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DIRAC OPERATORS ON THE FEFFERMAN SPIN SPACES IN ALMOST CR-GEOMETRY

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Abstract

A spin structure on a contact Riemannian manifold carries a spin structure on a circle bundle over the manifold. We have interest in the Dirac operators associated with those structures. In terms of a modified Tanno connection, relations between them are studied and some kinds of their explicit expressions are offered.

1. Introduction

Let (M,θ) be a $(2n+1)$-dimensional contact manifold with a contact form θ. There is a unique vector field ξ such that $\xi\rfloor\theta = 1$ and $\xi\rfloor d\theta = 0$. Let us equip M with a Riemannian metric g and a $(1,1)$-tensor field J which satisfy $g(\xi, X) = \theta(X)$, $g(X, JY) = -d\theta(X,Y) := -X(\theta(Y)) + Y(\theta(X)) + \theta([X,Y])$ and $J^2X = -X + \theta(X)\xi$ for any vector fields X, Y. If the contact Riemannian manifold (M,θ,g,J) has a spin structure, then it carries canonically a spin structure on a circle bundle over the manifold (cf. §3). The total space denoted by $\sqrt{F(M)}$ with the spin structure is called the Fefferman spin space. We notice Petit's study ([13]) on Dirac-type operators, Lichnerowicz-type formulas and vanishing theorems on M, and Baum's study ([3], [4, §2.7]) on the Dirac operator, the twistor spinors and the holonomy theorem, etc., on $\sqrt{F(M)}$ in the case J is integrable, i.e., $[\Gamma(H_+),\Gamma(H_+)] \subset \Gamma(H_+)$, where we set $H = \ker\theta$, $H_\pm = \{X \in H\otimes\mathbb{C} \mid JX = \pm iX\}$. The author wishes to contribute to those investigations by applying an idea employed in a series of our works [9], [6], [10], [11], [12]. In this paper, our study focuses on the Dirac operators on M and $\sqrt{F(M)}$ consistently with no assumption that J is integrable. Relations between them are studied and some kinds of their explicit expressions will be offered.

Here let us explain the idea. In CR-geometry, in the case J is integrable (as in [3]) the Tanaka-Webster connection will be the main tool, and in the case J may be non-integrable (as in [13]) so will be a generalized one ${}^*\nabla$ introduced by Tanno ([14]), called the Tanno connection in this paper, defined by

$$ {}^*\nabla_X Y = \nabla^g_X Y - \frac{1}{2}\theta(X)JY - \theta(Y)\nabla^g_X\xi + (\nabla^g_X\theta)(Y)\xi, $$

where ∇^g is the Levi-Civita connection of g. As stated above the latter case is considered in this paper, and as the main tool we employ not the Tanno connection but a connection modified as follows, however: In general, the action of the Tanno connection does not commute with that of J. In fact, Tanno indicated

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$$({}^*\nabla_X J)Y = Q(Y,X) := (\nabla^g_X J)Y + (\nabla^g_X \theta)(JY)\,\xi + \theta(Y) J \nabla^g_X \xi$$

and showed that the tensor Q vanishes if and only if J is integrable. We consider now the hermitian part ${}^\sharp\nabla$, called the hermitian Tanno connection, that is, we set

$$ {}^\sharp\nabla_X Y = {}^*\nabla_X Y - \frac{1}{2} J Q(Y,X) = \begin{cases} {}^*\nabla_X (f\xi) & : Y = f\xi \ \ (f \in C^\infty(M)), \\ \frac{1}{2}\Big({}^*\nabla_X Y - J\,{}^*\nabla_X JY\Big) & : Y \in \Gamma(H), \end{cases}$$

so that ${}^\sharp\nabla J = 0$ obviously. (Note that the connections ${}^*\nabla$, ${}^\sharp\nabla$ and the Tanaka-Webster connection coincide if J is integrable.) The commutativity rather simplifies investigation and computation, and there seem to be many results in the case of integrable J which will be generalized to the case of general J just by changing the Tanaka-Webster connection to the hermitian Tanno connection.

In Theorem 2.3 (or (2.16)) we express the Dirac operator on M explicitly in terms of the hermitian Tanno connection, and accordingly we deduce another type of explicit expression in Theorem 2.6 (or (2.18)) by utilizing the exterior covariant differentiation and its dual. The hermitian Tanno connection works effectively also in the study of the Fefferman spin space. The merit to be mentioned first is that the curvature of an Ehresmann-type connection of the bundle $\sqrt{F(M)} \to M$, called the Fefferman connection, can be expressed explicitly in terms of the pseudohermitian Ricci and scalar curvatures $\mathrm{Ric}^{{}^\sharp\nabla}$, $s^{{}^\sharp\nabla}$ of ${}^\sharp\nabla$ (cf. (3.2)). Consequently we obtain explicit expressions (Theorems 4.3 and 4.4) of the Dirac operator on $\sqrt{F(M)}$. The author feels it difficult to reach such results without the concept of hermitian Tanno connection.

2. Dirac operator on the contact Riemannian manifold

In this section, for the sake of later use, we examine mainly the case where M has a spin structure. It is easily generalized to the case M has a Spin^c structure, which will be mentioned briefly in §2.1.

First, let us recall quickly basic properties of the connections ${}^*\nabla$, ${}^\sharp\nabla$ and explain notational rules. Refer to [14], [9], [12] for more detailed explanation. We have ${}^*\nabla\theta = {}^\sharp\nabla\theta = 0$, ${}^*\nabla g = {}^\sharp\nabla g = 0$, $T({}^*\nabla)(Z,W) = 0$, $T({}^*\nabla)(Z,\overline{W}) = ig(Z,\overline{W})\xi$, $T({}^\sharp\nabla)(Z,W) = [J,J](Z,W)/4 := (-[Z,W] + [JZ,JW] - J[JZ,W] - J[Z,JW])/4$, $T({}^\sharp\nabla)(Z,\overline{W}) = ig(Z,\overline{W})\xi$ $(Z,W \in \Gamma(H_+))$, where $T({}^*\nabla)$, etc., are the torsion tensors. Obviously we have $T({}^*\nabla)(\xi,X) = T({}^\sharp\nabla)(\xi,X)$, which we denote by τX. Note that $\tau \circ J + J \circ \tau = 0$. Next, a local frame $\xi_\bullet = (\xi_1,\ldots,\xi_n,\xi_{\bar 1},\ldots,\xi_{\bar n},\xi_0 = \xi)$ $(\xi_{\bar\alpha} := \overline{\xi_\alpha} \in H_-)$ of the bundle $TM \otimes \mathbb{C} = H_+ \oplus H_- \oplus \mathbb{C}\xi$ is always assumed to be unitary, i.e., $g(\xi_\alpha,\xi_\beta) = 0$, $g(\xi_\alpha,\xi_{\bar\beta}) = \delta_{\alpha\beta}$ $(1 \le \alpha,\beta \le n)$, and its dual frame is denoted by $\theta^\bullet = (\theta^1,\ldots,\theta^n,\theta^{\bar 1},\ldots,\theta^{\bar n},\theta^0 = \theta)$. Setting $e_{2\alpha-1} = (\xi_\alpha + \xi_{\bar\alpha})/\sqrt{2}$ and $e_{2\alpha} = Je_{2\alpha-1} = (\xi_{\bar\alpha} - \xi_\alpha)/\sqrt{-2}$, we have a positively oriented orthonormal frame, or an $SO(2n+1)$-frame $e_\bullet = (e_1,e_2,\ldots,e_{2n},e_0)$, and denote its dual frame by $e^\bullet = (e^1,e^2,\ldots,e^{2n},e^0)$. As usual the Greek indices $\alpha,\beta,\ldots$ vary from 1 to n and the block Latin indices A, B, … vary in $\{0,1,\ldots,n,\bar 1,\ldots,\bar n\}$, so that

$$\tau = \sum \xi_\alpha \otimes \theta^{\bar\gamma} \cdot \tau^\alpha_{\bar\gamma} + \sum \xi_{\bar\alpha} \otimes \theta^\gamma \cdot \tau^{\bar\alpha}_\gamma \quad (\tau^{\bar\alpha}_\gamma = \tau^{\bar\gamma}_\alpha),$$

$$Q = \sum \xi_\alpha \otimes \theta^{\bar\beta} \otimes \theta^{\bar\gamma} \cdot Q^\alpha_{\bar\beta\bar\gamma} + \sum \xi_{\bar\alpha} \otimes \theta^\beta \otimes \theta^\gamma \cdot Q^{\bar\alpha}_{\beta\gamma} \ \ (Q^{\bar\alpha}_{\beta\gamma} = -Q^{\bar\beta}_{\alpha\gamma} = -Q^{\bar\beta}_{\gamma\alpha} - Q^{\bar\gamma}_{\alpha\beta}).$$

If we set ${}^*\nabla\xi_B = \sum \xi_A \cdot \omega({}^*\nabla)^A_B$, ${}^\sharp\nabla\xi_B = \sum \xi_A \cdot \omega({}^\sharp\nabla)^A_B$, then

$$\omega({}^*\nabla)^\alpha_\beta = \omega({}^\sharp\nabla)^\alpha_\beta,\ \ \omega({}^*\nabla)^{\bar\alpha}_{\bar\beta} = \omega({}^\sharp\nabla)^{\bar\alpha}_{\bar\beta},\ \ \omega({}^*\nabla)^{\bar\alpha}_\beta(\xi_\gamma) = -\frac{i}{2}Q^{\bar\alpha}_{\beta\gamma},\ \ \omega({}^*\nabla)^\alpha_{\bar\beta}(\xi_{\bar\gamma}) = \frac{i}{2}Q^\alpha_{\bar\beta\bar\gamma}$$

and the others vanish.

Now, we suppose that M has a spin structure

$$\rho : \mathrm{Spin}(TM) \to SO(TM), \tag{2.1}$$

where $SO(TM)$ is the principal $SO(2n+1)$-bundle consisting of $SO(2n+1)$-frames of TM and $\mathrm{Spin}(TM)$ is a principal $\mathrm{Spin}(2n+1)$-bundle together with a 2-sheeted covering map ρ. This naturally reduces to a spin structure of $H = \ker\theta$,

$$\rho_H : \mathrm{Spin}(H) \to SO(H). \tag{2.2}$$

Indeed, we embed $SO(H)$ into $SO(TM)$ by the map $(e_1, \ldots, e_{2n}) \mapsto (e_1, \ldots, e_{2n}, e_0 = \xi)$ and set $\mathrm{Spin}(H) = \rho^{-1}SO(H)$. The almost hermitian vector bundle $(H, g|_H, J|_H)$ carries, on the other hand, a canonical Spin^c structure (e.g. [7, Example D.6]), which is related to (2.2) as follows: The canonical line bundle

$$\begin{aligned} K &= \wedge^{n+1,0}(M) := \{\omega \in \wedge^{n+1}T^*M \otimes \mathbb{C} \mid\ X\rfloor\omega = 0\ (X \in H_-)\} \\ &= \wedge^{n,0}_H(M) := \{\omega \in \wedge^{n}T^*M \otimes \mathbb{C} \mid\ X\rfloor\omega = 0\ (X \in H_- \cup \mathbb{C}\xi)\} \end{aligned}$$

has a globally defined square root $K^{1/2}$ so that the canonical one is expressed as

$$\rho^c_H : \mathrm{Spin}(H) \times_{\mathbb{Z}_2} U(K^{-1/2}) \to SO(H) \times U(K^{-1}). \tag{2.3}$$

Here it will be proper to regard K as $\wedge^{n,0}_H(M)$, and $U(K^{-1})$, etc., are the principal $U(1)$-bundles associated with K^{-1}, etc. Referring to [7, p.395], we have the associated spinor bundle

$$\begin{gathered} \$^c_H = \$_H \otimes K^{-1/2} = \wedge^{0,*}_H(M) := \{\omega \in \wedge^* T^*M \otimes \mathbb{C} \mid X\rfloor\omega = 0\ (X \in H_+ \cup \mathbb{C}\xi)\} \\ (\$_H := \mathrm{Spin}(H) \times_{\Delta_{2n}} \wedge^*\mathbb{C}^n) \end{gathered} \tag{2.4}$$

with the Clifford action of $\mathbb{C}l(H)$ given by

$$\xi_\alpha\circ = \sqrt{2}\,\theta^{\bar\alpha}\wedge, \qquad \xi_{\bar\alpha}\circ = -\sqrt{2}\,\theta^{\bar\alpha}\vee, \tag{2.5}$$

where Δ_{2n} is the standard spinor representation and $\theta^{\bar\alpha}\vee = \xi_{\bar\alpha}\rfloor$ is the interior production.

The structure (2.3) naturally induces a Spin^c structure of TM with the determinant line bundle $K^{-1} = (\wedge^{n+1,0}(M))^{-1}$

$$\rho^c : \mathrm{Spin}(TM) \times_{\mathbb{Z}_2} U(K^{-1/2}) \to SO(TM) \times U(K^{-1}).$$

From now on, let us regard (2.4) as the associated spinor bundle with the Clifford action of $\mathbb{C}l(TM)$ given by (2.5) and

$$\xi\circ = (-1)^{q+1}i \quad (\text{on } \wedge^{0,q}_H(M)\,(\subset \$^c_H))$$

(e.g. [13, Propositions 3.1 and 3.2]). Accordingly, let us regard

$$\$_H = \$^c_H \otimes K^{1/2} = \wedge^{0,*}_H(M) \otimes K^{1/2} \tag{2.6}$$

naturally as the spinor bundle associated with (2.1) as well. Then, obviously we have the following decompositions.

