Fukuhara, S., Kawazumi, N. and Kuno, Y.
Osaka J. Math.
55 (2018), 761–768

SELF-INTERSECTIONS OF CURVES ON A SURFACE AND BERNOULLI NUMBERS

Shinji FUKUHARA, Nariya KAWAZUMI and Yusuke KUNO

(Received May 12, 2017, revised June 2, 2017)

Abstract

We study an operation which measures self-intersections of curves on an oriented surface. It turns out that a certain computation on this topological operation is related to the Bernoulli numbers B_m, and our study yields a family of explicit formulas for B_m. As a special case, this family contains the celebrated formula for B_m due to Kronecker.

1. Introduction

The Bernoulli numbers B_m ($m \geq 0$) are defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} x^m.$$

We have: $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_4 = -1/30, \ldots$, and $B_m = 0$ for all odd $m \geq 3$. The appearance of the Bernoulli numbers is ubiquitous in mathematics, and a large number of identities involving the Bernoulli numbers has been known [3] [4] [9] [10].

In this article, we show that the Bernoulli numbers arise naturally from the topology of surfaces, i.e., 2-manifolds. In more detail, by studying self-intersections of curves on an oriented surface, we obtain the following family of explicit formulas for B_m:

Theorem 1. *Let $m \geq 2$. For any integers a and n satisfying $0 \leq a \leq m \leq n$, we have*

$$B_m = (-1)^a \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n+1}{k} \sum_{i=1}^{k-1} i^a (k-i)^{m-a}. \tag{1}$$

Notice that the formula above has two parameters a and n. When $a = 0$ and $n = m$, the formula (1) reduces to the celebrated formula for B_m due to Kronecker ([7], see also [4] [5] [9] [10]]): for $m \geq 2$,

$$B_m = \sum_{k=1}^{m+1} \frac{(-1)^{k+1}}{k} \binom{m+1}{k} \sum_{i=1}^{k-1} i^m. \tag{2}$$

In fact, using the classical formula for the sum of powers (known as Faulhaver's formula) and a property of binomial coefficients (see Lemma 2), one can derive the formula (1) from

---

2010 Mathematics Subject Classification. 11B68, 57N05, 57M99.

The authors would like to thank Professor Noriko Yui for useful comments to an earlier version of this paper. S.F. is supported by JSPS KAKENHI (No.26400098). N.K. is partially supported by JSPS KAKENHI (No.24224002), (No.24340010) and (No.15H03617). Y.K. is supported by JSPS KAKENHI (No.26800044).

the Kronecker formula (2). However, our derivation of the formula (1) is self-contained and more direct.

Our proof of Theorem 1 is motivated by a topological consideration on an oriented surface. In §2, we consider an operation μ to a curve on the surface. This operation was introduced in [6] inspired by a construction of Turaev [11], and, among other things, it computes *self-intersections* of curves. The key is to compute $\mu(\log\gamma)$ for a simple loop γ and we find that it involves the Bernoulli numbers (Theorem 2). Here, we work with a suitable completion to be able to consider $\log\gamma$. In §3, we formalize the topological argument in §2 and prove the main results. In §4, we give another self-contained proof of Theorem 1 by introducing a certain generating function.

The Bernoulli numbers have already appeared in the study of intersections of *two curves* on an oriented surface [8]. Our formula provides yet another evidence for a close connection between the topology of surfaces and the Bernoulli numbers. This connection has been developed in [1] to an unexpected connection between the operation μ, or equivalently, the Turaev cobracket, and the Kashiwara-Vergne problem in the formulation by Alekseev-Torossian [2].

2. Self-intersection map and Bernoulli numbers

Let S be a compact connected oriented surface with $\partial S \neq \emptyset$. Fix a basepoint $* \in \partial S$ and set $\pi_1(S) := \pi_1(S, *)$. We denote by $\hat{\pi}(S)$ the set of free homotopy classes of oriented loops on S. For any $p \in S$, we denote by $|\ |\colon \pi_1(S, p) \to \hat{\pi}(S)$ the forgetful map of the basepoint.

