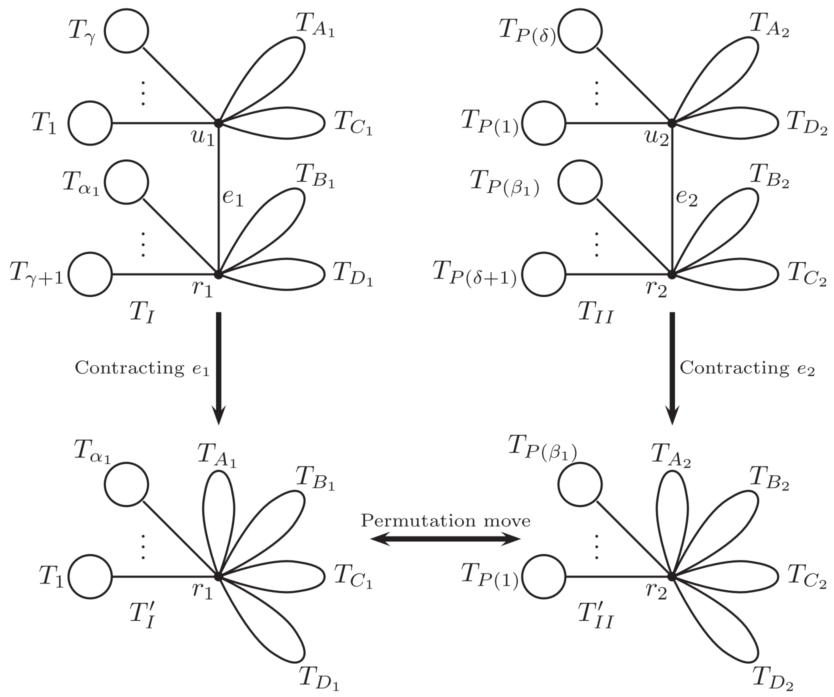


Fig. 10. The relations between $T_I, T_{II}, T'_I, T'_{II}$.

the permutation move, there are four subtrees corresponding to $T_{A_1} \bigvee T_{B_1} \bigvee T_{C_1} \bigvee T_{D_1}$ in $T'_{II}$, say $T_{A_2} \bigvee T_{B_2} \bigvee T_{C_2} \bigvee T_{D_2}$. Without loss of generality, we assume in $T_{II}$ the subtree $T_{A_2} \bigvee T_{D_2}$ is attached to $u_2$ and the other subtree $T_{B_2} \bigvee T_{C_2}$ is attached to $r_2$, see Figure 10. We note that since the permutation move was applied, $T_{A_1}$ and $T_{A_2}$ may represent different rooted trees, but they have the same number of edges, i.e. $|E(T_{A_1})| = |E(T_{A_2})|$. This equality also holds for the other three pairs of subtrees.

Observe that, due to our choices, $d(u_1) = d(u_2)$. On the other hand, we know that

$$d(u_1) = \sum_{i=1}^{\gamma} |E(T_i)| + |E(T_{A_1})| + |E(T_{C_1})| \text{ and}$$

$$d(u_2) = \sum_{i=1}^{\delta} |E(T_{P(i)})| + |E(T_{A_2})| + |E(T_{D_2})|.$$

Together with $|E(T_{A_1})| = |E(T_{A_2})|$, $|E(T_{D_1})| = |E(T_{D_2})|$ and $\sum_{i=1}^{\alpha_1} |E(T_i)| = \sum_{i=1}^{\beta_1} |E(T_{P(i)})|$ (by the definition of the permutation move), we deduce that

$$\sum_{i=1}^{\gamma} |E(T_i)| + |E(T_{C_1})| = \sum_{i=1}^{\delta} |E(T_{P(i)})| + |E(T_{D_1})|, \text{ and}$$

$$\sum_{i=\gamma+1}^{\alpha_1} |E(T_i)| + |E(T_{D_1})| = \sum_{i=\delta+1}^{\beta_1} |E(T_{P(i)})| + |E(T_{C_1})|.$$

Consider the the permutation move on $T_I$ as shown in Figure 11. Roughly speaking, this

