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Osaka J. Math.
55 (2018), 769–775

THE VERTICES OF THE COMPONENTS OF THE PERMUTATION MODULE INDUCED FROM PARABOLIC GROUPS

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(Received October 25, 2016, revised June 16, 2017)

Abstract
We consider the permutation module $k_P\uparrow^{\mathrm{GL}_n(p^f)}$, where P is a parabolic group in the general linear group $\mathrm{GL}_n(p^f)$ and k is an algebraically closed field of prime characteristic p. The vertices of the components of these modules have been calculated in [9] by Tinberg, who studied these modules for all groups with split BN-pairs in characteristic p. In this paper we show that the idea of suitability is strong enough to find all p-groups that are vertex of some component of $k_P\uparrow^{\mathrm{GL}_n(p^f)}$. Furthermore using a result of Burry and Carlson we show that all components have a different vertex.

1. Introduction

Let G be a finite group with subgroup H. We are interested in the p-groups of G that are vertices of components of the permutation-module $k_H\uparrow^G$, where k is an algebraically closed field of characteristic p. In [7] the author introduces the idea of H-suitability and in an example shows how suitability could be applied to detect all p-groups that are vertices of some component of $k_H\uparrow^G$. Furthermore in that example suitability is at its most restrictive as every H-suitable group turns out to be the vertex of a component of $k_H\uparrow^G$. In the present paper we present another example where this concept of suitability unfolds its full potential by giving us only p-groups that are vertices of some component of the given permutation module $k_H\uparrow^G$.

Next we describe the notion of H-suitability. A p-subgroup V of G is called *H-suitable* (with respect to G and p), if for every $S \in \mathrm{Syl}_p(G)$ with $V \leq S$ there exists some $g \in G$ so that $V = S \cap H^g$. Here H^g denotes the conjugate $g^{-1}Hg$ of H. In [7] we show that H-suitability is a necessary condition for a p-group to be the vertex of a component of $k_H\uparrow^G$ and present an example in which it is also sufficient. However in general an H-suitable group is not necessarily the vertex of some component of $k_H\uparrow^G$. For instance with respect to the symmetric group S_5 and $p = 2$ the trivial group is D-suitable, for the dihedral group $D := \langle(1234), (13)\rangle \leq S_5$, but $k_D\uparrow^{S_5}$ has no projective summand. Nevertheless one can use the concept of suitability to produce a list of potential candidates for a vertex by finding all H-suitable groups. If this list is short enough one may then deal with each candidate individually.

In [7] we see that if V is H-suitable then so is every G-conjugate of V. Hence given a Sylow-p-group S of G the set $\mathcal{A} := \{S \cap H^g : g \in G\}$ contains a representative of each

2010 Mathematics Subject Classification. 20G40.

conjugacy class of H-suitable groups. The following statement presents a method to shorten the list given by $\mathcal{A}$ by finding groups that are not H-suitable. A more general version of the statement with proof can be found in [7, Theorem 2.6.].

Lemma 1.1. *Let $S \in \mathrm{Syl}_p(G)$ and $K \leq G$. Also let $T \subseteq G$ such that*

(*a*) *For every $g \in G$ there is $\alpha \in T$ so that $S \cap H^g =_K S \cap H^{\alpha^{-1}}$.*

(*b*) *For all $\alpha \in T$ we have $S^\alpha \cap H \leq S$.*

(*c*) *If $S^\alpha \cap H =_K S^{\alpha'} \cap H$, for $\alpha, \alpha' \in T$, then $\alpha = \alpha'$.*

Furthermore let $V \leq S$ be H-suitable. Then there is some $\alpha \in T$ such that $V =_K S^\alpha \cap H$.

Observe that Lemma 1.1 implies that every H-suitable group V that is contained in S is a K-conjugate of both $S \cap H^{\alpha^{-1}}$ and $S^\beta \cap H$, for some $\alpha, \beta \in T$. In particular any group in the set $\{S \cap H^{\alpha^{-1}} : \alpha \in T\}$ which has no K-conjugate in the set $\{S^\alpha \cap H : \alpha \in T\}$ fails to be H-suitable.