Proposition 2.1 (cf. [13, p.234], [3, Proposition 22]). *We have*

$$\$_H = \$_H^+ \oplus \$_H^- = \bigoplus_{q=0}^{n} \$_H^{n-2q},$$
$$\$_H^\pm := \{\omega \in \$_H \mid i\xi \circ \omega = \pm\omega\} = \wedge_H^{0,\mathrm{even/odd}}(M) \otimes K^{1/2},$$
$$\$_H^{n-2q} := \{\omega \in \$_H \mid i\, d\theta \circ \omega = (n-2q)\omega\} = \wedge_H^{0,q}(M) \otimes K^{1/2},$$

where the Clifford action of $d\theta = i\sum \theta^\alpha \wedge \theta^{\bar\alpha} = \sum e^{2\alpha-1} \wedge e^{2\alpha}$ *is defined by* $d\theta \circ \omega = \sum e_{2\alpha-1} \circ e_{2\alpha} \circ \omega = -i(n + \sum \xi_\gamma \circ \xi_{\bar\gamma} \circ)\,\omega = -i(n - 2\sum \theta^{\bar\gamma} \wedge \theta^{\bar\gamma} \vee)\,\omega.$

The purpose in the following is to offer two kinds of explicit expressions of the Dirac operator $D^{\$_H}$ of (2.1). First, we recall

$$\omega(\nabla^{\$_H}) = \omega(\nabla^{(\$_H:\nabla^g)}) := \frac{1}{4}\sum e_i \circ e_j \circ \cdot\, \omega(\nabla^g; e_\bullet)_i^j = \frac{1}{4}\sum \xi_B \circ \xi_A \circ \cdot\, \omega(\nabla^g)_{\bar B}^A,$$
$$D^{\$_H} = D^{(\$_H:\nabla^g)} := \sum e_k \circ \nabla_{e_k}^{\$_H} = \xi \circ \nabla_\xi^{\$_H} + \sum \xi_\gamma \circ \nabla_{\xi_{\bar\gamma}}^{\$_H} + \sum \xi_{\bar\gamma} \circ \nabla_{\xi_\gamma}^{\$_H},$$

where $\omega(\nabla^{\$_H})$ is the connection 1-form of the spinor connection $\nabla^{\$_H}$ of the bundle $\$_H$ ($=$ $\mathrm{Spin}(TM) \times_{\Delta_{2n+1}} \wedge^*\mathbb{C}^n$), and $\omega(\nabla^g; e_\bullet)$, etc., are those of ∇^g. By definition,

$$\nabla_X^g Y = {}^*\nabla_X Y - g((\tau + \frac{1}{2}J)X, Y)\xi$$
$$+ \theta(Y)(\tau + \frac{1}{2}J)X + \theta(X)(\tau + \frac{1}{2}J)Y - \theta(X)\,\tau Y,$$

which implies the following formulas.

Lemma 2.2. *We have*

$$\omega(\nabla^g)_\beta^\alpha = \omega({}^*\nabla)_\beta^\alpha + \frac{i}{2}\delta_{\alpha\beta}\,\theta = \omega({}^\sharp\nabla)_\beta^\alpha + \frac{i}{2}\delta_{\alpha\beta}\,\theta,$$
$$\omega(\nabla^g)_\beta^{\bar\alpha} = \omega({}^*\nabla)_\beta^{\bar\alpha} = -\frac{i}{2}\sum Q_{\beta\gamma}^{\bar\alpha}\,\theta^\gamma,$$
$$\omega(\nabla^g)_\beta^0 = \frac{i}{2}\theta^{\bar\beta} - \tau^{\bar\beta}, \quad \omega(\nabla^g)_0^\alpha = \frac{i}{2}\theta^\alpha + \tau^\alpha, \quad \omega(\nabla^g)_0^0 = 0.$$

Let us consider the connection $\nabla^{(\$_H:{}^\sharp\nabla)}$ given by

$$\omega(\nabla^{(\$_H:{}^\sharp\nabla)}) := \frac{1}{4}\sum \xi_B \circ \xi_A \circ \cdot\, \omega({}^\sharp\nabla)_{\bar B}^A$$
$$= \frac{1}{4}\sum \xi_{\bar\beta} \circ \xi_\alpha \circ \cdot\, \omega({}^\sharp\nabla)_\beta^\alpha + \frac{1}{4}\sum \xi_\beta \circ \xi_{\bar\alpha} \circ \cdot\, \omega({}^\sharp\nabla)_{\bar\beta}^{\bar\alpha}.$$

Then we have the first expression.

Theorem 2.3. *We have*

$$D^{\$_H} = \sum \xi_C \circ \nabla_{\xi_{\bar C}}^{(\$_H:{}^\sharp\nabla)} - \frac{1}{4}\xi \circ d\theta \circ . \tag{2.7}$$

Remark. The following proof says that the formula with $\nabla^{(\$_H:{}^\sharp\nabla)}$ replaced by $\nabla^{(\$_H:{}^*\nabla)}$ given below also holds, which has been shown in [13, Proposition 3.4].

Proof. Let us consider another connection $\nabla^{(\$_H:{}^*\nabla)}$ given by

$$\omega(\nabla^{(\$_H:{}^*\nabla)}) = \frac{1}{4}\sum \xi_B \circ \xi_A \circ \cdot\, \omega({}^*\nabla)^A_{\bar{B}}.$$

Then, Lemma 2.2 implies

$$\begin{aligned}&\omega(\nabla^{(\$_H:\nabla^g)}) - \omega(\nabla^{(\$_H:{}^*\nabla)})\\ &\quad = \frac{1}{4}d\theta \circ \cdot\theta + \frac{1}{2}\sum \xi_\beta \circ \xi_0 \circ \cdot\Big\{-\frac{i}{2}\theta^\beta - \tau^\beta\Big\} + \frac{1}{2}\sum \xi_{\bar\beta} \circ \xi_0 \circ \cdot\Big\{\frac{i}{2}\theta^{\bar\beta} - \tau^{\bar\beta}\Big\},\end{aligned}$$

$$\sum \xi_C \circ \cdot\Big\{\omega(\nabla^{(\$_H:\nabla^g)})(\xi_{\bar C}) - \omega(\nabla^{(\$_H:{}^*\nabla)})(\xi_{\bar C})\Big\} = -\frac{1}{4}\,\xi \circ d\theta \circ .$$

Further, since $\omega(\nabla^{(\$_H:{}^*\nabla)}) = \omega(\nabla^{(\$_H:{}^\sharp\nabla)}) + \frac{i}{8}\sum \xi_\beta \circ \xi_\alpha \circ \cdot\, Q^\alpha_{\bar\beta\bar\gamma}\theta^{\bar\gamma} - \frac{i}{8}\sum \xi_{\bar\beta} \circ \xi_{\bar\alpha} \circ \cdot\, Q^{\bar\alpha}_{\beta\gamma}\theta^\gamma$ and $Q^{\bar\alpha}_{\beta\gamma} + Q^{\bar\beta}_{\gamma\alpha} + Q^{\bar\gamma}_{\alpha\beta} = 0$, obviously we have

$$\begin{aligned}\xi_0 \circ \cdot\{\omega(\nabla^{(\$_H:{}^*\nabla)})(\xi_0) - \omega(\nabla^{(\$_H:{}^\sharp\nabla)})(\xi_0)\} &= 0,\\ \sum \xi_\gamma \circ \cdot\{\omega(\nabla^{(\$_H:{}^*\nabla)})(\xi_{\bar\gamma}) - \omega(\nabla^{(\$_H:{}^\sharp\nabla)})(\xi_{\bar\gamma})\} &= 0,\\ \sum \xi_{\bar\gamma} \circ \cdot\{\omega(\nabla^{(\$_H:{}^*\nabla)})(\xi_\gamma) - \omega(\nabla^{(\$_H:{}^\sharp\nabla)})(\xi_\gamma)\} &= 0.\end{aligned}$$

Thus we obtain (2.7). □

Next, we investigate the Dirac operator on $\$^c_H$, from which we will deduce another expression of $D^{\$_H}$. To the line bundles K^{-1}, $K^{-1/2}$, attach the unitary connections $\mathbb{A}^{K^{-1}} := \sum \omega({}^\sharp\nabla)^\alpha_\alpha = \sum \omega({}^*\nabla)^\alpha_\alpha$, $\mathcal{A}(K^{-1/2}) := \frac{1}{2}\mathbb{A}^{K^{-1}}$, and define the connections $\nabla^{\$^c_H} = \nabla^{(\$^c_H;\nabla^g,\mathcal{A}(K^{-1/2}))}$ and $\nabla^{(\$^c_H;{}^\sharp\nabla,\mathcal{A}(K^{-1/2}))}$ of $\$^c_H = \$_H \otimes K^{-1/2}$ by

$$\omega(\nabla^{(\$^c_H:\nabla^g,\mathcal{A}(K^{-1/2}))}) = \omega(\nabla^{(\$_H:\nabla^g)}) + \mathcal{A}(K^{-1/2}),$$

etc. Then, obviously Theorem 2.3 yields that the Dirac operator

$$D^{\$^c_H} = D^{(\$^c_H;\nabla^g,\mathcal{A}(K^{-1/2}))} = \sum \xi_C \circ \nabla^{(\$^c_H:\nabla^g,\mathcal{A}(K^{-1/2}))}_{\xi_{\bar C}}$$

is described as

$$D^{\$^c_H} = \sum \xi_C \circ \nabla^{(\$^c_H:{}^\sharp\nabla,\mathcal{A}(K^{-1/2}))}_{\xi_{\bar C}} - \frac{1}{4}\,\xi \circ d\theta \circ . \tag{2.8}$$

Proposition 2.4. *The connection $\nabla^{(\$^c_H;{}^\sharp\nabla,\mathcal{A}(K^{-1/2}))}$ coincides with the hermitian Tanno connection ${}^\sharp\nabla$ itself, i.e., $\nabla^{(\$^c_H:{}^\sharp\nabla,\mathcal{A}(K^{-1/2}))} = {}^\sharp\nabla$ on $\$^c_H = \wedge^{0,*}_H(M)$.*

Proof. We have

$$\begin{aligned}&\frac{1}{4}\sum \xi_{\bar\alpha} \circ \xi_\alpha \circ \cdot\,\omega({}^\sharp\nabla)^\alpha_\alpha + \frac{1}{4}\sum \xi_\alpha \circ \xi_{\bar\alpha} \circ \cdot\,\omega({}^\sharp\nabla)^{\bar\alpha}_{\bar\alpha}\\ &\quad = \frac{1}{2}\sum \omega({}^\sharp\nabla)^\alpha_\alpha \cdot (-\theta^{\bar\alpha} \vee \theta^{\bar\alpha} \wedge + \theta^{\bar\alpha} \wedge \theta^{\bar\alpha} \vee)\\ &\quad = \sum \omega({}^\sharp\nabla)^\alpha_\alpha \cdot \theta^{\bar\alpha} \wedge \theta^{\bar\alpha} \vee - \frac{1}{2}\sum \omega({}^\sharp\nabla)^\alpha_\alpha\end{aligned}$$

and

$$\frac{1}{4}\sum_{\alpha\neq\beta}\xi_{\bar\beta}\circ\xi_\alpha\circ\cdot\,\omega({}^\sharp\nabla)^\alpha_\beta+\frac{1}{4}\sum_{\alpha\neq\beta}\xi_\beta\circ\xi_{\bar\alpha}\circ\cdot\,\omega({}^\sharp\nabla)^{\bar\alpha}_{\bar\beta}$$
$$=-\frac{1}{2}\sum_{\alpha\neq\beta}\omega({}^\sharp\nabla)^\alpha_\beta\cdot\theta^{\bar\beta}\vee\theta^{\bar\alpha}\wedge-\frac{1}{2}\sum_{\alpha\neq\beta}\omega({}^\sharp\nabla)^{\bar\alpha}_{\bar\beta}\cdot\theta^{\bar\beta}\wedge\theta^{\bar\alpha}\vee\;=\sum_{\alpha\neq\beta}\omega({}^\sharp\nabla)^\alpha_\beta\cdot\theta^{\bar\alpha}\wedge\theta^{\bar\beta}\vee\,.$$

□

Let $\bar\partial_H:\Omega^{0,*}_H(M):=\Gamma(\wedge^{0,*}_H(M))\to\Omega^{0,*+1}_H(M)$ be the exterior differentiation $d:\Omega^{0,*}_H(M)\to\Gamma(\wedge^{*+1}T^*M\otimes\mathbb{C})$ followed by the projection to $\Omega^{0,*+1}_H(M)$ and $\bar\partial^*_H$ be the formal adjoint with respect to the natural inner product. Then, by Proposition 2.4 and [9, Proposition 1.3(Weizenböck-type formula for the Kohn-Rossi Laplacian)], we have

$$\sum\xi_\gamma\circ\nabla^{(\mathcal{S}^c_H:{}^\sharp\nabla,\mathcal{A}(K^{-1/2}))}_{\xi_{\bar\gamma}}=\sqrt{2}\sum\theta^{\bar\gamma}\wedge{}^\sharp\nabla_{\xi_{\bar\gamma}}=\sqrt{2}\,\bar\partial_H,$$
$$\sum\xi_{\bar\gamma}\circ\nabla^{(\mathcal{S}^c_H:{}^\sharp\nabla,\mathcal{A}(K^{-1/2}))}_{\xi_\gamma}=-\sqrt{2}\sum\theta^{\bar\gamma}\vee{}^\sharp\nabla_{\xi_\gamma}=\sqrt{2}\,\bar\partial^*_H,$$

which, together with (2.8), imply the following.

