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LAGRANGIAN SUBMANIFOLDS IN STRICT NEARLY KÄHLER 6-MANIFOLDS

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Abstract

Lagrangian submanifolds in strict nearly Kähler 6-manifolds are related to special Lagrangian submanifolds in Calabi-Yau 6-manifolds and coassociative cones in G_2-manifolds. We prove that the mean curvature of a Lagrangian submanifold L in a nearly Kähler manifold (M, J, g) is symplectically dual to the Maslov 1-form on L. Using relative calibrations, we derive a formula for the second variation of the volume of a Lagrangian submanifold L^3 in a strict nearly Kähler manifold (M^6, J, g) and compare it with McLean's formula for special Lagrangian submanifolds. We describe a finite dimensional local model of the moduli space of compact Lagrangian submanifolds in a strict nearly Kähler 6-manifold. We show that there is a real analytic atlas on (M^6, J, g) in which the strict nearly Kähler structure (J, g) is real analytic. Furthermore, w.r.t. an analytic strict nearly Kähler structure the moduli space of Lagrangian submanifolds of M^6 is a real analytic variety, whence infinitesimal Lagrangian deformations are smoothly obstructed if and only if they are formally obstructed. As an application, we relate our results to the description of Lagrangian submanifolds in the sphere S^6 with the standard nearly Kähler structure described in [34].

Contents

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1. Introduction

Nearly Kähler manifolds first appeared in Gray's work [15] in connection with Gray's notion of weak holonomy. Nearly Kähler manifolds represent an important class in the 16 classes of almost Hermitian manifolds (M, J, g) classified by Gray and Hervella [17]. Let us recall the definition of a nearly Kähler manifold (M, J, g). Let ∇^{LC} denote the Levi-Civita covariant derivative associated with the Riemannian metric g.

Definition 1.1([15, §1, Proposition 3.5], [16]). An almost Hermitian manifold (M, g, J) is called *nearly Kähler* if $(\nabla^{LC}_X J)X = 0$ for all $X \in TM$. A nearly Kähler manifold is called *strict* if we have $\nabla^{LC}_X J \neq 0$ for all $X \in TM \setminus \{0\}$, and it is called of *constant type* if there exists a positive constant λ such that

$$||(\nabla^{LC}_X J)Y||^2 = \lambda^2(||X||^2||Y||^2 - \langle X, Y\rangle^2 - \langle JX, Y\rangle^2)$$

for every $x \in M$ and $X, Y \in T_xM$.

Remark 1.2. 1. It is known that any complete simply connected nearly Kähler manifold is a Riemannian product $M_1 \times M_2$ where M_1 and M_2 are Kähler and strict nearly Kähler, respectively [23, 42]. Furthermore, a de Rham type decomposition of a strictly nearly Kähler manifold was found by Nagy [43], where the factors of the decomposition are of the following types: 3-symmetric spaces, twistor spaces over quaternionic Kähler manifolds of positive scalar curvature, and strict nearly Kähler 6-manifolds.

2. It is easy to see that if (M, J, g) is a nearly Kähler manifold of constant type λ, then $(M, J, \lambda^{-1}g)$ and $(M, -J, \lambda^{-1}g)$ are nearly Kähler manifolds of constant type 1.

3. According to [16, Theorem 5.2], a strict nearly Kähler manifold of dimension 6 is always of constant type.

On an almost Hermitian manifold (M, J, g) the *fundamental 2-form* ω, defined by $\omega(X, Y) := g(JX, Y)$, measures the connection between the almost complex structure J and the Riemannian metric g. A submanifold $L \subset (M, J, g)$ whose dimenion is half the dimension of M is called *Lagrangian*, if $\omega|_L = 0$. As in symplectic geometry, the graph of a diffeomorphism of M that preserves ω is a Lagrangian submanifold in the almost Hermitian manifold $(M \times M, J \oplus (-J), g \oplus g)$. If (M, J, g) is Kähler, then ω is symplectic. Lagrangian submanifolds in Kähler manifolds have been studied in the context of calibrated geometry [18] and of relative calibrations [26], [27], in the investigation of the Maslov class [29], [40], of the variational problem [26], [44], [50], [52], and of the deformation problem/ moduli spaces [5], [6], [20], [30], [36], [52], etc. The literature on the subject is vast, and the authors omit the name of many important papers in the field.

The relation between nearly Kähler manifolds (M, J, g) and Riemannian manifolds with special holonomy is best manifested in dimension 6. In this dimension, a nearly Kähler manifold is either a Kähler manifold or a strict nearly Kähler manifold [16, Theorem 5.2]. It is known from Bär's work [4] that a cone without singular point over a strict nearly Kähler manifold (M^6, J, g) is a 7-manifold with G_2-holonomy. It is not hard to see that the cone over a Lagrangian submanifold L^3 in a strict nearly Kähler manifold (M^6, J, g) is a coassociative cone in CM^6. Thus the study of strict nearly Kähler 6-manifolds and their Lagrangian submanifolds are essential for the study of singular points of G_2-manifolds as well as for the study of singular points of coassociative 4-folds. Furthermore, special Lagrangian submanifolds in Calabi-Yau 6-manifolds could be treated as a limit case of Lagrangian submanifolds in nearly Kähler manifolds when the type constant λ goes to zero (Remarks 2.7, 3.15). We also note that Lagrangian submanifolds in the standard nearly Kähler manifold S^6 are found to be intimately related to holomorphic curves in $\mathbb{CP}^2$ and they present extremely rich geometry [8], [10], [33].

In this paper we study Lagrangian submanifolds L^3 in strict nearly Kähler 6-manifolds (M^6, J, g) in two aspects: the variation of the volume functional and Lagrangian deformations of L^3. Since L^3 are minimal submanifolds in (M^6, J, g) (Corollary 3.6), these two aspects are related to each other. In particular, results from theory of minimal submanifolds are applicable to Lagrangian submanifolds in strict nearly Kähler 6-manifolds, for instance see Remark 5.1. To study variation of the volume functional of L^3 we extend the method of relative calibrations developed by the first named author in [26, 27]. To study deformations of Lagrangian submanifolds in (M^6, J, g) we develop several methods. First we reduce the overdetermined equation for Lagrangian deformations to an elliptic equation (Proposition 4.4). Since the Fredholm index of the elliptic equation is zero (Proposition 4.6) and, on the other hand, most interesting examples of Lagrangian submanifolds have nontrivial deformations, the usual elliptic method yields only limited results. Thanks to our result on the analyticity of a strict nearly Kähler structure (Proposition 2.8), we reduce the smooth Lagrangian deformation problem to the deformation problem in the analytic category. We prove that the moduli space of Lagrangian deformation is locally an analytic variety and hence an infinitesimal Lagrangian deformation is smoothly unobstructed iff it is formally unobstructed (Theorem 4.9, Corollary 4.12).

Our paper is organized as follows. In section 2 we collect some important results on the canonical Hermitian connection on nearly Kähler manifolds. Then we prove the existence of a real analytic structure on any strict nearly Kähler 6-manifold (M^6, J, g) in which both J and g are real analytic (Proposition 2.8). In section 3, using a result of the first named author [26], we establish a relation between the Maslov 1-form and the mean curvature of a Lagrangian submanifold in a nearly Kähler manifold (M, J, g) (Proposition 3.3) and show its consequences (Corollaries 3.4, 3.6). If (M^6, J, g) is a strictly nearly Kähler 6-manifold, we derive a simple formula for the second variation of a Lagrangian submanifold in (M^6, J, g) using relative calibrations (Theorem 3.8). We compare this formula with the formula obtained by McLean for special Lagrangian submanifolds (Corollary 3.12, Remarks 3.13, 3.15, 3.16).

In section 4 we show that the moduli space of closed Lagrangian submanifolds $L^3 \subset M^6$ of a strict nearly Kähler manifold in the C^1-topology is locally a real analytic variety. That is, the set of C^1-small deformations of a compact Lagrangian submanifold can be described as the inverse image of a point of a real analytic map between open domains in a finite dimensional vector space, whence any smooth Lagrangian deformation of a Lagrangian submanifold L^3 in a strict nearly Kähler 6-manifold can be written as a convergent power series (Theorem 4.9).

In section 5, we apply our results to deformations of homogeneous Lagrangian submanifolds in the standard nearly Kähler sphere S^6 which have been considered by Lotay in [34] and give an explicit description of the moduli space for some of these cases.

2. Geometry of nearly Kähler manifolds

In this section we collect some important results on the canonical Hermitian connection on nearly Kähler manifolds (Propositions 2.1, 2.2) and derive an important consequence (Corollary 2.3), which plays a central role in the geometry of strict nearly Kähler 6-manifolds (Proposition 2.4, Remark 2.7). At the end of this section we prove the existence of a real analytic structure on M^6, in which both the complex structure J and the metric g are analytic (Proposition 2.8).

2.1. The canonical Hermitian connection. Let $U(M)$ denote the principal bundle consisting of unitary frames $(e_1, Je_1, \cdots, e_n, Je_n)$ over an almost Hermitian manifold (M, J, g). Denote by $\{e_i^*, (Je_i)^*\}$ the dual frames. Then $\{\theta^i := e_i^* + \sqrt{-1}(Je_i)^*\}$ is the canonical $\mathbb{C}^n$-valued 1-form on $U(M)$.

Let $\tilde{\omega}$ be a unitary connection 1-form on $U(M)$ and T its torsion 2-form. The Cartan equation for $\tilde{\omega}$, and T [22, Chapter IX, §3] [26, §3] is expressed as follows

$$d\theta^i = -\tilde{\omega}^i_j \wedge \theta^j + T^i_{jk}\theta^j \wedge \theta^k + T^i_{\bar{j}k}\bar{\theta}^j \wedge \theta^k + T^i_{\bar{j}\bar{k}}\bar{\theta}^j \wedge \bar{\theta}^k,$$

$$d\tilde{\omega}^i_j = -\tilde{\omega}^i_k \wedge \tilde{\omega}^k_j + \Omega^i_j,$$

where Ω is the curvature tensor of $\tilde{\omega}$.

Proposition 2.1 ([31, Chapter IV, §112]). *Let (M, J, g) be an almost Hermitian manifold. Then there exists a unique unitary connection 1-form $\tilde{\omega}$ on $U(M)$ such that its torsion tensor T is a two-form of type $(2,0) + (0,2)$, i.e.,*

$$T(JX, Y) = T(X, JY).$$

We shall denote the Levi-Civita connection of g and the canonical connection from this proposition by ∇^{LC} and ∇^{can}, respectively. If the almost Hermitian manifold is nearly Kähler, then the following is known.

Proposition 2.2 ([16], [23, Theorem 1]). *Suppose that (M, J, g) is a nearly Kähler manifold.*

(1) *Then $T(X, Y) = -J(\nabla^{LC}_X J)Y$.*
(2) *The associated torsion form $T^*(X, Y, Z) := \langle T(X, Y), Z\rangle$ is skew-symmetric.*
(3) $\nabla^{can} T^* = 0$.

The skew-symmetry of the torsion of the canonical connection of a nearly Kähler manifold (M, J, g) will play an important rôle in our study of (M, J, g).

We shall derive from Proposition 2.2 the following

Corollary 2.3. *On a nearly Kähler manifold (M, J, g) we have $d\omega(X, Y, Z) = -3T^*(X, Y, JZ)$. Furthermore, $d\omega$ is a 3-form of type $(3,0) + (0,3)$, that is,*

$$d\omega(JX, Y, Z) = d\omega(X, JY, Z) = d\omega(X, Y, JZ).$$

In particular, $\nabla^{can}(d\omega) = 0$.

Proof. We use the fact that the nearly Kähler condition is equivalent to the following condition [17, Theorem 3.1]

$$X \lrcorner d\omega = 3\nabla^{LC}_X \omega \tag{2.1}$$

for all $X \in TM$. A straightforward calculation shows that (2.1) implies

$$d\omega(X, Y, Z) = 3\langle(\nabla^{LC}_X J)Y, Z\rangle. \tag{2.2}$$

Since $T(X, Y) = -J(\nabla^{LC}_X J)Y$, we obtain immediately the first assertion of Corollary 2.3. The second assertion follows from the first one, taking into account the fact that T is a 2-form of type $(2, 0) + (0, 2)$. Finally, the parallelity of $d\omega$ follows from Proposition 2.2 (3) and since ∇^{can} is unitary. □

2.2. Strict nearly Kähler 6-manifolds. Among nearly Kähler manifolds the class of strict nearly Kähler 6-manifolds is most well-studied. By [16, Theorem 5.2 (1)], any 6-dimensional nearly Kähler manifold which is not Kähler, is of constant type, i.e., there is a constant $\lambda > 0$ such that

$$||\nabla^{LC}_X(J)Y||^2 = \lambda^2(||X||^2||Y||^2 - \langle X, Y\rangle^2 - \langle X, JY\rangle^2). \tag{2.3}$$

Throughout this section, $M = M^6$ will be 6-dimensional.