Let us now turn to our example for which we need the following notation. For the remainder of this paper let $n, f \geq 1$ and set $q = p^f$, for some prime number p. Furthermore let $G := \mathrm{GL}_n(q)$ be the general linear group, that is, the group of invertible $n \times n$-matrices over the finite field $\mathbb{F}_q$ with q elements. Finally let k be an algebraically closed field of characteristic p.

For any integer $j \geq 1$ we define $\mathrm{GL}_j := \mathrm{GL}_j(q)$. By B_j we denote the group of upper-triangular matrices in GL_j. Those elements in B_j with 1's on the main diagonal form the group U_j. Recall that U_j is a Sylow-p-group of GL_j and B_j is the normalizer of U_j in GL_j.

Furthermore $\mathcal{W}_j$ denotes the group of permutation matrices in GL_j. There is a bijection between the symmetric group $\mathrm{Sym}(j)$ and $\mathcal{W}_j$, in the sense that the permutation $\omega \in \mathrm{Sym}(j)$ corresponds to the permutation matrix $(\delta_{r,\omega(s)})_{r,s}$, where δ denotes the Kronecker-symbol.

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a *composition* of n, that is, $\lambda_1 \ldots, \lambda_r$ are positive integers such that $\lambda_1 + \ldots + \lambda_r = n$. Then for $X_t \in \mathrm{GL}_{\lambda_t}$, let $D_n(X_1, X_2, \ldots, X_r)$ be the matrix in G, which has $X_1, \ldots, X_r$ along the main diagonal and zeros otherwise. Similarly $D_n(X_1\bullet, X_2\bullet, \ldots, X_r)$ denotes any matrix in G which has $X_1, \ldots, X_r$ along the main diagonal, arbitrary elements in $\mathbb{F}_q$ above and zeros below that diagonal. Finally we define

$$P_\lambda = \{D_n(X_1\bullet, X_2\bullet, \ldots, X_r) : \; X_t \in \mathrm{GL}_{\lambda_t}\}.$$

Then P_λ is a *parabolic subgroup* of G. For instance if $\lambda = (1^n)$ then $P_\lambda = B_n$.

The permutation module $k_{P_\lambda}\!\uparrow^G$, where λ is a composition of n, has been studied before and in more generality. Tinberg [9] studied these modules for all groups with split BN-pairs in characteristic p. Further work was done by Canbanes [3] and Sin [8]. In particular the vertices of all components are known as well as their Green correspondents. In this paper we revisit $k_{P_\lambda}\!\uparrow^G$ and show that P_λ-suitability is a strong enough tool to find all p-groups that are vertices of components of $k_{P_\lambda}\!\uparrow^G$. In section 2 we find all P_λ-suitable groups up to G-conjugation and in section 3 we show that all P_λ-suitable groups are vertex of exactly one component of $k_{P_\lambda}\!\uparrow^G$.

Next let $1 \leq k, l \leq n$ such that $k \neq l$. By $F_{k,l}$ we denote the subgroup of G that consists of exactly those matrices that have ones on the main diagonal and zeros everywhere else except in the (k, l)-entry, which is arbitrary in $\mathbb{F}_q$. One checks easily that for $\omega \in \mathcal{W}_n$ and $k \neq l$ we have $F_{k,l} = (F_{\omega(k),\omega(l)})^{\omega}$. Finally by [5] a group $V \leq U_n$ is called a *pattern group* if $X_{k,l} \neq 0$, for some $X \in V$ and $1 \leq k < l \leq n$, implies that $F_{k,l} \leq V$. Note that a pattern group which contains both $F_{k,l}$ and $F_{l,m}$, for $k < l < m$, also contains $F_{k,m}$.

Lemma 1.2. *Let $V, W \leq U_n$ be pattern groups that are B_n-conjugate. Then $V = W$.*

Proof. It is enough to show that $F_{k,l} \leq V$ implies $F_{k,l} \leq W$. So let $F_{k,l} \leq V$, for some $k < l$, and let $g \in F_{k,l}$ be non-trivial. By assumption there is $h \in B_n$ so that $g' := g^h \in W$. As $g'_{k,l} \neq 0$ it now follows that $F_{k,l} \leq W$. □

2. P_λ-suitable groups

In this section let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a composition of n. Also set $s_i := \sum_{k=1}^{i-1} \lambda_k$, for $i = 1, \ldots, r$, and $s_{r+1} := n$.