Proposition 2.5. *We have*

$$D^{\mathcal{S}^c_H}=\sqrt{2}\,(\bar\partial_H+\bar\partial^*_H)+\xi\circ{}^\sharp\nabla_\xi-\frac{1}{4}\,\xi\circ d\theta\circ\,. \tag{2.9}$$

We have twisted $D^{\mathcal{S}_H}$ with the unitary connection $\mathcal{A}(K^{-1/2})$ of $K^{-1/2}$ to get the expression of $D^{\mathcal{S}^c_H}$. Hence, if we twist it now with the connection

$$\mathcal{A}=\mathcal{A}(K^{1/2}):=-\mathcal{A}(K^{-1/2})=-\frac{1}{2}\sum\omega({}^\sharp\nabla)^\alpha_\alpha$$

of $K^{1/2}$, we obtain an expression of $D^{\mathcal{S}_H}$. That is, we define $\bar\partial^{\mathcal{A}}_H:\Omega^{0,*}_H(M;K^{1/2}):=\Gamma(\wedge^{0,*}_H(M)\otimes K^{1/2})\to\Omega^{0,*+1}_H(M;K^{1/2})$ to be the exterior covariant differentiation $d^{\mathcal{A}}:\Omega^{0,*}_H(M;K^{1/2})\to\Gamma(\wedge^{*+1}T^*M\otimes K^{1/2})$ followed by the projection to $\Omega^{0,*+1}_H(M;K^{1/2})$, and define $\bar\partial^{\mathcal{A}*}_H$ to be the formal adjoint. Note that, considering the twisted hermitian Tanno connection

$${}^\sharp\nabla^{\mathcal{A}}:={}^\sharp\nabla+\mathcal{A},$$

we have

$$\bar\partial^{\mathcal{A}}_H=\sum\theta^{\bar\gamma}\wedge{}^\sharp\nabla^{\mathcal{A}}_{\xi_{\bar\gamma}},\qquad\bar\partial^{\mathcal{A}*}_H=-\sum\theta^{\bar\gamma}\vee{}^\sharp\nabla^{\mathcal{A}}_{\xi_\gamma}. \tag{2.10}$$

Theorem 2.6. *We have*

$$D^{\mathcal{S}_H}=\sqrt{2}\,(\bar\partial^{\mathcal{A}}_H+\bar\partial^{\mathcal{A}*}_H)+\xi\circ{}^\sharp\nabla^{\mathcal{A}}_\xi-\frac{1}{4}\,\xi\circ d\theta\circ\,. \tag{2.11}$$

Remark. A similar formula in the case J is integrable has been mentioned in [13, p.237].

Last, let us offer Lichnerowicz-type formulas.

Theorem 2.7. *On* $\Omega^{0,q}_H(M;K^{1/2})$: *We have*

$$\square^{\mathcal{A}}_H:=\bar\partial^{\mathcal{A}*}_H\bar\partial^{\mathcal{A}}_H+\bar\partial^{\mathcal{A}}_H\bar\partial^{\mathcal{A}*}_H \tag{2.12}$$

$$= -\frac{1}{2}\sum_{A\neq 0}\left({}^\sharp\nabla^{\mathcal{A}}_{\xi_A}{}^\sharp\nabla^{\mathcal{A}}_{\xi_{\bar{A}}} - {}^\sharp\nabla^{\mathcal{A}}_{{}^\sharp\nabla_{\xi_A}\xi_{\bar{A}}}\right) + \frac{i(n-2q)}{2}\,{}^\sharp\nabla^{\mathcal{A}}_{\xi} + \frac{1}{4}s^{\sharp\nabla}$$
$$+\frac{1}{8}\sum Q^{\lambda}_{\bar{\mu}\bar{\beta}}Q^{\bar{\nu}}_{\beta\lambda}\cdot\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee - \frac{1}{8}\sum Q^{\alpha}_{\bar{\mu}\bar{\lambda}}Q^{\bar{\lambda}}_{\beta\nu}\cdot\theta^{\bar{\alpha}}\wedge\theta^{\bar{\beta}}\vee\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee,$$

where we set $s^{\sharp\nabla} = \sum \mathrm{Ric}^{\sharp\nabla}(\xi_\alpha,\xi_{\bar{\alpha}}) = \sum g(F({}^\sharp\nabla)(\xi_\alpha,\xi_{\bar{\alpha}})\xi_\beta,\xi_{\bar{\beta}})$, *called the pseudohermitian scalar curvature. And we have*

$$(2.13)\qquad (D^{g_H})^2 = -\sum\left(\nabla^{g_H}_{\xi_A}\nabla^{g_H}_{\xi_{\bar{A}}} - \nabla^{g_H}_{\nabla^g_{\xi_A}\xi_{\bar{A}}}\right) + \frac{1}{4}s(\nabla^g)$$
$$= -\sum\left({}^\sharp\nabla^{\mathcal{A}}_{\xi_A}{}^\sharp\nabla^{\mathcal{A}}_{\xi_{\bar{A}}} - {}^\sharp\nabla^{\mathcal{A}}_{{}^\sharp\nabla_{\xi_A}\xi_{\bar{A}}}\right) + \frac{i(n-2q)}{2}\,{}^\sharp\nabla^{\mathcal{A}}_{\xi} + \frac{(n-2q)^2}{16} + \frac{1}{2}s^{\sharp\nabla}$$
$$+\frac{1}{4}\sum Q^{\lambda}_{\bar{\mu}\bar{\beta}}Q^{\bar{\nu}}_{\beta\lambda}\cdot\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee - \frac{1}{4}\sum Q^{\alpha}_{\bar{\mu}\bar{\lambda}}Q^{\bar{\lambda}}_{\beta\nu}\cdot\theta^{\bar{\alpha}}\wedge\theta^{\bar{\beta}}\vee\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee$$
$$+\sum\theta^{\bar{\alpha}}\wedge(-1)^q\sqrt{2}\left\{\frac{1}{2}\,{}^\sharp\nabla^{\mathcal{A}}_{\xi_{\bar{\alpha}}} - i\,{}^\sharp\nabla^{\mathcal{A}}_{\tau\xi_{\bar{\alpha}}} + i\,F({}^\sharp\nabla^{\mathcal{A}},\theta^\bullet)(\xi,\xi_{\bar{\alpha}})\right\}$$
$$+\sum\theta^{\bar{\alpha}}\vee(-1)^q\sqrt{2}\left\{\frac{1}{2}\,{}^\sharp\nabla^{\mathcal{A}}_{\xi_\alpha} + i\,{}^\sharp\nabla^{\mathcal{A}}_{\tau\xi_\alpha} - i\,F({}^\sharp\nabla^{\mathcal{A}},\theta^\bullet)(\xi,\xi_\alpha)\right\}$$
$$+\frac{i}{2}\sum\theta^{\bar{\alpha}}\wedge\theta^{\bar{\beta}}\wedge\left\{\sum Q^{\alpha}_{\bar{\beta}\bar{\gamma}}\,{}^\sharp\nabla^{\mathcal{A}}_{\xi_\gamma} - \sum({}^\sharp\nabla_{\xi_\nu}Q)^{\alpha}_{\bar{\beta}\bar{\mu}}\,\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee\right\}$$
$$-\frac{i}{2}\sum\theta^{\bar{\alpha}}\vee\theta^{\bar{\beta}}\vee\left\{\sum Q^{\bar{\alpha}}_{\beta\gamma}\,{}^\sharp\nabla^{\mathcal{A}}_{\xi_{\bar{\gamma}}} + \sum({}^\sharp\nabla_{\xi_{\bar{\mu}}}Q)^{\bar{\alpha}}_{\beta\nu}\,\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee\right\},$$

where $F({}^\sharp\nabla^{\mathcal{A}},\theta^\bullet)(\xi_A,\xi_B)$ *is the curvature of* ${}^\sharp\nabla^{\mathcal{A}}$ *acting on* $\Omega^{0,*}_H(M;K^{1/2})$, *i.e.,*

$$F({}^\sharp\nabla^{\mathcal{A}},\theta^\bullet)(\xi,\xi_{\bar{\alpha}}) = \sum F({}^\sharp\nabla)^{\mu}_{\nu}(\xi,\xi_{\bar{\alpha}})\,\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee + F(\mathcal{A})(\xi,\xi_{\bar{\alpha}})$$
$$= \sum\left\{({}^\sharp\nabla_{\xi_\nu}\tau)^{\alpha}_{\bar{\mu}} - \frac{i}{2}\sum\tau^{\bar{\lambda}}_{\nu}Q^{\alpha}_{\bar{\lambda}\bar{\mu}}\right\}\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee + \frac{1}{2}\sum\left\{({}^\sharp\nabla_{\xi_\mu}\tau)^{\alpha}_{\bar{\mu}} - \frac{i}{2}\sum\tau^{\bar{\lambda}}_{\mu}Q^{\alpha}_{\bar{\lambda}\bar{\mu}}\right\},$$
$$F({}^\sharp\nabla^{\mathcal{A}},\theta^\bullet)(\xi,\xi_\alpha) = \sum F({}^\sharp\nabla)^{\mu}_{\nu}(\xi,\xi_\alpha)\,\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee + F(\mathcal{A})(\xi,\xi_\alpha)$$
$$= -\sum\left\{({}^\sharp\nabla_{\xi_{\bar{\mu}}}\tau)^{\bar{\alpha}}_{\nu} + \frac{i}{2}\sum\tau^{\lambda}_{\bar{\mu}}Q^{\bar{\alpha}}_{\lambda\nu}\right\}\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee - \frac{1}{2}\sum\left\{({}^\sharp\nabla_{\xi_{\bar{\mu}}}\tau)^{\bar{\alpha}}_{\mu} + \frac{i}{2}\sum\tau^{\lambda}_{\bar{\mu}}Q^{\bar{\alpha}}_{\lambda\mu}\right\}.$$

Proof. We refer to the calculation of the curvature $F({}^\sharp\nabla)$ in the proof of [12, Proposition 1.2]. As for (2.12): By (2.10) and in a way similar to the proof of [9, (1.15)], we know

$$\Box^{\mathcal{A}}_H = -\sum\left({}^\sharp\nabla^{\mathcal{A}}_{\xi_\alpha}{}^\sharp\nabla^{\mathcal{A}}_{\xi_{\bar{\alpha}}} - {}^\sharp\nabla^{\mathcal{A}}_{{}^\sharp\nabla_{\xi_\alpha}\xi_{\bar{\alpha}}}\right) - i\,q\,{}^\sharp\nabla^{\mathcal{A}}_{\xi} - \sum\theta^{\bar{\alpha}}\wedge\theta^{\bar{\beta}}\vee F({}^\sharp\nabla^{\mathcal{A}},\theta^\bullet)(\xi_{\bar{\alpha}},\xi_\beta)$$
$$= -\frac{1}{2}\sum_{A\neq 0}\left({}^\sharp\nabla^{\mathcal{A}}_{\xi_A}{}^\sharp\nabla^{\mathcal{A}}_{\xi_{\bar{A}}} - {}^\sharp\nabla^{\mathcal{A}}_{{}^\sharp\nabla_{\xi_A}\xi_{\bar{A}}}\right) + \frac{i(n-2q)}{2}\,{}^\sharp\nabla^{\mathcal{A}}_{\xi}$$
$$+\frac{1}{2}\sum F({}^\sharp\nabla)^{\mu}_{\nu}(\xi_{\bar{\alpha}},\xi_\alpha)\cdot\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee - \sum F({}^\sharp\nabla)^{\mu}_{\nu}(\xi_{\bar{\alpha}},\xi_\beta)\cdot\theta^{\bar{\alpha}}\wedge\theta^{\bar{\beta}}\vee\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee$$
$$+\frac{1}{2}\sum F(\mathcal{A})(\xi_{\bar{\alpha}},\xi_\alpha) - \sum F(\mathcal{A})(\xi_{\bar{\alpha}},\xi_\beta)\cdot\theta^{\bar{\alpha}}\wedge\theta^{\bar{\beta}}\vee.$$

In addition, it is easy to show

$$\sum F({}^\sharp\nabla)^{\mu}_{\nu}(\xi_{\bar{\alpha}},\xi_\beta)\cdot\theta^{\bar{\alpha}}\wedge\theta^{\bar{\beta}}\vee\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee = \sum F({}^\sharp\nabla)^{\mu}_{\nu}(\xi_{\bar{\alpha}},\xi_\alpha)\cdot\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee$$
$$+\frac{1}{8}\sum Q^{\mu}_{\bar{\beta}\bar{\lambda}}Q^{\bar{\lambda}}_{\beta\nu}\cdot\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee + \frac{1}{8}\sum Q^{\alpha}_{\bar{\mu}\bar{\lambda}}Q^{\bar{\lambda}}_{\beta\nu}\cdot\theta^{\bar{\alpha}}\wedge\theta^{\bar{\beta}}\vee\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee,$$

$$\sum F({}^\sharp\nabla)^\mu_\nu(\xi_{\bar\alpha},\xi_\alpha)\cdot\theta^{\bar\mu}\wedge\theta^{\bar\nu}\vee$$
$$=\sum F({}^\sharp\nabla)^\alpha_\alpha(\xi_{\bar\mu},\xi_\nu)\cdot\theta^{\bar\mu}\wedge\theta^{\bar\nu}\vee+\frac{1}{4}\sum\Big\{\mathcal{Q}^\alpha_{\bar\lambda\bar\mu}\mathcal{Q}^{\bar\lambda}_{\alpha\nu}-\mathcal{Q}^\mu_{\bar\lambda\bar\alpha}\mathcal{Q}^{\bar\lambda}_{\nu\alpha}\Big\}\cdot\theta^{\bar\mu}\wedge\theta^{\bar\nu}\vee.$$