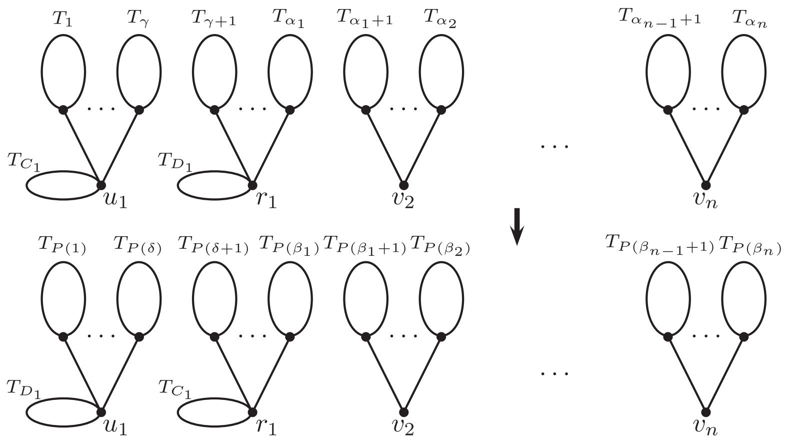


Fig. 11. A permutation move connecting $T_I$ and $T_{II}$.

permutation move is obtained from the permutation move on $T'_I$ by splitting the children of $v_1$ into the children of $u_1$ and the children of $r_1$, and all other subtrees $T_{\alpha_1+1}, \cdots, T_{\alpha_n}$ on $v_2, \cdots, v_n$ are the same as that of the permutation move on $T'_I$. Since

$$\sum_{i=1}^{\gamma} |E(T_i)| + |E(T_{C_1})| = \sum_{i=1}^{\delta} |E(T_{P(i)})| + |E(T_{D_1})| \text{ and}$$
$$\sum_{i=\gamma+1}^{\alpha_1} |E(T_i)| + |E(T_{D_1})| = \sum_{i=\delta+1}^{\beta_1} |E(T_{P(i)})| + |E(T_{C_1})|,$$

we find that this move satisfies the definition of permutation move. It is not difficult to find that under this permutation move $T_I$ will be transformed into $T_{II}$.

If $v_i \neq r_1$ for any $1 \leq i \leq n$, the proof is similar. One just needs to notice that in this case $\alpha_1 = \beta_1 = 0$, in other words, $\bigvee_{i=1}^{\alpha_1} T_i = \bigvee_{i=1}^{\beta_1} T_{P(i)} = \emptyset$. □

**Corollary 4.1.** *Let $T_I$ and $T_{II}$ be two reduced rooted trees. $D(T_I) = D(T_{II})$ if and only if they can be connected by one permutation move.*

As an example, we will show how to use a permutation move to connect the two rooted trees described in Figure 5. Choose three vertices $v_1, v_2$ and $v_3$ from $T_1$ (see Figure 5, notice that here $v_1$ is a descendant of $v_2$) and $P = (632541) = (16)(23)(45) \in \mathcal{S}_6$. Now the following permutation move (see Figure 12) can be used to transform $T_1$ into $T_2$. Note that, as we have already shown in Section 3, $T_1$ cannot be connected to $T_2$ via exchange moves only.

Finally, we would like to note that, for two rooted trees connected by a permutation move there might be many different permutation moves which also connect them.

## 5. The realization problem revisited

In this section we revisit the first question mentioned in Section 1, i.e. for a given polynomial $f(q) \in \mathbb{Z}[q]$, can we find a rooted tree $T$ such that $Q(T) = f(q)$? According to Proposition 2.4, it is equivalent to ask the following question from the viewpoint of graph

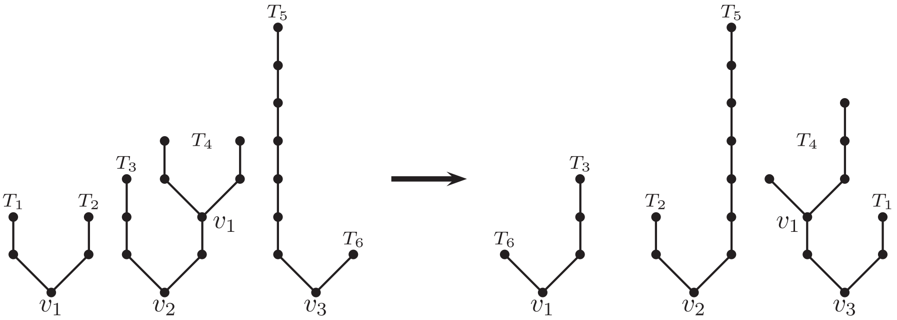


Fig. 12. A permutation move connecting $T_1$ and $T_2$ in Figure 5.

theory.

Question 5.1. For a given $n$-multiset $\{a_1, a_2, \cdots, a_n\}$ (without loss of generality, we suppose $0 \le a_1 \le a_2 \le \cdots \le a_n$), is there a rooted tree $T$ with $D(T) = \{a_1, a_2, \cdots, a_n\}$?