Proposition 2.4. *Assume that (M, g, J) is a strict nearly Kähler manifold of constant type λ (cf. (2.3)). Then $\frac{1}{3\lambda}d\omega$ is a special Lagrangian calibration. In particular, (M, g, J) has an $SU(3)$-structure.*

Proof. It is an immediate consequence of (2.2) and (2.3) that $\frac{1}{3\lambda}d\omega$ is a special Lagrangian calibration, i.e., $\omega|_\Sigma = 0$ on any 3-plane $\Sigma \subset T_pM$ with $d\omega|_\Sigma = 3\lambda vol_\Sigma$.

Since $d\omega$ is parallel w.r.t. ∇^{can} and is of type $(3, 0) + (0, 3)$ by Corollary 2.3, it follows that the complex linear $(3, 0)$-form $\Phi \in \Omega^3(M, \mathbb{C})$ given as

$$\begin{aligned}\Phi(X, Y, Z) &:= \frac{1}{3\lambda}(d\omega(X, Y, Z) - \sqrt{-1}\, d\omega(X, Y, JZ)) \\ &= \frac{1}{3\lambda}(d\omega(X, Y, Z) + 3\sqrt{-1}\, T^*(X, Y, Z))\end{aligned} \tag{2.4}$$

is parallel w.r.t. ∇^{can} and nowhere vanishing, so that a strict nearly Kähler 6-manifold carries a canonical $SU(3)$-structure. □

The above argument also shows that for any 3-dimensional subspace $\Sigma \subset T_pM$ we have (cf. [18, Chapter III])

$$|\Phi_\Sigma|^2 \le |vol_\Sigma|^2 \text{ with equality if and only if } \omega|_\Sigma = 0. \tag{2.5}$$

Namely, the first estimate and that equality holds only if $\omega_\Sigma = 0$ follows immediately from (2.2) and (2.3). For the converse, let $(e_i)_{i=1,2,3}$ be an orthonormal basis of a Lagrangian plane. Then (2.2) implies that $\nabla^{LC}_{e_1}(J)e_2$ is orthogonal to e_1, e_2, Je_1, Je_2, so that $e_3 \in span(\nabla^{LC}_{e_1}(J)e_2, J\nabla^{LC}_{e_1}(J)e_2)$. From this and (2.3), equality in (2.5) follows.

Evidently, there are no calibrated submanifolds of $d\omega$, since on such a manifold $L \subset M$ we would have $\omega|_L = 0$ by (2.5) and hence, $3\lambda vol_L = d\omega|_L = 0$. Thus, by (2.5) on a Lagrangian submanifold $L \subset M$, $-\mathrm{Im}\,(\Phi)|_L = -\lambda^{-1}T^*|_L$ is a volume form on L, i.e., it is calibrated by the non-closed 3-form $\mathrm{Im}\,(\Phi)$, see also [26], [27].

REMARK 2.5. For the remainder of our paper we shall choose the orientation on a Lagrangian submanifold L such that

$$vol_L = -\mathrm{Im}\,(\Phi)|_L. \tag{2.6}$$

This orientation agrees with the natural orientation of the Lagrangian sphere $S^3(1) = \mathbb{H} \cap S^6$ in the standard strictly nearly Kähler sphere $S^6 \subset \mathrm{Im}\,\mathbb{O}$, see also subsection 5 below. Note that our choice of the orientation of L agrees with that in [33, p. 2309], but differs from that in [52, p. 18].

Lemma 2.6. *For the orientation of L given by (2.6) we have for any $\beta \in \Omega^1_{C^0}(L)$*

$$3 * \beta = -(s_\beta \rfloor d\omega)|_L. \tag{2.7}$$

Proof. Given $x \in L$, pick a unitary basis $(e_i, Je_i)_{i=1,2,3}$ of T_xL with dual basis $((e_i)^*, (Je_i)^*)_{i=1,2,3}$ and complex dual basis $\theta^i := (e_i)^* + \sqrt{-1}(Je_i)^*$ such that $d\omega(x) = 3\mathrm{Re}\,(\theta^1 \wedge \theta^2 \wedge \theta^3)$ and $\omega = -\mathrm{Im}\,(\theta^1 \wedge \bar\theta^1 + \theta^2 \wedge \bar\theta^2 + \theta^3 \wedge \bar\theta^3)$. Since special Lagrangian planes in T_xM are transitive under the $SU(3)$-action on T_xM [18], we can assume that T_xL is spanned by (Je_1, Je_2, Je_3). Since $\omega(x)$, $d\omega(x)$ and T_xL are invariant under the action of $SO(3) \subset SU(3) \subset Aut(T_xM)$, we can assume furthermore that $\beta_x = c\cdot(Je_1)^*$ and hence $s_\beta = c\cdot L_\omega^{-1}(Je_1)^* = c\cdot e_1$ for some $c \ge 0$. By (2.6) on L,

$$\mathrm{Im}\,(\theta^1 \wedge \theta^2 \wedge \theta^3)|_L = -3vol|_L,$$

i.e. (Je_1, Je_2, Je_3) is an oriented frame. Then

$$3 * (Je_1)^* = 3(Je_2)^* \wedge (Je_3)^* = -(e_1 \rfloor d\omega)|_L,$$

and multiplication by c yields (2.7) and hence completes the proof. □

REMARK 2.7. By the above discussion, a nearly Kähler 6-manifold (M, J, g, ω) of constant type λ satisfies the following equation (cf. [7, §4])

$$d\omega = 3\lambda\,\mathrm{Re}\,(\Phi), \qquad d\,\mathrm{Im}\,(\Phi) = -2\lambda\,\omega \wedge \omega. \tag{2.8}$$

Thus, a Calabi-Yau 6-manifold can be regarded as an almost strict nearly Kähler manifold with $\lambda = 0$.

In principle, one could verify (2.8) by a direct calculation, but there is a more elegant way to do this, due to C. Bär. Namely, first of all, by rescaling the metric g (Remark 1.2) we can assume that the metric is of constant type $\lambda = 1$.

In [4, §7] Bär constructed a 3-form φ on the cone $CM = M \times_{r^2} \mathbb{R}^+$ equipped with the warped Riemannian metric $\bar g = r^2 g + dr^2$ over a strict nearly Kähler 6-manifold (M, J, g) of constant type 1. We identify M with $M \times \{1\} \subset CM$. The form φ on CM is defined by [4, §7]

$$\varphi(r, x) = \frac{r^3}{3} d\omega + r^2 dr \wedge \omega. \tag{2.9}$$

For $x \in M$, pick a unitary basis $(e_i, Je_i)_{i=1,2,3}$ of T_xM with dual frames $((e_i)^*, (Je_i)^*)_{i=1,2,3}$ and the complex coframe $\theta^i := (e_i)^* + \sqrt{-1}(Je_i)^*$.

Then $(\hat{e}_i)_{i=1,\ldots,7}$ with $\hat{e}_i := r^{-1}e_i$ and $\hat{e}_{i+3} := r^{-1}Je_i$ for $i = 1, 2, 3$ and $\hat{e}_7 := dr$ form an orthonormal basis of $T_{(x,r)}CM$. If we set $\varepsilon_{ijk} = \hat{e}_i \wedge \hat{e}_j \wedge \hat{e}_k$, then φ is written by

$$\varphi(r, x) = (\varepsilon_{135} - \varepsilon_{146} - \varepsilon_{236} - \varepsilon_{245}) + \varepsilon_{127} + \varepsilon_{347} + \varepsilon_{567}, \tag{2.10}$$

and φ defines a G_2-structure on CM. By (2.9), $d\varphi = 0$. Bär also showed that $d^*\varphi = 0$, so that this G_2-structure is torsion free. In particular, φ (resp. $*\varphi$) is an associative (resp. coassociative) calibration on CM. Furthermore, $d^*\varphi = 0$ implies the second relation in (2.8) for $\lambda = 1$. From this, we also deduce the following result.

Proposition 2.8. *Let (M, J, g) be a strict nearly-Kähler manifold. Then there is a real analytic structure on M in which both the complex structure J and the metric g are real analytic.*

Proof. It is known that a strict nearly Kähler metric on a 6-manifold M is an Einstein metric [16, Lemma 4.8]. By the DeTurck-Kazdan theorem [9], M possesses an analytic atlas in which g is an analytic metric. It follows that in the induced real analytic structure on CM the aforementioned cone metric $\bar{g} := dr^2 + r^2 g$ on CM is analytic and the vector field ∂_r on CM is analytic. Since the form $\varphi \in \Omega^3(CM, \bar{g})$ of (2.9) defining the G_2-structure on CM is harmonic, it is analytic as well. Thus, $\partial_r \rfloor \phi = r^2\omega$ is analytic, and so is its restriction to the analytic submanifold $M \times \{1\} \subset CM$.

Therefore, $\omega \in \Omega^2(M)$ is analytic, and J is defined by contraction of ω with the real analytic metric g and hence analytic as well. □

3. Variation of the volume of Lagrangian submanifolds

In this section we introduce the notion of the Maslov 1-form $\mu(L)$ of a Lagrangian submanifold L in a Hermitian manifold (M, J, g) and relate this notion with the classical notion of the Maslov class of a Lagrangian submanifold in $(\mathbb{R}^{2n}, \omega_0)$ (Remark 3.2). Then we prove that $\mu(L)$ is symplectically dual to the twice of the mean curvature H_L of a Lagrangian submanifold L in a nearly Kähler manifold (M, J, g) (Proposition 3.3) and derive its consequences (Corollaries 3.4, 3.6). Using relative calibrations, we prove a simple formula for the second variation of the volume of a Lagrangian submanifold in a strictly nearly Kähler 6-manifolds (Theorem 3.8) and discuss its consequences (Corollary 3.12, Remarks 3.13, 3.15, 3.16). We discuss the relation between the obtained results with known results (Remark 3.7, 3.15, 3.16).

3.1. Maslov 1-form and minimality of a Lagrangian submanifold in a nearly Kähler manifold. Let L be a Lagrangian submanifold in an almost Hermitian manifold (M, J, g) and $(\tilde{\omega}^i_j)$ the canonical Hermitian connection 1-form on $U(M, J, g)$. The Gaussian map g_L sends L to the Lagrangian Grassmanian $Lag(M)$ of Lagrangian subspaces in the tangent bundle of M. Denote by $p : U(M) \to Lag(M)$ the projection defined by

$$(v_1, Jv_1, \cdots, v_n, Jv_n) \mapsto [v_1 \wedge \cdots \wedge v_n].$$

Set

$$\gamma := -\sqrt{-1} \sum_i \tilde{\omega}^i_i.$$

We recall the following fact

Lemma 3.1 (cf. [5],[26, Proposition 3.1])**.** *There exists a 1-form $\bar{\gamma}$ on $Lag(M)$ whose pull-back to the unitary frame bundle $U(M)$ is equal to γ.*

We call $2\bar{\gamma}$ *the universal Maslov 1-form* and the induced 1-form $g_L^*(2\bar{\gamma})$ on L *the Maslov 1-form of L*. We also denote $g_L^*(2\bar{\gamma})$ by $\mu(L)$.

REMARK 3.2. For $M = \mathbb{R}^{2n}$ we have $Lag(M) = \mathbb{R}^{2n} \times U(n)/O(n)$. In this case it is well-known that the Maslov 1-form $\mu(L)$ is a closed 1-form and represents its Maslov index of a Lagrangian submanifold L [40].