2.1. Good Permutations. We call $\omega \in \mathcal{W}_n$ *good* with respect to λ, if $\omega(s_i + 1) < \ldots < \omega(s_i + \lambda_i)$, for all $i = 1, \ldots, r$.

Lemma 2.1. *Every (B_n, P_λ)-double coset in G contains a permutation matrix in $\mathcal{W}_n$ that is good with respect to λ.*

Proof. Let $g \in G$. Then by the Bruhat decomposition, (see [1]), and since $B_n \leq P_\lambda$, there exists $\mu \in \mathcal{W}_n$ such that $B_n \cdot g \cdot P_\lambda = B_n \cdot \mu \cdot P_\lambda$. Next let

$$\mu(\{s_i + 1, \ldots, s_i + \lambda_i\}) = \{k_{s_i+1}, \ldots, k_{s_i+\lambda_i}\}, \text{ where } k_{s_i+1} < \ldots < k_{s_i+\lambda_i},$$

for all $i = 1, \ldots, r$. Set $\gamma(j) := \mu^{-1}(k_j)$, for all $j = 1, \ldots, n$. Then $\gamma \in \mathcal{W}_n$. Also observe that γ acts on the sets $\{s_i + 1, \ldots, s_i + \lambda_i\}$. Hence $\gamma \in P_\lambda$.

Next set $\omega := \mu \cdot \gamma \in \mathcal{W}_n$. Since $\omega(j) = \mu(\mu^{-1}(k_j)) = k_j$, for all $j = 1, \ldots, n$, it follows that ω is good. As now $B_n \cdot g \cdot P_\lambda = B_n \cdot \omega \cdot P_\lambda$, the proof is complete. □

For any $g \in G$ we define $V(g) := U_n{}^g \cap P_\lambda$. The following lemma is then an easy exercise.

Lemma 2.2. *Let $\omega \in \mathcal{W}_n$ be good with respect to λ. Then*

$$\begin{aligned} V(\omega) = \{X = D_n(A_1\bullet, A_2\bullet, \ldots, A_r) :\ & A_i \in U_{\lambda_i}, \text{ and} \\ & X_{k,l} = 0, \text{ for all } k, l = 1, \ldots, n \text{ so that } \omega(k) > \omega(l)\}. \end{aligned} \tag{1}$$

In particular $V(\omega) \leq U_n$, and $V(\omega)$ is a pattern group.

Observe that Lemma 2.1 implies the following

Corollary 2.3. *Every P_λ-suitable group is G-conjugate to some $V(\omega)$, where $\omega \in \mathcal{W}_n$ is good with respect to λ.*

Lemma 2.4. *The set T of all $\omega \in \mathcal{W}_n$ that are good with respect to λ satisfies the properties (a)-(c) in Lemma 1.1, where $H = P_\lambda$, $S = U_n$ and $K = B_n$.*

Proof. Property (a) is a consequence of Lemma 2.1, property (b) follows since $V(\omega) \leq U_n$ and property (c) is a consequence of Lemma 1.2 and the structure of $V(\omega)$, as given in (1). □

Proposition 2.5. *Let $V \leq U_n$ be a pattern group. If V is P_λ-suitable, then $V = V(\omega)$, where $\omega \in \mathcal{W}_n$ is good with respect to λ.*

Proof. By Lemma 2.4 and Lemma 1.1 we have $V =_{B_n} V(\omega)$, where $\omega \in \mathcal{W}_n$ is good with respect to λ. Now $V = V(\omega)$ follows from Lemma 1.2. □