We recall the operation $\mu\colon \mathbb{Q}\pi_1(S) \to \mathbb{Q}\pi_1(S) \otimes (\mathbb{Q}\hat{\pi}(S)/\mathbb{Q}\mathbf{1})$, which has been introduced in [6] inspired by a construction of Turaev [11]. Here, $\mathbf{1}$ is the class of a constant loop. Let $\gamma\colon [0,1] \to S$ be an immersed based loop. We arrange so that the pair of tangent vectors $(\dot{\gamma}(0), \dot{\gamma}(1))$ is a positive basis of the tangent space T_*S, and that the self-intersections of γ (except for the base point $*$) lie in the interior $\mathrm{Int}(S)$ and consist of transverse double points. Let Γ be the set of such double points of γ. For $p \in \Gamma$ we denote $\gamma^{-1}(p) = \{t_1^p, t_2^p\}$, so that $0 < t_1^p < t_2^p < 1$. We define

$$\mu(\gamma) := -\sum_{p\in\Gamma} \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p))\,(\gamma_{0t_1^p}\gamma_{t_2^p 1}) \otimes |\gamma_{t_1^p t_2^p}| \in \mathbb{Q}\pi_1(S) \otimes (\mathbb{Q}\hat{\pi}(S)/\mathbb{Q}\mathbf{1}).$$

Here,

- the sign $\varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p))$ is $+1$ if the pair $(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p))$ is a positive basis of T_pS, and is -1 otherwise,
- the based loop $\gamma_{0t_1^p}\gamma_{t_2^p 1}$ is the conjunction of the paths $\gamma|_{[0,t_1^p]}$ and $\gamma|_{[t_2^p,1]}$,
- the element $\gamma_{t_1^p t_2^p} \in \pi_1(S, p)$ is the restriction of γ to $[t_1^p, t_2^p]$ and we understand that $|\gamma_{t_1^p t_2^p}| = 0$ if the loop $\gamma_{t_1^p t_2^p}$ is homotopic to a constant loop.

REMARK 1. The operation μ is essentially the same as Turaev's operation $\mu^T : \pi_1(S) \to \mathbb{Q}\pi_1(S)$ in [11]. In fact, we have $\mu^T(\gamma)\gamma = -(\mathrm{id} \otimes \varepsilon)\mu(\gamma)$ for any $\gamma \in \pi_1(S)$, where $\varepsilon(\alpha) = 1$ for any $\alpha \in \hat{\pi}(S) \setminus \{\mathbf{1}\}$. Conversely, one can express μ in terms of μ^T. The alternating part of $(|\ | \otimes 1)\mu(\gamma)$ is exactly the Turaev cobracket [12] of the free loop $|\gamma|$.

We observe that if γ is simple and the pair $(\dot{\gamma}(0), \dot{\gamma}(1))$ is a positive basis of T_*S, then for any integer $k \in \mathbb{Z}$,

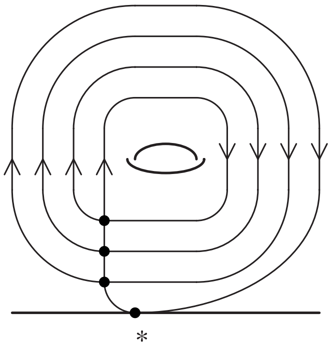

Fig. 1. computation of $\mu(\gamma^k)$ for a simple γ ($k = 4$).

(3)
$$\mu(\gamma^k) = \begin{cases} -\sum_{i=1}^{k-1} \gamma^i \otimes |\gamma^{k-i}| & (k > 0) \\ 0 & (k = 0) \\ \sum_{i=0}^{|k|-1} \gamma^{-i} \otimes |\gamma^{k+i}| & (k < 0). \end{cases}$$

See Fig. 1.

In [6] §4, it was shown that the map μ extends to a map between completions $\mu\colon \widehat{\mathbb{Q}\pi_1(S)} \to \widehat{\mathbb{Q}\pi_1(S)} \widehat{\otimes} \widehat{\mathbb{Q}\hat{\pi}(S)}$. Here $\widehat{\mathbb{Q}\pi_1(S)}$ and $\widehat{\mathbb{Q}\hat{\pi}(S)}$ are the completions of the group ring $\mathbb{Q}\pi_1(S)$ and the Goldman-Turaev Lie bialgebra $\mathbb{Q}\hat{\pi}(S)/\mathbb{Q}\mathbf{1}$, respectively, with respect to the augmentation ideal of $\mathbb{Q}\pi_1(S)$. Then we can consider $\log\gamma = \sum_{i=1}^{\infty}((-1)^{i+1}/i)(\gamma - 1)^i \in \widehat{\mathbb{Q}\pi_1(S)}$.