Hence, we have

$$\square^{\mathcal{A}}_H+\frac{1}{2}\sum_{A\neq 0}\Big({}^\sharp\nabla^{\mathcal{A}}_{\xi_A}{}^\sharp\nabla^{\mathcal{A}}_{\xi_{\bar A}}-{}^\sharp\nabla^{\mathcal{A}}_{{}^\sharp\nabla_{\xi_A}\xi_{\bar A}}\Big)-\frac{i(n-2q)}{2}\,{}^\sharp\nabla^{\mathcal{A}}_\xi$$
$$=-\frac{1}{2}\sum F({}^\sharp\nabla)^\alpha_\alpha(\xi_{\bar\mu},\xi_\nu)\cdot\theta^{\bar\mu}\wedge\theta^{\bar\nu}\vee-\frac{1}{8}\sum\Big\{\mathcal{Q}^\alpha_{\bar\lambda\bar\mu}\mathcal{Q}^{\bar\lambda}_{\alpha\nu}-\mathcal{Q}^\mu_{\bar\lambda\bar\alpha}\mathcal{Q}^{\bar\lambda}_{\nu\alpha}\Big\}\cdot\theta^{\bar\mu}\wedge\theta^{\bar\nu}\vee$$
$$-\frac{1}{8}\sum\mathcal{Q}^\mu_{\bar\beta\bar\lambda}\mathcal{Q}^{\bar\lambda}_{\beta\nu}\cdot\theta^{\bar\mu}\wedge\theta^{\bar\nu}\vee-\frac{1}{8}\sum\mathcal{Q}^\alpha_{\bar\mu\bar\lambda}\mathcal{Q}^{\bar\lambda}_{\beta\nu}\cdot\theta^{\bar\alpha}\wedge\theta^{\bar\beta}\vee\theta^{\bar\mu}\wedge\theta^{\bar\nu}\vee$$
$$+\frac{1}{2}\sum F(\mathcal{A})(\xi_{\bar\alpha},\xi_\alpha)-\sum F(\mathcal{A})(\xi_{\bar\alpha},\xi_\beta)\cdot\theta^{\bar\alpha}\wedge\theta^{\bar\beta}\vee$$
$$=\frac{1}{2}\sum F(\mathcal{A})(\xi_{\bar\alpha},\xi_\alpha)-\frac{1}{2}\sum\Big\{\mathrm{Ric}^{{}^\sharp\nabla}(\xi_{\bar\mu},\xi_\nu)+2F(\mathcal{A})(\xi_{\bar\mu},\xi_\nu)\Big\}\cdot\theta^{\bar\mu}\wedge\theta^{\bar\nu}\vee$$
$$+\frac{1}{8}\sum\mathcal{Q}^\lambda_{\bar\mu\bar\beta}\mathcal{Q}^{\bar\nu}_{\beta\lambda}\cdot\theta^{\bar\mu}\wedge\theta^{\bar\nu}\vee-\frac{1}{8}\sum\mathcal{Q}^\alpha_{\bar\mu\bar\lambda}\mathcal{Q}^{\bar\lambda}_{\beta\nu}\cdot\theta^{\bar\alpha}\wedge\theta^{\bar\beta}\vee\theta^{\bar\mu}\wedge\theta^{\bar\nu}\vee$$
$$=\frac{1}{4}s^{{}^\sharp\nabla}+\frac{1}{8}\sum\mathcal{Q}^\lambda_{\bar\mu\bar\beta}\mathcal{Q}^{\bar\nu}_{\beta\lambda}\cdot\theta^{\bar\mu}\wedge\theta^{\bar\nu}\vee-\frac{1}{8}\sum\mathcal{Q}^\alpha_{\bar\mu\bar\lambda}\mathcal{Q}^{\bar\lambda}_{\beta\nu}\cdot\theta^{\bar\alpha}\wedge\theta^{\bar\beta}\vee\theta^{\bar\mu}\wedge\theta^{\bar\nu}\vee.$$

The last line follows from $2F(\mathcal{A})=2\,d\mathcal{A}=-\mathrm{Ric}^{{}^\sharp\nabla}$. Next let us show (2.13). (2.11) implies

$$(D^{\mathcal{g}_H})^2=2\square^{\mathcal{A}}_H-{}^\sharp\nabla^{\mathcal{A}}_\xi{}^\sharp\nabla^{\mathcal{A}}_\xi-\frac{i(n-2q)}{2}\,{}^\sharp\nabla^{\mathcal{A}}_\xi+\frac{(n-2q)^2}{16}\quad(\Omega^{0,q}_H\to\Omega^{0,q}_H)$$
$$+(-1)^q\sqrt{2}\,i\Big({}^\sharp\nabla^{\mathcal{A}}_\xi\,\bar\partial^{\mathcal{A}}_H-\bar\partial^{\mathcal{A}}_H\,{}^\sharp\nabla^{\mathcal{A}}_\xi\Big)+(-1)^q\frac{\sqrt{2}}{2}\,\bar\partial^{\mathcal{A}}_H\quad(\Omega^{0,q}_H\to\Omega^{0,q+1}_H)$$
$$+(-1)^q\sqrt{2}\,i\Big({}^\sharp\nabla^{\mathcal{A}}_\xi\,\bar\partial^{\mathcal{A}*}_H-\bar\partial^{\mathcal{A}*}_H{}^\sharp\nabla^{\mathcal{A}}_\xi\Big)-(-1)^q\frac{\sqrt{2}}{2}\,\bar\partial^{\mathcal{A}*}_H\quad(\Omega^{0,q}_H\to\Omega^{0,q-1}_H)$$
$$+2(\bar\partial^{\mathcal{A}}_H)^2\quad(\Omega^{0,q}_H\to\Omega^{0,q+2}_H)$$
$$+2(\bar\partial^{\mathcal{A}*}_H)^2\quad(\Omega^{0,q}_H\to\Omega^{0,q-2}_H).$$

As for the line $(\Omega^{0,q}_H\to\Omega^{0,q}_H)$: By (2.12), it is equal to

$$-\sum\Big({}^\sharp\nabla^{\mathcal{A}}_{\xi_A}{}^\sharp\nabla^{\mathcal{A}}_{\xi_{\bar A}}-{}^\sharp\nabla^{\mathcal{A}}_{{}^\sharp\nabla_{\xi_A}\xi_{\bar A}}\Big)+\frac{i(n-2q)}{2}\,{}^\sharp\nabla^{\mathcal{A}}_\xi+\frac{(n-2q)^2}{16}+\frac{1}{2}s^{{}^\sharp\nabla}$$
$$+\frac{1}{4}\sum\mathcal{Q}^\lambda_{\bar\mu\bar\beta}\mathcal{Q}^{\bar\nu}_{\beta\lambda}\cdot\theta^{\bar\mu}\wedge\theta^{\bar\nu}\vee-\frac{1}{4}\sum\mathcal{Q}^\alpha_{\bar\mu\bar\lambda}\mathcal{Q}^{\bar\lambda}_{\beta\nu}\cdot\theta^{\bar\alpha}\wedge\theta^{\bar\beta}\vee\theta^{\bar\mu}\wedge\theta^{\bar\nu}\vee.$$

As for the line $(\Omega^{0,q}_H\to\Omega^{0,q+1}_H)$: Since

$$\bar\partial^{\mathcal{A}}_H\,{}^\sharp\nabla^{\mathcal{A}}_\xi=\sum\theta^{\bar\alpha}\wedge\cdot{}^\sharp\nabla^{\mathcal{A}}_{\xi_{\bar\alpha}}{}^\sharp\nabla^{\mathcal{A}}_\xi=\sum\theta^{\bar\alpha}\wedge\cdot\Big\{{}^\sharp\nabla^{\mathcal{A}}_\xi{}^\sharp\nabla^{\mathcal{A}}_{\xi_{\bar\alpha}}+F({}^\sharp\nabla^{\mathcal{A}},\theta^\bullet)(\xi_{\bar\alpha},\xi)+{}^\sharp\nabla^{\mathcal{A}}_{[\xi_{\bar\alpha},\xi]}\Big\}$$
$$={}^\sharp\nabla^{\mathcal{A}}_\xi\,\bar\partial^{\mathcal{A}}_H-\sum\theta^{\bar\alpha}\wedge\cdot F({}^\sharp\nabla^{\mathcal{A}},\theta^\bullet)(\xi,\xi_{\bar\alpha})+\sum\theta^{\bar\alpha}\wedge\cdot{}^\sharp\nabla^{\mathcal{A}}_{\tau\xi_{\bar\alpha}},$$

it is equal to

$$(-1)^q\sqrt{2}\Big(\frac{1}{2}\,\bar\partial^{\mathcal{A}}_H-\sum\theta^{\bar\alpha}\wedge\cdot i\,{}^\sharp\nabla^{\mathcal{A}}_{\tau\xi_{\bar\alpha}}+\sum\theta^{\bar\alpha}\wedge\cdot i\,F({}^\sharp\nabla^{\mathcal{A}},\theta^\bullet)(\xi,\xi_{\bar\alpha})\Big)$$
$$=\sum\theta^{\bar\alpha}\wedge\cdot(-1)^q\sqrt{2}\Big\{\frac{1}{2}{}^\sharp\nabla^{\mathcal{A}}_{\xi_{\bar\alpha}}-i\,{}^\sharp\nabla^{\mathcal{A}}_{\tau\xi_{\bar\alpha}}+i\,F({}^\sharp\nabla^{\mathcal{A}},\theta^\bullet)(\xi,\xi_{\bar\alpha})\Big\}.$$

As for the line $(\Omega_H^{0,q} \to \Omega_H^{0,q-1})$: Since

$$\begin{aligned}
\bar\partial_H^{\mathcal{A}*}\, {}^\sharp\nabla^{\mathcal{A}}_{\xi} &= -\sum \theta^{\bar\alpha} \vee \cdot {}^\sharp\nabla^{\mathcal{A}}_{\xi_\alpha} {}^\sharp\nabla^{\mathcal{A}}_{\xi} \\
&= -\sum \theta^{\bar\alpha} \vee \cdot \big\{ {}^\sharp\nabla^{\mathcal{A}}_{\xi} {}^\sharp\nabla^{\mathcal{A}}_{\xi_\alpha} + F({}^\sharp\nabla^{\mathcal{A}}, \theta^\bullet)(\xi_\alpha, \xi) + {}^\sharp\nabla^{\mathcal{A}}_{[\xi_\alpha, \xi]} \big\} \\
&= {}^\sharp\nabla^{\mathcal{A}}_{\xi}\, \bar\partial_H^{\mathcal{A}*} + \sum \theta^{\bar\alpha} \vee \cdot F({}^\sharp\nabla^{\mathcal{A}}, \theta^\bullet)(\xi, \xi_\alpha) - \sum \theta^{\bar\alpha} \vee \cdot {}^\sharp\nabla^{\mathcal{A}}_{\tau\xi_\alpha},
\end{aligned}$$

it is equal to

$$\begin{aligned}
&(-1)^q\sqrt{2}\Big(-\frac{1}{2}\,\bar\partial_H^{\mathcal{A}*} + \sum \theta^{\bar\alpha} \vee \cdot i\, {}^\sharp\nabla^{\mathcal{A}}_{\tau\xi_\alpha} - \sum \theta^{\bar\alpha} \vee \cdot i\, F({}^\sharp\nabla^{\mathcal{A}}, \theta^\bullet)(\xi, \xi_\alpha) \Big) \\
&\qquad = \sum \theta^{\bar\alpha} \vee\ \cdot (-1)^q \sqrt{2} \Big\{ \frac{1}{2} {}^\sharp\nabla^{\mathcal{A}}_{\xi_\alpha} + i\, {}^\sharp\nabla^{\mathcal{A}}_{\tau\xi_\alpha} - i\, F({}^\sharp\nabla^{\mathcal{A}}, \theta^\bullet)(\xi, \xi_\alpha) \Big\}.
\end{aligned}$$

As for the line $(\Omega_H^{0,q} \to \Omega_H^{0,q+2})$: We have calculated $(\bar\partial_H)^2$ in [9, Corollary 1.4], which implies that it is equal to

$$\begin{aligned}
&-\frac{1}{4} \sum \theta^{\bar\alpha} \wedge \theta^{\bar\beta} \wedge \cdot {}^\sharp\nabla^{\mathcal{A}}_{[J,J](\xi_{\bar\alpha}, \xi_{\bar\beta})} - i \sum \frac{({}^\sharp\nabla_{\xi_\nu} Q)^{\alpha}_{\bar\beta\bar\mu}}{2} \cdot \theta^{\bar\alpha} \wedge \theta^{\bar\beta} \wedge \theta^{\bar\mu} \wedge \theta^{\bar\nu} \vee \\
&\qquad = \frac{i}{2} \Big\{ \sum Q^{\alpha}_{\bar\beta\bar\gamma}\, \theta^{\bar\alpha} \wedge \theta^{\bar\beta} \wedge \cdot {}^\sharp\nabla^{\mathcal{A}}_{\xi_\gamma} - \sum ({}^\sharp\nabla_{\xi_\nu} Q)^{\alpha}_{\bar\beta\bar\mu} \cdot \theta^{\bar\alpha} \wedge \theta^{\bar\beta} \wedge \theta^{\bar\mu} \wedge \theta^{\bar\nu} \vee \Big\}.
\end{aligned}$$

As for the line $(\Omega_H^{0,q} \to \Omega_H^{0,q-2})$: Similarly it is equal to

$$\begin{aligned}
&-\frac{1}{4} \sum \theta^{\bar\alpha} \vee \theta^{\bar\beta} \vee \cdot {}^\sharp\nabla^{\mathcal{A}}_{[J,J](\xi_\alpha, \xi_\beta)} - i \sum \frac{({}^\sharp\nabla_{\xi_{\bar\mu}} Q)^{\bar\alpha}_{\beta\nu}}{2} \cdot \theta^{\bar\alpha} \vee \theta^{\bar\beta} \vee \theta^{\bar\mu} \wedge \theta^{\bar\nu} \vee \\
&\qquad = \frac{i}{2} \Big\{ -\sum Q^{\bar\alpha}_{\beta\gamma}\, \theta^{\bar\alpha} \vee \theta^{\bar\beta} \vee\ \cdot {}^\sharp\nabla^{\mathcal{A}}_{\xi_{\bar\gamma}} - \sum ({}^\sharp\nabla_{\xi_{\bar\mu}} Q)^{\bar\alpha}_{\beta\nu} \cdot \theta^{\bar\alpha} \vee \theta^{\bar\beta} \vee \theta^{\bar\mu} \wedge \theta^{\bar\nu} \vee \Big\}.
\end{aligned}$$

Thus we get the formula (2.13). □

2.1. Dirac operator for general Spin^c structure. In this subsection, we suppose that M has a Spin^c structure with a determinant line bundle L

$$\rho^c_{(L)} : \mathrm{Spin}(TM) \times_{\mathbb{Z}_2} U(L^{1/2}) \to SO(TM) \times U(L). \tag{2.14}$$

We want to show that the associated Dirac operator $D^{\$^c_{(L)}}$ has similar expressions. Compare the formulas (2.16), (2.18), (2.19), (2.20) for $D^{\$^c_{(L)}}$ in the following, with (2.7), (2.11), (2.12), (2.13) for $D^{\$_H}$.