It is easy to find some obstacles for a $n$-multiset $\{a_1, \cdots, a_n\}$ being the set $D(T)$ of some rooted tree $T$. For example, it is evident to see that $a_n = n - 1$ and $a_{n-1} < a_n$. Hence both $\{0, 0, 1, 1, 2\}$ and $\{0, 0, 1, 2, 2\}$ are not realizable. On the other hand, $a_1$ must be 0. Actually, it is easy to observe that the number of 0's in $\{a_1, \cdots, a_n\}$ is not less than the number of 1's, since each vertex $v$ with $d(v) = 1$ has a child $w$ with $d(w) = 0$. Therefore, for instance $\{0, 1, 1, 3, 4\}$ is also not realizable. However, even if $\{a_1, \cdots, a_n\}$ satisfies all conditions above, it is not always realizable. As an example, one can easily find that although $\{0, 1, 2, 2, 4\}$ satisfies all conditions above, it cannot be realized as the set $D(T)$ for some rooted tree $T$.

If there exists a rooted tree $T$ with $D(T) = \{a_1, \cdots, a_n\}$, then we know that the plucking polynomial of $T$ equals $\frac{[a_n]_q!}{\prod_{i=1}^{n-1}[a_i+1]_q}$. Since the plucking polynomial of a rooted tree can be written as the product of some $q$-binomial coefficients, it follows that $\frac{[a_n]_q!}{\prod_{i=1}^{n-1}[a_i+1]_q}$ can be written as the product of some $q$-binomial coefficients. As the main result of this section, we will show that this condition is not only necessary but also sufficient. Hence it offers a complete answer to Question 5.1.

**Theorem 5.2.** *For a given n-multiset* $\{a_1, a_2, \cdots, a_n\}$ *where* $0 \le a_1 \le a_2 \le \cdots \le a_n$, *there exists a rooted tree T such that* $D(T) = \{a_1, a_2, \cdots, a_n\}$ *if and only if* $\frac{[a_n]_q!}{\prod_{i=1}^{n-1}[a_i+1]_q}$ *can be written as the product of some q-binomial coefficients.*

Before giving the proof of this theorem, let us take a brief review of our main result in [1]. Assume that a polynomial $f(q)$ is a product of some $q$-binomial coefficients, i.e.

$$f(q) = \prod_{i=1}^{k} \binom{m_i+n_i}{m_i, n_i}_q = \prod_{i=1}^{k} \frac{[m_i+n_i]_q!}{[m_i]_q![n_i]_q!}.$$

If a $q$-number $[p]_q$ appears both in the numerator and denominator of $f(q)$, then we delete both of them. Finally we will obtain a fraction $\frac{[a_1]_q \cdots [a_s]_q}{[b_1]_q \cdots [b_t]_q}$ and $a_i \neq b_j$. We call this fraction the *reduced form* of $f(q)$. It is not difficult to observe that the reduced form is unique.

**Theorem 5.3** ([1]). *Consider a product of q-binomial coefficients $f(q) = \prod_{i=1}^{k} \binom{m_i+n_i}{m_i,n_i}_q$, then $f(q)$ can be realized as the plucking polynomial of some rooted trees if and only if each q-number appears at most once in the numerator of the reduced form of $f(q)$.*

We would like to remark that we can always find a binary rooted tree $T$ to realize $f(q)$, and $\prod_{i=1}^{k} \binom{m_i+n_i}{m_i,n_i}_q$ coincides with the state product formula (Proposition 2.3) of the plucking polynomial of $T$ if we ignore the contributions from those vertices that have only one child (note that these contributions are trivial). The readers are referred to [1] for more details.

Now we give the proof of Theorem 5.2. Proof. The "only if" part has been explained in Section 2, therefore it suffices to prove the"if" part.

First note that if $a_{n-1} = a_n$ then obviously $\frac{[a_n]_q!}{\prod_{i=1}^{n-1}[a_i+1]_q}$ cannot be written as a product of some $q$-binomial coefficients. We claim that actually we can assume that $a_{n-1} \leq a_n - 2$. This is because, if $a_{n-1} = a_n - 1$ then we have

$$\frac{[a_n]_q!}{\prod_{i=1}^{n-1}[a_i+1]_q} = \frac{[a_{n-1}]_q!}{\prod_{i=1}^{n-2}[a_i+1]_q}.$$

If $\{a_1, a_2, \cdots, a_{n-1}\}$ can be realized as the set $D(T)$ for some rooted tree $T$, then we can find a rooted tress $T'$ realizing $\{a_1, a_2, \cdots, a_n\}$ by applying a stabilization on $T$. Hence from now on let us assume that $a_{n-1} \leq a_n - 2$.