Now we relate the Maslov 1-form $\mu(L) := g_L^*(2\bar{\gamma})$ with the mean curvature of a Lagrangian submanifold L. We define a linear isomorphism $L_\omega : TM \to T^*M$ as follows.

$$L_\omega(V) := V \rfloor \omega. \tag{3.1}$$

Proposition 3.3. *The Maslov 1-form $\mu(L)$ is symplectic dual to the minus twice of the mean curvature H_L of a Lagrangian submanifold L in a nearly Kähler manifold (M, J, g), that is,*

$$-2L_\omega(H_L) = \mu(L).$$

Proof. By Proposition 2.2(2) the 1-form $\sum_{ik} T^i_{\bar{i}\bar{k}}\bar{\theta}^k$ vanishes, where T is the torsion of the connection form $\tilde{\omega}$. Using [26, Lemmas 2.1, 3.1 and (3.6)], we obtain for any normal vector X to L

$$\langle -H_L, X\rangle = (\mu(L)/2, JX). \tag{3.2}$$

Since $\omega(-H_L, JX) = \langle -H_L, X\rangle$, we derive Proposition 3.3 immediately from (3.2). □

Since the curvature $d\gamma$ form of the connection form γ is the first Chern form of a nearly Kähler manifold we obtain immediately

Corollary 3.4. *Assume that a Lagrangian submanifold L in a nearly Kähler manifold (M, J, g) is minimal. Then the restriction of the first Chern form to L vanishes.*

In the remainder of this section we assume that L is a Lagrangian submanifold in a strict nearly Kähler manifold (M, J, g). We also need to fix some notations. Recall the definition of the ∇^{can}-parallel complex volume form $\Phi = \theta^1 \wedge \theta^2 \wedge \theta^3$ of (2.4), and recall that

$$\mathrm{Re}\,\Phi = (3\lambda)^{-1} d\omega, \qquad \mathrm{Im}\,\Phi = -\lambda^{-1} T^*.$$

Lemma 3.5. *Let ξ be a simple 3-vector in $\mathbb{R}^6 = \mathbb{C}^3$ and ω the standard compatible symplectic form on $\mathbb{R}^6$. Then*

(1) ([18, Chapter III Theorem 1.7]) $|\Phi(\xi)|^2 = \mathrm{Re}\,\Phi(\xi)^2 + \mathrm{Im}\,\Phi(\xi)^2$.
(2) ([18, Chapter III (1.17)]) $|\Phi(\xi)|^2 + \sum_{i=1}^3 |\theta^i \wedge \omega(\xi)|^2 = |\xi|^2$.

We choose the canonical orientation (2.6) on L, i.e., $\mathrm{Im}\,\Phi|_L = -vol_L$. For $x \in L$ let $\xi(x)$ denote the unit simple 3-vector associated with T_xL. By [26, Lemma 2.1], [27, Lemma 1.1] for any $V \in NL$ we obtain

$$\langle -H_L, V\rangle = (V\rfloor d \pm \operatorname{Im}\Phi)(\xi). \tag{3.3}$$

(In [26] Lê showed that the formula (3.3) is equivalent to the formula (3.2).) Using (2.8), we obtain immediately that $H_L = 0$.

Corollary 3.6. *Any Lagrangian submanifold L in a strict nearly Kähler 6-manifold (M, J, g) is orientable and minimal. Hence its Maslov 1-form vanishes.*

REMARK 3.7. The relation between the Maslov class and the minimality of Lagrangian submanifolds has been found for Lagrangian submanifolds in various classes of Hermitian manifolds [40], [29], [26]. Corollary 3.4 extends a previous result by Bryant [5, Proposition 1] and partially extends a result by Lê in [26, Corollary 3.1]. The minimality of Lagrangian submanifolds in a strict nearly Kähler 6-manifolds has been proved by Schäfer and Smoczyk by studying the second fundamental form of L in M [52, §4], extending a previous result by Ejiri [13] for $M = S^6$. The minimality of a Lagrangian submanifold L in a strict nearly Kähler manifold M can be also obtained from the minimality of the coassociative cone $CL \subset CM$.

3.2. Second variation of the volume of Lagrangian submanifolds. The second variation of the volume of a minimal submanifold N in a Riemannian manifold M has been expressed by Simons [51] in terms of an elliptic second order operator $I(N, M)$ that depends on the second fundamental form of N and the Riemannian curvature on M, see also [28], [45]. If L is a Lagrangian submanifold in a strict nearly Kähler manifold M, we shall derive a simple formula for $I(L, M)$ that depends entirely on the intrinsic geometry of L supplied with the induced Riemannian metric.

Theorem 3.8. *Assume that (M, J, g) is a strict nearly Kähler manifold of constant type λ. Let V be a normal vector field with compact support on a Lagrangian submanifold $L \subset M$. Then the second variation of the volume of L with the variation field V is given by*

$$\frac{d^2}{dt^2}|_{t=0} vol(L_t) = \int_L \langle d(L_\omega(V)) - 3\lambda * L_\omega(V), d(L_\omega(V)) + \lambda * L_\omega(V)\rangle + \int_L \|d * L_\omega(V)\|^2. \tag{3.4}$$

Proof. Let $\phi_t : L \to M$ be a variation of L generated by the vector field V. Set

$$\xi_t(x) := (\phi_t)_*(\xi(x)).$$

We observe that, to compute the second variation of the volume of L, using Lemma 3.5 and the minimality of L, it suffices to compute the second variation of the integral over L of $\sum_{i=1}^3 |\theta^i \wedge \omega(\xi)|^2$, $(\operatorname{Re}\Phi(\xi))^2$ and $(\operatorname{Im}\Phi(\xi))^2$. By the observation that for all $x \in L$

$$|\xi_0(x)| = 1, \qquad \frac{d}{dt}|_{t=0}|(\xi_t(x))| = 0 \tag{3.5}$$

we obtain

$$\frac{d^2}{dt^2}|_{t=0} vol(\phi_t(L)) = \int_L \frac{d^2}{dt^2}|_{t=0}|(\xi_t(x)|\, d\, vol_x \tag{3.6}$$

$$= \frac{1}{2}\int_L \frac{d^2}{dt^2}|_{t=0}(|\xi_t(x)|^2)\, d\, vol_x.$$

Lemma 3.9. *For any $x \in L$ we have*

$$\frac{d^2}{dt^2}|_{t=0}\sum_{i=1}^{3}((\theta^i \wedge \omega), \xi_t(x))^2 = 2|dL_\omega(V) - 3\lambda * L_\omega(V)|^2(x).$$

Proof. Since $\omega|_L = 0$ we have for all i

$$\frac{d^2}{dt^2}|_{t=0}|\theta^i \wedge \omega(\xi)|^2 = 2[\frac{d}{dt}|_{t=0}(\theta^i \wedge \omega(\xi))]^2. \tag{3.7}$$

By Lemma 2.6, taking into account the rescaling factor λ, see also [52, Theorem 8.1], we have

$$\frac{d}{dt}|_{t=0}\phi_t^*(\omega)(x) = d(L_\omega(V))(x) - 3(\lambda * L_\omega(V))(x). \tag{3.8}$$

Since the RHS of (3.8) is a 2-form on L, there exists an orthonormal basis f^1, f^2, f^3 of T_x^*L and a number $c \in \mathbb{R}$ such that

$$d(L_\omega(V))(x) - 3(\lambda * L_\omega(V))(x) = c \cdot f^1 \wedge f^2.$$

Using $\omega|_L = 0$ and the expression of the RHS of (3.8) in this basis, we obtain of (3.8)

$$\frac{d}{dt}|_{t=0}\sum_{i=1}^{3}\phi_t^*(\theta^i \wedge \omega) = c \cdot f^1 \wedge f^2 \wedge f^3. \tag{3.9}$$

Using again $\omega|_L = 0$, we obtain Lemma 3.9 immediately from (3.7) and (3.9). □

Lemma 3.10. *For all $x \in L$ we have*

$$\frac{d^2}{dt^2}|_{t=0}(\mathrm{Re}\,\Phi(\xi_t(x))^2) = 2|d * L_\omega(V)|^2(x).$$

Proof. By Lemma 2.6 we have

$$\frac{dt}{dt}|_{t=0}(\mathrm{Re}\,\Phi(\xi_t(x)) = (d * L_\omega(V))(x). \tag{3.10}$$

Since $\mathrm{Re}\,\Phi(\xi(x)) = 0$, we obtain Lemma 3.10 from (3.10) immediately. □

Lemma 3.11. *We have*

$$\frac{d^2}{dt^2}|_{t=0}\int_L \mathrm{Im}\,\Phi(\xi_t)^2\, dvol_x = 8\lambda \int_L \langle *L_\omega(V), d(L_\omega(V)) - 3\lambda * L_\omega(V)\rangle\, d\, vol_x.$$

Proof. Since $(V\rfloor \mathrm{Im}\,\Phi)|_L = 0$, (see e.g. [26, Proposition 2.2.(ii)], [27, Proposition 1.2.ii], which is also now called the first cousin principle), using the Cartan formula we have

$$\frac{d}{dt}|_{t=0}(\mathrm{Im}\,\Phi(x), \xi_t(x)) = (V\rfloor d\mathrm{Im}\,\Phi, \xi(x)), \tag{3.11}$$

for all $x \in L$.

By (3.3) the RHS of (3.11) vanishes. Since $\mathrm{Im}\,\Phi(\xi(x)) = -1$ for all $x \in L$, we obtain

$$\frac{d^2}{dt^2}|_{t=0}(\mathrm{Im}\,\Phi(\xi_t(x))^2) = -2\frac{d^2}{dt^2}|_{t=0}(\mathrm{Im}\,\Phi(\xi_t(x)). \tag{3.12}$$

It follows that

$$\frac{d^2}{dt^2}|_{t=0}\int_L \mathrm{Im}\,\Phi(\xi_t(x))^2\,d\,vol_x = -2\frac{d^2}{dt^2}|_{t=0}\int_L(\phi_t^*(\mathrm{Im}\,\Phi),\xi)\,d\,vol_x. \tag{3.13}$$

Using the Cartan formula, we derive of (3.13)

$$\frac{d^2}{dt^2}|_{t=0}\int_L \mathrm{Im}\,\Phi(\xi_t(x))^2\,d\,vol_x = -2\int_L \mathcal{L}_V((V\rfloor d\mathrm{Im}\,\Phi) + d(V\rfloor \mathrm{Im}\,\Phi)). \tag{3.14}$$

Since $\mathcal{L}_V(d(V\rfloor\mathrm{Im}\,\Phi)) = d(\mathcal{L}_V(V\rfloor\mathrm{Im}\,\Phi))$, we obtain of (3.14), taking into account that $d\mathrm{Im}\,\Phi$ $= -2\lambda\,\omega\wedge\omega$

$$\frac{d^2}{dt^2}|_{t=0}\int_L \mathrm{Im}\,\Phi(\xi_t(x))^2\,d\,vol_x = 4\lambda\int_L \mathcal{L}_V(V\rfloor(\omega\wedge\omega)). \tag{3.15}$$

Taking into account $V\rfloor(\omega\wedge\omega) = 2(V\rfloor\omega)\wedge\omega$ and $\omega|_L = 0$ we obtain of (3.15)

$$\frac{d^2}{dt^2}|_{t=0}\int_L \mathrm{Im}\,\Phi(\xi_t(x))^2\,d\,vol_x = 8\lambda\int_L (V\rfloor\omega)\wedge\mathcal{L}_V(\omega). \tag{3.16}$$

Since $(V\rfloor\omega) = L_\omega(V)$ and $\mathcal{L}_V(\omega) = dL_\omega(V) - 3\lambda * L_\omega(V)$, we obtain Lemma 3.11 immediately from (3.16). □

Now let us complete the proof of Theorem 3.8. By Lemma 3.5 and equation (3.6), we obtain

$$\begin{aligned} 2\frac{d^2}{dt^2}|_{t=0}vol(\phi_t(L)) &= \int_L \frac{d^2}{dt^2}|_{t=0}\sum_{i=1}^3(\theta^i\wedge\omega,\xi_t)^2\,d\,vol_x \\ &+ \int_L\frac{d^2}{dt^2}|_{t=0}(\mathrm{Re}\,\Phi,\xi_t)^2\,d\,vol_x + \int_L\frac{d^2}{dt^2}|_{t=0}(\mathrm{Im}\,\Phi,\xi_t)^2\,d\,vol_x. \end{aligned} \tag{3.17}$$

Theorem 3.8 now follows from (3.17) and Lemmas 3.9, 3.10, 3.11. □

Corollary 3.12. *Assume that L is a compact Lagrangian submanifold in a strict nearly Kähler manifold (M, J, g) and $H^1(L,\mathbb{R}) \neq 0$. Let β be a non-zero harmonic 1-form on L. Then the variation generated by $L_\omega^{-1}(\beta)$ decreases the volume of L.*

REMARK 3.13. There are many known examples of Lagrangian submanifolds L in the manifold S^6 supplied with the standard nearly Kähler structure induced from $\mathbb{R}^7 = \mathrm{Im}\,\mathbb{O}$ such that $\dim H^1(L)$ is arbitrary large. For instance, L is obtained by composing the Hopf lifting of a holomorphic curve Σ_g of genus g in the projective plane $\mathbb{CP}^2$ to S^5 with a geodesic embedding $S^5 \to S^6$ [10, Theorem 1], see also [33, Example 6.11].