The structure (2.14) reduces to the Spin^c structure of H

$$\rho^c_{(L)H} : \mathrm{Spin}(H) \times_{\mathbb{Z}_2} U(L^{1/2}) \to SO(H) \times U(L). \tag{2.15}$$

Let us employ the (locally defined) bundle Spin(H) to describe the canonical Spin^c structure of the almost hermitian vector bundle $(H, g|_H, J|_H)$ as

$$\rho^c_H : \mathrm{Spin}(H) \times_{\mathbb{Z}_2} U(K^{-1/2}) \to SO(H) \times U(K^{-1}).$$

Then, $\mathcal{L} := K^{1/2} \otimes L^{1/2}$ is a globally defined square root of $K \otimes L$ and the spinor bundle $\$^c_{(L)}$ associated with (2.15) is expressed as

$$\$^c_{(L)} = \$_H \otimes L^{1/2} = \$^c_H \otimes \mathcal{L} = \wedge^{0,*}_H(M) \otimes \mathcal{L}$$

where $\mathcal{S}_H = \mathrm{Spin}(H) \times_{\Delta_{2n}} \wedge^* \mathbb{C}^n$ may be locally defined. As before, let us regard it as the one associated with (2.14) naturally, and take a unitary connection $\mathbb{A}^L$ of L. We attach the connection $\mathcal{A}(L^{1/2}) = \frac{1}{2}\mathbb{A}^L$ to $L^{1/2}$ and consider the connections $\nabla^{\mathcal{S}^c_{(L)}} = \nabla^{(\mathcal{S}^c_{(L)}; \nabla^g, \mathcal{A}(L^{1/2}))}$, etc., on $\mathcal{S}^c_{(L)}$ defined by

$$\omega(\nabla^{(\mathcal{S}^c_{(L)}:\nabla^g, \mathcal{A}(L^{1/2}))}) = \omega(\nabla^{(\mathcal{S}_{(L)}:\nabla^g)}) + \mathcal{A}(L^{1/2}),$$

etc. Then, by (2.7), the Dirac operator

$$D^{\mathcal{S}^c_{(L)}} = D^{(\mathcal{S}^c_{(L)}; \nabla^g, \mathcal{A}(L^{1/2}))} = \sum \xi_C \circ \nabla^{(\mathcal{S}^c_{(L)}:\nabla^g, \mathcal{A}(L^{1/2}))}_{\xi_{\bar{C}}}$$

is expressed as

$$D^{\mathcal{S}^c_{(L)}} = \sum \xi_C \circ \nabla^{(\mathcal{S}^c_{(L)}:{}^\sharp\nabla, \mathcal{A}(L^{1/2}))}_{\xi_{\bar{C}}} - \frac{1}{4}\xi \circ d\theta \circ . \tag{2.16}$$

Next, let us consider the unitary connection $\mathcal{A}(\mathcal{L}) = \mathcal{A}(K^{1/2}) + \mathcal{A}(L^{1/2})$ of $\mathcal{L}$, which provides the bundle $\mathcal{S}^c_{(L)} = \wedge^{0,*}_H(M) \otimes \mathcal{L}$ with the twisted hermitian Tanno connection

$${}^\sharp\nabla^{\mathcal{A}(\mathcal{L})} = {}^\sharp\nabla + \mathcal{A}(\mathcal{L}).$$

Then, we have

$$\nabla^{(\mathcal{S}^c_{(L)}:{}^\sharp\nabla, \mathcal{A}(L^{1/2}))} = {}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}.$$

Further, if we set

$$\bar{\partial}^{\mathcal{A}(\mathcal{L})}_H = \sum \theta^{\bar{\gamma}} \wedge \cdot {}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}_{\xi_{\bar{\gamma}}}, \qquad \bar{\partial}^{\mathcal{A}(\mathcal{L})*}_H = -\sum \theta^{\bar{\gamma}} \vee \cdot {}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}_{\xi_\gamma}, \tag{2.17}$$

then we have another formula

$$D^{\mathcal{S}^c_{(L)}} = \sqrt{2}\,(\bar{\partial}^{\mathcal{A}(\mathcal{L})}_H + \bar{\partial}^{\mathcal{A}(\mathcal{L})*}_H) + \xi \circ {}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}_\xi - \frac{1}{4}\xi \circ d\theta \circ . \tag{2.18}$$

Note that, as before, the operator $\bar{\partial}^{\mathcal{A}(\mathcal{L})}_H : \Omega^{0,*}_H(M;\mathcal{L}) \to \Omega^{0,*+1}_H(M;\mathcal{L})$ given at (2.17) can be interpreted also as the exterior covariant differentiation $d^{\mathcal{A}(\mathcal{L})} : \Omega^{0,*}_H(M;\mathcal{L}) \to \Gamma(\wedge^{*+1}T^*M \otimes \mathcal{L})$ followed by the projection to $\Omega^{0,*+1}_H(M;\mathcal{L})$ and $\bar{\partial}^{\mathcal{A}(\mathcal{L})*}_H$ coincides with the formal adjoint.

Further, we have also Lichnerowicz-type formulas: On $\Omega^{0,q}_H(M;\mathcal{L})$, we have

$$\begin{aligned}
\square^{\mathcal{A}(\mathcal{L})}_H &= -\sum \Big({}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}_{\xi_\alpha}{}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}_{\xi_{\bar{\alpha}}} - {}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}_{{}^\sharp\nabla_{\xi_\alpha}\xi_{\bar{\alpha}}}\Big) - i\,q\,{}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}_\xi \\
&\quad - \sum \theta^{\bar{\alpha}} \wedge \theta^{\bar{\beta}} \vee \Big\{F({}^\sharp\nabla)^\mu_\nu(\xi_{\bar{\alpha}}, \xi_\beta)\,\theta^{\bar{\mu}} \wedge \theta^{\bar{\nu}} \vee + F(\mathcal{A}(\mathcal{L}))(\xi_{\bar{\alpha}}, \xi_\beta)\Big\} \\
&= -\frac{1}{2}\sum_{A\neq 0} \Big({}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}_{\xi_A}{}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}_{\xi_{\bar{A}}} - {}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}_{{}^\sharp\nabla_{\xi_A}\xi_{\bar{A}}}\Big) + \frac{i(n-2q)}{2}\,{}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}_\xi \\
&\quad + \frac{1}{4}s^{{}^\sharp\nabla} - \frac{1}{4}s^{\mathbb{A}^L} - \frac{1}{2}\sum F(\mathbb{A}^L)(\xi_{\bar{\mu}}, \xi_\nu)\,\theta^{\bar{\mu}} \wedge \theta^{\bar{\nu}} \vee \\
&\quad + \frac{1}{8}\sum Q^\lambda_{\bar{\mu}\bar{\beta}} Q^{\bar{\nu}}_{\beta\lambda}\,\theta^{\bar{\mu}} \wedge \theta^{\bar{\nu}} \vee - \frac{1}{8}\sum Q^\alpha_{\bar{\mu}\bar{\lambda}} Q^{\bar{\lambda}}_{\beta\nu}\,\theta^{\bar{\alpha}} \wedge \theta^{\bar{\beta}} \vee \theta^{\bar{\mu}} \wedge \theta^{\bar{\nu}} \vee
\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}
&(D^{\mathcal{S}^c_{(L)}})^2 \\
&= -\sum \Big(\nabla^{\mathcal{S}^c_{(L)}}_{\xi_A}\nabla^{\mathcal{S}^c_{(L)}}_{\xi_{\bar{A}}} - \nabla^{\mathcal{S}^c_{(L)}}_{\nabla^g_{\xi_A}\xi_{\bar{A}}}\Big) + \frac{1}{4}s(\nabla^g) + \frac{1}{4}\sum F(\mathbb{A}^L)(\xi_A, \xi_B)\,\xi_{\bar{A}} \circ \xi_{\bar{B}} \circ
\end{aligned} \tag{2.20}$$

$$
\begin{aligned}
&= -\sum\left({}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}_{\xi_A}{}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}_{\xi_{\bar{A}}} - {}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}_{{}^\sharp\nabla_{\xi_A}\xi_{\bar{A}}}\right) + \frac{i(n-2q)}{2}\,{}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}_{\xi} + \frac{(n-2q)^2}{16}\\
&\quad + \frac{1}{2}s^{\sharp\nabla} - \frac{1}{2}s^{\mathbb{A}^L} - \sum F(\mathbb{A}^L)(\xi_{\bar{\mu}},\xi_\nu)\,\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee\\
&\quad + \frac{1}{4}\sum Q^{\lambda}_{\bar{\mu}\bar{\beta}}Q^{\bar{\nu}}_{\beta\lambda}\,\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee - \frac{1}{4}\sum Q^{\alpha}_{\bar{\mu}\bar{\lambda}}Q^{\bar{\lambda}}_{\beta\nu}\,\theta^{\bar{\alpha}}\wedge\theta^{\bar{\beta}}\vee\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee\\
&\quad + \sum\theta^{\bar{\alpha}}\wedge(-1)^q\sqrt{2}\left\{\frac{1}{2}\,{}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}_{\xi_{\bar{\alpha}}} - i\,{}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}_{\tau\xi_{\bar{\alpha}}} + i\,F({}^\sharp\nabla^{\mathcal{A}(\mathcal{L})},\theta^\bullet)(\xi,\xi_{\bar{\alpha}})\right\}\\
&\quad + \sum\theta^{\bar{\alpha}}\vee(-1)^q\sqrt{2}\left\{\frac{1}{2}\,{}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}_{\xi_{\alpha}} + i\,{}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}_{\tau\xi_{\alpha}} - i\,F({}^\sharp\nabla^{\mathcal{A}(\mathcal{L})},\theta^\bullet)(\xi,\xi_{\alpha})\right\}\\
&\quad + \frac{i}{2}\sum\theta^{\bar{\alpha}}\wedge\theta^{\bar{\beta}}\wedge\left\{\sum Q^{\alpha}_{\bar{\beta}\bar{\gamma}}\,{}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}_{\xi_\gamma} - \sum({}^\sharp\nabla_{\xi_\nu}Q)^{\alpha}_{\bar{\beta}\bar{\mu}}\,\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee\right\}\\
&\quad - \frac{i}{2}\sum\theta^{\bar{\alpha}}\vee\theta^{\bar{\beta}}\vee\left\{\sum Q^{\bar{\alpha}}_{\beta\gamma}\,{}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}_{\xi_{\bar{\gamma}}} + \sum({}^\sharp\nabla_{\xi_{\bar{\mu}}}Q)^{\bar{\alpha}}_{\beta\nu}\,\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee\right\},
\end{aligned}
$$

where $F({}^\sharp\nabla^{\mathcal{A}(\mathcal{L})},\theta^\bullet)(\xi_A,\xi_B)$ is the curvature of ${}^\sharp\nabla^{\mathcal{A}(\mathcal{L})}$ acting on $\Omega^{0,*}_H(M;\mathcal{L})$, i.e.,

$$
\begin{aligned}
F({}^\sharp\nabla^{\mathcal{A}(\mathcal{L})},\theta^\bullet)(\xi,\xi_{\bar{\alpha}}) &= \sum F({}^\sharp\nabla)^{\mu}_{\nu}(\xi,\xi_{\bar{\alpha}})\,\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee + F(\mathcal{A}(\mathcal{L}))(\xi,\xi_{\bar{\alpha}})\\
&= \sum\left\{(\nabla_{\xi_\nu}\tau)^{\alpha}_{\bar{\mu}} - \frac{i}{2}\sum\tau^{\bar{\lambda}}_{\nu}\,Q^{\alpha}_{\bar{\lambda}\bar{\mu}}\right\}\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee\\
&\qquad + \frac{1}{2}\sum\left\{({}^\sharp\nabla_{\xi_\mu}\tau)^{\alpha}_{\bar{\mu}} - \frac{i}{2}\sum\tau^{\bar{\lambda}}_{\mu}\,Q^{\alpha}_{\bar{\lambda}\bar{\mu}}\right\} + \frac{1}{2}F(\mathbb{A}^L)(\xi,\xi_{\bar{\alpha}}),\\
F({}^\sharp\nabla^{\mathcal{A}(\mathcal{L})},\theta^\bullet)(\xi,\xi_{\alpha}) &= \sum F({}^\sharp\nabla)^{\mu}_{\nu}(\xi,\xi_{\alpha})\,\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee + F(\mathcal{A}(\mathcal{L}))(\xi,\xi_{\alpha})\\
&= -\sum\left\{(\nabla_{\xi_{\bar{\mu}}}\tau)^{\bar{\alpha}}_{\nu} + \frac{i}{2}\sum\tau^{\lambda}_{\bar{\mu}}\,Q^{\bar{\alpha}}_{\lambda\nu}\right\}\theta^{\bar{\mu}}\wedge\theta^{\bar{\nu}}\vee\\
&\qquad - \frac{1}{2}\sum\left\{({}^\sharp\nabla_{\xi_{\bar{\mu}}}\tau)^{\bar{\alpha}}_{\mu} + \frac{i}{2}\sum\tau^{\lambda}_{\bar{\mu}}\,Q^{\bar{\alpha}}_{\lambda\mu}\right\} + \frac{1}{2}F(\mathbb{A}^L)(\xi,\xi_{\alpha}).
\end{aligned}
$$