2.2. λ-permutations. Recall that I_j denotes the identity matrix in GL_j. By $\overline{I_j}$ we mean the permutation matrix in GL_j that has all its ones on the anti-diagonal. Also let $\omega = D_n(K_1, \dots, K_u) \in \mathcal{W}_n$, where $K_i \in \{I_{n_i}, \overline{I_{n_i}}\}$, and $n_1 + \dots + n_u = n$. We call such ω a *λ-permutation* if for all $t = 1, \dots, r$ so that $\lambda_t \geq 2$ we have that ω acts on each of the following sets

$$\{1, \dots, s_t + 1\}, \{s_t + 2\}, \dots, \{s_{t+1} - 1\}, \{s_{t+1}, \dots, n\}.$$

Observe that every λ-permutation is good with respect to λ. In the following we show that if $\omega \in \mathcal{W}_n$ is a λ-permutation, then $V(\omega)$ is P_λ-suitable.

Lemma 2.6. *Let $\omega \in \mathcal{W}_n$ be a λ-permutation. Then*

$$V(\omega) = \{D_n(A_1\bullet, A_2\bullet, \dots, A_u) : \text{ where } A_i \in U_{n_i}, \text{ if } K_i = I_{n_i}, \text{ and } A_i = I_{n_i}, \text{ if } K_i = \overline{I_{n_i}}\}.$$

Proof. Let S be the set on the right hand side. Then $S = {U_n}^\omega \cap U_n \leq P_\lambda$. Since $V(\omega) \leq U_n$ we get $S \leq {U_n}^\omega \cap P_\lambda = V(\omega) \leq {U_n}^\omega \cap U_n = S$. Thus $V(\omega) = S$. □

Lemma 2.6 implies the following

Corollary 2.7. *Let $\omega \in \mathcal{W}_n$ be a λ-permutation. Then*
(1) $B_n \leq \mathrm{N}_G(V(\omega))$
(2) *If $\mu \in \mathcal{W}_n$ such that $V(\omega)^\mu \leq U_n$, then $\mu = D_n(\mu_1, \dots, \mu_u)$ where $\mu_i = I_{n_i}$, if $K_i = I_{n_i}$, and $\mu_i \in \mathcal{W}_{n_i}$, if $K_i = \overline{I_{n_i}}$.*

Lemma 2.8. *Let $\omega \in \mathcal{W}_n$ be a λ-permutation. Also let N be the set of all matrices of the from $D_n(A_1\bullet, A_2\bullet, \dots, A_u)$ where $A_i \in B_{n_i}$, if $K_i = I_{n_i}$, and $A_i \in \mathrm{GL}_{n_i}$, if $K_i = \overline{I_{n_i}}$. Moreover let $g \in G$ such that $V(\omega)^g \leq U_n$. Then $g \in N$. In particular $\mathrm{N}_G(V(\omega)) = N$.*

Proof. By the Bruhat decomposition there are $A, B \in B_n$ and $\mu \in \mathcal{W}_n$ such that $g = A\mu B$. As $B_n \subseteq \mathrm{N}_G(V(\omega))$ we have $V(\omega)^\mu \leq U_n$, and thus $\mu \in N$, by Corollary 2.7. Since $B_n \subseteq N$, we get $g \in N$. Now $\mathrm{N}_G(V(\omega)) \subseteq N$ follows. As one checks easily that N normalizes $V(\omega)$, the proof is complete. □

Corollary 2.9. *Let $\omega_1, \omega_2 \in \mathcal{W}_n$ be λ-permutations so that $V(\omega_1) =_G V(\omega_2)$. Then $\omega_1 = \omega_2$.*

Proof. As $V(\omega_1)^g = V(\omega_2) \leq U_n$, for some $g \in G$, we have $g \in \mathrm{N}_G(V(\omega_1))$, by Lemma 2.8. Thus $V(\omega_1) = V(\omega_2)$. Now $\omega_1 = \omega_2$ follows from (1). □

Proposition 2.10. *Let $\omega \in \mathcal{W}_n$ be a λ-permutation. Then $V(\omega)$ is P_λ-suitable.*

Proof. Since $V(\omega)^{\omega^{-1}} \leq U_n$, it follows $\omega^{-1} \in \mathrm{N}_G(V(\omega))$, by Lemma 2.8. Thus $V(\omega) = U_n \cap P_\lambda{}^{\omega^{-1}}$. Note that $U_n \in \mathrm{Syl}_p(\mathrm{N}_G(V(\omega)))$. So the statement follows from [7, Lemma 2.2.(3)]. □

2.3. The P_λ-suitable $V(\omega)$. In the following let $\omega \in \mathcal{W}_n$ be good with respect to λ so that $V(\omega)$ is P_λ-suitable. We aim to show that ω is a λ-permutation.