As the following result shows, if γ is simple then one can compute $\mu(\log\gamma)$ explicitly and the formula involves the Bernoulli numbers.

Theorem 2. *Let $\gamma \in \pi$ be represented by a simple loop, and assume that the pair $(\dot{\gamma}(0), \dot{\gamma}(1))$ is a positive basis of the tangent space T_*S. Then we have*

(4)
$$\mu(\log\gamma) = -\sum_{m=0} \frac{B_m}{m!} \sum_{p=0}^{m} (-1)^p \binom{m}{p} (\log\gamma)^p \widehat{\otimes} |(\log\gamma)^{m-p}|.$$

3. Proof of Theorem 1 and Theorem 2

First of all, we describe a preliminary construction.

Let $\mathbb{Q}[[Z]]$ (resp. $\mathbb{Q}[[X, Y]]$) be the commutative ring of formal power series in an indeterminate Z (resp. in indeterminates X and Y). For a non-negative integer p, let F_p^Z (resp. $F_p^{X,Y}$) be the set of formal power series in $\mathbb{Q}[[Z]]$ (resp. $\mathbb{Q}[[X, Y]]$) which has only terms of (total) degree $\geq p$. We have natural isomorphisms $\mathbb{Q}[[Z]] \cong \varprojlim_p \mathbb{Q}[[Z]]/F_p^Z$ and $\mathbb{Q}[[X, Y]] \cong \varprojlim_p \mathbb{Q}[[X, Y]]/F_p^{X,Y}$.

Set $z := e^Z = \sum_{i=0}^{\infty}(1/i!) Z^i$. Then the Laurent polynomial ring $\mathbb{Q}[z, z^{-1}]$ is a subring of $\mathbb{Q}[[Z]]$. The augmentation ideal I is defined by

$$I = \mathrm{Ker}(\mathbb{Q}[z, z^{-1}] \to \mathbb{Q}, \sum_j a_j z^j \mapsto \sum_j a_j).$$

Then I gives a filtration $\{I^p\}_p$ of $\mathbb{Q}[z, z^{-1}]$. By the inclusion map $\mathbb{Q}[z, z^{-1}] \hookrightarrow \mathbb{Q}[[Z]]$, the filtration $\{F_p^Z\}_p$ restricts to $\{I^p\}_p$. Moreover, we have a natural isomorphism $\mathbb{Q}[[Z]] \cong \varprojlim_p \mathbb{Q}[z, z^{-1}]/I^p$.

Motivated by the formula (3), we define a $\mathbb{Q}$-linear map $\hat{\mu}\colon \mathbb{Q}[z, z^{-1}] \to \mathbb{Q}[[X, Y]]$ by

$$\hat{\mu}(z^k) = \begin{cases} -\sum_{i=1}^{k} e^{iX} e^{(k-i)Y} & (k > 0) \\ 0 & (k = 0) \\ \sum_{i=0}^{|k|-1} e^{-iX} e^{(k+i)Y} & (k < 0). \end{cases} \tag{5}$$

From the definition of $\hat{\mu}$ it is easy to see that

$$(e^{-X} e^{Y} - 1)\hat{\mu}(z^k) = e^{kX} - e^{kY}, \quad k \in \mathbb{Z}.$$

Therefore, we have

$$(e^{-X} e^{Y} - 1)\hat{\mu}(f(z)) = f(e^X) - f(e^Y) \tag{6}$$

for any Laurent polynomial $f(z) \in \mathbb{Q}[z, z^{-1}]$. Consider

$$\Phi(X, Y) := \sum_{i=0}^{\infty} \frac{B_i}{i!}(-X + Y)^i.$$

Then we have $(e^{-X} e^{Y} - 1)\Phi(X, Y) = -X + Y$. Multiplying $\Phi(X, Y)$ to both sides of (6), we have

$$(-X + Y)\hat{\mu}(f(z)) = (f(e^X) - f(e^Y))\Phi(X, Y) \tag{7}$$

for any $f(z) \in \mathbb{Q}[z, z^{-1}]$.