Note that if $\frac{[a_n]_q!}{\prod_{i=1}^{n-1}[a_i+1]_q}$ can be written as the product of some $q$-binomial coefficients $\prod_{i=1}^{k} \binom{m_i+n_i}{m_i,n_i}_q$, then the numerator of the reduced form does not repeat any $q$-number, since the numerator equals $[a_n]_q! = [1]_q[2]_q \cdots [a_n]_q$. According to Theorem 5.3 we know that there exists a rooted tree $T$ such that $Q(T) = \prod_{i=1}^{k} \binom{m_i+n_i}{m_i,n_i}_q = \frac{[a_n]_q!}{\prod_{i=1}^{n-1}[a_i+1]_q}$. With some destabilizations we can assume that $T$ is reduced.

Let us assume that $D(T) = \{b_1, \cdots, b_m\}$ and $0 \leq b_1 \leq b_2 \leq \cdots \leq b_m$. Then we have

$$\frac{[b_m]_q!}{\prod_{i=1}^{m-1}[b_i+1]_q} = \frac{[a_n]_q!}{\prod_{i=1}^{n-1}[a_i+1]_q}.$$

Since $a_{n-1} \leq a_n - 2$ and $b_{m-1} \leq b_m - 2$, it follows that $b_m = a_n$. Now we have

$$\frac{1}{\prod_{i=1}^{m-1}[b_i+1]_q} = \frac{1}{\prod_{i=1}^{n-1}[a_i+1]_q},$$

which implies $\{b_1 + 1, \cdots, b_{m-1} + 1\} = \{a_1 + 1, \cdots, a_{n-1} + 1\}$ and $m = n$. Therefore $D(T) = \{b_1, \cdots, b_m\} = \{a_1, \cdots, a_n\}$, this completes the proof. □

ACKNOWLEDGEMENTS. The first author would like to thank Hao Zheng for helpful discussion. The first author is supported by NSFC 11771042 and NSFC 11571038. The third author is supported by the Simons Foundation Collaboration Grant for Mathematicians–316446 and CCAS Dean's Research Chair award.

## References


[1] Z. Cheng, S. Mukherjee, J.H. Przytycki, X. Wang and S.Y. Yang: *Realization of plucking polynomials*, J. Knot Theory Ramifications **26** (2017), 1740016, 9pp.

[2] Z. Cheng, S. Mukherjee, J.H. Przytycki, X. Wang and S.Y. Yang: *Strict unimodality of plucking polynomials of rooted trees*, J. Knot Theory Ramifications **27** (2018), 1841009, 19pp.

[3] V. Kac and P. Cheung: Quantum calculus, Universitext, Springer, New York, 2002.

[4] M.K. Dabkowski, C. Li and J.H. Przytycki: *Catalan states of lattice crossing*, Topology Appl. **182** (2015), 1–15.

[5] M.K. Dabkowski and J.H. Przytycki: *Catalan States of Lattice Crossing: Application of Plucking Polynomial*, Topology Appl. **254** (2019), 12–28.

[6] J.H. Przytycki: *Skein modules of 3-manifolds*, Bull. Polish. Acad. Sci. Math. **39** (1991), 91–100.

[7] J.H. Przytycki: *q-polynomial invariant of rooted trees*, Arnold Math. J. **2** (2016), 449–461.



Zhiyun Cheng
School of Mathematical Sciences
Laboratory of Mathematics and Complex Systems
Beijing Normal University
Beijing 100875
China
e-mail: czy@bnu.edu.cn

Sujoy Mukherjee
Department of Mathematics
The George Washington University
Washington, DC 20052
USA
e-mail: sujoymukherjee@gwu.edu

Józef H. Przytycki
Department of Mathematics
The George Washington University
Washington, DC 20052
USA and University of Gdańsk
e-mail: przytyck@gwu.edu

Xiao Wang
Department of Mathematics
Jilin University
Changchun 130012
China and Department of Mathematics
The George Washington University
Washington, DC 20052
USA
e-mail: wangxiao@gwu.edu

Seung Yeop Yang
Department of Mathematics
Kyungpook National University
Daegu, 41566
Republic of Korea
e-mail: seungyeop.yang@knu.ac.kr