REMARK 3.14. In [48] Palmer derived a simple formula for the second variation of Lagrangian submanifolds in the standard nearly Kähler 6-sphere by simplifying the classical second variation formula with help of (relative) calibrations.

REMARK 3.15. Letting λ go to zero, we obtain the formula for the second variation of the volume of a special Lagrangian submanifold L in a Calabi-Yau manifold M with a variation field V which is normal to L:

$$\frac{d^2}{dt^2}|_{t=0} vol(L_t) = \int_L ||d(L_\omega V)||^2 + \int_L ||d^*(L_\omega V)||^2. \tag{3.18}$$

Formula (3.18) has been obtained by McLean in [36, Theorem 3.13] for special Lagrangian submanifolds in Calabi-Yau manifolds of dimension $2n$ as a consequence of his formula for the second variation of the volume of calibrated submanifolds, using moving frame method. Note that our proof of Theorem 3.8 can be easily adapted to give (3.18) for special Lagrangian submanifolds $L^n \subset M^{2n}$. Here we use the full version of Lemma 3.5 given in [18, Chapter III, Theorem 1.7, (1.17)]. The first summand in RHS of (3.18) is the second variation of the term $(|\xi|^2 - \Phi(\xi)^2)/2$. The second summand in the RHS of (3.18) is the second variation of the term $(\mathrm{Re}\,\Phi(\xi))^2/2$. By [26, (4.11)], the second variation of the term $\mathrm{Re}\,\Phi(\xi)$ vanishes, if M^{2n} is a Calabi-Yau manifold. This proves (3.18) for any dimension n. Note that (3.18) also follows from Oh's second variation formula for Lagrangian minimal submanifolds in Kähler manifolds [44, Theorem 3.5].

REMARK 3.16. Using the strategy of the proof of Theorem 3.8, we can have a (new simple proof of a) formula for the second variation of the volume of ϕ-calibrated submanifolds N^n in a manifold M^m provided with a relative calibration ϕ such that a generalized version of Lemma 3.5 is valid, that expresses $|\xi|^2$ as a sum $|\phi(\xi)|^2 + \sum_{i=1}^k |\mathrm{Re}\,\Phi_k(\xi)|^2$. Generalized versions of Lemma 3.5 have been found for Kähler $2p$-vectors, coassociative 4-vectors, ect. in [18].

4. Deformations of Lagrangian submanifolds in strict nearly Kähler 6-manifolds

In this section we consider the moduli space of closed Lagrangian submanifolds $L = L^3 \subset M = M^6$ of a strict nearly-Kähler 6-manifold. As the dimensions are fixed throughout this section, we again omit the superscripts. We shall show that any C^1-small Lagrangian deformation of L in M is a solution of an elliptic first order PDE of Fredholm index 0 (Propositions 4.4, 4.6). Furthermore, a closed Lagrangian submanifold L is analytic and any smooth deformation of L is analytic. Moreover, the moduli space of smooth Lagrangian deformations of L locally is a finite dimensional analytic variety, hence any formally unobstructed infinitesimal deformation is smoothly unobstructed (Theorem 4.9, Corollary 4.12).

Our notation on forms will be as follows. By $\Omega^*(L)$ we denote smooth differential forms on L. If we wish to specify the degree of regularity, we write $\Omega^*_{C^k}(L)$ for the space of C^k-regular forms.

4.1. Deformations of Lagrangian submanifolds. Let L be a submanifold in a Riemannian manifold (M, g). Then the normal exponential mapping $Exp_L : NL \to M$ identifies a neighborhood of the 0-section in NL with a tubular neighborhood $U(L) \subset M$ of L. With this, Exp_L , which we shall also denote by Exp if no confusion arises, identifies C^1-small deformations of L with C^1-small section $s : L \to NL$.

Now assume that (M, J, g) is a Hermitian manifold and ω is the associated fundamental 2-form. If $L \subset M$ is a Lagrangian submanifold, then the isomorphism L_ω of (3.1) identifies a covector in T^*L (resp. a 1-form $\alpha \in \Omega^1(L)$) with a vector in NL (resp. a section $s_\alpha \in \Gamma(NL)$). Since we are interested in Lagrangian deformations of $L \subset M$, we therefore consider the map

$$F : \Omega^1_{C^1}(L) \longrightarrow \Omega^2_{C^0}(L), \qquad \alpha \longmapsto (Exp(s_\alpha))^*(\omega). \tag{4.1}$$

Evidently, $F(0) = 0$ as L is Lagrangian, and the space of C^1-small Lagrangian deformations of L can be identified with a neighborhood of $0 \in F^{-1}(0)$.

Now we shall compute the linearization of F at $\alpha = 0$.

Proposition 4.1. *Let $(\alpha_r)_{r\in(-\epsilon,\epsilon)}$ be a C^1-regular variation of $\alpha_0 = 0 \in \Omega^1(L)$, i.e. such that the map*

$$(-\varepsilon, \varepsilon) \times L \longrightarrow T^*L, \qquad (r, x) \longmapsto \alpha_r(x) \in T^*_x L$$

is C^1, and let $\dot\alpha_0(x) = \partial_r(\alpha_r(x))|_{r=0}$ be the pointwise derivative. Then

$$\left.\frac{d}{dr}\right|_{r=0} F(\alpha_r) = d\dot\alpha_0 - 3 * \dot\alpha_0,$$

whence

$$\partial F|_0(\beta) = d\beta - 3 * \beta \tag{4.2}$$

for all $\beta \in \Omega^1_{C^1}(L)$.

Proof. We define the C^1-map

$$D : (-\varepsilon, \varepsilon) \times L \longrightarrow M, \qquad (r, x) \longmapsto Exp(s_{\alpha_r})_x =: D_r(x).$$

Note that $D_0 = Id_L$ and $dD(\partial_r)|_{\{0\}\times L} = s_{\dot\alpha_0}$. Also, if we let Φ_r denote the flow of ∂_r on $(-\varepsilon, \varepsilon) \times L$, then $D_{r+t} = \Phi_r D_t$, whence by definition,

$$\begin{aligned}
\left.\frac{d}{dr}\right|_{r=0} F(\alpha_r) &= \left.\frac{d}{dr}\right|_{r=0} D_r^*(\omega) = \left.\left(\left.\frac{d}{dr}\right|_{r=0} \Phi_r(D^*(\omega))\right)\right|_{\{0\}\times L} \\
&= (\mathfrak{L}_{\partial_r} D^*(\omega))\big|_{\{0\}\times L} = (\partial_r \lrcorner D^*(d\omega) + d(\partial_r \lrcorner D^*(\omega))|_{\{0\}\times L} \\
&= D_0^*(s_{\dot\alpha_0} \lrcorner d\omega) + d(D_0^*(s_{\dot\alpha_0} \lrcorner \omega)) = s_{\dot\alpha_0} \lrcorner d\omega + d\dot\alpha_0.
\end{aligned}$$

Here, we used Cartan's formula for the Lie derivative as well as the fact that by (3.1), $s_{\dot\alpha_0} \lrcorner \omega = \dot\alpha_0$. Now the formula follows since $s_{\dot\alpha_0} \lrcorner d\omega = -3 * \dot\alpha_0$ by (2.7). □

Recall that the Laplace operator on forms is defined as $\triangle = (d + d^*)^2$, where $d^*|_{\Omega^k(L)} = (-1)^k * d*$ is the adjoint of d. Proposition 4.1 yields immediately the following Corollary 4.2, which has been obtained by Schäfer-Smoczyk by a different method.

Corollary 4.2 (cf. [52, Theorem 8.1])**.** *Let $(L_r)_{r\in(-\varepsilon,\varepsilon)}$ be a C^1-regular family of Lagrangian submanifolds of M, such that $L_0 = L$ and $L_r = Exp(s_{\alpha_r})$ for a family $(\alpha_r)_{r\in(-\varepsilon,\varepsilon)}$ in $\Omega^1(L)$. Then the derivative $\beta := \dot\alpha_0 = \partial_r \alpha_r|_{r=0}$ is a solution of*

$$*d\beta - 3\beta = 0. \tag{4.3}$$

In particular, $d^*\beta = 0$ and $\triangle\beta = 9\beta$.

We call the map $\partial F|_0(\beta)$ of (4.2) the *linearization of the equation $F = 0$ at* 0. We set

$$\Omega^1_a(L) := \{\alpha \in \Omega^1(L) | \triangle(\alpha) = a \cdot \alpha\}.$$

All eigenvalues a are nonnegative and the eigenspaces $\Omega^1_a(L)$ are finite dimensional, as $\triangle$ is an elliptic differential operator [49].

Lemma 4.3. *The map $*d : \Omega^1(L) \to \Omega^1(L)$ is selfadjoint, and its kernel is $dC^\infty(L) \oplus \Omega^1_0$. Moreover, for each $a > 0$ we have the L^2-orthogonal decomposition*

$$\Omega^1_a(L) \cap \ker d^* = K_{\sqrt{a}}(L) \oplus K_{-\sqrt{a}}(L),$$

where $K_{\pm\sqrt{a}}(L)$ is the $(\pm\sqrt{a})$-eigenspace of $*d$.

Proof. The Hodge-$*$ operator is self adjoint satisfying $*^2 = 1$, whereas for the adjoint of d we have

$$d^*|_{\Omega^1(L)} = - * d*, \qquad d^*|_{\Omega^2(L)} = *d * .$$

Thus, for the restriction of $*d$ to $\Omega^1(L)$ we obtain

$$(*d)^* = d^** = *d*^2 = *d.$$

Moreover, since $*d = d^**$, it follows that the image of $*d$ equals the image of $d^* : \Omega^2(L) \to \Omega^1(L)$, so that the kernel of $*d$ is the orthogonal complement of this image, which by Hodge decomposition equals $dC^\infty(L) \oplus \Omega^1_0$ as claimed.

Since $*d$ commutes with $\triangle$, it follows that $*d$ preserves Ω^1_a and hence, $\Omega^1_a(L) \cap \ker d^*$. We have $(*d)^2 = d^*d = \triangle$ on $\ker d^*$, and $(*d)^2 = a\, Id$ on $\Omega^1_a(L) \cap \ker d^*$. Hence these subspaces can be decomposed into $K_{\sqrt{a}} \oplus K_{-\sqrt{a}}$ as claimed. □

It follows from (4.2) that

$$T_L := \ker \partial F|_0 = K_3(L). \tag{4.4}$$

The equation $F(\alpha) = 0$ with F of (4.1) is overdetermined. In fact, one of the technical problems we wish to overcome is the fact that $\partial F|_0$ of (4.2) is not an elliptic operator, but only the restriction of an elliptic first order operator to a subspace as we shall show now.

For this purpose, we extend F by its prolongation dF and add another parameter. Namely, we extend the map F to

$$\begin{aligned} \hat{F} : \Omega^1(L) \oplus C^\infty(L) &\longrightarrow \Omega^1(L) \oplus C^\infty(L) \\ (\alpha, f) &\longmapsto \big(* F(\alpha) + df, \tfrac{1}{3} * dF(\alpha) + 3f\big). \end{aligned} \tag{4.5}$$

Proposition 4.4. *A pair (α, f) is a solution of the equation $\hat{F}(\alpha, f) = 0$ if and only if α is a solution of the equation $F(\alpha) = 0$ and $f = 0$.*

Proof. By definition (4.5), (α, f) is a solution of $\hat{F}(\alpha, f) = 0$ iff

$$F(\alpha) = - * df \qquad \text{and} \qquad * dF(\alpha) = -9f. \tag{4.6}$$

Substituting the first equation into the second implies

$$-9f = - * d * df = d^*df = \triangle f,$$

and as $\triangle$ is nonnegative, this implies $f = 0$ and $F(\alpha) = - * df = 0$. □

REMARK 4.5. From this proof, we can also conclude that

$$\hat{F}(\alpha, f) \in T_L \Leftrightarrow f = 0 \text{ and } F(\alpha) = \frac{1}{3} d * F(\alpha). \tag{4.7}$$

Here, we use the obvious inclusion $T_L \cong T_L \oplus \{0\} \subset \Omega^1(L) \oplus C^\infty(L)$.