Lemma 1. *There is a unique continuous extension* $\hat{\mu}\colon \mathbb{Q}[[Z]] \to \mathbb{Q}[[X, Y]]$ *of the map* $\hat{\mu}$ *in* (5).

Proof. It is sufficient to prove that $\hat{\mu}(I^p) \subset F^{X,Y}_{p-1}$ for any $p \geq 1$. Suppose $f(z) \in I^p$. Then $f(e^X)$ and $f(e^Y)$ lie in $F^{X,Y}_p$. This means that the right hand side of (7) is an element of $F^{X,Y}_p$. Therefore, $\hat{\mu}(f(z)) \in F^{X,Y}_{p-1}$. □

Now for each $k \geq 1$ we can put $f(z) = (\log z)^k = Z^k$ in (7), and we obtain

$$(-X + Y)\hat{\mu}(Z^k) = (X^k - Y^k)\Phi(X, Y).$$

This shows that $\hat{\mu}(Z^k) \in F^{X,Y}_{k-1}$. Setting $k = 1$, we have

$$\hat{\mu}(Z) = -\Phi(X, Y) = -\sum_{i=0}^{\infty} \frac{B_i}{i!} \sum_{j=0}^{i} (-1)^j \binom{i}{j} X^j Y^{i-j}. \tag{8}$$

This formula is essentially the same as the assertion of Theorem 2:

Proof of Theorem 2. We identify the ring $\mathbb{Q}[[X, Y]]$ with the complete tensor product $\mathbb{Q}[[Z]]\widehat{\otimes}\mathbb{Q}[[Z]]$ by the map $X \mapsto Z\widehat{\otimes}1$ and $Y \mapsto 1\widehat{\otimes}Z$. Then the computation (8) implies

$$\hat{\mu}(\log z) = -\sum_{m=0}^{\infty} \frac{B_m}{m!} \sum_{p=0}^{m} (-1)^p \binom{m}{p} (\log z)^p \,\widehat{\otimes}\, (\log z)^{m-p}. \tag{9}$$

From (3) and (5) it follows that the substitution $z \mapsto \gamma$ commutes with μ and $\hat{\mu}$. Thus we obtain (4). □

Further, by expanding the left hand side of (8) in terms of $\hat{\mu}(z^k)$'s modulo higher degree terms, we have the following:

Proposition 1. *Let m, n, a be integers satisfying $0 \leq a \leq m \leq n$. Then it holds that*

$$B_m = (-1)^a \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n+1}{k} \left[\sum_{i=1}^{k-1} i^a (k-i)^{m-a} + \delta_{a,m} k^m \right].$$

Here $\delta_{a,m}$ is the Kronecker delta.

Proof. In what follows, $\equiv$ means an equality in $\mathbb{Q}[[X, Y]]$ modulo $F_{n+1}^{X,Y}$. For $k = 1, \ldots, n+1$, we have

$$\hat{\mu}(z^k) = \hat{\mu}(e^{kZ}) = \sum_{i=1}^{\infty} \frac{k^i}{i!} \hat{\mu}(Z^i) \equiv \sum_{i=1}^{n+1} \frac{k^i}{i!} \hat{\mu}(Z^i). \tag{10}$$

Consider the square matrix $D = (D_{ki})_{k,i}$ of order $n+1$, where $D_{ki} = k^i/i!$. Then D is invertible since $\det D$ is a non-zero multiple of Vandermonde's determinant $\det(k^{i-1})_{k,i}$. The inverse matrix of D has the first row $(a_1, \ldots, a_{n+1})$, where

$$a_k = \frac{(-1)^{k+1}}{k} \binom{n+1}{k}.$$

(To see this, for instance, one can use Lemma 2 below to get $(a_1, \ldots, a_{n+1})D = (1, \ldots, 0)$.) From (10) we have

$$\hat{\mu}(Z) \equiv \sum_{k=1}^{n+1} a_k \hat{\mu}(z^k) = \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n+1}{k} \hat{\mu}(z^k). \tag{11}$$

Furthermore, for $k = 1, \ldots, n+1$, from (5) we have

$$\hat{\mu}(z^k) = -\sum_{i=1}^{k-1} \sum_{a,b=0}^{\infty} \frac{i^a (k-i)^b}{a!b!} X^a Y^b - \sum_{a=0}^{\infty} \frac{k^a}{a!} X^a. \tag{12}$$