Lemma 2.11. *Let $F_{k,l} \leq V(\omega)$, that is, $\omega(k) < \omega(l)$, for some $k < l$. Then*

(1) *$F_{t,l} \leq V(\omega)$, that is, $\omega(t) < \omega(l)$, for all $t = 1, \ldots, k$*

(2) *$F_{k,t} \leq V(\omega)$, that is, $\omega(k) < \omega(t)$, for all $t = l, \ldots, n$.*

Proof. We prove part (1) by contradiction. Without lose of generality we may assume that $F_{k-1,l} \nleq V(\omega)$. In particular $F_{k-1,k} \nleq V(\omega)$. Next let $\mu \in \mathcal{W}_n$ correspond to the permutation $(k-1, k) \in \mathrm{Sym}(n)$, and set $V := V(\omega)^\mu$. Then $V \leq U_n$ is a P_λ-suitable pattern group, and so, by Proposition 2.5, there is $\alpha \in \mathcal{W}_n$ that is good with respect to λ such that $V = V(\alpha)$.

As $F_{k,k-1} \nleq V(\omega)$, $F_{k-1,l} \nleq V(\omega)$ and $F_{k,l} \leq V(\omega)$, we have $F_{k-1,k} \nleq V(\alpha)$, $F_{k,l} \nleq V(\alpha)$ and $F_{k-1,l} \leq V(\alpha)$. That forces $\alpha(k) < \alpha(k-1)$, $\alpha(l) < \alpha(k)$ and $\alpha(k-1) < \alpha(l)$, respectively. This contradiction proves part (1). A similar argument proves part (2). □

Corollary 2.12. *If $\omega(k) < \omega(k+1)$, then ω acts on the sets $\{1, \ldots, k\}$ and $\{k+1, \ldots, n\}$.*

Proof. Let $r \in \{1, \ldots, k\}$. Then $\omega(r) < \omega(k+1)$, by Lemma 2.11 (1). Hence $\omega(r) < \omega(t)$, for all $t = k+1, \ldots, n$, by Lemma 2.11 (2). In particular $\omega(\{1, \ldots, k\}) \subseteq \{1, \ldots, k\}$, and the statement follows. □

Proposition 2.13. *Let $\omega \in \mathcal{W}_n$ be good with respect to λ such that $V(\omega)$ is P_λ-suitable. Then ω is a λ-permutation.*

Proof. First we show that $\omega = D_n(K_1, \ldots, K_u)$, where $K_i \in \{I_{n_i}, \overline{I_{n_i}}\}$, and $n_1 + \ldots + n_u = n$. Clearly there is some $k \in \{0, 1, \ldots, n\}$ such that $\omega = D_n(K_1, \ldots, K_u, \omega')$, where $K_i \in \{I_{n_i}, \overline{I_{n_i}}\}$, $n_1 + \ldots + n_u = k$ and $\omega' \in \mathcal{W}_{n-k}$. Let us choose k maximal with this property and suppose $k < n$. Observe that ω acts on the sets $\{1, \ldots, k\}$ and $\{k+1, \ldots, n\}$. Also the maximality of k implies that $\omega(k+1) \neq k+1$. Next let $l \geq k+1$ be maximal such that $\omega(k+1) > \omega(k+2) \ldots > \omega(l)$. Since $l = n$ or $\omega(l) < \omega(l+1)$, it follows from Corollary 2.12 that ω acts on the set $\{k+1, \ldots, l\}$ and it does so like the permutation matrix $\overline{I_{l-k}}$. But this contradicts the maximality of k, and thus $k = n$.