It follows from (4.2) and (4.5) that the differential

$$\partial \hat{F}|_{(0,0)} : \Omega^1(L) \oplus C^\infty(L) \longrightarrow \Omega^1(L) \oplus C^\infty(L)$$

has the following form

$$\partial \hat{F}|_{(0,0)}(\beta, f) = (*d\beta - 3\beta + df, d^*\beta + 3f). \tag{4.8}$$

Proposition 4.6. *$\partial \hat{F}|_{(0,0)}$ is a self adjoint elliptic first order differential operator, and*

$$\ker \partial \hat{F}|_{(0,0)} = T_L,$$

that is, $(\beta, f) \in \ker \partial \hat{F}|_{(0,0)}$ iff $*d\beta = 3\beta$ and $f = 0$.

Proof. The symbol of $\partial \hat{F}|_{(0,0)}$ coincides with that of $(*d + d^*)|_{\Omega^1(L)} + d|_{C^\infty(L)}$, and it is straightforward to see that the square of the latter operator is $\triangle|_{\Omega^1(L)\oplus C^\infty(L)}$. From this, the ellipticity of $\partial \hat{F}|_{(0,0)}$ follows.

We have already seen in Lemma 4.3 that $*d : \Omega^1(M) \to \Omega^1(M)$ is self adjoint, whence so is the map $(\beta, f) \mapsto (*d\beta - 3\beta, 3f)$. Thus, we have to show that the map $(\beta, f) \mapsto (df, d^*\beta)$ is self adjoint. But this is evident as d and d^* are adjoint maps.

To compute the kernel, let (β, f) be such that $*d\beta - 3\beta + df = 0$ and $d^*\beta = -3f$. Then, applying d^* to the first equation and using the second, it follows that $0 = 9f + d^*df = 9f + \triangle f$, and since $\triangle$ is nonnegative, this implies that $f = 0$. Thus, $*d\beta - 3\beta = 0$, so that $\beta \in T_L \subset \ker d^*$. □

REMARK 4.7. Propositions 4.4, 4.6 imply that the expected dimension of the moduli space of Lagrangian submanifolds is zero. On the other hand, most interesting examples of strict nearly Kähler manifolds possess a non-trivial symmetry group which acts on Lagrangian submanifolds, so that in this case, the moduli space of these is of positive dimension, cf. section 5 below.

Observe that the differential at the origin of the restriction $\hat{F} : \Omega^1(L) \to \Omega^1(L) \oplus C^\infty(L)$ is a Fredholm operator from $\Omega^1(L)$ to $\{(\beta, f) \in \Omega^1(L) \oplus C^\infty(L) \mid d^*\beta + 3f = 0\}$. This can be shown by a direct calculation, or, more elegantly, as follows. Setting $D_1 := \partial_0 \hat{F}$ and $D_2(\beta, f) := d^*\beta + 3f$, it easily follows that both $D_1^* D_1$ and $D_1 D_1^* + D_2 D_2^*$ are elliptic. Then [41, Proposition 2.2] implies that D_1 is a Fredholm operator into the kernel of D_2, and $\ker D_1 = \ker D_1^* D_1$.

However, using Fredholm theory in this situation bears further technical difficulties, see e.g. [34] for a related consideration. Thus, we shall mainly be concerned with the map $\hat{F}$ defined on all of $\Omega^1(L) \oplus C^\infty(L)$ and exploit the analyticity of the strict nearly Kähler structure (J, g) on M in the subsequent sections.

4.2. Analyticity of Lagrangian deformations and its consequences. As we pointed out in Proposition 2.8, a nearly-Kähler manifold (M, g, J, ω) is real analytic. Since any Lagrangian submanifold $L \subset (M, J, g)$ is a minimal submanifold in (M, g) [52, Theorem A], the Morrey regularity theorem for vector solutions of class C^1 of a regular variational problem [37], [38] (see also [39], [18, IV.2.B]) implies that $L \subset M$ is a real analytic submanifold of M. Thus, the normal exponential

$$Exp : T^*_\varepsilon L \overset{L_\omega}{\cong} N_\varepsilon L \longrightarrow M$$

is an analytic diffeomorphism onto a tubular neighborhood of L for sufficiently small $\varepsilon > 0$, where $T^*_\varepsilon L$ and $N_\varepsilon L$ denote the ε-disc bundles in the cotangent bundle and the normal bundle of L, respectively.

DEFINITION 4.8. Let $L \subset M$ be a closed Lagrangian submanifold.

(1) An element $\alpha \in T_L$ is called *smoothly unobstructed* or *smoothly integrable*, if there is a smooth Lagrangian deformation $s(t) : L \to M$ such that $s(0) = L$ and $\dot{s}(0) = \alpha$. Otherwise, α is called *smoothly obstructed*.

(2) An element $\alpha \in T_L$ is called *formally unobstructed* or *formally integrable*, if there exists a sequence $\alpha_1 = \alpha, \alpha_2, \dots \in \Omega^1(L)$ such that the formal power series

$$\alpha_t := \sum_{n=1}^{\infty} \alpha_n t^n \in \Omega^1(L)[[t]] \tag{4.9}$$

satisfies

$$F(\alpha_t) = 0 \in \Omega^2(L)[[t]]$$

as a formal power series in t.

(3) We call L *regular* if every $\alpha \in T_L$ is formally unobstructed.

Clearly, if $\alpha \in T_L$ is smoothly unobstructed, then it is formally unobstructed. Indeed, if $s(t)$ is a smooth Lagrangian deformation with $s(0) = L$, then $s(t) = Exp(L_\omega(\alpha(t)))$ for some curve $\alpha(t) \in \Omega^1(L)$ with $\alpha(0) = 0$ such that $F(\alpha_t) \equiv 0$. Let $\alpha_t := \sum_{n=1}^\infty \alpha_n t^n \in \Omega^1(L)[[t]]$ be the Taylor series of $\alpha(t)$ at $t = 0$. Then $F(\alpha_t)$ is the Taylor series at $t = 0$ of the function $0 \equiv F(\alpha(t))$ and hence vanishes.

The main purpose of this section is to show the converse: every formally unobstructed element $\alpha \in T_L$ is smoothly unobstructed, and this condition is equivalent to the smooth or formal unobstructedness of α w.r.t. an analytic function $\tau : U \to T_L$ with $U \subset T_L$ a neighborhood of the origin, i.e. of an analytic function in finitely many variables (cf. Corollary 4.12 below). More precisely, we show the following

Theorem 4.9. *The moduli space of closed Lagrangian submanifolds of a* 6*-dimensional nearly-Kähler manifold in the* C^1*-topology locally is a finite dimensional analytic variety.*

More concretely, for a closed Lagrangian $L \subset M$ *there is an open neighborhood* $U \subset T_L$ *of the origin and a real analytic map* $\tau : U \to T_L$ *with* $\tau(0) = 0$ *and* $\partial\tau|_0 = 0$*, as well as a* C^∞*-map* $\Phi : L \times U \to M$ *such that* $L_\alpha := \Phi(L \times \{\alpha\})$ *satisfies:*

$L_0 = L$*, and any closed submanifold* $L' \subset M$ C^1*-close to* L *is Lagrangian if and only if* $L' = L_\alpha$ *for some* $\alpha \in \tau^{-1}(0)$.

In order to work towards the proof, we let $\tilde{\omega} := (Exp \circ L_\omega)^*(\omega) \in \Omega^2(T^*L)$, so that $F(\alpha) = \alpha^*(\tilde{\omega})$. We associate to each $\alpha \in \Omega^1(L)$ the vector field ξ_α on T^*L which on each fiber T^*_pL is constant equal to α_p.

If $\alpha \in \Omega^*_{C^k}(L)$ is C^k-regular, we define the C^k-norms at $p \in L$ and on L as

$$\|\alpha\|_{C^k;p} := \sum_{|I|\le k} \|(D^I\alpha)_p\|, \qquad \|\alpha\|_{C^k} := \sup_{p\in L} \|\alpha\|_{C^k;p}, \tag{4.10}$$

where the sum is taken over all multi-indices I in a coordinate system of L which is orthogonal at p.

Lemma 4.10. *There are constants $A, K > 0$ such that for each C^k-regular $\alpha \in \Omega^1_{C^k}(L)$, $k \ge 1$, and each $p \in L$ we have*

$$\left\| \frac{d^n}{dt^n}\bigg|_{t=0} (F(t\alpha)) \right\|_{C^{k-1};p} \le n!AK^n\|\alpha\|^n_{C^k;p} \tag{4.11}$$

$$\left\| \frac{d^n}{dt^n}\bigg|_{t=0} (d(F(t\alpha))) \right\|_{C^{k-1};p} \le n!AK^n\|\alpha\|^n_{C^k;p}. \tag{4.12}$$

In particular, at each $p \in L$, the maps $t \mapsto F(t\alpha)_p \in \Lambda^2T^*_pL$ and $t \mapsto d(F(t\alpha))_p \in \Lambda^3T^*_pL$ are real analytic at $t = 0$, and all their derivatives w.r.t. t are C^{k-1}-regular 2-forms and 3-forms on L, respectively.

Proof. Let us describe this situation in local coordinates. Namely, we let $x = (x^i)$ be analytic coordinates on L and let $(x; y) = (x^i; y^r)$ be the corresponding bundle coordinates on $T^*L \to L$. With this, we can write

$$\tilde\omega = f_{ij}dx^i \wedge dx^j + g_{ir}dx^i \wedge dy^r + h_{rs}dy^r \wedge dy^s,$$

where the coefficients are analytic functions.

After shrinking this coordinate neighborhood, we may assume that the coefficients of the Riemannian metric g are uniformely bounded. Moreover, by [24, Proposition 2.2.10], there exist positive constants A_1, K_1 satisfying the pointwise estimate

$$|X^n(\phi)| \le n!A_1K_1^n|X|^n \tag{4.13}$$

for $\phi \in \{f_{ij}, g_{ir}, h_{rs}\}$ and a vector field $X = a^r(x)\partial_{y^r}$ with arbitrary continuous coefficients $a^r(x)$.

Let $\alpha \in \Omega^1_{C^k}(L)$ be C^k-regular, and suppose that $\|\alpha\|_{C^0}$ is sufficiently small, so that its graph is given in these coordinates by $y = \hat\alpha(x)$ for C^k-regular functions $\hat\alpha(x) = (\hat\alpha^r(x))_{r=1,2,3}$. Thus, in these coordinates

$$\xi_\alpha = \hat\alpha_r(x)\frac{\partial}{\partial y^r},$$

and for $|t|$ small

$$\begin{aligned} F(t\alpha) &= f_{ij}(x; t\hat\alpha(x))\, dx^i \wedge dx^j + tg_{ir}(x; t\hat\alpha(x))\, dx^i \wedge d\hat\alpha^r(x) \\ &\qquad + t^2h_{rs}(x; t\hat\alpha(x))\, d\hat\alpha^r(x) \wedge d\hat\alpha^s(x) \\ &= \Big(f_{ij}(x; t\hat\alpha(x)) + tg_{[ir}(x; t\hat\alpha(x))\frac{\partial\hat\alpha^r}{\partial x_{j]}} \\ &\qquad + t^2h_{rs}(x; t\hat\alpha(x))\frac{\partial\hat\alpha^r}{\partial x_{[i}}\frac{\partial\hat\alpha^s}{\partial x_{j]}}\Big) dx^i \wedge dx^j. \end{aligned}$$

Thus, for fixed x, the map $t \mapsto F(t\alpha)_x$ yields an analytic curve in $\Lambda^2T^*_xL$. For the derivatives w.r.t. t of the coefficient functions, note that for $\phi \in \{f_{ij}, g_{ir}, h_{rs}\}$ and $m \in \mathbb{N}$

$$\frac{d^m}{dt^m}\Big|_{t=0} \phi(x; t\hat{\alpha}(x)) = (\xi_\alpha)^m(\phi)_{(x;0)},$$

whence

$$\begin{aligned}
\frac{d^n}{dt^n}\Big|_{t=0} f_{ij}(x; t\hat{\alpha}(x)) &= (\xi_\alpha)^n(f_{ij})_{(x;0)}\\
\frac{d^n}{dt^n}\Big|_{t=0} (tg_{ir}(x; t\hat{\alpha}(x))) &= n(\xi_\alpha)^{n-1}(g_{ir})_{(x;0)}\\
\frac{d^n}{dt^n}\Big|_{t=0} (t^2 h_{rs}(x; t\hat{\alpha}(x))) &= n(n-1)(\xi_\alpha)^{n-2}(h_{rs})_{(x;0)}.
\end{aligned}$$