By (11) and (12), the coefficient of $X^a Y^{m-a}$ in $\hat{\mu}(Z)$ is

$$\sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} \left[\sum_{i=1}^{k-1} \frac{i^a (k-i)^{m-a}}{a!(m-a)!} + \delta_{m,a} \frac{k^m}{m!} \right].$$

On the other hand, by (8), this coincides with

$$(-1)^{a+1} \frac{B_m}{m!} \binom{m}{a} = \frac{(-1)^{a+1}}{a!(m-a)!} B_m.$$

This completes the proof. □

Now, we can derive Theorem 1 from Proposition 1 by applying the following lemma. Although it might be well known, we give its proof for the sake of completeness.

Lemma 2. *Let m, n be integers satisfying $0 \leq m \leq n$. Then it holds that*

$$\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} k^m = \begin{cases} 0 & \text{if } m \geq 1, \\ -1 & \text{if } m = 0. \end{cases}$$

Proof. Set $f(x) := (e^x - 1)^{n+1}$. Since $m \leq n$, the coefficient of x^m in the series expansion of $f(x)$ is zero.

On the other hand, we compute

$$\begin{aligned} f(x) &= \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} e^{kx} \\ &= (-1)^{n+1} \left[\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} e^{kx} + 1 \right] \\ &= (-1)^{n+1} \left[\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \sum_{a=0}^{\infty} \frac{k^a}{a!} x^a + 1 \right]. \end{aligned}$$

Since the coefficient of x^m in the last expression is equal to

$$\begin{cases} \dfrac{(-1)^{n+1}}{m!} \displaystyle\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} k^m & \text{if } m \geq 1, \\ (-1)^{n+1} \left[\displaystyle\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} + 1 \right] & \text{if } m = 0, \end{cases}$$

the assertion follows. □

4. Another proof of Theorem 1

Introducing a generating function of two variables, we give another self-contained proof of Theorem 1. Since we have Lemma 2, it is sufficient to prove Proposition 1.

Let $f(x, y)$ and $g(x, y)$ be functions in variables x and y defined by

$$f(x, y) := \int_x^y (e^t - 1)^{n+1} dt, \quad \text{and} \quad g(x, y) := \frac{f(x, y)}{e^{y-x} - 1}.$$

We will examine the coefficient of $x^a y^{m-a}$ in the series expansion of $g(x, y)$.

First we compute $f(x, y)$ as follows:

$$\begin{aligned} f(x, y) &= \int_x^y (e^t - 1)^{n+1} dt \\ &= \int_x^y \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} e^{kt} dt \\ &= (-1)^{n+1} \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} (e^{ky} - e^{kx}) + (-1)^{n+1}(y - x). \end{aligned}$$

Since

$$\frac{e^{ky} - e^{kx}}{e^{y-x} - 1} = \frac{e^{kx}(e^{k(y-x)} - 1)}{e^{y-x} - 1} = \sum_{i=1}^{k-1} e^{ix} e^{(k-i)y} + e^{kx},$$

we can compute $g(x, y)$ as follows:

$$\begin{aligned}
g(x,y) =& \frac{f(x,y)}{e^{y-x}-1} \\
=& (-1)^{n+1} \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} \frac{(e^{ky}-e^{kx})}{e^{y-x}-1} + (-1)^{n+1} \frac{y-x}{e^{y-x}-1} \\
=& (-1)^{n+1} \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} \left[\sum_{i=1}^{k-1} e^{ix} e^{(k-i)y} + e^{kx} \right] \\
&+ (-1)^{n+1} \sum_{b=0}^{\infty} \frac{B_b}{b!} (y-x)^b.
\end{aligned}$$

Then using the identities:

$$e^{ix} e^{(k-i)y} = \sum_{b,c=0}^{\infty} \frac{i^b (k-i)^c}{b! c!} x^b y^c \quad \text{and} \quad e^{kx} = \sum_{b=0}^{\infty} \frac{k^b}{b!} x^b,$$

we see that the coefficient of $x^a y^{m-a}$ in $g(x, y)$ is given by

$$\begin{aligned}
&(-1)^{n+1} \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} \left[\sum_{i=1}^{k-1} \frac{i^a}{a!} \frac{(k-i)^{m-a}}{(m-a)!} + \delta_{a,m} \frac{k^m}{m!} \right] \\
&\quad + (-1)^{n+1+a} \frac{B_m}{m!} \binom{m}{a}.
\end{aligned}$$