Now let $i \in \{1, \ldots, r\}$ so that $\lambda_i \geq 2$. As ω is good with respect to λ we have $\omega(s_i + 1) < \ldots < \omega(s_{i+1})$. Now Corollary 2.12 implies that ω acts on the sets $\{1, \ldots, s_i + 1\}, \{s_i + 2\}, \ldots, \{s_{i+1} - 1\}, \{s_{i+1}, \ldots, n\}$. In particular ω is a λ-permutation. □

Theorem 2.14. *The set $\mathcal{C} := \{V(\omega) : \omega \in \mathcal{W}_n$ is a λ-permutation$\}$ provides a full set of representatives for the G-conjugacy classes of P_λ-suitable groups without repetitions.*

Proof. In Proposition 2.10 we have established that $V(\omega)$, where $\omega \in \mathcal{W}_n$ is a λ-permutation, is P_λ-suitable. Furthermore any two different such groups lie in different conjugacy classes, by Corollary 2.9.

Now let V be P_λ-suitable. Then V is G-conjugate to some $V' \leq U_n$. Hence $V' = U_n \cap P_\lambda{}^g$, for some $g \in G$. By Lemma 2.1 it follows that V' is G-conjugate to $V(\omega)$, where $\omega \in \mathcal{W}_n$ is good with respect to λ. In particular ω is a λ-permutation, by Proposition 2.13. □

3. The number of components of $k_{P_\lambda}\!\uparrow^{\mathrm{GL}_n}$ and their vertices

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a composition of n. In the previous section we found all P_λ-suitable groups up to G-conjugation. In fact we have shown that V is P_λ-suitable if and only if $V =_G V(\omega)$, where ω is a λ-permutation. Hence the set $\mathcal{C}$ from Theorem 2.14 contains a vertex for each component of $k_{P_\lambda}\!\uparrow^{\mathrm{GL}_n}$. In this section we want to determine how many components have each of the groups in $\mathcal{C}$ as a vertex. So for the remainder of this paper let $\omega \in \mathcal{W}_n$ be a λ-permutation and set $V := V(\omega)$.

By a result of Burry and Carlson [2] the kG-module $k_{P_\lambda}\!\uparrow^G$ and the $k\,\mathrm{N}_G(V)$-module $\bigoplus_{g\in A} k_{P_\lambda{}^g\cap \mathrm{N}_G(V)}\!\uparrow^{\mathrm{N}_G(V)}$ have the same number of components with vertex V, where A is a set of representatives g for those $(P_\lambda, \mathrm{N}_G(V))$-double cosets of G with $V \leq P_\lambda{}^g$.

Next suppose that $V \leq P_\lambda{}^g$, for some $g \in G$. Since U_n is a Sylow-p-group of P_λ it follows that $V^{g^{-1}h} \leq U_n$, for some $h \in P_\lambda$. Now by Lemma 2.8 we have $g^{-1}h \in \mathrm{N}_G(V)$, and thus $g \in P_\lambda \cdot \mathrm{N}_G(V)$. Hence by the above paragraph we obtain that $k_{P_\lambda}\!\uparrow^G$ has the same number of components with vertex V as $k_{P_\lambda\cap \mathrm{N}_G(V)}\!\uparrow^{\mathrm{N}_G(V)}$.

Lemma 3.1. *We have $P_\lambda \cap \mathrm{N}_G(V(\omega)) = B_n$.*

Proof. Clearly $B_n \subseteq P_\lambda \cap \mathrm{N}_G(V)$. Next let $X \in P_\lambda \cap \mathrm{N}_G(V)$, such that $X \notin B_n$. Hence $X_{l,k} \neq 0$, for some $k < l$. Since $X \in \mathrm{N}_G(V)$, Lemma 2.8 implies that there is some $j \in \{1, \ldots, u\}$ such that $K_j = \overline{I_{n_j}}$ and $n_1 + \ldots + n_{j-1} < k < l \leq n_1 + \ldots + n_j$. In particular $\omega(k) > \omega(l)$.