Thus, for the derivatives we get

$$\begin{aligned}
\frac{d^n}{dt^n}\Big|_{t=0} F(t\alpha)_x = &\Big((\xi_\alpha)^n(f_{ij})_{(x;0)} + n(\xi_\alpha)^{n-1}(g_{[ir})_{(x;0)}\frac{\partial\hat{\alpha}^r}{\partial x_{j]}}\\
&+n(n-1)(\xi_\alpha)^{n-2}(h_{rs})_{(x;0)}\frac{\partial\hat{\alpha}^r}{\partial x_{[i}}\frac{\partial\hat{\alpha}^s}{\partial x_{j]}}\Big)dx^i \wedge dx^j.
\end{aligned} \tag{4.14}$$

Now (4.13) implies that

$$|(\xi_\alpha)^m(\phi)_{(x;0)}| \le m! A_1 K_1^n |\alpha_x|^m$$

for $\phi \in \{f_{ij}, g_{ir}, h_{rs}\}$. Furthermore, since the coefficients of the metric are bounded, there is a constant $K_2 > 0$ such that at every x and for all i, r

$$\left\|\frac{\partial\hat{\alpha}^r}{\partial x^i}\right\|_{C^{k-1};x} \le K_2 \|\alpha\|_{C^k;x}.$$

Since also $\|dx^i \wedge dx^j\|_{C^k;x}$ is uniformely bounded, and as $|\alpha_x| \le \|\alpha\|_{C^k;x}$ for all $k \ge 0$, (4.11) follows for all $p \in L$ parametrized by this coordinate system and for constants A, K depending on these coordinates. As L is compact, it can be covered by finitely many such neighborhoods, whence (4.11) follows for all $p \in L$ for uniform constants $A, K > 0$.

The estimate on the derivatives of $d(F(t\alpha))_p = (t\alpha)^*(d\tilde{\omega})$ follows analogously. □

It follows from Lemma 4.10 that for each $n \in \mathbb{N}_0$, there are symmetric tensors

$$\tilde{\omega}_n : \odot^n\Omega^1_{C^1}(L) \longrightarrow \Omega^2_{C^0}(L), \qquad d\tilde{\omega}_n : \odot^n\Omega^1_{C^1}(L) \longrightarrow \Omega^3_{C^0}(L), \tag{4.15}$$

satisfying

$$\|\langle\tilde{\omega}_n;\alpha\rangle\|_{C^{k-1}} \le n!AK^n\|\alpha\|_{C^k} \qquad \text{and} \qquad \|\langle d\tilde{\omega}_n;\alpha\rangle\|_{C^{k-1}} \le n!AK^n\|\alpha\|_{C^k}$$

by (4.11) and (4.12), respectively, such that for all $\alpha \in \Omega^1_{C^1}(L)$ with $|\alpha_p| < K^{-1}$ with $K > 0$ of (4.11) we have the power series expansion

$$F(\alpha) = \sum_{n=1}^{\infty}\frac{1}{n!}\langle(\tilde{\omega}_n);\alpha\rangle, \qquad dF(\alpha) = \sum_{n=1}^{\infty}\frac{1}{n!}\langle(d\tilde{\omega}_n);\alpha\rangle. \tag{4.16}$$

In particular, $\langle(d\tilde{\omega}_n);\alpha\rangle = d\langle(\tilde{\omega}_n);\alpha\rangle$. It is important to point out that for $\alpha \in \Omega^1_{C^k}(L)$ we have $d\langle(\tilde{\omega}_n);\alpha\rangle \in \Omega^3_{C^{k-1}}(L)$, even though it is the exterior differential of $\langle(\tilde{\omega}_n);\alpha\rangle \in \Omega^2_{C^{k-1}}(L)$.

For $\alpha \in \Omega^1_{C^1}(L)$ the flow of ξ_α on T^*L is given by the formula

$$\Phi_t^{\xi_\alpha}(\beta) = \beta + t\alpha,$$

whence

$$(\Phi_t^{\xi_\alpha})^*(\tilde{\omega}_{\Phi_t^{\xi_\alpha}(p)}) = F(t\alpha)_p,$$

so that

$$\langle(\tilde{\omega}_n);\alpha\rangle = \frac{d^n}{dt^n}\Big|_{t=0} F(t\alpha) = ((\mathcal{L}_{\xi_\alpha})^n(\tilde{\omega})). \tag{4.17}$$

Since the vector fields ξ_α, ξ_β for $\alpha, \beta \in \Omega^1_{C^1}(L)$ commute, it follows that the symmetrization of $(\tilde{\omega}_n)_p$ is given by

$$\langle(\tilde{\omega}_n);\alpha_1,\ldots,\alpha_n\rangle = \mathcal{L}_{\xi_{\alpha_1}}\cdots\mathcal{L}_{\xi_{\alpha_n}}(\tilde{\omega}). \tag{4.18}$$

Proof of Theorem 4.9. Observe that by (4.17) the description of $\langle\tilde{\omega}_n;\alpha\rangle$ is given in (4.14) and hence a first oder differential operator, and the same holds for the description of $\langle d\tilde{\omega}_n;\alpha\rangle$. For a vector bundle $E \to L$ we denote by $L^2_k(E)$ the Sobolev space of sections of E with regularity $(2;k)$. Thus, the maps $\tilde{\omega}_n$ and $d\tilde{\omega}_n$ of (4.15) extend to a symmetric n-linear map

$$\begin{aligned} \tilde{\omega}_n : \odot^n L^2_k(T^*L) &\longrightarrow L^2_{k-1}(\Lambda^2 T^*L),\\ d\tilde{\omega}_n : \odot^n L^2_k(T^*L) &\longrightarrow L^2_{k-1}(\Lambda^3 T^*L), \end{aligned}$$

By (4.11) we obtain the estimates

$$\begin{aligned} \|\langle\tilde{\omega}_n;\alpha\rangle\|_{L^2_{k-1}(\Lambda^2T^*L)} &\leq n!AK^n\|\alpha\|_{L^2_k(T^*L)} \qquad \text{and}\\ \|\langle d\tilde{\omega}_n;\alpha\rangle\|_{L^2_{k-1}(\Lambda^3T^*L)} &\leq n!AK^n\|\alpha\|_{L^2_k(T^*L)}. \end{aligned}$$

In particular, for $\alpha \in L^2_k$ with $\|\alpha\|_{L^2_k} < K^{-1}$, the power series $\sum_{n=0}^\infty \frac{1}{n!}\langle\tilde{\omega}_n;\alpha\rangle$ converges in $L^2_{k-1}(\Lambda^2T^*L)$, thus defining the maps

$$\begin{aligned} F_k : B_{K^{-1}}(0) &\longrightarrow L^2_{k-1}(\Lambda^2T^*L), & \alpha &\longmapsto \textstyle\sum_{n=0}^\infty \frac{1}{n!}\langle\tilde{\omega}_n;\alpha\rangle\\ dF_k : B_{K^{-1}}(0) &\longrightarrow L^2_{k-1}(\Lambda^3T^*L), & \alpha &\longmapsto \textstyle\sum_{n=0}^\infty \frac{1}{n!}\langle d\tilde{\omega}_n;\alpha\rangle \end{aligned}$$

where $B_{K^{-1}}(0) \subset L^2_k(T^*L)$ is the ball centered at 0. Clearly, F_k and dF_k are analytic maps between Banach spaces in the sense of Definition 6.1, and moreover, because of (4.16) and (4.17), F_k and dF_k extend the maps $F : \Omega^1_{C^k}(L) \to \Omega^2_{C^{k-1}}(L)$ and $dF : \Omega^1_{C^k}(L) \to \Omega^3_{C^{k-1}}(L)$ of before.

Thus, $\hat{F}$ of (4.5) extends for all $k \geq 1$ to a map

$$\begin{aligned} \hat{F}_k : \quad B_{K^{-1}}(0) &\longrightarrow L^2_{k-1}(T^*L \oplus \mathbb{R})\\ (\alpha, f) &\longmapsto \Big(*F_k(\alpha) + df, \frac{1}{3} * dF_k(\alpha) + 3f\Big) \end{aligned}$$

which is analytic at $(0,0)$ as F_k and dF_k are analytic. Observe that $\partial\hat{F}_k|_{(0,0)}$ is the extension of $\partial\hat{F}|_{(0,0)}$ which by Proposition 4.6 is self adjoint and elliptic. Thus, for all $k \geq 1$

$$\ker(\partial\hat{F}_k|_{(0,0)}) = \operatorname{coker}(\partial\hat{F}_k|_{(0,0)}) = T_L.$$

Denote by $\pi = \pi_k : L^2_k(T^*L \oplus \mathbb{R}) \to T_L$ the orthogonal projection which is continuous as T_L is finite dimensional. Then

$$\underline{\hat{F}}_k := \pi - \hat{F}_k : B_{K^{-1}}(0) \longrightarrow L^2_{k-1}(T^*L \oplus \mathbb{R}) \tag{4.19}$$

is again analytic at $(0,0)$, and its differential at $(0,0)$ is an isomorphism. Therefore, the inverse function theorem for analytic maps of Banach spaces (Proposition 6.3) implies that there is an analytic inverse of $\underline{\hat{F}}_k$:

$$G_k : U_k \longrightarrow V_k,$$

where U_k and V_k are open neighborhoods of the origin in $L^2_{k-1}(T^*L \oplus \mathbb{R})$ and $L^2_k(T^*L \oplus \mathbb{R})$, respectively.

Let $U := U_k \cap T_L$ which is an open neighborhood of the origin and independent of k as $T_L \subset \cap_{k\geq 1} L^2_k(T^*L \oplus \mathbb{R})$, and define the map

$$\tau : U \longrightarrow T_L, \qquad \tau(\alpha) := \pi(G_k(\alpha)) - \alpha. \tag{4.20}$$

Then τ is again analytic, and clearly, $\tau(0) = 0$. Moreover, for $\alpha \in T_L$ we have

$$\partial\tau|_0(\alpha) = \pi(\partial G_k|_{(0,0)}(\alpha)) - \alpha = \pi((\partial\underline{\hat{F}}_k|_{(0,0)})^{-1}(\alpha)) - \alpha = 0,$$

so that $\partial\tau|_0 = 0$.

Observe that for $\alpha \in U$ we have $G_k(\alpha) \in \Omega^1_{C^\infty}(L)$ for all $k \geq 1$, so that we may omit the subscript k. Indeed, the smoothness of $G(\alpha)$ follows since $U \subset T_L$ consists of smooth (in fact analytic) forms; moreover, if $G(\alpha) = (\tilde{\alpha}, \tilde{f})$, then $\hat{F}_k(\tilde{\alpha}, \tilde{f}) = \alpha \in T_L$. As $\hat{F}_k(\tilde{\alpha}, \tilde{f}) = \hat{F}(\tilde{\alpha}, \tilde{f})$ by the smoothness of $(\tilde{\alpha}, \tilde{f})$, (4.7) implies that $\tilde{f} = 0$ and for $\tilde{\alpha} = G(\alpha)$ we have

$$F(G(\alpha)) = \frac{1}{3} d * F(G(\alpha)).$$

Let us define the C^∞-map

$$\Phi : L \times U \longrightarrow M, \qquad (p, \alpha) \longmapsto Exp_p(\xi_{G(\alpha)}),$$

and let $L_\alpha := \Phi(L \times \{\alpha\})$. Evidently, $L_0 = L$ as $\Phi(p, 0) = p$. If $L' \subset M$ is a closed submanifold which is C^1-close to L, then we may write L' as the image of $p \mapsto Exp_p(\xi_\beta)$ for some $\beta \in \Omega^1_{C^1}(L) \subset L^2_1(T^*L)$ with $\|\beta\|_{C^1}$ sufficiently small so that $\beta \in V_1$. We remark that L' is Lagrangian if and only if $\hat{F}_1(\beta) = 0$. Let α be the image $\underline{\hat{F}}_1(\beta) \in U_1$. The condition $\hat{F}_1(\beta) = 0$ is equal that $\alpha = \pi(\beta)$ by (4.19). Then $\alpha (= \pi(\beta))$ is in $U (= U_1 \cap T_L)$. Hence $\beta = G(\alpha)$ and $\tau(\alpha) = \pi(G(\alpha)) - \alpha = 0$, and it completes the proof. □

Theorem 4.9 reduces our consideration of smooth deformations of Langrangian submanifolds to the one in analytic category as follows. Namely, in analogy to Definition 4.8 we define

DEFINITION 4.11. Let $L \subset M$ be a closed Lagrangian submanifold, and let $\tau : U \to T_L$ be the real analytic map from Theorem 4.9.