3. Spinor bundle on the Fefferman spin space

As in §2, we suppose M has a spin structure and consider the associated globally defined square root $K^{1/2} = \sqrt{K(M)}$ of the canonical line bundle $K = K(M) = \wedge^{n+1,0}(M)$. In a way similar to the construction of the ordinary Fefferman space for $K(M)$ (cf. [5], [8]), Baum ([3, §5]) constructed the Fefferman (spin) space for $\sqrt{K(M)}$ in the case J is integrable. It is easy to generalize it to the case of general J (cf. [11, §2], [1, §6]), just by changing the Tanaka-Webster connection to the hermitian Tanno connection, as follows: We set $\sqrt{K(M)}^0 = \{\omega^{\surd} \in \sqrt{K(M)} \mid \omega^{\surd} \neq 0\}$ and consider the canonical $U(1)$-bundle

$$\sqrt{\pi} : \sqrt{F(M)} := \sqrt{K(M)}^0/\mathbb{R}_+ \to M.$$

Given unitary frames $\theta^\bullet$, there are the trivializations

$$\sqrt{\pi}^{-1}(U) \;\cong\; U\times[0,2\pi),\quad \sqrt{[\theta\wedge\theta^1\wedge\cdots\wedge\theta^n]}(p)\cdot e^{i\Theta^{\surd}} \leftrightarrow (p,\Theta^{\surd}) \tag{3.1}$$

and the bundle has an Ehresmann-type connection $i(n+2)\sigma^{\surd} \in \Gamma(\mathfrak{u}(1)\otimes T^*\sqrt{F(M)})$, called the Fefferman connection, given by

$$\sigma^{\surd} = \frac{1}{n+2}\Big\{d\Theta^{\surd} + \sqrt{\pi}^{*}\frac{1}{2}\Big(i\sum\omega({}^{\sharp}\nabla)^{\alpha}_{\alpha} - \frac{s^{\sharp\nabla}}{2(n+1)}\theta\Big)\Big\} \quad (\text{on } \sqrt{\pi}^{-1}(U)),$$

whose curvature $F(i(n+2)\sigma^{\surd})$ is described as

$$\begin{aligned} &F(i(n+2)\sigma^{\surd}) = d(i(n+2)\sigma^{\surd}) = i(n+2)\sqrt{\pi}^{*}\mathcal{F}(\sigma^{\surd}), \\ &\mathcal{F}(\sigma^{\surd}) := \frac{i}{2(n+2)}\Big(\mathrm{Ric}^{\sharp\nabla} + \frac{i\,d(s^{\sharp\nabla}\theta)}{2(n+1)}\Big) = \overline{\mathcal{F}(\sigma^{\surd})} \in \Gamma(\wedge^2 T^*M), \end{aligned} \tag{3.2}$$

where $\mathrm{Ric}^{\sharp\nabla}$ is the pseudohermitian Ricci curvature (cf. Theorem 2.7). Via the trivialization (3.1), the horizontal lift of $X \in TM$ is written as

$$\sqrt{\pi}^{*}_{\mathcal{H}}X := X - \frac{i}{2}\Big\{\sum\omega({}^{\sharp}\nabla)^{\alpha}_{\alpha}(X) + \frac{i\,s^{\sharp\nabla}\,\theta(X)}{2(n+1)}\Big\}\partial/\partial\Theta^{\surd}$$

and the dual frame of the local frame $(\sqrt{\pi}^{*}\theta^{1}, \dots, \sqrt{\pi}^{*}\theta^{n}, \sqrt{\pi}^{*}\theta^{\bar 1}, \dots, \sqrt{\pi}^{*}\theta^{\bar n}, \sqrt{\pi}^{*}\theta, \sigma^{\surd})$ is

$$(\sqrt{\pi}^{*}_{\mathcal{H}}\xi_1, \dots, \sqrt{\pi}^{*}_{\mathcal{H}}\xi_n, \sqrt{\pi}^{*}_{\mathcal{H}}\xi_{\bar 1}, \dots, \sqrt{\pi}^{*}_{\mathcal{H}}\xi_{\bar n}, N := \sqrt{\pi}^{*}_{\mathcal{H}}\xi, \Sigma^{\surd} := (n+2)\partial/\partial\Theta^{\surd}). \tag{3.3}$$

Note that the horizontal and vertical components of the bracket $[\sqrt{\pi}^{*}_{\mathcal{H}}X, \sqrt{\pi}^{*}_{\mathcal{H}}Y]$ are expressed as

$$[\sqrt{\pi}^{*}_{\mathcal{H}}X, \sqrt{\pi}^{*}_{\mathcal{H}}Y]_{\mathcal{H}} = \sqrt{\pi}^{*}_{\mathcal{H}}[X, Y], \quad [\sqrt{\pi}^{*}_{\mathcal{H}}X, \sqrt{\pi}^{*}_{\mathcal{H}}Y]_{\mathcal{V}} = -\mathcal{F}(\sigma^{\surd})(X, Y)\,\Sigma^{\surd}.$$

Now, let us equip the Fefferman space $\sqrt{F(M)}$ with the Fefferman metric

$$h^{\surd} = \sum(\sqrt{\pi}^{*}\theta^{\alpha}\otimes\sqrt{\pi}^{*}\theta^{\bar\alpha} + \sqrt{\pi}^{*}\theta^{\bar\alpha}\otimes\sqrt{\pi}^{*}\theta^{\alpha}) + 4(\sqrt{\pi}^{*}\theta\otimes\sigma^{\surd} + \sigma^{\surd}\otimes\sqrt{\pi}^{*}\theta),$$

i.e., a Lorentzian metric of type $(2n+1, 1)$. This carries canonically a spin structure

$$\rho_{T\sqrt{F(M)}} : \mathrm{Spin}(T\sqrt{F(M)}) \to SO(T\sqrt{F(M)})$$

as follows: Here $SO(T\sqrt{F(M)})$ $(= SO_0(T\sqrt{F(M)}))$ is the connected component of the set of $SO(2n+1, 1)$-frames to which the frame

$$(s_1, \dots, s_{2n+1}; s_{2n+2}) = \Big(\sqrt{\pi}^{*}_{\mathcal{H}}e_1, \dots, \sqrt{\pi}^{*}_{\mathcal{H}}e_{2n}, \frac{N+\Sigma^{\surd}}{2\sqrt{2}}; \frac{N-\Sigma^{\surd}}{2\sqrt{2}}\Big) \tag{3.4}$$

belongs. We embed $\sqrt{\pi}^{*}SO(H)$ into $SO(T\sqrt{F(M)})$ by the map which sends $(P, (e_1, \dots, e_{2n})_{\sqrt{\pi}(P)})$ to (3.4) at P, via which $SO(T\sqrt{F(M)}) = \sqrt{\pi}^{*}SO(H)\times_{inc.}SO_0(2n+1, 1)$. We define now $\mathrm{Spin}(T\sqrt{F(M)})$ to be $\sqrt{\pi}^{*}\mathrm{Spin}(H)\times_{inc.}\mathrm{Spin}_0(2n+1, 1)$.

For concrete calculation Baum ([3, §2], [2]) introduced a realization of the spinor representation $\Delta_{2n+1,1}$ of $\mathrm{Spin}_0(2n+1, 1)$. Here we want to adopt another one, which will be more suitable to relate the spin structure of $\sqrt{F(M)}$ to that of M.

On an irreducible representation of the Clifford algebra $\mathbb{C}l(2n+1, 1) = Cl(\mathbb{R}^{2n+1,1}, (e_1, \dots, e_{2n}, e_0; \varepsilon) : e_j \circ e_j = -1,\ \varepsilon\circ\varepsilon = 1)\otimes\mathbb{C}$: We recall the standard representations (e.g. [7, (5.27) and (4.0)])

$$\begin{aligned} &r_{2n} : Cl(2n) = Cl(\mathbb{R}^{2n}, (e'_1, \dots, e'_{2n}) : e'_j\circ e'_j = -1) \to \mathrm{End}(\wedge^*\mathbb{C}^n), \\ &r_{1,1} : Cl(1, 1) = Cl(\mathbb{R}^{1,1}, (e''_1; \varepsilon''_1) : e''_1\circ e''_1 = -1,\ \varepsilon''_1\circ\varepsilon''_1 = 1) \cong \mathbb{R}(2) \end{aligned}$$

$$e_1'' \mapsto \sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \varepsilon_1'' \mapsto \sigma_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ \varepsilon_1'' \circ e_1'' \mapsto \sigma_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

to get the irreducible one

$$r_{2n+1,1} : \mathbb{C}l(2n+1,1) \cong \mathbb{C}l(2n) \otimes \mathbb{C}l(1,1) \overset{r_{2n}\otimes r_{1,1}}{\to} \mathrm{End}(\wedge^*\mathbb{C}^n \otimes \mathbb{C}^2)$$
$$\left(e_j \leftrightarrow e_j' \otimes \varepsilon_1'' \circ e_1''\ (j \geq 1),\ e_0 \leftrightarrow 1 \otimes e_1'',\ \varepsilon \leftrightarrow 1 \otimes \varepsilon_1''\right).$$

Let us decompose the representation space $S_{2n+1,1} := \wedge^*\mathbb{C}^n \otimes \mathbb{C}^2$ into

$$
\begin{aligned}
&S_{2n+1,1} = S^+_{2n+1,1} \oplus S^-_{2n+1,1}, \\
&S^{\pm}_{2n+1,1} := \left\{\phi \in S_{2n+1,1} \mid r_{2n+1,1}(i^n e_1 \circ \cdots \circ e_{2n} \circ e_0 \circ \varepsilon)\phi = \pm\phi\right\} \\
&\quad = S^{\pm}_{2n} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus S^{\mp}_{2n} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \left((-\sigma_3)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = \pm\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}\right) \\
&\quad = S^{\pm}_{2n} \oplus S^{\mp}_{2n} = S_{2n}, \\
&S^{\pm}_{2n} := \left\{\psi \in \wedge^*\mathbb{C}^n \mid r_{2n}(i^n e_1 \circ \cdots \circ e_{2n})\psi = \pm\psi\right\} = \wedge^{0,\,\mathrm{even/odd}}\mathbb{C}^n.
\end{aligned}
\tag{3.5}
$$

Then, since $r_{2n+1,1}(e_j),\ r_{2n+1,1}(\varepsilon) : S^{\pm}_{2n+1,1} \to S^{\mp}_{2n+1,1}$, the spinor representation $\Delta_{2n+1,1} = r_{2n+1,1}|\mathrm{Spin}_0(2n+1,1)$ splits into the sum $\Delta_{2n+1,1} = \Delta^+_{2n+1,1} \oplus \Delta^-_{2n+1,1}$ via the splitting (3.5).

On the spinor bundle $\not{S} = \not{S}(T\sqrt{F(M)})$ with splitting (cf. [3, Proposition 18]) : We have

$$\not{S}^{\pm} := \mathrm{Spin}(T\sqrt{F(M)}) \times_{\Delta^{\pm}_{2n+1,1}} S^{\pm}_{2n+1,1} \cong \sqrt{\pi}^*\not{S}^{\pm}_H \oplus \sqrt{\pi}^*\not{S}^{\mp}_H = \sqrt{\pi}^*\not{S}_H,$$

which, we want to emphasize, are, thus, identified with the pull-backs of the spinor bundle $\not{S}_H$ (cf. (2.6)) of (2.1). Consequently the spinor bundle $\not{S}$ is identified with the sum

$$\not{S} = \sqrt{\pi}^*\not{S}_H \oplus \sqrt{\pi}^*\not{S}_H \tag{3.6}$$

and it is obvious that the Clifford action of $T\sqrt{F(M)}$ on the left hand side can be read on the right hand side as

$$
\begin{aligned}
&\sqrt{\pi}^*_{\mathcal{H}} e_j \circ (\varphi, \psi) = (-\sqrt{\pi}^*_{\mathcal{H}} e_j \circ \varphi, \sqrt{\pi}^*_{\mathcal{H}} e_j \circ \psi), \\
&\frac{N + \Sigma^{\sqrt{}}}{2\sqrt{2}} \circ (\varphi, \psi) = (-\psi, \varphi), \quad \frac{N - \Sigma^{\sqrt{}}}{2\sqrt{2}} \circ (\varphi, \psi) = (-\psi, -\varphi), \\
&\text{hence,}\ \frac{N}{2\sqrt{2}} \circ (\varphi, \psi) = (-\psi, 0), \quad \frac{\Sigma^{\sqrt{}}}{2\sqrt{2}} \circ (\varphi, \psi) = (0, \varphi).
\end{aligned}
\tag{3.7}
$$

From our realization, for example [3, Proposition 22] will be obvious: It says that $\not{S}_H$ has the decompositions having appeared in Proposition 2.1, and, in particular, $\sqrt{\pi}^*\not{S}^n_H = \sqrt{\pi}^*K^{1/2}$, $\sqrt{\pi}^*\not{S}^{-n}_H = \sqrt{\pi}^*(\wedge^{0,n}_H(M) \otimes K^{1/2})$ are trivial line bundles with global cross-sections $\psi_{\pm}$ given by

$$
\begin{aligned}
&\sqrt{F(M)} \ni \omega^{\sqrt{}} = \sqrt{[\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n]}(p) \cdot e^{i\Theta^{\sqrt{}}}\ (\in K^{1/2}), \\
&\psi_+(\omega^{\sqrt{}}) := [\omega^{\sqrt{}}, \omega^{\sqrt{}}] \in \sqrt{\pi}^*K^{1/2}, \\
&\psi_-(\omega^{\sqrt{}}) := [\omega^{\sqrt{}}, (\theta^{\bar{1}} \wedge \cdots \wedge \theta^{\bar{n}}) \cdot e^{-i2\Theta^{\sqrt{}}} \otimes \omega^{\sqrt{}}] \in \sqrt{\pi}^*(\wedge^{0,n}_H(M) \otimes K^{1/2}).
\end{aligned}
$$

4. Dirac operator on the Fefferman spin space $\sqrt{F(M)}$

The spinor connection $\nabla^{\mathcal{S}}$ for the spinor bundle $\mathcal{S}$ is defined by

$$\begin{aligned}
(4.1)\qquad \omega(\nabla^{\mathcal{S}}) &= \frac{1}{2}\sum_{k<\ell}\epsilon_\ell\epsilon_k\, h^{\surd}(\nabla^{h^{\surd}} s_\ell, s_k)\, s_\ell \circ s_k \circ \\
&= \sum_{A,B\neq 0}\frac{1}{4}\,\omega(\nabla^{h^{\surd}})^A_{\bar B}\,\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_B \circ \sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_A \circ \\
&\quad + \sum_{A\neq 0}\frac{1}{2}\,\omega(\nabla^{h^{\surd}})^A_{(N\Sigma^{\surd}_+)}\,\frac{N+\Sigma^{\surd}}{2\sqrt{2}} \circ \sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_A \circ \\
&\quad - \sum_{A\neq 0}\frac{1}{2}\,\omega(\nabla^{h^{\surd}})^A_{(N\Sigma^{\surd}_-)}\,\frac{N-\Sigma^{\surd}}{2\sqrt{2}} \circ \sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_A \circ,
\end{aligned}$$

where $\epsilon_\ell := h^{\surd}(s_\ell, s_\ell) = \pm 1$ (cf. (3.4)), $\omega(\nabla^{h^{\surd}})^A_{\bar B} = h^{\surd}(\nabla^{h^{\surd}}\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_{\bar B}, \sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_{\bar A})$ and $\omega(\nabla^{h^{\surd}})^A_{(N\Sigma^{\surd}_\pm)} = h^{\surd}(\nabla^{h^{\surd}}\frac{N\pm\Sigma^{\surd}}{2\sqrt{2}}, \sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_{\bar A})$. The Dirac operator is given by

$$\begin{aligned}
D^{\mathcal{S}} = \sum \epsilon_j\, s_j \circ \nabla^{\mathcal{S}}_{s_j} &= \sum \sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_\gamma \circ \nabla^{\mathcal{S}}_{\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_{\bar\gamma}} + \sum \sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_{\bar\gamma} \circ \nabla^{\mathcal{S}}_{\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_\gamma} \\
&\quad + \frac{N+\Sigma^{\surd}}{2\sqrt{2}} \circ \nabla^{\mathcal{S}}_{\frac{N+\Sigma^{\surd}}{2\sqrt{2}}} - \frac{N-\Sigma^{\surd}}{2\sqrt{2}} \circ \nabla^{\mathcal{S}}_{\frac{N-\Sigma^{\surd}}{2\sqrt{2}}}.
\end{aligned}$$

The purpose of the section is to offer their explicit expressions.