This is equal to $((-1)^{n+1+a}/m!)\binom{m}{a}$ times

$$(-1)^a \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} \left[\sum_{i=1}^{k-1} i^a (k-i)^{m-a} + \delta_{a,m} k^m \right] + B_m. \tag{13}$$

Secondly, we expand $g(x, y)$ in a different way. Put $g_1(x, y) = f(x, y)/(y - x)$. Then we have

$$g(x,y) = \frac{f(x,y)}{y-x} \frac{y-x}{e^{y-x}-1} = g_1(x,y) \sum_{b=0}^{\infty} \frac{B_b}{b!} (y-x)^b.$$

Writing $(e^t - 1)^{n+1} = \sum_{i \geq n+1} a_i t^i$, we have

$$f(x,y) = \int_x^y (e^t-1)^{n+1} dt = \sum_{i \geq n+1} \frac{a_i}{i+1} (y^{i+1} - x^{i+1}).$$

Thus the series expansion of $g_1(x, y)$ has all terms of degree $\geq n + 1$, so does that of $g(x, y)$. In particular, the coefficient of $x^a y^{m-a}$ in this expansion is zero. Therefore, the expression (13) is zero, and we obtain Proposition 1. □

References

[1] A. Alekseev, N. Kawazumi, Y. Kuno and F. Naef: *Higher genus Kashiwara-Vergne problems and the Goldman-Turaev Lie bialgebra,* C.R. Math. Acad. Sci. Paris, **355** (2017), 123–127.
[2] A. Alekseev and C. Torossian: *The Kashiwara-Vergne conjecture and Drinfeld's associators,* Ann. of Math. **175** (2012), 415–463.
[3] K. Dilcher, L. Skula and I. Slavutskii: Bernoulli Numbers Bibliography (1713–1990), Queen's Papers in Pure and Applied Mathematics, Kingston, Ontario, 1987.
[4] H. Gould: *Explicit formulas for Bernoulli numbers,* Amer. Math. Monthly **79** (1972), 44–51.
[5] J. Higgins: *Double series for the Bernoulli and Euler numbers,* J. London Math. Soc. (2) **2** (1970), 722–726.
[6] N. Kawazumi and Y. Kuno: *Intersections of curves on surfaces and their applications to mapping class groups,* Ann. Inst. Fourier (Grenoble) **65** (2015), 2711–2762.
[7] L. Kronecker: *Ueber die Bernouillischen Zahlen,* J. Reine Angew. Math. **94** (1883), 268–269.
[8] G. Massuyeau and V. Turaev: *Fox pairings and generalized Dehn twists,* Ann. Inst. Fourier (Grenoble) **63** (2013), 2403–2456.
[9] N. Nielsen: Traité élémentaire des nombres de Bernoulli, Gauthier-Villars et Cie, Paris, 1923.
[10] L. Saalschütz: Vorlesungen über die Bernoullischen Zahlen, ihren Zusammenhang mit den Secanten-Coefficienten und ihre wichtigeren Anwendungen, Julius Springer, Berlin, 1893.
[11] V. Turaev: *Intersections of loops in two-dimensional manifolds,* Math. USSR Sbornik **35** (1979), 229–250.
[12] V. Turaev: *Skein quantization of Poisson algebras of loops on surfaces,* Ann. sci. École Norm. Sup. (4) **24** (1991), 635–704.

Shinji Fukuhara
Department of Mathematics
Tsuda University
2–1–1 Tsuda–machi, Kodaira-shi Tokyo 187–8577
Japan
e-mail: fukuhara@tsuda.ac.jp

Nariya Kawazumi
Department of Mathematical Sciences
University of Tokyo
3–8–1 Komaba, Meguro-ku Tokyo 153–8914
Japan
e-mail: kawazumi@ms.u-tokyo.ac.jp

Yusuke Kuno
Department of Mathematics
Tsuda University
2–1–1 Tsuda-machi, Kodaira-shi Tokyo 187–8577
Japan
e-mail: kunotti@tsuda.ac.jp