(1) An element $\alpha \in T_L$ is called *smoothly unobstructed* or *smoothly integrable* w.r.t. τ, if there is a smooth curve $\alpha(t)$ in T_L such that $\alpha(0) = 0$ and $\dot{\alpha}(0) = \alpha$, and such that $\tau(\alpha(t)) \equiv 0$. Otherwise, α is called *smoothly obstructed*.
(2) An element $\alpha \in T_L$ is called *formally unobstructed* or *formally integrable* w.r.t. τ, if there exists a sequence $\alpha_1 = \alpha, \alpha_2, \cdots \in T_L$ such that the formal power series

$$\alpha_t := \sum_{n=1}^{\infty} \alpha_n t^n \in R[[t]] \otimes_{\mathbb{R}} T_L \tag{4.21}$$

satisfies

$$\tau(\alpha_t) = 0 \in R[[t]] \otimes_{\mathbb{R}} T_L$$

as a formal power series in t.

Corollary 4.12. *Let $L \subset M$ be as above. Then for $\alpha \in T_L$ the following are equivalent.*

(1) *α is smoothly obstructed.*
(2) *α is formally obstructed.*
(3) *α is smoothly obstructed w.r.t. τ.*
(4) *α is formally obstructed w.r.t. τ.*

In particular, $L \subset M$ is regular iff $\tau \equiv 0$.

Proof. According to Theorem 4.9, a smooth family $s(t)$ of Lagrangian submanifolds with $s(0) = L$ and $\dot{s}(0) = \alpha$ must be of the form $s(t) = L_{\alpha(t)}$ for a smooth curve $\alpha(t) \in T_L$ with $\alpha(0) = 0$ and $\dot{\alpha}(0) = \alpha$ such that $\tau(\alpha(t)) \equiv 0$. That is, α is smoothly obstructed iff it is smoothly obstructed w.r.t. τ. Likewise, working on the level of formal power series, it follows from Theorem 4.9 that α is formally obstructed iff it is formally obstructed w.r.t. τ. But as $\tau : U \to T_L$ is an analytic function in finitely many variables, the equivalence of smooth and formal obstruction of $\alpha \in T_L$ follows immediately from Artin's approximation theorem [3, Theorem 1.2].

To show the last statement, let $\tau = \sum_{n=2}^{\infty} \tau_n$ be the analytic expansion of τ with $\tau_n : \odot^n T_L \to T_L$. If $\tau \not\equiv 0$, there exists a minimal number $n \geq 2$ such that $\tau_n \neq 0$, so that there is some $\alpha \in T_L$ such that $\tau_n^\alpha := \tau_n(\alpha, \ldots, \alpha) \neq 0$. Let $\alpha_t := \sum_{n=1}^{\infty} \alpha_n t^n$ be a formal power series with $\alpha_1 = \alpha$. Then $\tau(\alpha_t) = t^n \tau_n^\alpha \mod t^{n+1}$, so that $\tau(\alpha_t) \neq 0$. Hence, α is formally obstructed.

Thus, if all $\alpha \in T_L$ are formally unobstructed, then $\tau \equiv 0$. □

Corollary 4.13. *Each connected component of the moduli space of closed regular Lagrangian submanifolds of M, equipped with the C^1-topology, is an analytic manifold whose tangent space at each L may be canonically identified with T_L.*

Proof. Let $L \subset M$ be a regular Lagrangian submanifold, so that the map $\tau : U \to T_L$ from Theorem 4.9 vanishes identically. Then the map $\Phi : L \times U$ induces an analytic parametrization of all C^1-close Lagrangian submanifolds of L, given by $\alpha \mapsto L_\alpha$. □

Let us now describe the analytic expansion of $\tau : U \to T_L$ from Theorem 4.9.

Proposition 4.14. *For $\alpha \in T_L$, define $\hat{\tau}_n^\alpha \in \Omega^1(L)$ for $n \in \mathbb{N}$ recursively by*

$$\begin{aligned} \hat{\tau}_1^\alpha &= \alpha \\ \hat{\tau}_n^\alpha &= \frac{n!}{3} \sum_{r=2}^{n} \sum_{|I|=n} \frac{1}{r! I!} \left(\left(\partial \underline{\hat{F}}|_{(0,0)} \right)^{-1} * (3-d) \langle \omega_r; \hat{\tau}_{i_1}^\alpha, \ldots, \hat{\tau}_{i_r}^\alpha \rangle \right)_{\Omega^1(L)} , \end{aligned} \tag{4.22}$$

summing over all multi-indices $I = (i_1, \ldots, i_r)$, setting $|I| := i_1 + \ldots + i_r$ and $I! := i_1! \cdots i_r!$.

Then

$$\tau(\alpha) = \sum_{n=2}^{\infty} \pi(\hat{\tau}_n^\alpha) =: \sum_{n=2}^{\infty} \tau_n^\alpha \tag{4.23}$$

with the orthogonal projection $\pi : \Omega^1(L) \to T_L$. *In fact,*

$$\tau_n^\alpha = n! \sum_{r=2}^{n} \sum_{|I|=n} \frac{1}{r!I!} \pi\left(*\langle \omega_r; \hat{\tau}_{i_1}^\alpha, \dots, \hat{\tau}_{i_r}^\alpha \rangle\right). \tag{4.24}$$

Proof. For $\beta \in \Omega^1(L)$ and any $k \geq 1$, the map $t \mapsto \underline{\hat{F}}_k(t\beta)$ of (4.19) is real analytic at $t = 0$ with the expansion

$$\underline{\hat{F}}_k(t\beta) = t\, \partial \underline{\hat{F}}_k|_{(0,0)}(\beta) - \sum_{n=2}^{\infty} \frac{t^n}{3n!} * (3-d)\langle \tilde{\omega}_n; \beta, \dots, \beta \rangle. \tag{4.25}$$

using the definition of $\hat{F}$ in (4.5) and the expansions in (4.16). Since $\partial \underline{\hat{F}}_k|_{(0,0)}$ is invertible, the map $t \mapsto G_k(t\alpha)$ is also analytic at $t = 0$. We expand $G_k(t\alpha)$ as

$$G_k(t\alpha) = \sum_{n=1}^{\infty} \frac{t^n}{n!} \langle g_n; \alpha \dots \alpha \rangle =: \sum_{n=1}^{\infty} \frac{t^n}{n!} g_n^\alpha, \tag{4.26}$$

for n-multilinear maps $g_n : \odot^n T_L \longrightarrow \Omega^1(L) \oplus C^\infty(L)$ which we decompose as

$$g_n^\alpha = \hat{\tau}_n^\alpha + f_n^\alpha.$$

By applying $\underline{\hat{F}}$ to the equation (4.26), and using (4.25), g_n^α must be solutions of the equation

$$\begin{aligned} t\alpha \;&=\; t\, \partial \underline{\hat{F}}_k|_{(0,0)}(g_1^\alpha) + \sum_{n=2}^{\infty} t^n \Big(\frac{1}{n!} \partial \underline{\hat{F}}_k|_0(g_n^\alpha) \\ &\qquad - \frac{1}{3} \sum_{r=2}^{n} \sum_{|I|=n} \frac{1}{r!I!} * (3-d)\langle \omega_r; \hat{\tau}_{i_1}^\alpha, \dots, \hat{\tau}_{i_r}^\alpha \rangle \Big). \end{aligned} \tag{4.27}$$

Comparing the t^n-coefficient and using that $\partial \underline{\hat{F}}_k|_{(0,0)}$ is the identity on T_L, we obtain the equation $g_1^\alpha = \alpha$, so that $\hat{\tau}_1^\alpha = \alpha$ and $f_1^\alpha = 0$, and

$$g_n^\alpha \;=\; \frac{n!}{3} \sum_{r=2}^{n} \sum_{|I|=n} \frac{1}{r!I!} \left(\partial \underline{\hat{F}}|_{(0,0)}\right)^{-1} * (3-d)\langle \omega_r; \hat{\tau}_{i_1}^\alpha, \dots, \hat{\tau}_{i_r}^\alpha \rangle.$$

Hence, $\hat{\tau}_n^\alpha$ of (4.22) is the $\Omega^1(L)$-component of g_n^α.

Now by (4.20), the series expansion of τ is

$$\tau(t\alpha) = \pi(G_k(t\alpha)) - t\alpha = \sum_{n=1}^{\infty} t^n \pi(g_n^\alpha) - t\alpha = \sum_{n=2}^{\infty} t^n \pi(\hat{\tau}_n^\alpha),$$

using that $g_1^\alpha = \alpha$. This shows (4.23), and (4.24) follows as π is the projection onto the $(+1)$-eigenspace of $\underline{\hat{F}}_k|_{(0,0)}$ by definition, so that $\pi\left(\underline{\hat{F}}_k|_{(0,0)}\right)^{-1} = \pi$. □

DEFINITION 4.15. Let $L \subset M$ be a closed Lagrangian submanifold. The *Kuranishi map of* L is the symmetric bilinear map

$$K : T_L \times T_L \longrightarrow \Omega^2(L), \qquad K(\alpha_1, \alpha_2) := \langle \omega_2; \alpha_1, \alpha_2 \rangle = \mathcal{L}_{\xi_{\alpha_1}} \mathcal{L}_{\xi_{\alpha_2}}(\omega).$$

Thus, by (4.24) we have

$$\tau_2^\alpha = \pi * K(\alpha, \alpha),$$

whence the we obtain the following result.

Proposition 4.16. *Assume that $L \subset M$ is a Lagrangian submanifold, and let $\alpha \in T_L$. If α is smoothly unobstructed, then $\pi * K(\alpha, \alpha) = 0$, i.e.,*

$$\int_L K(\alpha, \alpha) \wedge \beta = 0 \qquad \textit{for all } \beta \in T_L.$$

Proof. If $\alpha \in T_L$ is smoothly unobstructed, then by Theorem 4.9 there is a curve $\alpha(t)$ in T_L with $\alpha(0) = 0$ and $\dot\alpha(0) = \alpha$ such that $\tau(\alpha(t)) \equiv 0$. As $\partial\tau|_0 = 0$, we have

$$0 = \left.\frac{d^2}{dt^2}\right|_{t=0} \tau(\alpha(t)) = 2\tau_2(\dot\alpha, \dot\alpha)|_{t=0} = 2\tau_2^\alpha = 2\pi * K(\alpha, \alpha).$$

From this, the claim follows. □

Evidently, with increasing n, the the n-th order formal obstructions of an element $\alpha \in T_L$ become increasingly involved.

5. Examples

In this section we wish to apply our results to Lagrangian submanifolds of the standard nearly Kähler sphere (S^6, J_0, g_0) and put our work into the context of the deformation results in [34].

Let $\mathbb{O}$ denote the octonions, which is the unique 8-dimensional normed division algebra. It may be orthogonally decomposed into $\mathbb{O} = \mathbb{R} \cdot 1 \oplus \operatorname{Im} \mathbb{O}$, and there is a vector cross product $\times$ on $\operatorname{Im} \mathbb{O}$, defined as the imaginary part of the octonion multiplication, i.e.

$$x \times y = \frac{1}{2}(x \cdot y - y \cdot x) \qquad \text{for all } x, y \in \operatorname{Im} \mathbb{O},$$

where $\langle .; . \rangle$ denotes the scalar product on $\mathbb{O}$. Then the automorphism group G_2 of $\mathbb{O}$ preserves the inner product and acts on the 7-dimensional space $\operatorname{Im} \mathbb{O}$, and it clearly preserves the cross product $\times$ on $\operatorname{Im} \mathbb{O}$. Furthermore, we define the 3-form φ on $\operatorname{Im} \mathbb{O}$ by

$$\varphi(x, y, z) := \langle x \times y; z \rangle,$$

so that φ is invariant under the action of G_2; indeed, G_2 can also be described as the group of automorphisms on $\operatorname{Im} \mathbb{O}$ preserving ϕ.