First, we will calculate the connection form $\omega(\nabla^{h^{\surd}})$. In [11], the author calculated that of the Fefferman metric of the Fefferman space for $K(M)$. The idea adopted there is effective also here. That is, since $\nabla^{h^{\surd}}$ is invariant under $U(1)$-action, it descends to a connection $\sqrt{\pi}_*\nabla^{h^{\surd}}$ on M, which is well-defined by $(\sqrt{\pi}_*\nabla^{h^{\surd}})_X Y = \sqrt{\pi}_*\big(\nabla^{h^{\surd}}_{\sqrt{\pi}^{\,*}_{\mathcal{H}}X}\sqrt{\pi}^{\,*}_{\mathcal{H}}Y\big)$. And we have the following.

Lemma 4.1 (cf. [11, Propositions 3.1 and 3.2(1)]). *The torsion of $\sqrt{\pi}_*\nabla^{h^{\surd}}$ vanishes and we have*

$$\begin{aligned}
(4.2)\qquad (\sqrt{\pi}_*\nabla^{h^{\surd}})_X Y &= {}^*\nabla_X Y + \frac{1}{2}g(X, JY)\xi + \theta(Y)\tau X \\
&\quad - \theta(X)\big(2\mathcal{F}^{\sigma^{\surd}}(Y) + 2\mathcal{F}(\sigma^{\surd})(Y,\xi)\,\xi\big) - \theta(Y)\big(2\mathcal{F}^{\sigma^{\surd}}(X) + 2\mathcal{F}(\sigma^{\surd})(X,\xi)\,\xi\big),
\end{aligned}$$

where $\mathcal{F}^{\sigma^{\surd}}(Y)$ is the vector defined by $g(Z, \mathcal{F}^{\sigma^{\surd}}(Y)) = \underline{\mathcal{F}(\sigma^{\surd})(Z, Y)}$ for any vector Z. Set $(\sqrt{\pi}_*\nabla^{h^{\surd}})\xi_B = \sum \xi_A \cdot \omega(\sqrt{\pi}_*\nabla^{h^{\surd}})^A_B$. Then $\omega(\sqrt{\pi}_*\nabla^{h^{\surd}})^{\bar A}_{\bar B} = \overline{\omega(\sqrt{\pi}_*\nabla^{h^{\surd}})^A_B}$ and

$$\begin{aligned}
\omega(\sqrt{\pi}_*\nabla^{h^{\surd}})^\alpha_\beta &= \omega({}^*\nabla)^\alpha_\beta + 2\mathcal{F}(\sigma^{\surd})(\xi_\beta, \xi_{\bar\alpha})\theta, \\
\omega(\sqrt{\pi}_*\nabla^{h^{\surd}})^{\bar\alpha}_\beta &= \omega({}^*\nabla)^{\bar\alpha}_\beta + 2\mathcal{F}(\sigma^{\surd})(\xi_\beta, \xi_\alpha)\theta, \\
\omega(\sqrt{\pi}_*\nabla^{h^{\surd}})^0_\beta &= \frac{i}{2}\theta^{\bar\beta}, \\
\omega(\sqrt{\pi}_*\nabla^{h^{\surd}})^\alpha_0 &= -\sum 2\mathcal{F}(\sigma^{\surd})(\xi_{\bar\alpha}, \xi_\gamma)\theta^\gamma \\
&\quad - \sum\big(2\mathcal{F}(\sigma^{\surd})(\xi_{\bar\alpha}, \xi_{\bar\gamma}) - \tau^\alpha_{\bar\gamma}\big)\theta^{\bar\gamma} - 4\mathcal{F}(\sigma^{\surd})(\xi_{\bar\alpha}, \xi)\theta, \\
\omega(\sqrt{\pi}_*\nabla^{h^{\surd}})^0_0 &= 0.
\end{aligned}$$

Proof. As for (4.2): By definition, we have

$$g({}^*\nabla_X Y, Z) = g(\nabla^g_X Y, Z) - g(\theta(Y)\,\tau X, Z) + g(g(\tau X, Y)\xi, Z) + \frac{1}{2}\Big\{ - g(\theta(X) JY, Z) - g(\theta(Y) JX, Z) - g(g(X, JY)\xi, Z)\Big\}.$$

Hence, for a vector Z with $Z_0 := \theta(Z)\xi = 0$,

$$\begin{aligned}
g((\sqrt{\pi}_*\nabla^{h^{\surd}})_X Y, Z) &= h^{\surd}(\nabla^{h^{\surd}}_{\sqrt{\pi}^*_{\mathcal{H}}X}\sqrt{\pi}^*_{\mathcal{H}}Y, \sqrt{\pi}^*_{\mathcal{H}}Z) \\
&= \frac{1}{2}\Big\{\sqrt{\pi}^*_{\mathcal{H}}X h^{\surd}(\sqrt{\pi}^*_{\mathcal{H}}Y, \sqrt{\pi}^*_{\mathcal{H}}Z) + \sqrt{\pi}^*_{\mathcal{H}}Y h^{\surd}(\sqrt{\pi}^*_{\mathcal{H}}X, \sqrt{\pi}^*_{\mathcal{H}}Z) \\
&\quad - \sqrt{\pi}^*_{\mathcal{H}}Z h^{\surd}(\sqrt{\pi}^*_{\mathcal{H}}X, \sqrt{\pi}^*_{\mathcal{H}}Y) + h^{\surd}([\sqrt{\pi}^*_{\mathcal{H}}X, \sqrt{\pi}^*_{\mathcal{H}}Y], \sqrt{\pi}^*_{\mathcal{H}}Z) \\
&\quad + h^{\surd}([\sqrt{\pi}^*_{\mathcal{H}}Z, \sqrt{\pi}^*_{\mathcal{H}}X], \sqrt{\pi}^*_{\mathcal{H}}Y) - h^{\surd}(\sqrt{\pi}^*_{\mathcal{H}}X, [\sqrt{\pi}^*_{\mathcal{H}}Y, \sqrt{\pi}^*_{\mathcal{H}}Z])\Big\} \\
&= g(\nabla^g_X Y, Z) + \frac{1}{2}\Big\{Zg(X_0, Y_0) - g([Z, X], Y_0) - g(X_0, [Z, Y])\Big\} \\
&\quad - 2\mathcal{F}(\sigma^{\surd})(Z, X)\theta(Y) - 2\mathcal{F}(\sigma^{\surd})(Z, Y)\theta(X) \\
&= g(\nabla^g_X Y, Z) + \frac{1}{2}\Big\{ - g(\theta(X) JY, Z) - g(\theta(Y) JX, Z)\Big\} \\
&\quad - \theta(X)\, 2\mathcal{F}(\sigma^{\surd})(Z, Y) - \theta(Y)\, 2\mathcal{F}(\sigma^{\surd})(Z, X) \\
&= g({}^*\nabla_X Y, Z) + \theta(Y) g(\tau X, Z) - \theta(X)\, 2\mathcal{F}(\sigma^{\surd})(Z, Y) - \theta(Y)\, 2\mathcal{F}(\sigma^{\surd})(Z, X)
\end{aligned}$$

and

$$\begin{aligned}
4g((\sqrt{\pi}_*\nabla^{h^{\surd}})_X Y, \xi) &= h^{\surd}(\nabla^{h^{\surd}}_{\sqrt{\pi}^*_{\mathcal{H}}X}\sqrt{\pi}^*_{\mathcal{H}}Y, \Sigma^{\surd}) \\
&= \frac{1}{2}\Big\{\sqrt{\pi}^*_{\mathcal{H}}X h^{\surd}(\sqrt{\pi}^*_{\mathcal{H}}Y, \Sigma^{\surd}) + \sqrt{\pi}^*_{\mathcal{H}}Y h^{\surd}(\sqrt{\pi}^*_{\mathcal{H}}X, \Sigma^{\surd}) \\
&\quad - \Sigma^{\surd} h^{\surd}(\sqrt{\pi}^*_{\mathcal{H}}X, \sqrt{\pi}^*_{\mathcal{H}}Y) + h^{\surd}([\sqrt{\pi}^*_{\mathcal{H}}X, \sqrt{\pi}^*_{\mathcal{H}}Y], \Sigma^{\surd}) \\
&\quad + h^{\surd}([\Sigma^{\surd}, \sqrt{\pi}^*_{\mathcal{H}}X], \sqrt{\pi}^*_{\mathcal{H}}Y) - h^{\surd}(\sqrt{\pi}^*_{\mathcal{H}}X, [\sqrt{\pi}^*_{\mathcal{H}}Y, \Sigma^{\surd}])\Big\} \\
&= 2\Big\{X\theta(Y) + Y\theta(X) + \theta([X, Y])\Big\} \\
&= 4g({}^*\nabla_X Y, \xi) - 2g(J(X - X_0), Y) = 4g({}^*\nabla_X Y, \xi) - 2g(JX, Y).
\end{aligned}$$

Thus we get the formula. It is easy to check $T(\sqrt{\pi}_*\nabla^{h^{\surd}}) = 0$. The formulas for $\omega(\sqrt{\pi}_*\nabla^{h^{\surd}})$ are easily deduced from (4.2). □

Proposition 4.2 (cf. [11, Proposition 3.2(2)]). *Denote the frame* (3.3) *by* $(W_1, \ldots, W_{\bar{1}}, \ldots, W_0, W_{2n+1})$ *and set* $\nabla^{h^{\surd}} W_B = \sum W_A \cdot \omega(\nabla^{h^{\surd}})^A_B$. *Then,* $\omega(\nabla^{h^{\surd}})^{\bar{A}}_{\bar{B}} = \overline{\omega(\nabla^{h^{\surd}})^A_B}$ $(\bar{0} := 0,$ $\overline{2n+1} := 2n+1)$ *and*

$$\begin{aligned}
&\omega(\nabla^{h^{\surd}})^{\alpha}_{\beta} = \sqrt{\pi}^*\omega(\sqrt{\pi}_*\nabla^{h^{\surd}})^{\alpha}_{\beta} + i\,2\delta_{\alpha\beta}\,\sigma^{\surd}, \quad \omega(\nabla^{h^{\surd}})^{\bar{\alpha}}_{\beta} = \sqrt{\pi}^*\omega(\sqrt{\pi}_*\nabla^{h^{\surd}})^{\bar{\alpha}}_{\beta}, \\
&\omega(\nabla^{h^{\surd}})^{(N)}_{\beta} = \sqrt{\pi}^*\omega(\sqrt{\pi}_*\nabla^{h^{\surd}})^{0}_{\beta}, \quad \omega(\nabla^{h^{\surd}})^{\alpha}_{(N)} = \sqrt{\pi}^*\omega(\sqrt{\pi}_*\nabla^{h^{\surd}})^{\alpha}_{0}, \\
&\omega(\nabla^{h^{\surd}})^{(\Sigma^{\surd})}_{\beta} = -\frac{1}{4}\,\overline{\sqrt{\pi}^*\omega(\sqrt{\pi}_*\nabla^{h^{\surd}})^{\beta}_{0}}, \quad \omega(\nabla^{h^{\surd}})^{\alpha}_{(\Sigma^{\surd})} = -4\,\overline{\sqrt{\pi}^*\omega(\sqrt{\pi}_*\nabla^{h^{\surd}})^{0}_{\alpha}}, \\
&\omega(\nabla^{h^{\surd}})^{(N)}_{(N)} = \omega(\nabla^{h^{\surd}})^{(\Sigma^{\surd})}_{(N)} = \omega(\nabla^{h^{\surd}})^{(N)}_{(\Sigma^{\surd})} = \omega(\nabla^{h^{\surd}})^{(\Sigma^{\surd})}_{(\Sigma^{\surd})} = 0,
\end{aligned}$$

where we put $\omega(\nabla^{h^{\surd}})^{(N)}_B = \omega(\nabla^{h^{\surd}})^0_B$, $\omega(\nabla^{h^{\surd}})^{(\Sigma^{\surd})}_B = \omega(\nabla^{h^{\surd}})^{2n+1}_B$, *etc.*

Proof. We have

$$\begin{aligned}\omega(\nabla^{h\surd})^{\alpha}_{\beta}(\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_C) &= h^{\surd}(\nabla^{h\surd}_{\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_C}\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_\beta, \sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_{\bar\alpha})\\ &= \sqrt{\pi}^{\,*}g((\sqrt{\pi}_*\nabla^{h\surd})_{\xi_C}\xi_\beta, \xi_{\bar\alpha}) = \sqrt{\pi}^{\,*}\omega(\sqrt{\pi}_*\nabla^{h\surd})^{\alpha}_{\beta}(\xi_C),\\ \omega(\nabla^{h\surd})^{\alpha}_{\beta}(\Sigma^{\surd}) &= h^{\surd}(\nabla^{h\surd}_{\Sigma^\surd}\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_\beta, \sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_{\bar\alpha}) = h^{\surd}(\nabla^{h\surd}_{\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_\beta}\Sigma^{\surd}, \sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_{\bar\alpha})\\ &= -h^{\surd}(\nabla^{h\surd}_{\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_\beta}\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_{\bar\alpha}, \Sigma^{\surd}) = 2g(J\xi_\beta, \xi_{\bar\alpha}) = 2i\delta_{\alpha\beta},\end{aligned}$$

which yield the formula for $\omega(\nabla^{h\surd})^{\alpha}_{\beta}$. The other ones are shown similarly. □

Now, by (4.1) and Proposition 4.2, the connection form $\omega(\nabla^{\mathcal{S}})$ is described as follows.