On the other hand as $X \in P_\lambda$, then $s_i + 1 \leq k < l \leq s_{i+1}$, for some $i \in \{1, \ldots, r\}$. But then $\omega(k) < \omega(l)$, as ω is good with respect to λ. This contradiction completes the proof. □

Lemma 3.2. *Let $\omega \in \mathcal{W}_n$ be a λ-permutation. Then there is exactly one component in $k_{P_\lambda}\!\uparrow^{\mathrm{GL}_n}$ with vertex $V(\omega)$.*

Proof. Let $V = V(\omega)$. By the introduction of this section and Lemma 3.1 we know that the number of components of $k_{P_\lambda}\!\uparrow^{\mathrm{GL}_n}$ with vertex V coincides with the number of components of $k_{B_n}\!\uparrow^{\mathrm{N}_G(V)}$ with vertex V. This number in turn equals the number of projective components of $k_{B_n/V}\!\uparrow^{\mathrm{N}_G(V)/V}$. We have

$$\begin{aligned} B_n/V \cong \{D_n(A_1, \ldots, A_u) :\ & \text{where } A_i \in D_{n_i}, \text{ if } K_i = I_{n_i}, \\ & \text{and } A_i \in B_{n_i}, \text{ if } K_i = \overline{I_{n_i}}\} \end{aligned}$$

and

$$\begin{aligned} \mathrm{N}_G(V)/V \cong \{D_n(A_1, \ldots, A_u) :\ & \text{where } A_i \in D_{n_i}, \text{ if } K_i = I_{n_i}, \\ & \text{and } A_i \in \mathrm{GL}_{n_i}, \text{ if } K_i = \overline{I_{n_i}}\}, \end{aligned}$$

where D_{n_i} is the group of diagonal matrices in GL_{n_i}. Hence

$$k_{B_n/V}\!\uparrow^{\mathrm{N}_G(V)/V} \cong \bigotimes_{i:K_i=\overline{I_{n_i}}} k_{B_{n_i}}\!\uparrow^{\mathrm{GL}_{n_i}}.$$

But every $k_{B_j}\uparrow^{\mathrm{GL}_j}$ contains exactly one projective component, known as the Steinberg-module, (see [4] for more details). Now the statement follows by [6, Proposition 1.2]. □

Finally we can state our main result.

Theorem 3.3. *The number of components of* $k_{P_\lambda}\uparrow^{\mathrm{GL}_n}$ *coincides with the number of* λ-*permutations in* $\mathcal{W}_n$, *and* $\{V(\omega) : \omega \in \mathcal{W}_n$ *is a* λ-*permutation*$\}$ *gives a full set of the different vertices of the components of* $k_{P_\lambda}\uparrow^{\mathrm{GL}_n}$.

Proof. We have seen that every component of $k_{P_\lambda}\uparrow^{\mathrm{GL}_n}$ has a vertex of the form $V(\omega)$, where $\omega \in \mathcal{W}_n$ is a λ-permuation. By Corollary 2.9 we know that two such groups are not G-conjugate. The result of Lemma 3.2 completes the proof. □

We conclude our paper with a specific example. Let $\lambda = (3, 1, 2)$ be a composition of $n = 6$. Then $s_1 = 0$, $s_2 = 3$, $s_3 = 4$ and $s_4 = 6$. Next observe that a λ-permutation ω acts on the sets $\{1\}$, $\{2\}$, $\{3, 4, 5\}$ and $\{6\}$. Hence there are exactly four λ-permutations and they are $\omega_1 = I_6$, $\omega_2 = D_6(I_2, \overline{I_3}, I_1)$, $\omega_3 = D_6(I_2, \overline{I_2}, I_2)$ and $\omega_4 = D_6(I_3, \overline{I_2}, I_1)$. In particular $k_{P_\lambda}\uparrow^{\mathrm{GL}_6}$ has exactly four components with the respective vertices

$$\begin{aligned} V(\omega_1) &= U_6 \\ V(\omega_2) &= \{X \in U_6 : X_{3,4} = X_{3,5} = X_{4,5} = 0\}, \\ V(\omega_3) &= \{X \in U_6 : X_{3,4} = 0\}, \\ V(\omega_4) &= \{X \in U_6 : X_{4,5} = 0\}. \end{aligned}$$

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