Let $S^6 \subset \operatorname{Im} \mathbb{O}$ denote the unit sphere with the round metric g_0 induced by the inner product $\langle .; . \rangle$ on $\operatorname{Im} \mathbb{O}$. Then there is an orthogonal almost complex structure J_0 on S^6, defined as

$$J_0|_p(u) := p \times u$$

for $p \in S^6$ and $u \in T_pS^6 \subset \operatorname{Im}\mathbb{O}$. Since the cone metric over S^6 is the flat metric on $\operatorname{Im}\mathbb{O}$ which is clearly a (flat) G_2-manifold, the result of Bär [4, §7] already mentioned in section 2.2 implies that (S^6, g_0, J_0) is a strict nearly-Kähler manifold. It follows that the action of G_2 on $S^6 \subset \operatorname{Im}\mathbb{O}$ preserves the nearly-Kähler structure and is in fact the invariance group of this structure. Moreover, this action of G_2 on S^6 is transitive, with stabilizer $SU(3) \subset G_2$, whence we may write the sphere as a homogeneous space

$$S^6 = G_2/SU(3).$$

We call a Lagrangian submanifold $L \subset S^6$ *linearly full*, if it is not contained in a totally geodesic sphere $S^5 \subset S^6$. For instance (cf. [33, Example 6.11.]), if $\Sigma \subset \mathbb{CP}^2$ is a holomorphic curve, then the inverse image of Σ under the Hopf fibration $S^5 \to \mathbb{CP}^2$ yields a Lagrangian submanifold $L_\Sigma \subset S^5 \subset S^6$ which is not linearly full. In fact, any Lagrangian submanifold $L \subset S^6$ which is not linearly full is of this type [33, Theorem 1.1].

In [35], Mashimo gave a complete classification of homogeneous Lagrangian sumbmanifolds $L \subset S^6$, i.e., L is the orbit of some subgroup $H \subset G_2$. Indeed, there are, up to G_2-equivalence, five inequivalent Lagrangian submanifolds; the description of the induced metric is given in [8].

(1) The totally geodesic Lagrangian sphere $L_0 := S^3 \subset S^6$, given as the intersection of S^6 with a coassociative subspace $V^4 \subset \mathbb{O}$, i.e., such that $\varphi|_V \equiv 0$.

(2) The "squashed sphere"

$$L_1 := \left\{ \frac{\sqrt{5}}{3} q i \bar{q} + \frac{2}{3} \bar{q} \epsilon \ : \ q \in Sp(1) \right\},$$

on the decomposition $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\epsilon$ for some unit octonion $\epsilon \in \mathbb{H}^\perp$. Clearly, L_1 is again a sphere, and the metric induced by this embedding is a Berger metric, invariant under $U(2) \subset G_2$. That is, every oriented isometry of L_1 extends to S^6.

(3) The space $L_2 := L_\Sigma$ with the notation from above, where $\Sigma \subset \mathbb{CP}^2$ is the quadric

$$z_1^2 + z_2^2 + z_3^2 = 0.$$

Then $L_2 \subset S^5(1) \subset S^6(1)$ is not linearly full and diffeomorphic to $\mathbb{RP}^3$. In fact, it is acted on simply transitively by the subgroup $SO(3) \subset SU(3) \subset G_2$, where $SU(3)$ is the stabilizer of $(S^5)^\perp$. The induced metric on L_2 is again a Berger metric, but the only oriented isometries which extend to S^6 are the elements of $SO(3)$.

(4) There is a (unique) subgroup $SO(3) \subset G_2$ which acts irreducibly on $\operatorname{Im}\mathbb{O}$, thus identifying $\operatorname{Im}\mathbb{O}$ with $\mathcal{H}^3(\mathbb{R}^3)$, the space of harmonic cubic polynomials in the three variables x, y, z, as an $SO(3)$-module.

If $p \in \mathcal{H}^3(\mathbb{R}^3)$ is completely reducible, then – up to a multiple – it is contained in the $SO(3)$-orbit of one of the two non-equivalent polynomials

$$p_3(x, y, z) = x(x^2 - 3y^2) \qquad \text{or} \qquad p_4(x, y, z) := xyz,$$

and we let $L_k := SO(3) \cdot \frac{p_k}{|p_k|} \subset S^6$ for $k = 3, 4$ be the $SO(3)$-orbit of these polynomials. Then $L_3 = SO(3)/D_3$ and $L_4 = SO(3)/A_4$, where D_3 and A_4 denote the dihedral and the tetrahedral group, respectively.

The induced metric on L_3 is a Berger metric, whereas the induced metric on L_4 is a metric of constant curvature.

Observe that among the homogeous examples given above, the two Lagrangian subspaces L_0 and L_2 are not full. Indeed, L_0 and L_2 coincide with L_Σ, where $\Sigma \subset \mathbb{CP}^2$ is the degree 1 curve $z_1 = 0$ or the degree 2 curve $z_1^2 + z_2^2 + z_3^2 = 0$, respectively.

The action of G_2 on S^6 is transitive on totally geodesic Lagrangian 3-spheres, whence the moduli space of deformations of L_0 contains its orbit under G_2 which is $G_2/SO(4)$ and hence 10-dimensional. On the other hand, in [34], Lotay calculated that the formal tangent space T_{L_0} is 10-dimensional, whence for each element of T_{L_0} there is a local deformation. From this, it follows immediately that $L_0 \subset S^6$ is regular, and the moduli space of its deformation is the manifold $G_2/SO(4)$.

For $d = 2$, the sets of smooth conics in $\mathbb{CP}^3$ is the homogeneous space $SL(3,\mathbb{C})/SO(3,\mathbb{C})$, whence the moduli space of the Hopf lifts of curves of degree $d = 2$ is given as

$$\mathcal{M}_{d=2} = G_2 \times_{SU(3)} SL(3,\mathbb{C})/SO(3,\mathbb{C}).$$

In particular, $\mathcal{M}_{d=2}$ is a manifold of dimension 16, and the moduli space of deformations of the smooth conic $L_2 \subset S^6$ must contain $\mathcal{M}_{d=2}$. On the other hand, according to [34], the formal tangent space T_{L_2} is 16-dimensional, whence for each element of T_{L_2} there is a local deformation, namely the corresponding curve in $\mathcal{M}_{d=2}$. From this, it follows again that $L_2 \subset S^6$ is regular, and the moduli space of local deformation is $\mathcal{M}_{d=2}$.

The Lagrangian $L_1 \subset S^6$ is full, and stabilized by the action of the subgroup $U(2) \subset G_2$ stabilizing the decomposition $\mathrm{Im}\,\mathbb{O} = \mathrm{Im}\,\mathbb{H} \oplus \mathbb{H}$. Thus, the space of its deformations contains its orbit under the G_2-action which is $G_2/U(2)$ and hence 10-dimensional. Again, T_{L_1} is 10-dimensional by [34], whence as in the preceding cases, L_1 is regular with local deformation space $G_2/U(2)$.

For the remaining two homogeneous Lagrangian submanifolds $L_3, L_4 \subset S^6$, the dimension of T_{L_i} has also been calculated in [34], but in these two cases, it is not evident from the description if these are regular submanifolds nor what the local deformation spaces would look like. We summarize our discussion in the following table.

Table. Properties of homogeneous Lagrangian subspaces of S^6.

	$\dim T_{L_i}$	L_i regular?	deformation space of L_i
$i = 0$	8	yes	$\mathcal{M}_{d=1} = G_2/SO(4)$
$i = 1$	10	yes	$G_2/U(2)$
$i = 2$	16	yes	$\mathcal{M}_{d=2} = G_2 \times_{SU(3)} SL(3,\mathbb{C})/SO(3,\mathbb{C})$
$i = 3$	41	?	
$i = 4$	22	?	

Remark 5.1. (1) The rigidity of the Lagrangian sphere $S^3(1)$ also follows from the Simons rigidity theorem which states that each geodesic sphere in S^n is rigid as minimal submanifold up to the motion of the isometry group $SO(n+1)$ [51, Theorem 5.2.3].

(2) The term rigidity we use here is equivalent to the notion of rigidity in [34, Definition 4.12, p. 28]. Our notation of regularity corresponds to Lotay's notion of Jacobi integrability [34, Definition 3.18, p.18]. Lotay's notion of stability corresponds to a special case of our notion of regularity [34, Definition 4.12, p. 27-28].

6. Appendix. Real analytic Banach manifolds and implicit function theorem

In this Appendix we recall the notion of a real analytic Banach manifold, following Eells [12], see also [11, §2], and the analytic inverse function theorem, following Douady [11, §6]. Then we prove a simple criterion for a smooth mapping to be analytic (Lemma 6.2). We also derive the analytic implicit function theorem (Proposition 6.4) from the analytic inverse function theorem. We always work over the field $\mathbb{R}$ of real numbers, if not specified otherwise.

Let E and F be real Banach spaces, and U an open subset of E. Denote by $L(E, F)$ the vector space of all continuous linear maps $u : E \to F$. Let us recall that a map $\phi : U \to F$ is called *Fréchet differentiable* at $x_0 \in U$ if there is an element $\Phi \in L(E, F)$ such that

$$\lim_{v\to 0} \frac{|\phi(x_0 + v) - \phi(x) - \Phi(v)|_F}{|v|_E} = 0.$$

In this case $\Phi(v)$ is unique and also denoted by $\phi_*(x, v)$ or $d\phi(x; v)$. We regard $d\phi$ as a mapping from U to $L(E, F)$.

Denote by $SL^r(E, F)$ the class of continuous symmetric r-linear maps $E \times_{r\, times} \times E \to F$. Inductively, ϕ is of class C^r, if $d^r\phi : U \to SL^r(E, F)$ is continuous.

DEFINITION 6.1. Let E and F be two Banach spaces and U an open subset in E. A smooth map $h : U \to F$ is called *analytic at a point $a \in U$*, if there exists $r > 0$ such that for all $|x| < r$ we have $(a + x) \in U$ and

$$h(a + x) = \sum_{k=0}^{\infty} \frac{d^k h(a; x, \cdots, x)}{k!}. \tag{6.1}$$

To recognize analytic maps among smooth maps we use the following Lemma.

Lemma 6.2. *Let U be an open subset of a Banach space E. A smooth mapping f from U to a Banach space F is analytic at a point $x \in U$ iff there exists a positive number r depending on x such that the following holds. For any affine line l through x the restriction of f to $l \cap U$ is analytic at x with radius of convergence at least r.*

Proof. The "only if" assertion of Lemma 6.2 is straightforward. Now let us prove the "if" assertion of Lemma 6.2. Since we do not assume any condition on f, w.l.o.g. we can assume that $x = 0 \in E$. By the assumption the sphere $S(r)$ of radius r and with center at $x = 0$ lies in U. Let $s \in S(r)$. Set $g(t) := f(ts)$. By the assumption $g(t)$ is analytic at 0 with the convergence radius at least r. Since

$$\frac{dg}{dt} = df(ts; s)$$

and hence

$$\frac{d^n g}{dt^n} = d^n f(ts; s, \cdots, s)$$

we have the Taylor expansion of g at zero is

$$g(t) = f(0) + \sum_{k=0}^{\infty} \frac{d^k f(0, s, \cdots, s)t^k}{k!}. \tag{6.2}$$

Comparing (6.2) with (6.1), we obtain immediately Lemma 6.2. □

Having the notion of an analytic mapping between Banach vector spaces, it is straightforward to define the notion of an analytic Banach manifold. Now we formulate the analytic inverse function theorem that has been proved in [11].

Proposition 6.3 ([11, Theorem 1]). *Let X and Y be two analytic Banach manifolds and $f : X \to Y$ an analytic map. Assume that $b = f(a)$ and $T_a f : T_a X \to T_b Y$ is an isomorphism. Then f is a local isomorphism.*

Now we combine the implicit function theorem for Banach spaces as formulated in [25, Chapter I, Theorem 5.9] and the analytic inverse function theorem to prove the following.

Proposition 6.4. *Let U, V be an open sets in Banach spaces E and F respectively, and let $f : U \times V \to G$ be an analytic mapping. Let $(a, b) \in U \times V$ and assume that the restriction of the differential Df at (a, b) to $(0, F) \subset E \times F$ to G is a topological isomorphism. Let $f(a, b) = 0$. Then there exist a small neighborhood U_0 of a in U and an analytic mapping $g : U_0 \to V$ such that*

$$f(x, g(x)) = 0$$

for all $x \in U_0$.

Proof. The proof of Proposition 6.4 repeats the proof of the implicit function theorem given in [25, p. 19]. It reduces to the analytic inverse function theorem 6.3 by considering the new map $\phi : U \times V \to E \times F$, $(x, y) \mapsto (x, f(x, y))$, so we omit the detail of the proof. □

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