Theorem 4.3. *We have*

$$\begin{aligned}\omega(\nabla^{\mathcal{S}})(\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_\gamma) &= \sum_{A,B\neq 0}\frac{1}{4}\sqrt{\pi}^{\,*}\omega({}^*\nabla)^{A}_{\bar B}(\xi_\gamma)\,\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_B\circ\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_A\,\circ\\ &\quad+\frac{i\sqrt{2}}{2}\,\frac{N}{2\sqrt{2}}\circ\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_\gamma\,\circ-\sum_{A\neq 0}\frac{\sqrt{2}}{4}\sqrt{\pi}^{\,*}\Big(2\mathcal{F}(\sigma^\surd)(\xi_{\bar A},\xi_\gamma)-\tau^{A}_{\gamma}\Big)\,\frac{\Sigma^\surd}{2\sqrt{2}}\circ\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_A\circ,\\ \omega(\nabla^{\mathcal{S}})\Big(\frac{N\pm\Sigma^\surd}{2\sqrt{2}}\Big) &= \pm\frac{1}{2\sqrt{2}}\sqrt{\pi}^{\,*}d\theta\,\circ\\ &\quad+\sum_{A,B\neq 0}\frac{1}{8\sqrt{2}}\sqrt{\pi}^{\,*}\Big(\omega({}^*\nabla)^{A}_{\bar B}(\xi)+2\mathcal{F}(\sigma^\surd)(\xi_{\bar B},\xi_{\bar A})\Big)\,\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_B\circ\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_A\,\circ\\ &\quad-\sum_{A\neq 0}\frac{1}{2}\sqrt{\pi}^{\,*}\mathcal{F}(\sigma^\surd)(\xi_{\bar A},\xi)\,\frac{\Sigma^\surd}{2\sqrt{2}}\circ\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_A\circ\end{aligned}$$

and $\omega(\nabla^{\mathcal{S}})(\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_{\bar\gamma}) = \overline{\omega(\nabla^{\mathcal{S}})(\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_\gamma)}$, *where we set* $\sqrt{\pi}^{\,*}d\theta\circ = -i(n+\sum\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_\alpha\circ\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_{\bar\alpha}\circ)$ (*cf. Proposition 2.1*).

Proof. We have

$$\begin{aligned}\omega(\nabla^{\mathcal{S}})(\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_\gamma) &= \sum_{A,B\neq 0}\frac{1}{4}\,\sqrt{\pi}^{\,*}\omega({}^*\nabla)^{A}_{\bar B}(\xi_\gamma)\,\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_B\circ\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_A\,\circ\\ &\quad-\sum\frac{1}{2}\,\sqrt{\pi}^{\,*}\Big\{\frac{\sqrt{2}}{2}\mathcal{F}(\sigma^\surd)(\xi_{\bar\alpha},\xi_\gamma)+\frac{-i\sqrt{2}}{2}\,\delta_{\alpha\gamma}\Big\}\frac{N+\Sigma^\surd}{2\sqrt{2}}\circ\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_\alpha\,\circ\\ &\quad-\sum\frac{1}{2}\,\sqrt{\pi}^{\,*}\frac{\sqrt{2}}{4}\Big(2\mathcal{F}(\sigma^\surd)(\xi_\alpha,\xi_\gamma)-\tau^{\bar\alpha}_{\gamma}\Big)\,\frac{N+\Sigma^\surd}{2\sqrt{2}}\circ\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_{\bar\alpha}\,\circ\\ &\quad-\sum\frac{1}{2}\,\sqrt{\pi}^{\,*}\Big\{-\frac{\sqrt{2}}{2}\mathcal{F}(\sigma^\surd)(\xi_{\bar\alpha},\xi_\gamma)+\frac{-i\sqrt{2}}{2}\,\delta_{\alpha\gamma}\Big\}\frac{N-\Sigma^\surd}{2\sqrt{2}}\circ\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_\alpha\,\circ\\ &\quad-\sum\frac{1}{2}\,\sqrt{\pi}^{\,*}\Big\{-\frac{\sqrt{2}}{4}\Big(2\mathcal{F}(\sigma^\surd)(\xi_\alpha,\xi_\gamma)-\tau^{\bar\alpha}_{\gamma}\Big)\Big\}\frac{N-\Sigma^\surd}{2\sqrt{2}}\circ\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_{\bar\alpha}\,\circ\,.\end{aligned}$$

We sort it out to get the formula for $\omega(\nabla^{\mathcal{S}})(\sqrt{\pi}^{\,*}_{\mathcal{H}}\xi_\gamma)$. The others are shown similarly. □

Theorem 4.4. *The action of the Dirac operator* $D^{\mathcal{S}}$ *on* $(\varphi, \psi) \in \Gamma(\sqrt{\pi}^*\mathcal{S}_H) \oplus \Gamma(\sqrt{\pi}^*\mathcal{S}_H)$ (*cf.* (3.6)) *is read as follows: The left element of* $D^{\mathcal{S}}(\varphi, \psi)$ *is*

$$-\sqrt{2}\,\sqrt{\pi}^*(\bar{\partial}_H^{\mathcal{A}} + \bar{\partial}_H^{\mathcal{A}*})\varphi - \frac{1}{\sqrt{2}}\Big(\Sigma^{\surd} - \sqrt{\pi}^* d\theta \circ\Big)\psi$$

and the right one is

$$\sqrt{2}\,\sqrt{\pi}^*(\bar{\partial}_H^{\mathcal{A}} + \bar{\partial}_H^{\mathcal{A}*})\psi + \frac{1}{\sqrt{2}}\sqrt{\pi}^*\Big({}^\sharp\nabla_\xi^{\mathcal{A}} - \frac{1}{2}\sum_{A,B\neq 0}\mathcal{F}(\sigma^{\surd})(\xi_{\bar{B}}, \xi_{\bar{A}})\,\xi_B \circ \xi_A \circ\Big)\varphi.$$

In addition, we have

$$\begin{aligned}(4.3)\qquad &\sum_{A,B\neq 0}\mathcal{F}(\sigma^{\surd})(\xi_{\bar{B}}, \xi_{\bar{A}})\,\xi_B \circ \xi_A \circ \\ &\quad = \frac{1}{4(n+2)}\Big\{i\sum_{A,B\neq 0}\mathrm{Ric}^{\sharp\nabla}(\xi_{\bar{B}}, \xi_{\bar{A}})\,\xi_B \circ \xi_A \circ - \frac{s^{\sharp\nabla}}{n+1}\,d\theta \circ\Big\}.\end{aligned}$$

Proof. $\sum \epsilon_j\, s_j \circ\, \omega(\nabla^{\mathcal{S}})(s_j)$ is equal to

$$\begin{aligned}
&\sum_{A,B,C\neq 0}\frac{1}{4}\sqrt{\pi}^*\omega({}^*\nabla)_{\bar{B}}^{A}(\xi_{\bar{C}})\,\sqrt{\pi}^*_{\mathcal{H}}\xi_C \circ \sqrt{\pi}^*_{\mathcal{H}}\xi_B \circ \sqrt{\pi}^*_{\mathcal{H}}\xi_A \circ - \frac{1}{\sqrt{2}}\frac{N}{2\sqrt{2}} \circ \sqrt{\pi}^* d\theta \circ \\
&\quad + \sum_{A,B\neq 0}\frac{1}{4\sqrt{2}}\sqrt{\pi}^*\Big\{\omega({}^*\nabla)_{\bar{B}}^{A}(\xi) - 2\mathcal{F}(\sigma^{\surd})(\xi_{\bar{B}}, \xi_{\bar{A}})\Big\}\frac{\Sigma^{\surd}}{2\sqrt{2}} \circ \sqrt{\pi}^*_{\mathcal{H}}\xi_B \circ \sqrt{\pi}^*_{\mathcal{H}}\xi_A \circ \\
&= \sum_{C\neq 0}\sqrt{\pi}^*_{\mathcal{H}}\xi_C \circ \cdot\Big(\sqrt{\pi}^*\omega(\nabla^{(\mathcal{S}_H:{}^*\nabla)})(\xi_{\bar{C}}) \oplus \sqrt{\pi}^*\omega(\nabla^{(\mathcal{S}_H:{}^*\nabla)})(\xi_{\bar{C}})\Big) \\
&\quad + \frac{\Sigma^{\surd}}{2\sqrt{2}} \circ \cdot\Big(\frac{1}{\sqrt{2}}\sqrt{\pi}^*\omega(\nabla^{(\mathcal{S}_H:{}^*\nabla)})(\xi) \oplus \frac{1}{\sqrt{2}}\sqrt{\pi}^*\omega(\nabla^{(\mathcal{S}_H:{}^*\nabla)})(\xi)\Big) \\
&\quad - \frac{N}{2\sqrt{2}} \circ \frac{1}{\sqrt{2}}\sqrt{\pi}^* d\theta \circ - \frac{\Sigma^{\surd}}{2\sqrt{2}} \circ \frac{1}{2\sqrt{2}}\sqrt{\pi}^*\sum_{A,B\neq 0}\mathcal{F}(\sigma^{\surd})(\xi_{\bar{B}}, \xi_{\bar{A}})\,\xi_B \circ \xi_A \circ \\
&= \sum_{C\neq 0}\sqrt{\pi}^*_{\mathcal{H}}\xi_C \circ \cdot\Big(\sqrt{\pi}^*\omega({}^\sharp\nabla^{\mathcal{A}})(\xi_{\bar{C}}) \oplus \sqrt{\pi}^*\omega({}^\sharp\nabla^{\mathcal{A}})(\xi_{\bar{C}})\Big) \\
&\quad + \frac{\Sigma^{\surd}}{2\sqrt{2}} \circ \cdot\Big(\frac{1}{\sqrt{2}}\sqrt{\pi}^*\omega({}^\sharp\nabla^{\mathcal{A}})(\xi) \oplus \frac{1}{\sqrt{2}}\sqrt{\pi}^*\omega({}^\sharp\nabla^{\mathcal{A}})(\xi)\Big) \\
&\quad - \frac{N}{2\sqrt{2}} \circ \frac{1}{\sqrt{2}}\sqrt{\pi}^* d\theta \circ - \frac{\Sigma^{\surd}}{2\sqrt{2}} \circ \frac{1}{2\sqrt{2}}\sqrt{\pi}^*\sum_{A,B\neq 0}\mathcal{F}(\sigma^{\surd})(\xi_{\bar{B}}, \xi_{\bar{A}})\,\xi_B \circ \xi_A \circ .
\end{aligned}$$

Hence,

$$\begin{aligned}
D^{\mathcal{S}} &= \sum_{C\neq 0}\sqrt{\pi}^*_{\mathcal{H}}\xi_C \circ \cdot\Big(\sqrt{\pi}^*\,{}^\sharp\nabla_{\xi_{\bar{C}}}^{\mathcal{A}} \oplus \sqrt{\pi}^*\,{}^\sharp\nabla_{\xi_{\bar{C}}}^{\mathcal{A}}\Big) + \frac{N}{2\sqrt{2}} \circ \cdot\Big(\frac{\Sigma^{\surd}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\sqrt{\pi}^* d\theta \circ\Big) \\
&\quad + \frac{\Sigma^{\surd}}{2\sqrt{2}} \circ \cdot\Big(\frac{1}{\sqrt{2}}\sqrt{\pi}^*\,{}^\sharp\nabla_\xi^{\mathcal{A}} \oplus \frac{1}{\sqrt{2}}\sqrt{\pi}^*\,{}^\sharp\nabla_\xi^{\mathcal{A}}\Big) \\
&\quad - \frac{\Sigma^{\surd}}{2\sqrt{2}} \circ \cdot\frac{1}{2\sqrt{2}}\sqrt{\pi}^*\sum_{A,B\neq 0}\mathcal{F}(\sigma^{\surd})(\xi_{\bar{B}}, \xi_{\bar{A}})\,\xi_B \circ \xi_A \circ .
\end{aligned}$$

This and (3.7) together induce the formulas for the elements. (4.3) will be obvious from the definition of $\mathcal{F}(\sigma^{\surd})$. □

Last, we wish to record a formula for the ordinary scalar curvature of $\nabla^{h^{\surd}}$ which appears in the Lichnerowicz formula

$$(D^{\mathcal{S}})^2 = -\sum \epsilon_j \Big(\nabla^{\mathcal{S}}_{s_j} \nabla^{\mathcal{S}}_{s_j} - \nabla^{\mathcal{S}}_{\nabla^{h^{\surd}}_{s_j} s_j} \Big) + \frac{1}{4} s(\nabla^{h^{\surd}}).$$

Theorem 4.5. *We have*

$$s(\nabla^{h^{\surd}}) = \frac{2n}{n+1} \sqrt{\pi}^{\,*} s^{\sharp\nabla}.$$

This is confirmed by a lengthy calculation following Proposition 4.2, roughly the same as that of the Fefferman metric for $K(M)$ (cf. [11, Theorem 1.1]).

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